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Nonlinear anisotropic elasticity for laminate composites

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Abstract Many structural materials, which are preferred for the developing of advanced constructions, are inhomogeneous ones. These materials have complex internal structure and properties, which make them to be more effectual in the solution of special problems required for development engineering. On the other hand, in consequence of this internal heterogeneity, they exhibit complex mechanical properties. In this work, the analysis of some features of the behavior of composite materials under different loading conditions is carried out. The dependence of nonlinear elastic response of composite materials on loading conditions is studied. Several approaches to model elastic nonlinearity such as different stiffness for particular type of loadings and nonlinear shear stress–strain relations are considered. Instead of a set of constant anisotropy coefficients, the anisotropy functions are introduced. Eventually, the combined constitutive relations are proposed to describe simultaneously two types of physical nonlinearities, one of which characterizes the nonlinearity of shear stress–strain dependency and another one determines the stress state susceptibility of material properties. The method for experimental determination of material's functions is proposed. Quite satisfactory correlation between the theoretical dependencies and the results of experimental studies is demonstrated.

Keywords Structural composites · Non-linear behavior · Anisotropy · Stress state dependence · Constitutive relations · Non-linear shear diagram

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1 Introduction

Anisotropic heterogeneous materials are widely used in different modern constructions and the most challenging of them are composite materials. Experimental investigations of mechanical properties of composite materials reveal different types of nonlinearity in their behavior. There is a variety of mechanisms of deformation of these materials, which are dependent on the type of reinforcement, matrix properties, loading conditions, directions of loads with respect to reinforcement and some others. These mechanisms and their interactions determine the stress–strain behavior of materials. The physical nonlinearity of composite materials is defined by their heterogeneous nature and the presence of initial defects caused by imperfections of manufacturing technologies. This nonlinearity and shear stiffness loss under finite strains are usually referred to the damage accumulations in materials during the loading [1–7]. These processes include matrix microstructure variations, cracking, fiber/matrix splitting and some others, which influence considerably on stiffness, strength and fatigue properties of composite materials. One of the difficulties in model development for composite materials is their nonlinear response to shear loading [8–20]. The mechanical properties of composites are strong dependent on fiber/matrix interfacial adhesion and matrix properties, too. The stress–strain curves of neat polymer resins are nonlinear over the entire strain range and at very low strain levels [20]. Most composite materials have initial nonlinearity in stiffness characteristics, which cannot be explained only by the process of accumulation of new internal damage, material phase transformation or some other irreversible processes because this nonlinearity usually is observed from the very outset of deformation process when the elastic strains are dominant. This is visually displayed in the cases when the loading direction does not coincide with the reinforcement orientation [14, 20]. Another type of physical nonlinearity is that the mechanical properties of these materials are not invariant to the loading condition but depend on the stress state type that is caused by heterogeneous structure of composite materials, which particularly is displayed in stiffness difference of composites under tension and compression loadings and other loading conditions, too [21–27]. This effect is most prominent for fabric based composites and especially for woven carbon–carbon composites or carbon composites with three axial weave [28]. The elastic modulus of these composite materials under tension can be sometimes grater in comparison with elastic modulus under compression. The natural phenomena of this behavior are described in [21, 23]. The variety of deformation mechanisms and corresponding nonlinear behavior of composite materials impede the model development for the characterization of their behavior. A current and widely used method to perform the stress–strain analysis of structures is based on the use of averaged properties of the material in combination with linear elastic model. This could give a satisfactory for engineering practice estimations of stresses and strains, but always keeps some doubts about the reliability of results obtained with the use of simplified properties in the cases of complex loadings. That makes design cycle of a composite structure less clear and over expensive due to increased number of specimen's tests and full-scale experiments. Moreover in the case of essential nonlinearity of material used in structure, the experimental results of subcomponents cannot guarantee the same response of tested ones in complete assembly of components. Thus, the only analysis of the designing construction with well-tested mathematical model of the material could be the trusted engineering tool. Typical stress–strain curves for uniaxial tension, uniaxial compression and pure shear tests of composite material on the base of glass cloth and polyether matrix are shown in Figs. 1, 2 and 3 [8]. Widely used approaches to characterize the behavior of this type of materials are based on the study of composites constituents and their cohesion interphase properties with the use of multi-scale modelling. With corresponding homogenization methods, these approaches can describe the internal mechanisms of material nonlinearity with the variety of types of inclusions and defects [29–34]. Nevertheless, the phenomenological approaches, which analyze directly the experimental data for composite materials obtained under different loading conditions, and based on complex constitutive equations, can demonstrate well promising capabilities to characterize the material behavior and evaluate the stress and strain fields in structures. This paper presents the analysis of some features of the material behavior without separate consideration of the composite constituents and the formulation of corresponding phenomenological constitutive relations that can characterize them. It represents a continuation and extension of the research described in [21] to take additionally into consideration the nonlinear material behavior under conditions of shear loading. This analysis is confined to the consideration of deformation properties of composite materials under low strain levels when elastic deformations prevail to take into account different types of nonlinearity of material's behavior on initial stage of loading, which can promote more accurate results of calculations. The modelling of irreversible processes in composite materials under large deformations is in advance. The nonlinear models proposed in this research have a general form and they are suitable for any heterogeneous materials but mostly are directed to anisotropic laminate composites subjected to loadings under plane stress conditions.

