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Similarity solutions for unsteady flow behind an exponential shock in an axisymmetric rotating non-ideal gas

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Abstract One-dimensional self-similar unsteady isothermal and adiabatic flows behind a strong exponential shock wave driven out by a cylindrical piston moving with time according to an exponential law in a rotational axisymmetric non-ideal gas is investigated. The medium is assumed to be non-ideal gas rotating about the axis of symmetry. The fluid velocities in the ambient medium are assumed to be varying with time according to an exponential law. Similarity solutions exist only when the surrounding medium is of constant density. Solutions are obtained, in both the cases, when the flow between the shock and the piston is isothermal or adiabatic by taking into account components of vorticity vector. It is found that the assumption of zero temperature gradient brings a profound change in the density and compressibility distributions as compared to that of the adiabatic case. The effect of an increase in the value of the parameter of the non-idealness of the gas is investigated. Also, a comparison between the solutions in the cases of isothermal and adiabatic flows is made. Further, it is shown that the consideration of zero temperature gradient and the effect of variation of the parameter of non-idealness of the gas decrease the shock strength and widens the disturbed region between the shock and piston. The shock waves in non-ideal gas can be important for description of shocks in supernova explosions, in the study of a flare produced shock in solar wind, central part of star burst galaxies, nuclear explosion, rupture of pressurized vessel, in the analysis of data from exploding wire experiments, and cylindrically symmetric hypersonic flow problems associated with meteors or reentry vehicles, etc. The findings of the present work provided a clear picture of whether and how the non-idealness of the gas and consideration of zero temperature gradient affect the propagation of shock and the flow behind it.

Keywords Similarity solutions · Shock wave · Mechanics of fluid · Rotating medium · Interstellar medium · Non-ideal gas · Isothermal and adiabatic flows

1 Introduction

Perturbation of different kinds may produce discontinuities in astrophysical fluid flow. Discontinuities in fluid flow are said to take place over one or more surfaces when any dynamical and/or thermodynamic quantity changes discontinuously as such surfaces are crossed; the corresponding surfaces are called surfaces of discontinuity. Certain boundary conditions are to satisfied across such surfaces, and according to those conditions, surfaces of discontinuities are classified

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into various categories, the most important being the shock or shocks. Shock waves are generated in various kinds of supersonic astrophysical flows having intrinsic angular momentum, resulting in a flow that becomes subsonic. This is because the repulsive centrifugal potential barrier experience by such flows can be sufficiently strong to brake the motion, and a stationary solution could be introduced only through a shock. Rotating, transonic astrophysical fluid flows are thus believed to be "prone" to shock formation phenomena. It has been established in recent years that in order to satisfy the inner boundary conditions imposed by the event horizon, accretion onto black hole should exhibit transonic properties in general, which further indicates that the formation of shock waves is possible in astrophysical fluid flows onto galactic and extragalactic black holes. It might also be expected that shock formation in black hole accretion disks is a general phenomenon, because shock waves in rotating astrophysical flows are convincingly able to provide an important and efficient mechanism for converting a significant amount of the gravitational energy (available from deep potential wells created by these massive compact accretors) into radiation by randomizing the directed in fall motion of the accreting fluid. Hence shocks possibly play an important role in governing the overall dynamical and radiative processes taking place in astrophysical fluids and plasmas accreting onto black holes (Das [1] and Das et al. [2]).

Shock processes can naturally occur in various astrophysical situations for example, photo-ionized gas, stellar winds, supernova explosions, collisions between high velocity clumps of interstellar gas etc. Shock phenomena such as a global shock resulting from a stellar pulsation or supernova explosion passing outward through a stellar envelope or perhaps a shock emanating from a point source such as a man-made explosion in the Earth's atmosphere or an impulsive flare in the Sun's atmosphere, have tremendous importance in astrophysics and space sciences. Shocks are ubiquitous throughout the observed universe and are thought to play a crucial role in the transportation of energy into the interstellar medium, setting in motion processes observed in nebulae that eventually could lead to the creation of new stars. Shock waves are common in the interstellar medium because of a great variety of supersonic motions and energetic events, such as cloud–cloud collision, bipolar outflow from young proto-stellar objects, powerful mass losses by massive stars in a late stage of their evolution (stellar winds), supernova explosions, central part of star burst galaxies, etc. Shock waves are also associated with spiral density waves, radio galaxies and quasars. Similar phenomena also occur in laboratory situations, for example, when a piston is driven rapidly into a tube of gas (a shock tube), when a projectile or aircraft moves supersonically through the atmosphere, in the blast wave produced by a strong explosion, or when rapidly owing gas encounters a constriction in a flow channel or runs into a wall.

Self-propagating star formation theory uses the shock waves from supernova explosions to shape the spiral pattern. When a supernova shock wave reaches a gas cloud, it compresses the cloud to stimulate the formation of stars. Some of them will be massive enough to produce their own supernova explosions to keep the cycle going. Coupled with the differential rotation of the disk, the shock wave will keep the spiral arms visible.

Thus the study of shock waves in rotating transonic and supersonic astrophysical fluid flows and black hole accretion has acquired a very important status in recent years. The formation of self-similar problems and examples describing the adiabatic motion of non-rotating gas mode of stars are considered by Sedov [3], Zel'dovich and Raizer [4], Lee and Chen [5] and Summers [6]. The experimental studies and astrophysical observations show that the outer atmosphere of the planets and stars rotates due to rotation of the planets and stars. Macroscopic motion with super-sonic speed occurs in an interplanetary atmosphere and shock waves are generated. Shock waves often arise in nature because of a balance between wave breaking non-linear and wave damping dissipative forces (Zel'dovich and Raizer [4]). Thus the rotation of planets and stars significantly affects the process taking place in their outer layers; therefore question connected with the explosions in rotating gas atmospheres are of definite astrophysical interest. Chaturani [7] studied the propagation of cylindrical shock wave through a gas having solid body rotation, and obtained the solutions by a similarity method adopted by Sakurai [8]. Nath et al. [9] obtained the similarity solutions for the flow behind spherical shock waves propagating in a non-uniform rotating interplanetary atmosphere with increasing energy. Vishwakarma and Nath [10] obtained the similarity solution for the propagation of a cylindrical shock wave in a rotating dust gas with heat conduction and radiation heat flux. Nath [11] obtained the non-similarity solution for the propagation of a strong cylindrical shock wave in a rotational axisymmetric dusty gas with exponentially varying density in both the cases when the flow behind the shock waves was isothermal or adiabatic. He assumed variable azimuthal and axial fluid velocity, and exponential time dependence for the velocity of the shock.

Sedov [3] (see, Ranga Rao and Ramana [12], Vishwakarma and Nath [13, 14]) indicated that a limiting case of a self-similar flow-field with a power law shock is the flow-field formed with an exponential shock. Ranga Rao and Ramana [12] obtained approximate analytic solutions for the problem of unsteady self-similar motion of a perfect gas displaced by a piston according to an exponential law. Vishwakarma and Nath [13, 14] obtained the similarity solutions for the problem of unsteady self-similar motion of non-ideal gas or dusty gas (a mixture of perfect gas and small solid particles) behind a strong shock driven out by a cylindrical (or spherical) piston moving with time according to an exponential law.