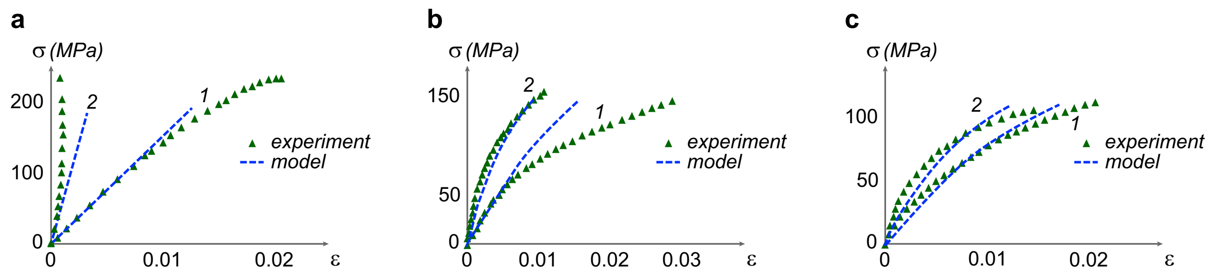

Fig. 1 The stress–strain diagrams for laminate composite material under conditions of tension at the angles 0° (**a**), 22.5° (**b**) and 45° (**c**) to the direction of the warp of the cloth. *1* for longitudinal strain, *2* for transverse strain

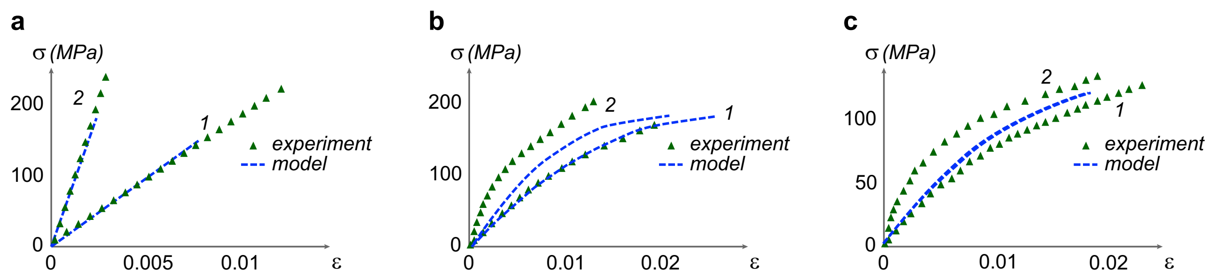

Fig. 2 The stress–strain diagrams under conditions of compression of the composite at the angles 0° (**a**), 22.5° (**b**) and 45° (**c**) to the direction of the warp of the cloth. *1* for longitudinal strain, *2* for transverse strain

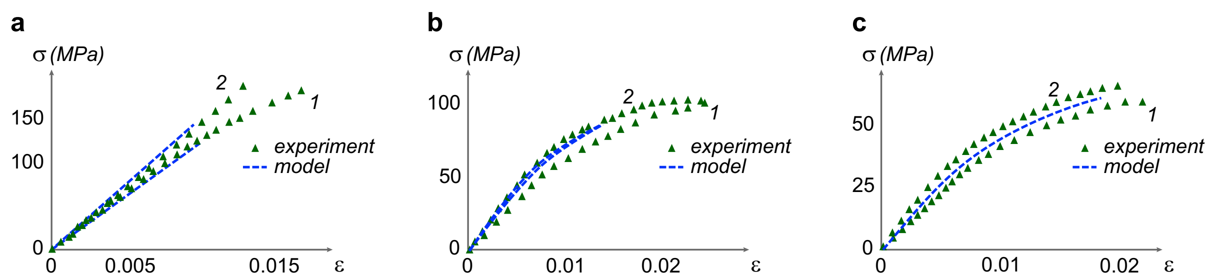

Fig. 3 The stress–strain diagrams for the conditions of shear with tension/compression directions 0°/90° (**a**), 22.5°/112.5° (**b**) and 45°/135° (**c**). *1* for longitudinal strain, *2* for transverse strain