At extreme conditions that prevail in most of the problems associated with shock waves, the assumption that the gas is ideal no longer valid when the flow takes at high temperature. Anisimov and Spiner [15] have taken an equation of state for non-ideal gases in simplified form, and investigated the effect of the parameter for non-idealness on the problem of strong point explosion, which describes the behavior of the medium satisfactory at low densities. Ranga Rao and Purohit [16] have studied the self-similar flow of a non-ideal gas driven by an expanding piston and obtained solutions by taking the equation of state suggested by Anisimov and Spiner [15]. Wu and Roberts [17] and Roberts and Wu [18] discussed the shock wave theory of sonoluminesence by taking a similar equation of state of the medium.

In the present work, we generalize the solution of Ranga Rao and Ramana [12] in gas (i.e. the solution of Vishwakarma and Nath [14] in non-ideal gas) to the case of rotational axisymmetric non-ideal gas, which has a variable azimuthal and axial fluid velocities (Nath [11], Levin and Skopina [19]). Here, we therefore investigate the one-dimensional unsteady self- similar flow of a rotational axisymmetric non-ideal gas behind a strong shock driven out by cylindrical piston moving with time according to an exponential law. The equation of state of non-ideal gas is taken in the form as suggested by Anisimov and Spiner [15]. The motion of piston is assumed to obey the law (Vishwakarma and Nath [13, 14], Ranga Rao and Ramana [12]), namely,

$$r_p = B\ \exp(\lambda t),\ \ \lambda > 0, \tag{1}$$

where r_p is the radius of the piston, B and λ are dimensional constants, t is the time and 'B' represents the initial radius of the piston. It may be, physically, the radius of the stellar corona or the condensed explosive or the diaphragm containing a very high-pressure driver gas, at $t = 0$. By sudden expansion of the stellar corona or the detonation products or the driver gas into the undisturbed ambient gas, a shock wave is produced in the ambient gas. The shocked gas is separated from the expanding surface which is a contact discontinuity. This contact surface acts as a 'piston' for the shock wave in the ambient medium (Rosenau and Frankenthal [20], Higashino [21], Liberman and Velikovich [22]).

The law of motion of the piston Eq. (1) implies a boundary condition on the gas speed at the piston, which is required in the determination of the problem. Since we are concerned with self-similar motions, we may postulate that

$$R = C\ \exp(\lambda t), \tag{2}$$

where R is the shock radius, and 'C' *is* a dimensional constant which depends on the constant 'B' and the non-dimensional position of the piston [see Eq. (33)]. As is often the case in problems of this type, it is more convenient to solve for the piston motion in terms of the shock motion, rather than vice versa. We shall therefore adopt this point of view forthwith, and consider 'C' a known parameter of the problem, rather than B (Vishwakarma and Nath [14], Rosenau and Frankenthal [20]).

Due to high temperature in the flow, intense radiation heat transfer take place behind a strong shock. For such flows the assumption of adiabaticity may not be valid. Therefore, an alternative assumption of zero temperature gradient throughout the flow (flows which satisfy this condition are also known as isothermal flow) may approximately be taken (as in Laumbach and Probstin [23], Sachdev and Ashraf [24], Korobeinikov [25], Gretler and Regenfelder [26], Vishwakarma and Nath [13, 14, 27], Nath [11, 28, 29]). With this assumption, we therefore obtain the similarity solutions in Sects. 2 and 3, the similarity solutions of the problem treated by Vishwakarma and Nath [14]. In Sect. 4, we present the solutions for the flow taken to be adiabatic. Solutions are obtained, in both the cases, when the flow between the shock and the piston is isothermal or adiabatic by taking into account the components of vorticity vector. In order to obtain the similarity solutions of the problem it is necessary to take the density of the ambient non-ideal gas to be a constant. The effect of an increase in the value of the parameter of the non-idealness of the gas is investigated. It is shown that the assumption of zero temperature gradient brings a profound change in the density and compressibility distribution as compared to that of the adiabatic case. Also, a comparison between the solutions in the case of isothermal and adiabatic flows is made. Further, it is shown that the consideration of zero temperature gradient and the effect of variation of the parameter of non-idealness of the gas decreases the shock strength and widens the disturbed region between the shock and piston. Effects of viscosity, magnetic field and gravitation are not taken into account.

2 Fundamental equations and boundary conditions-isothermal flow

The fundamental equations for one-dimensional, unsteady and cylindrically symmetric isothermal flow of non-ideal gas, which is rotating about the axis of symmetry, can be written as (c.f. Chaturani [7], Levin and Skopina [19], Vishwakarma and Nath [13, 14, 27], Laumbach and Probstein [23], Korobeinikov [25], Zhuravskaya and Levin [30], Nath [11, 28, 29])

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{u\rho}{r} = 0, \tag{3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{v^2}{r} = 0, \tag{4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r} = 0, \tag{5}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} = 0, \tag{6}$$

$$\frac{\partial T}{\partial r} = 0, \tag{7}$$

where p, ρ and T are the pressure, the density and the temperature, u; v and w are the radial, azimuthal and axial components of the fluid velocity $\vec{q}$ in the cylindrical coordinates (r, θ, z); r and t are the distance and time respectively.

The above system of equations should be supplemented with an equation of state. In most of the cases the propagation of shock waves arises in extreme conditions under which the assumption that the gas is ideal is not a sufficient accurate description. To discover how deviations from the ideal gas can affect the solutions, we adopt a simple model. We assume that the gas obey a simplified van der Waals equation of state of the form (Wu and Robert [17], Robert and Wu [18], Nath [32])

$$p = \frac{\Gamma T}{(v-b)}; \quad e = C_v T = \frac{p(v-b)}{(\gamma-1)} \tag{8}$$

where Γ is the gas constant, $v = \frac{1}{\rho}$ is the specific volume, $C_v = \frac{\Gamma}{(\gamma-1)}$ is the specific heat at constant volume, e is the internal energy per unit mass of the non-ideal gas and γ is the ratio of the specific heats, b is the van der Waals excluded volume, it places a limit, $\rho_{\max} = \frac{1}{b}$, on the density of the gas, and b is in general, a function of temperature T, but at high temperature range it tends to a constant value equal to the internal volume of the gas molecules which lies between 0.9×10^{-3} and 1.1×10^{-3} m^3/kg (Anisimov and Spiner [15], Wu and Roberts [17], Roberts and Wu [18], Landau and Lifshitz [33]). Real gas effects can be expressed in the fundamental equations according to Vishwakarma and Nath [14], Chandrasekhar [34] by two thermodynamic variables, namely by the sound velocity factor (the isotropic exponent) Γ^* and a factor K, which contains internal energy as follows:

$$\Gamma^* = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_S \text{ and } K = -\frac{\rho}{p}\left(\frac{\partial e}{\partial \ln \rho}\right)_P \tag{9}$$

where the subscript 'S' and 'P' refers to the process of constant entropy and pressure respectively. Using the first law of thermodynamics and Eq. (8), we obtain

$$\Gamma^* = \frac{\gamma}{(1-b\rho)} \text{ and } K = \frac{1}{(\gamma-1)} \tag{10}$$

This shows that the isentropic exponent Γ^* is non-constant in the shocked gas, but the factor K is constant for the simplified equation of state of the non-ideal gas in the form (8).

The isentropic speed of sound 'a' is given by

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_S^{\frac{1}{2}} = \Gamma^* \frac{p}{\rho}. \tag{11}$$

The deviation of the behavior of non-ideal gas from that of a perfect gas is indicated in Eq. (4) by the isothermal compressibility

$$\tau_{iso} = \frac{1}{a_{iso}^2 \rho}, \tag{12}$$

where $a_{iso} = \left(\frac{\partial p}{\partial \rho}\right)_T^{\frac{1}{2}} = \left[\frac{1-b\rho}{p}\right]^{\frac{1}{2}}$ expressing the isothermal sound speed and the subscript 'T' refers to the process of constant temperature.

Also,

$$v = Ar, \tag{13}$$

where 'A' is the angular velocity of the medium at radial distance r from the axis of symmetry. In this case the vorticity vector $\vec{\varsigma} = \frac{1}{2} Curl\, \vec{q}$ has the components (c.f. Levin and Skopina [19], Nath [28, 29, 31])

$$\varsigma_r = 0, \quad \varsigma_\theta = -\frac{1}{2}\frac{\partial w}{\partial r}, \quad \varsigma_z = \frac{1}{2r}\frac{\partial}{\partial r}(rv) \tag{14}$$

In order to obtain the solution, it is assumed that a strong cylindrical shock wave is propagating outwards from the axis of symmetry in the undisturbed medium (non-ideal gas) with constant density, which has zero radial velocity, and variable azimuthal and axial velocities. The flow variables immediately ahead of the shock front are

$$u = 0, \tag{15}$$

$$\rho = \rho_a = \text{constant}, \tag{16}$$

$$v_a = C^* \exp(\delta t), \tag{17}$$

$$w_a = E \exp(\alpha t), \tag{18}$$

where C^*, E, δ and α are dimensional constants and the subscript 'a' refer to the values in the initial state.

Ahead of the shock, the components of the vorticity vector, therefore, vary as

$$\varsigma_{r_a} = 0 \tag{19}$$

$$\varsigma_{\theta_a} = -\frac{E\alpha}{2\lambda R}\exp(\alpha t), \tag{20}$$

$$\varsigma_{z_a} = \frac{C^*(\lambda + \delta)}{2\lambda R}\exp(\delta t). \tag{21}$$

The initial angular velocity of the medium at radial distance R is given by, from Eq. (13)

$$A_a = \frac{v_a}{R}. \tag{22}$$

From Eqs. (22) and (17), we find that the initial angular velocity vary as

$$A_a = \frac{C^* \exp(\delta t)}{R}. \tag{23}$$

The law of conservation of mass, momentum and energy across the shock front propagating with velocity $V\left(=\frac{dR}{dt}\right)$ into the non-ideal gas give the following shock conditions

$$\begin{aligned}
&\rho_n(V - u_n) = \rho_a V = m_s(\text{say}),\\
&p_n + \rho_n(V - u_n)^2 = p_a + \rho_a V^2,\\
&e_n + \frac{p_n}{\rho_n} + \frac{1}{2}(V - u_n)^2 - \frac{F_n}{m_s} = e_a + \frac{p_a}{\rho_a} + \frac{1}{2}V^2 - \frac{F_a}{m_s},\\
&v_n = v_a,\\
&w_n = w_a
\end{aligned} \tag{24}$$

where the subscript 'n' denotes the conditions immediately behind the shock front, and 'F' is the radiation heat flux. The pressure ahead of a strong shock is very small in comparison to the pressure behind of the shock, and therefore it is neglected (Zel'dovich and Raizer [4])

$$p_a \approx 0, \quad e_a \approx 0. \tag{25}$$

Then the shock conditions (24) across a strong shock propagating into a rotating non-ideal gas reduce to

$$\begin{aligned}
&u_n = (1 - \beta)V,\\
&\rho_n = \frac{\rho_a}{\beta},\\
&p_n = (1 - \beta)\rho_a V^2,\\
&v_n = C^* \exp(\delta t),\\
&w_n = E \exp(\alpha t),
\end{aligned} \tag{26}$$

where the density ratio $\beta (0 < \beta < 1)$across the shock is given by the relation

$$\frac{(\beta-\bar{b})}{(\gamma-1)}+\beta-\frac{1}{2}(1+\beta)=\frac{(F_n-F_a)}{Vp_n}, \tag{27}$$

where $\bar{b}=b\rho_a$. As the shock is strong, we assume that (F_n-F_a) to be negligible in comparison with the product of p_n and V (Laumbach and Probstin [23], Vishwakarma and Nath [13, 14, 27], Nath [11, 29, 32]). Therefore, Eq. (27) reduces to

$$\beta=\frac{\gamma-1+2\bar{b}}{(\gamma+1)}. \tag{28}$$

Following Levin and Skopina [19] (see also, Nath [11, 29, 31]), we obtain the jump conditions for the components of vorticity vector across the shock front as

$$\varsigma_{\theta_n}=\frac{\varsigma_{\theta_a}}{\beta}, \tag{29}$$

$$\varsigma_{z_n}=\frac{\varsigma_{z_a}}{\beta}. \tag{30}$$

Equation (7) together with Eq. (8) give

$$\frac{p}{p_n}=\frac{\rho(1-b\rho_n)}{\rho_n(1-b\rho)}. \tag{31}$$

3 Similarity solutions

Zel'dovich and Raizer [4] shown that the gas dynamic equations admit similarity transformations, that there are possible different flows similar to each other which are derivable from each other by changing the basic scales of length, time, and density. The motion itself may be described by the most general functions of the two variables r and t, $\rho(r,t)$, $p(r,t)$, $u(r,t)$, $v(r,t)$ and $w(r,t)$. These functions also contain the parameters entering the initial and boundary conditions of the problem (and specific heat ratio γ).

However, there exist motions whose distinguishing property is the similarity in the motion itself. These motions are called self-similar (Sedov [3], Zel'dovich and Raizer [4]). The distribution as a function of position of any of the flow variables, such as the pressure p, evolves with time in a self-similar motion in such a manner that only the scale of the pressure $\Pi(t)$ and the length scale $R(t)$ of the region included in the motion change, but the shape of the pressure distribution remains unaltered. The $p(r)$ curves corresponding to different times t can be made the same by either expanding or contracting the Π and the R scales. The function $p(r,\ t)$ can be written in the form

$$p(r,t)=\Pi(t)P\left(\frac{r}{R}\right),$$

where the dimensional scales Π and R depend on time in some manner, and the dimensionless ratio $\frac{p}{\Pi}=P\left(\frac{r}{R}\right)$ is a "universal" (in the sense that it is independent of time) function of the new dimensionless coordinates $\eta=\frac{r}{R}$. Multiplying the variables $P\left(\frac{r}{R}\right)$ and η by the scale function $\Pi(t)$ and $R(t)$, we can obtain from the universal function $P(\eta)$ the true pressure distribution curve $p(r)$ as a function of position for any time t. The other flow variables, density and component of velocity are expressed similarly.