2 Stress state dependent anisotropic elastic material model

Considering different possible responses of materials of heterogeneous structure to the loading conditions, the characterization of stress state type is required to develop corresponding mathematical model. To describe this feature of the behavior of composite materials, it is possible to introduce the stress state parameter $\xi = \sigma/\sigma_{0,}$ where $\sigma = \sigma_{ii}/3$, is the hydrostatic stress component, and $\sigma_0 = \sqrt{3/2 S_{ij} S_{ij}}$ is the effective stress, where $S_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ is stress deviator [21]. The parameter ξ has clear mechanical sense characterizing on an average the ratio of normal stresses to shear stresses in a solid, and it has notable advantages such as an invariant nature and scalar simplicity. This introduced parameter can be found in the literature under the name of stress triaxiality. Using this parameter to describe the initial elastic behavior of anisotropic material of stress state dependent properties, the constitutive relations can be formulated on the base of potential represented in the following form:

$$\Phi = \frac{1}{2} a_{ijkl}(\xi)\sigma_{ij}\sigma_{kl} \quad (1)$$

The values of parameter ξ cover the entire numerical axis from $-\infty$ (uniform triaxial compression) to ∞ (uniform triaxial tension). These cases require special investigation of asymptotic behavior of material functions to satisfy the requirement of finite correspondence between stresses and strains. This can be achieved by corresponding analytical representations for anisotropic functions $a_{ijkl}(\xi)$. In case of plane stress conditions the values of the parameter are limited $-3/2 \le \xi \le 3/2$ and the constitutive equations obtained on the base of potential (1) can be represented in following form:

$$\begin{aligned}
\varepsilon_{11} &= a_{1111}(\xi)\sigma_{11} + a_{1122}(\xi)\sigma_{22} \\
&\quad + \left[\left(\frac{1}{3\xi} + \frac{3}{2}\xi\right)\sigma - \frac{3}{2}\xi\sigma_{11}\right]\Phi_1\sigma_0^{-2}, \\
\varepsilon_{22} &= a_{1122}(\xi)\sigma_{11} + a_{2222}(\xi)\sigma_{22} \\
&\quad + \left[\left(\frac{1}{3\xi} + \frac{3}{2}\xi\right)\sigma - \frac{3}{2}\xi\sigma_{22}\right]\Phi_1\sigma_0^{-2}, \\
\varepsilon_{12} &= \left[a_{1212}(\xi) - \frac{3}{2}\xi\Phi_1\sigma_0^{-2}\right]\sigma_{12},
\end{aligned} \quad (2)$$

$$\begin{aligned}
\Phi_1 = \frac{1}{2}&\left[a'_{1111}(\xi)\sigma_{11}^2 + a'_{2222}(\xi)\sigma_{22}^2 \right.\\
&\left. + 2a'_{1122}(\xi)\sigma_{11}\sigma_{22} + a'_{1212}(\xi)\sigma_{12}^2\right],
\end{aligned}$$

where prime denotes the derivative with respect to parameter ξ.

The coefficients $a_{1111}(\xi), a_{2222}(\xi), a_{1122}(\xi)$ and $a_{1212}(\xi)$ can be represented as a piecewise linear functions of ξ and the data used for determination of these functions should be obtained experimentally using the proposed procedure [21]. The constitutive Eq. (2) describe a special form of physical nonlinearity concerned the dependence of material properties on the stress state type.