For self-similar motions of the system of partial differential Eqs. (3)–(6) of gas dynamics reduces to a system of ordinary differential equations in new unknown functions of the similarity variable $\eta=\frac{r}{R}$. Let us derive these equations. To do this we represent the solution of the partial differential Eqs. (3)–(6) in terms of products of scale functions and the new unknown functions of the similarity variable η,

$$\eta=\frac{r}{R},\quad R=R(t)$$

The pressure, density, velocity, and length scales are not all independent of each other. If we choose R and ρ_a as the basic scales, then the quantity $\frac{dR}{dt}\equiv V$ can serve as the velocity scale, $\rho_a V^2$ as the pressure scale. This does not limit the generality of the solution, as scale is only defined to within a numerical coefficient which can always be included in the new unknown function. We seek a solution of the form (Vishwakarma and Nath [13, 14])

$$u=VU(\eta),v=V\phi(\eta),w=VW(\eta),\rho=\rho_a D(\eta),p=\rho_a V^2 P(\eta) \tag{32}$$

where U, ϕ, W, D and P are the function of the non-dimensional variable (similarity variable) $\eta=\frac{r}{R}$ only. The variable η assumes the value '1' at the shock front and η_p on the piston. Equations (1), (2) and (32) yields a relation between B and C in the form

$$C=\frac{B}{\eta_p}. \tag{33}$$

Equation (31) with the aid of Eqs. (32) and (26) yields a relation between P and D in the form

$$P(\eta) = \frac{(\beta - \bar{b})(1-\beta)D(\eta)}{(1 - \bar{b}D(\eta))}. \tag{34}$$

Using Eqs. (32) and (34), in fundamental Eqs. (3)–(6), we obtain

$$(U-\eta)\frac{dD}{d\eta} + D\frac{dU}{d\eta} + \frac{DU}{\eta} = 0, \tag{35}$$

$$(U-\eta)\frac{dU}{d\eta} + \frac{(1-\beta)(\beta-\bar{b})}{D(1-\bar{b}D)^2}\frac{dD}{d\eta} + U - \frac{\phi^2}{\eta} = 0, \tag{36}$$

$$(U-\eta)\frac{d\phi}{d\eta} + \frac{U(\phi+\eta)}{\eta} = 0, \tag{37}$$

$$(U-\eta)\frac{dW}{d\eta} + W = 0. \tag{38}$$

From Eqs. (35–38), we obtain

$$\frac{dU}{d\eta} = L, \tag{39}$$

$$\frac{dD}{d\eta} = -\frac{D}{(U-\eta)}\left[\frac{U}{\eta} + L\right], \tag{40}$$

$$\frac{d\phi}{d\eta} = -\frac{\phi(U+\eta)}{\eta(U-\eta)}, \tag{41}$$

$$\frac{dW}{d\eta} = -\frac{W}{(U-\eta)}. \tag{42}$$

where

$$L = L(\eta) = \frac{[(1-\bar{b}D)^2(U-\eta)(\phi^2 - U\eta) + (1-\beta)(\beta-\bar{b})U]}{[(1-\bar{b}D)^2(U-\eta)^2 - (1-\beta)(\beta-\bar{b})]\eta}.$$

Applying the similarity transformations (32) on Eq. (14), we obtain the non-dimensional components of the vorticity vector $l_r = \frac{\zeta_r}{V/R}$, $l_\theta = \frac{\zeta_\theta}{V/R}$, $l_z = \frac{\zeta_z}{V/R}$, in the flow-filed behind the shock as

$$l_r = 0, \tag{43}$$

$$l_\theta = \frac{W}{2(U-\eta)}, \tag{44}$$

$$l_z = -\frac{\phi}{(U-\eta)}. \tag{45}$$

The isothermal compressibility τ_{iso} can be expressed in the non-dimensional form as

$$(\tau_{iso})\rho_a V^2 = \frac{(1-\bar{b}D)}{P}. \tag{46}$$

Using the self-similarity transformations (32), Eq. (26) can be written as

$$U(1) = (1-\beta), \quad D(1) = \frac{1}{\beta}, \quad \phi(1) = \frac{C^*}{\lambda C}, \quad W(1) = \frac{E}{\lambda C}, \quad P(1) = (1-\beta), \tag{47}$$

where it was necessary to use $\lambda = \alpha = \delta$ to obtain the similarity solutions. In addition to shock conditions (47), the condition to be satisfied at the piston surface is that the velocity of the fluid is equal to the velocity of the piston itself. This kinematic condition from Eq. (32) can be written as

$$U(\eta_p) = \eta_p. \tag{48}$$

Now, Eqs. (39)–(42) can be numerically integrated, with boundary conditions (47) to obtain the solution of the problem.

4 Adiabatic flow

In this section, we present the similarity solutions for the adiabatic flow behind a strong shock driven out by a cylindrical piston moving according to the exponential law (1), in the case of axisymmetric rotating non-ideal gas, which is rotating about the axis of symmetry. The strong shock conditions, which serve as the boundary conditions for the problem, are given by

$$u_n = \frac{2(1-\bar{b})}{(\gamma+1)}V, \quad \rho_n = \frac{(\gamma+1)}{(2\bar{b}+\gamma-1)}\rho_a, \quad p_n = \frac{2(1-\bar{b})}{(\gamma+1)}\rho_a V^2, \quad v_n = C^*\exp(\delta t), \quad w_n = E\exp(\alpha t) \tag{49}$$

which are the same as given by Eqs. (26) and (28) in the case of isothermal flow.

For adiabatic flow, Eq. (7) is replaced by (Vishwakarma and Nath [13, 14, 27], Nath [11])

$$\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial r}-\frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}\right)=0. \quad (50)$$

The adiabatic compressibility of non-ideal gas may be calculated as (c.f. Moelwyn-Hughes [35], Nath [11])

$$C_{adi}=\frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_S=\frac{1}{\rho a^2}=\frac{(1-Z)}{\Gamma p}. \quad (51)$$

With the help of Eqs. (32) (3)–(6) and (50), can be transformed and simplified to

$$\frac{dU}{d\eta}=-\frac{1}{(U-\eta)}\left[\frac{1}{D}\frac{dP}{d\eta}+U-\frac{\phi^2}{\eta}\right], \quad (52)$$

$$\frac{dD}{d\eta}=\frac{2(1-\bar{b}D)D}{\gamma(U-\eta)}+\frac{(1-\bar{b}D)D}{\gamma P}\frac{dP}{d\eta}, \quad (53)$$

$$\frac{dP}{d\eta}=\frac{PD\left[\gamma U(2\eta-U)-2\eta(1-\bar{b}D)(U-\eta)-\phi^2\gamma\right]}{\left[(U-\eta)^2D(1-\bar{b}D)-P\gamma\right]\eta}, \quad (54)$$

$$\frac{d\phi}{d\eta}=-\frac{\phi(U+\eta)}{\eta(U-\eta)}, \quad (55)$$

$$\frac{dW}{d\eta}=-\frac{W}{(U-\eta)}. \quad (56)$$

The transformed shock conditions, the kinematic condition at the piston and the non-dimensional component of the vorticity vector will be same as in the case of isothermal flow.

The ordinary differential Eqs. (52)–(56) with the boundary conditions (47) can now be numerically integrated to obtain the solution for the adiabatic flow behind the shock front.

By using Eq. (32) in Eq. (51), we obtain the expression for the adiabatic compressibility C_{adi} as

$$(C_{adi})\rho_a V^2=\frac{(1-\bar{b}D)}{\gamma P}. \quad (57)$$

Normalizing the variables u, v, w, ρ and p with their respective values at the shock, we obtain

$$\frac{u}{u_n}=\frac{U(\eta)}{U(1)},\ \frac{v}{v_n}=\frac{\phi(\eta)}{\phi(1)},\ \frac{w}{w_n}=\frac{W(\eta)}{W(1)},\ \frac{\rho}{\rho_n}=\frac{D(\eta)}{D(1)},\ \frac{p}{p_n}=\frac{P(\eta)}{P(1)}. \quad (58)$$

Because of the dependence of the boundary conditions (47) and the Eqs. (34), (39), (40), (46) and (52)–(57) on the parameter of non-idealness of the gas $\bar{b}$ $(= b\ \rho_a)$, the similarity solutions exist only when $\bar{b}$ is constants. Therefore, for existence of similarity solutions it is necessary to take the initial density ρ_a to be a constant.

5 Results and discussion

The distribution of the flow variables between the shock front ($\eta = 1$)and the inner expanding surface or piston ($\eta = \eta_p$) is obtained by the numerical integration of Eqs. (39–42) for isothermal flow, and from Eqs. (48–56) for adiabatic flow with the boundary conditions (47) by the Runge–Kutta method of the fourth order. The typical values of physical quantities involved in the computation are taken as (Vishwakarma and Nath [10, 27], Nath [28, 31]) $\gamma = 1.4$; $\bar{b} = 0, 0.05, 0.075, 0.1, 0.2, 0.3$. The values $\vec{b} = 0$ corresponds to the perfect gas case (see curve −1 in Figs. 1, 2).

The GRP scheme was successfully employed for solving complex shock wave interactions in pure gas (see for example, Falcovtiz et al. [36], Igra et al. [37, 38], Falcovtiz and Ben-Artzi [39]). In these papers the numerical solutions were compared with experimental findings and excellent agreement was found between the two, confirming the reliability of the numerical solution obtained for the considered cases. We refer readers to Falcovitz and Ben-Artzi [39] for an extensive review of the GRP principles and its fluid dynamical implementations.

In the present case, the flow takes place in a axisymmetric rotating non-ideal gas. Unfortunately, to the best of our knowledge, there are no experimental results that can be used as a bench mark for the presently considered flows. The present study is the generalization of our earlier work (Vishwakarma and Nath [14]) by considering the presence of the azimuthal and axial fluid velocities and vorticity components (Figs. 1b, c, f, g, 2b, c, f, g). In the present work, we also studied the variation of isothermal and adiabatic compressibility with respect to the parameter $\bar{b}$, (Figs. 1h, 2h).

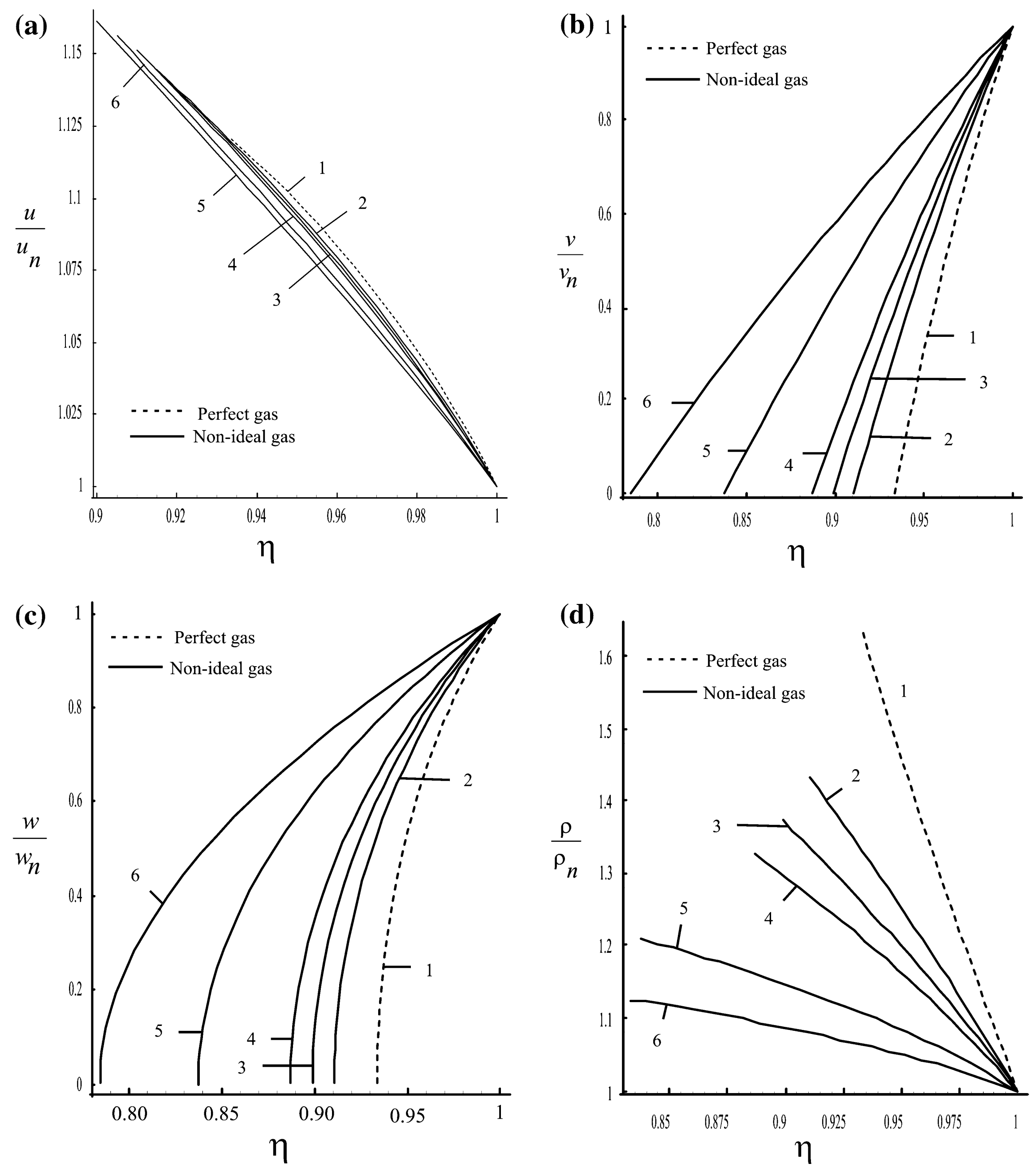

Fig. 1 Distribution of the flow variables in the region behind the shock front in the case of isothermal flow: **a** radial component of velocity $\frac{u}{u_n}$, **b** azimuthal component of velocity $\frac{v}{v_n}$, **c** axial component of velocity $\frac{w}{w_n}$, **d** density $\frac{\rho}{\rho_n}$, **e** pressure $\frac{p}{p_n}$, **f** non-dimensional azimuthal component of vorticity vector l_θ, **g** non-dimensional axial component of vorticity vector l_z, **h** isothermal compressibility $(\tau_{iso})\rho_a V^2$: 1. $\overline{b} = 0$ (Perfect gas); 2. $\overline{b} = 0.05$, 3. $\overline{b} = 0.075$; 4. $\overline{b} = 0.1$; 5. $\overline{b} = 0.2$; 6. $\overline{b} = 0.3$