In practice, it is a common situation when the curves obtained on the base of experimental data for different loading conditions are approximated by linear ones. This simplifies the calculations and might be dictated by a linear elastic model, which is supposed to be used for further structural analysis. It could be a straight line with initial modulus or averaged one. For some composite materials, the linear dependencies approximating experimental stress–strain diagrams can be significantly different in dependence on the type of loading: for example, the compression modulus could be essentially lower than one for tension. In this situation, the Eq. (2) can describe this effect. The formulated constitutive Eq. (2) represent a linear relation between stress and strain components in the case of proportional loading, where the type of loading is fixed and consequently the parameter ξ is constant. The form of proposed equations has some advantages in experimental determination of material functions and experimental validation of theoretical approach. For example, in the cases of uniaxial tension or uniaxial compression when the values of parameter $\xi = \pm 1/3$, the nonlinear parts of equations for ε_{11} and ε_{22} in square brackets are equal to zero. Consequently, one can reduce the system of Eq. (2) to the following form for tension:

$$\varepsilon_{11} = a_{1111}(0.33)\sigma_{11} + a_{1122}(0.33)\sigma_{22},$$

$$\varepsilon_{22} = a_{1122}(0.33)\sigma_{11} + a_{2222}(0.33)\sigma_{22};$$

for compression:

$$\varepsilon_{11} = a_{1111}(-0.33)\sigma_{11} + a_{1122}(-0.33)\sigma_{22},$$

$$\varepsilon_{22} = a_{1122}(-0.33)\sigma_{11} + a_{2222}(-0.33)\sigma_{22}.$$

Also in the case of pure shear, when parameter $\xi = 0$, the corresponding equation has the following form:

$$\varepsilon_{12} = a_{1212}(0)\sigma_{12},$$

which can be easily resolved.

In the case of proportional biaxial loading, the analysis of constitutive relations (2) shows that for the loading conditions when the directions of loads coincide with fiber orientation and transverse one, it is possible to obtain the analytical relation that can be checked for satisfaction to experimental data. If the loading time parameter is referred as t, the proposed Eq. (2) can be reduced to the following form:

$$\varepsilon_{11} = (a_{1111}(\xi) + a_{1122}(\xi)Const_1 + Const_2)t,$$

$$\varepsilon_{22} = (a_{1122}(\xi) + a_{2222}(\xi)Const_3 + Const_4)t,$$

where $Const_i$ and the value of ξ are determined by the ratio of applied loads.

These relations for determination of a set of coefficients $a_{1111}(\xi), a_{2222}(\xi)$ and $a_{1122}(\xi)$ are linear ones. Considering different sets of applied forces, which keep the same value of ξ, but have the exchanged values of σ_{11} and σ_{22}, it is possible to prove that the system of four equations for three unknown coefficients $a_{1111}(\xi), a_{2222}(\xi)$ and $a_{1122}(\xi)$ can be resolved in unambiguous way.

Thus, the analysis of the proposed mathematical model for the laminate composite material under elastic conditions shows that the data for any set of uniaxial, biaxial and in-plane shear tests, where experimental loading directions are coincident with the principal axes of anisotropy, can be satisfied by appropriate combination of functions $a_{1111}(\xi), a_{2222}(\xi)$ and $a_{1122}(\xi)$. Different off-axis tests can be supposed as the independent and verified ones.

3 Nonlinear shear anisotropic material model

Another regularity of the composites behavior that ought to be taken into account is the nonlinear deformation under the action of shear stresses. For the development of nonlinear shear material model, the parameter that represents the degree of shear stresses or deformations should be formulated. For this purposes, the parameter $q = D_{ij}\varepsilon_{ij}$ can be introduced in constitutive relations, where coefficients D_{ij} have the following representation in coordinate system coincident with the orientation of anisotropy axes in the cases of unidirectional or cross ply reinforcement:

$$D_{ij} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The tensor D_{ij} can be considered as an additional structural parameter characterizing the anisotropic properties of a material. Tensor D_{ij} is defined with respect to the same coordinate system and the dyads correspond to ones of stress or strain tensors. In the cases of coordinate system transformation, the correspondent transformation of components D_{ij} has to be done according to the equations of transformation of second rank tensor [35]. In coordinate system coincident with the orientation of material reinforcement, the introduced parameter is $q = \varepsilon_{12}$. The proposed parameter can be regarded as the scalar invariant and could be used for the formulation of intended constitutive equations.