Table 1 shows the variation of density ratio $\beta\left(= \frac{\rho_a}{\rho_n}\right)$ across the shock front and the position of the piston η_p for different values of $\bar{b}$ with $\gamma = 1.4$. Figures 1 and 2 show the variation of the flow variables $\frac{u}{u_n}, \frac{v}{v_n}, \frac{w}{w_n}, \frac{\rho}{\rho_n}, \frac{p}{p_n}$ the non-dimensional azimuthal component of vorticity vector l_θ, the non-

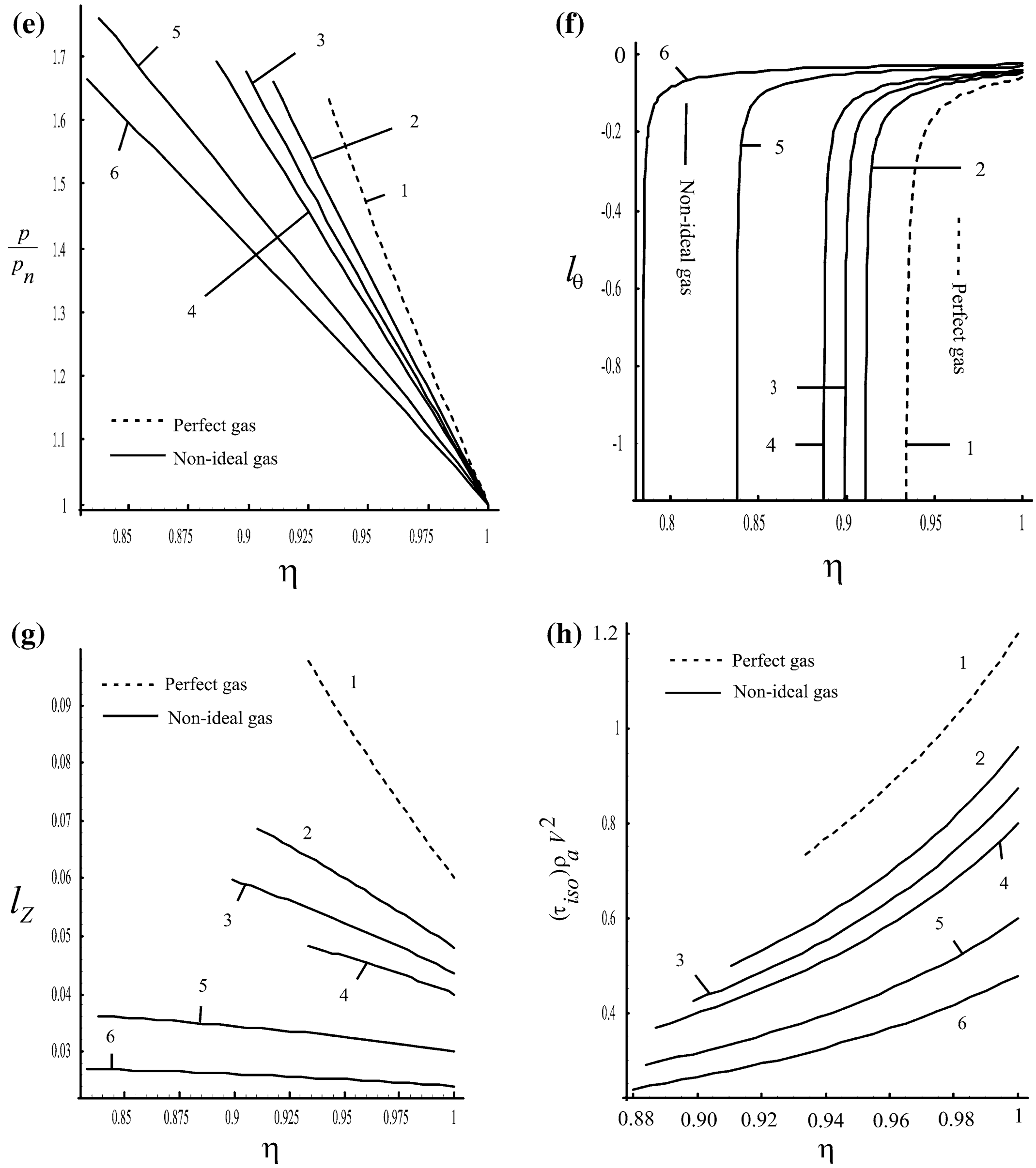

Fig. 1 continued

dimensional axial component of vorticity vector l_z and compressibility with η at various values of the parameter of non-idealness of the gas $\bar{b}$ in isothermal and adiabatic cases respectively.

These figures demonstrate that the flow variables $\frac{u}{u_n}, \frac{\rho}{\rho_n}, \frac{p}{p_n}$ and the non-dimensional axial component of vorticity vector l_z increase and the flow variables $\frac{v}{v_n}, \frac{w}{w_n}$, the non-dimensional azimuthal component of vorticity vector l_θ, the isothermal compressibility $(\tau_{iso})\rho_a V^2$ and adiabatic compressibility $(C_{adi})\rho_a V^2$ decrease from the shock front to the piston. The flow variables $\frac{u}{u_n}, \frac{\rho}{\rho_n}, \frac{p}{p_n}$ and the non-dimensional axial component of vorticity vector l_z have higher values at the piston than that at the shock front. In fact, since the total energy increases with time, the velocity of the piston is higher than the radial component of fluid velocity just behind the shock, therefore, most of the mass is concentrated near the piston.

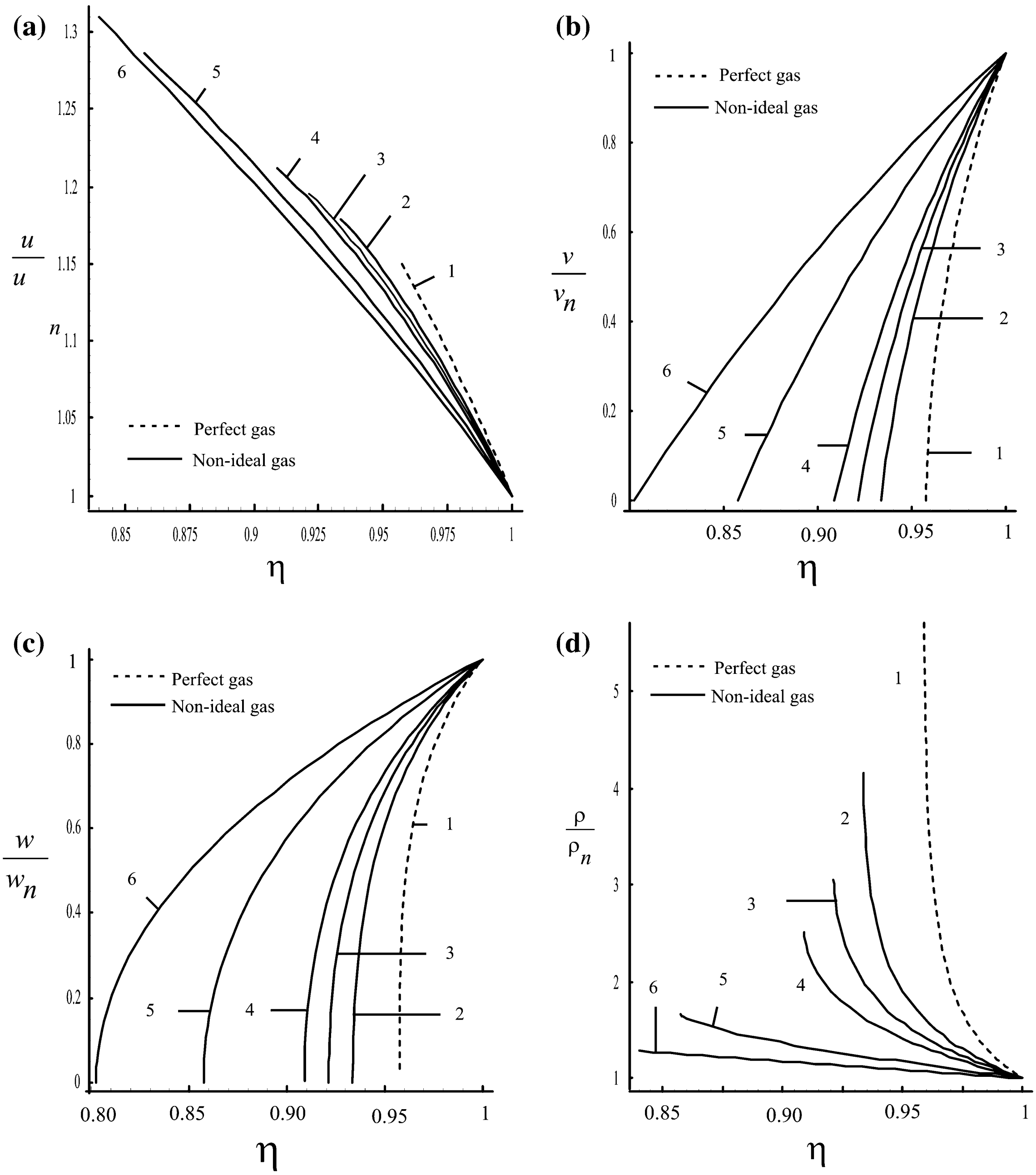

Fig. 2 Distribution of the flow variables in the region behind the shock front in the case of adiabatic flow: **a** radial component of velocity $\frac{u}{u_n}$, **b** azimuthal component of velocity $\frac{v}{v_n}$, **c** axial component of velocity $\frac{w}{w_n}$, **d** density $\frac{\rho}{\rho_n}$, **e** pressure $\frac{p}{p_n}$, **f** non-dimensional azimuthal component of vorticity vector l_θ, **g** non-dimensional axial component of vorticity vector l_z, **h** adiabatic compressibility $(C_{adi})\rho_a V^2$: 1. $\overline{b} = 0$ (Perfect gas); 2. $\overline{b} = 0.05$, 3. $\overline{b} = 0.075$; 4. $\overline{b} = 0.1$; 5. $\overline{b} = 0.2$; 6. $\overline{b} = 0.3$

Figure 2d shows that there is unbounded density distribution near the piston, in some cases, when the flow is adiabatic. This is quite acceptable and may be explained as follows. First of all, Sedov [3] (see also Ranga Rao Ramana [12], and Vishwakarma and Nath [13, 14]) indicating that a limiting case, as $n \to \infty$, of a self-similar flow-field with a power law shock,

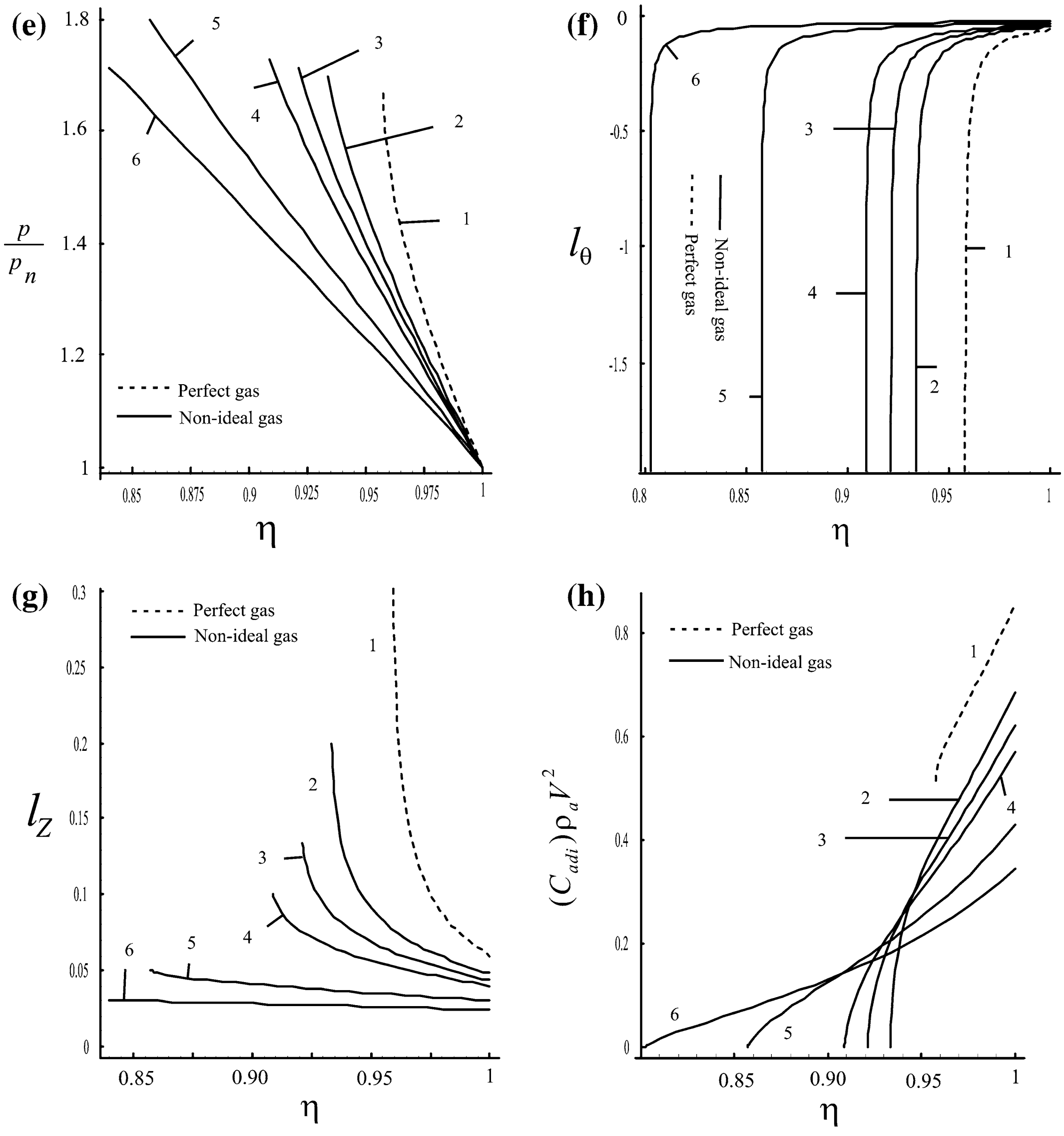

Fig. 2 continued

$$R \sim t^{n+1}, \tag{59}$$

is the flow field formed with an exponential shock described by Eq. (2). For such flow with a power law shock, in the adiabatic case, it can be easily seen from the asymptotic form of the adiabatic integral (see "Appendix")

$$\frac{D^{\gamma+\left(\frac{n}{n+1}\right)}}{(1-\bar{b}D)^{\gamma}} = \frac{P\eta^{\left(\frac{2}{n+1}\right)}}{C^1}[U-(n+1)]^{-\left(\frac{n}{n+1}\right)}, \tag{60}$$

that the density tends to infinity for $n > 0$ as the piston is approached, provided $(1-\bar{b}D)$ does not tends to zero. The density distribution exhibits such behavior in the case of a perfect gas $(\bar{b}=0)$ or in the case of non-ideal gas with $\bar{b} \leq 0.1$ (see Fig. 2d), when $\bar{b} = 0.2, 0.3$ this behavior of density is absent. This is perhaps due to the fact that, in this case, the expression $(1-\bar{b}D)$ in (60) tends to zero as the piston is approached for $n > 0$. This phenomenon can be physically interpreted as follows. In the case of perfect gas or in case of a non-ideal gas with $\bar{b} \leq 0.1$ the path of the piston converges with the path of the particle immediately ahead, thus compressing the gas to infinite density; whereas in the case of $\bar{b} = 0.2, 0.3$,