Following the approach similar to previous material model formulation, one can write the potential for the characterization of nonlinear shear elastic deformation of solids in the following form:

$$U = \frac{1}{2}E_{ijkl}(q)\varepsilon_{ij}\varepsilon_{kl} \tag{3}$$

Using the potential (3), the constitutive equations can be written as:

$$\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} = \frac{1}{2}\frac{\partial E_{mnkl}(q)}{\partial \varepsilon_{ij}}\varepsilon_{mn}\varepsilon_{kl} + E_{ijkl}(q)\varepsilon_{kl} \tag{4}$$

In order to describe shear nonlinearity, it is enough to keep the dependency on parameter q only in shear modulus G. Consequently, in the case when coordinate system coincides with principle axes of anisotropy, it is possible to show that the first part of the Eq. (4) can be reduced to the following form:

$$\frac{\partial E_{mnkl}(q)}{\partial \varepsilon_{12}}\varepsilon_{mn}\varepsilon_{kl} = \frac{dG(q)}{dq}\frac{dq}{d\varepsilon_{12}}\varepsilon_{12}\varepsilon_{12} = G'\varepsilon_{12}^2 \tag{5}$$

where prime denotes the derivative with respect to parameter q.

Thus, the Eq. (4) can be written for σ_{12} stress component as following one:

$$\sigma_{12} = \left(\frac{1}{2}G'\varepsilon_{12} + G\right)\varepsilon_{12} \tag{6}$$

One can assume that the shear modulus function $G(q)$ is represented by arbitrary polynomial of parameter q:

$$G(q) = \sum_n C_n q^n.$$

Then Eq. (6) can be reduced to the following one:

$$\begin{aligned}\sigma_{12} &= \left[\frac{1}{2}\left(\sum_n C_n n q^{n-1}\right)q + \sum_n C_n q^n\right]\varepsilon_{12} \\ &= \left[\sum_n C_n\left(\frac{1}{2}n + 1\right)q^n\right]\varepsilon_{12}.\end{aligned}$$

Using the substitution of $B_n = C_n(n/2 + 1)$, which keeps the arbitrariness of coefficients B_n, it is possible to obtain the following relation:

$$\sigma_{12} = \left[\sum_n B_n q^n\right]\varepsilon_{12}. \tag{7}$$

Eventually the Eq. (7) shows that the dependency between shear stress and shear strain could be described by arbitrary function that can be approximated by means of polynomial, and, what is practically important, it could be a piecewise linear function.

The constitutive relation with proposed potential for plane stress condition in Cartesian coordinates can be written in the following form:

$$\begin{aligned}\varepsilon_{11} &= \frac{\sigma_{11}}{E_1} - \nu\frac{\sigma_{22}}{E_1},\\ \varepsilon_{22} &= -\nu\frac{\sigma_{22}}{E_1} + \frac{\sigma_{22}}{E_2},\\ \varepsilon_{12} &= \frac{\sigma_{12}}{\tilde{G}(q)},\end{aligned} \tag{8}$$

where $\tilde{G}(q)$ characterizes the nonlinearity of shear stress–strain diagram.

Thus, the proposed constitutive Eq. (8) can describe any nonlinear in-plane shear test data in the case of loading directions coincident with material orientation axes. Similar to the previous material model, in general case of arbitrary loads directions, the experiments with off-axis specimens should be considered as verification ones. In this case, the coefficients of anisotropy are transformed according to the equations of transformation of components of a forth rank tensor [35]. The form of the developed material model is very close to the classical laminate theory that makes this approach mostly convenient for practical engineering applications.

4 Stress state dependent anisotropic elastic material model with shear nonlinearity

The useful development of proposed constitutive relations for practical applications consists in the formulation of the combined approach. Due to complexity of the first set of proposed system of Eq. (2) and the difficulties that are concerned the resolving it with regard to the stress components, the reasonable way of formulation of constitutive relations is to use system (2) as a basis, and to modify the formulation of the relations (8) into the stress components form. For this propose in Eq. (8) it is undoubtedly straightforward to replace the strain ε_{12} by the shear stress component σ_{12} in the statement of parameter q. Consequently, the new parameter of degree of shear loading could be defined as:

$$Q = D_{ij}\sigma_{ij},$$

where D_{ij} in coordinate system coincident with orientation of the material axes, can be represented in the following form similar to previous formulation of nonlinear shear model:

$$D_{ij} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The parameter Q equals to σ_{12} in the coordinate system coincident with principal anisotropy axes. According to the idea of taking into account the combination of effects considered above, the potential in this case have to include simultaneously the dependency on both parameters ξ and Q and it can be written in the following form:

$$\Phi = \frac{1}{2}A_{ijkl}(\xi, Q)\sigma_{ij}\sigma_{kl}. \tag{9}$$

For plane stress conditions of an orthotropic solid some of anisotropy coefficients $A_{ijkl}(\xi, Q)$ are equal to zero except for $A_{1111}, A_{1122}, A_{2222}$ and A_{1212}. Similar to the previous formulations, one can keep also the shear dependency on the parameter Q only for the coefficient A_{1212}. Further, taking into account that $\partial Q/\partial\sigma_{11} = \partial Q/\partial\sigma_{22} = 0$, new constitutive equations based on potential (9) can be represented in the following form:

$$\begin{aligned}\varepsilon_{11} =& A_{1111}(\xi)\sigma_{11} + A_{1122}(\xi)\sigma_{22}\\ &+ \left[\left(\frac{1}{3\xi} + \frac{3}{2}\xi\right)\sigma - \frac{3}{2}\xi\sigma_{11}\right]\Phi_1\sigma_0^{-2},\\ \varepsilon_{22} =& A_{1122}(\xi)\sigma_{11} + A_{2222}(\xi)\sigma_{22}\\ &+ \left[\left(\frac{1}{3\xi} + \frac{3}{2}\xi\right)\sigma - \frac{3}{2}\xi\sigma_{22}\right]\Phi_1\sigma_0^{-2},\\ \varepsilon_{12} =& \left[\left(A_{1212}(\xi,Q) + \frac{1}{2}\frac{\partial A_{1212}(\xi,Q)}{\partial Q}\right) - \frac{3}{2}\xi\Phi_1\sigma_0^{-2}\right]\sigma_{12},\end{aligned} \tag{10}$$

where Φ_1 has the same meaning as in (2):

$$\begin{aligned}\Phi_1 =& \frac{1}{2}\left[A'_{1111}(\xi)\sigma_{11}^2 + A'_{2222}(\xi)\sigma_{22}^2\right.\\ &\left. + 2A'_{1122}(\xi)\sigma_{11}\sigma_{22} + A'_{1212}(\xi, Q)\sigma_{12}^2\right].\end{aligned}$$

The prime denotes the derivative with respect to parameter ξ.

It is possible to see that in the system of Eq. (10) only the third equation differs from system (2). Following the same technique as in previous nonlinear shear model, let us assume that

$$A_{1212}(\xi, Q) = \sum_n C_n(\xi) Q^n,$$

Then, the part of the equation for ε_{12} containing the coefficient A_{1212} and its derivative can be written in the form:

$$\left(A_{1212}(\xi, Q) + \frac{1}{2}\frac{\partial A_{1212}(\xi, Q)}{\partial Q}\right) = \sum_n C_n(\xi)Q^n + \frac{1}{2}\left(\sum_n C_n(\xi) n Q^{n-1}\right) Q = \sum_n \left(1 + \frac{n}{2}\right) C_n(\xi) Q^n.$$

Using the substitution

$$B_n(\xi) = \left(1 + \frac{n}{2}\right) C_n(\xi),$$

it is possible to obtain

$$\left(A_{1212}(\xi, Q) + \frac{1}{2}\frac{\partial A_{1212}(\xi, Q)}{\partial Q}\right) = \sum_n B_n(\xi) Q^n.$$

Thus, the equation of system (10) for the shear strain component can be written in the following form:

$$\varepsilon_{12} = \left[B(\xi, Q) - \frac{3}{2}\xi \Phi_1 \sigma_0^{-2}\right] \sigma_{12}, \tag{11}$$

where $B(\xi, Q)$ is an arbitrary function that could be approximated by polynomial dependency. In spite of the complexity of relation (11), in the case of pure in-plane shear experiment, the stress state parameter $\xi = 0$, and the equation for ε_{12} is reduced to

$$\varepsilon_{12} = B(0, \sigma_{12})\sigma_{12}. \tag{12}$$

Consequently, due to the arbitrariness of function $B(0, \sigma_{12})$, it means that arbitrary nonlinear test data for in-plane shear test can be satisfied at any required precision.

Analyzing the proposed system of Eq. (10) one can see that it has an improvement in comparison with system (2), which is concerned the taking into consideration the nonlinear shear properties of composite materials. Nevertheless the material model represented by (2) has essential advantage, namely, the linear response in the case of proportional loading ($\xi = const$). The linear stress–strain relations remain in the cases of Eq. (10), too, when specimen orientation and the loading direction coincident with principle axes of anisotropy of a material. For these types of loadings the stress component σ_{12}, which adds nonlinearity, is equal to zero. Consequently, all goals demonstrated for system (2) in case of uniaxial and biaxial tests are preserved by a combined approach represented by Eq. (10).