Table 1 Variation of the density ratio $\beta\left(=\frac{\rho_a}{\rho_n}\right)$ across the shock front and the position of the piston surface η_p for different values of $\bar{b}$ with $\gamma = 1.4$

$\bar{b}$	β	Position of the piston η_p	
		Isothermal flow	Adiabatic flow
0	0.166667	0.933640	0.957585
0.05	0.208333	0.910618	0.933552
0.075	0.229167	0.898865	0.921293
0.10	0.250000	0.886944	0.908865
0.20	0.3333300	0.837428	0.857308
0.30	0.416667	0.784504	0.802335

the path of the piston is almost parallel to the path of the particle immediately ahead, the above behavior of the density distribution is not observed.

It is evident from Fig. 1d, g, h that in the case of isothermal flow the density, compressibility and the non-dimensional axial component of the vorticity vector l_z are finite at the piston for all values of $\bar{b}$. Thus, one may note that the feature of unbounded density, compressibility and the non-dimensional axial component of vorticity vector near the piston in the adiabatic flow, is absent when the flow is isothermal for all values of $\bar{b}$. This seems to be necessary because with an unbounded density near the piston the temperature there approaches zero, thus violating the basic assumption of zero temperature gradient throughout the flow. Therefore, it may be observed that the assumption of zero temperature gradient brings a profound change in the density, compressibility and non-dimensional axial component of vorticity vector distributions as compared to that of the adiabatic flow (except the case when $\bar{b} = 0.2$ and $\bar{b} = 0.3$); whereas the distributions of pressures, components of velocity and the azimuthal component of the vorticity vector are slightly affected.

Table 1 also shows that the distance of the piston from the shock front is less in the case of adiabatic flow in comparison with that in the case of isothermal flow.

The radial component of velocity, density, pressure, axial component of vorticity vector, isothermal and adiabatic compressibility in the disturbed region decrease, in general as we move inward from the shock front, whereas axial component of velocity, the azimuthal component of velocity and vorticity vector increase with an increase in the parameter of non-idealness $\bar{b}$ of the gas (see Figs. 1a–h, 2a–h). Also, the density ratio β across the shock front and the distance of the piston from the shock front increase with an increase in the parameter of non-idealness $\bar{b}$ of the gas i.e. there is a decrease in the shock strength and the flow field behind the shock become somewhat rarefied (see Table 1).

6 Conclusion

The present work investigates the self-similar flow behind a strong exponential cylindrical shock wave, propagating in a rotational axisymmetric non-ideal gas, in the case of isothermal and adiabatic flows. The shock wave is driven out by a piston moving with time according to an exponential law. The article concerns with the explosion problem in rotating medium, however the methodology and analysis presented here may be used to describe many other physical systems involving non-linear hyperbolic partial differential equations. The shock waves in rotational axisymmetric non-ideal gas can be important for description of shocks in supernova explosions, in the study of a flare produced shock in solar wind, central part of star burst galaxies, nuclear explosion, rupture of a pressurized vessel etc. The findings of the present work provided a clear picture of whether and how the non-idealness parameters affect the flow behind the shock front in rotating medium. On the basis of this work, one may draw the following conclusions:

1. The consideration of zero temperature gradient decreases the shock strength.
2. An increase in the parameter of non-idealness of the gas affects significantly the flow-variables, decreases the shock strength and widens the disturbed region between the shock and the piston.
3. The assumption of zero temperature gradient removes the singularities in the density, the non-dimensional axial component of vorticity vector and compressibility distributions near the piston (which arise in some cases of the adiabatic flow).
4. The potential applications of this study include analysis of data from exploding wire experiments, and cylindrically symmetric hypersonic flow problems associated with meteors or reentry vehicles (c.f. Hutchens [40]).

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Appendix

From the basic equations of continuity, momentum and energy in Eulearian co-ordinates for the rotational axisymmetric flow of non-ideal gas with the similarity transformations

$$\eta = \frac{r}{R}, \tag{61}$$

$$u = \frac{r}{t}U(\eta), v = \frac{r}{t}\phi(\eta), w = \frac{r}{t}W(\eta), \rho = \rho_a D(\eta),$$
$$p = \frac{r^2}{t^2}\rho_a P(\eta), \tag{62}$$

where the variable η assumes the value '1' at the shock front and η_p on the piston surface, such that the piston radius $r_p = \eta_p R$, $R(\sim t^{n+1})$being the shock radius, we obtain

$$[U - (n+1)]\frac{dD}{d\eta} + D\frac{dU}{d\eta} + \frac{2DU}{\eta} = 0 \tag{63}$$

$$[U - (n+1)]\frac{dU}{d\eta} + \frac{1}{D}\frac{dP}{d\eta} + \frac{U(U-1)}{\eta} + \frac{(2P - \phi^2 D)}{D\eta} = 0 \tag{64}$$

$$[U - (n+1)]\frac{d\phi}{d\eta} + \frac{(2U-1)\phi}{\eta} = 0 \tag{65}$$

$$[U - (n+1)]\frac{dW}{d\eta} + \frac{(U-1)W}{\eta} = 0 \tag{66}$$

$$\frac{dP}{d\eta} - \frac{P\gamma}{D(1-\bar{b}D)}\frac{dD}{d\eta} + \frac{2(U-1)P}{\eta[U-(n+1)]} = 0 \tag{67}$$

The boundary conditions for a strong shock in the rotating non-ideal gas at $\eta = 1$ are given by

$$U(1) = (1-\beta)(n+1),\ D(1) = \frac{1}{\beta},\ P(1) = (1-\beta)(n+1)^2,$$
$$\phi(1) = C^*(C)^{-\frac{1}{n+1}}, W(1) = E(C)^{-\frac{1}{n+1}} \tag{68}$$

where it was necessary to use $\alpha = \delta = \frac{n}{n+1}$ for existence of similarity solutions. In addition to the shock conditions (68), the kinematic condition $U(\eta_p) = (n+1)$ at the piston surface must be satisfied. From Eqs. (63) and (67), we obtain the relation

$$\frac{D^{\gamma + \left(\frac{n}{n+1}\right)}}{(1-\bar{b}D)^{\gamma}} = \frac{P\ \eta^{\left(\frac{2}{n+1}\right)}}{C_1}\ [U-(n+1)]^{-\left(\frac{n}{n+1}\right)} \tag{69}$$

where C_1 is a constant to be determined from (68).

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