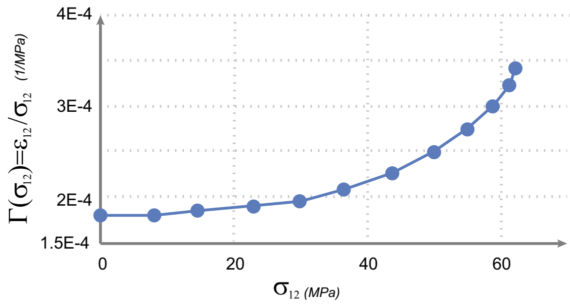

Fig. 4 Shear deformation parameter $\Gamma(\sigma_{12}) = \varepsilon_{12}/\sigma_{12}$ (1/MPa)

The possibilities of proposed nonlinear elastic constitutive relations (10) in the description of mechanical behavior of laminate composite materials can be demonstrated on the base of experimental data for laminate based on the glass cloth and polyether matrix, shown in Figs. 1, 2 and 3 [7]. These results of the tests have a particular interest, because they cover three different stress states: tension ($\xi = 1/3$), compression ($\xi = -1/3$), and shear ($\xi = 0$); and for each type of loadings there are experiments with three different orientations, which for constant values of parameter ξ display the anisotropic properties of composite material.

For the determination of coefficients or functional dependencies of proposed model, there are different ways of doing this. For the first step, some relatively simple method with reduced number of parameters might be used. In practical way, the piecewise linear functions could be used, but following the idea to keep less number of parameters in the model one can define A_{ijkl} and function B in the following form:

Table 1 Values of the model coefficients for experimental correlation

a_{11}^0 (1/MPa)	c_{11} (1/MPa)	a_{22}^0 (1/MPa)	c_{22} (1/MPa)	a_{12}^0 (1/MPa)	c_{12} (1/MPa)
6E−5	2E−5	6E−5	2E−5	−1.6E−5	−2E−5

$$\begin{aligned} &A_{1111}(\xi) = a_{11}^0 + c_{11}\xi, \\ &A_{2222}(\xi) = a_{22}^0 + c_{22}\xi, \\ &A_{1122}(\xi) = a_{12}^0 + c_{12}\xi, \\ &B(\xi, Q) = \Gamma(Q). \end{aligned} \tag{13}$$

The function $\Gamma(Q)$ can be determined by means of Eq. (12) $\Gamma(\sigma_{12}) = \varepsilon_{12}/\sigma_{12}$ and the results of shear test, which are presented in Fig. 3c. The corresponding dependency is shown in Fig. 4.

The rest of coefficients can be obtained by means of the analysis of results of uniaxial tests. The possible set of coefficients introduced in Eq. (13) for experimental data correlation is presented in Table 1, which can be used for experimental data correlation. The experimental and theoretical stress–strain dependencies determined with the use of obtained values for coefficients (Table 1) are shown in Figs. 1, 2 and 3.

One can obtain quite satisfactory correlation for test curves shown on Figs. 1, 2 and 3, because for each type of loading, there are different values of parameter ξ. It is possible to determine for each value of ξ the corresponding set of compliances and approximate this data by some functions, for example, piecewise linear ones. Furthermore, in the proposed definition of coefficients (13), the dependency on stress state parameter is not introduced into function $B(\xi, Q)$, which can modify and improve the shear dependency for each set of tests with corresponding type of loading. Nevertheless, taking into consideration the dependencies of material's functions on all parameters included in proposed mathematical model simultaneously makes essentially difficult the analysis of obtained results with the use of analytical methods.

5 Conclusions

The most of composite laminate materials, even under small elastic strains, exhibit different forms of the nonlinearity of mechanical properties. This work shows the possibilities of taking into account the elastic nonlinear behavior of laminate composite materials, which include the stress state dependence of their characteristics and the nonlinearity of shear stress–strain relations. The mathematical model for these anisotropic materials with dependence of properties on the type of loading is analyzed. It has been shown that for uniaxial, biaxial and in-plane shear tests this model can guarantee the experimental data satisfaction.

The capabilities of proposed approach to the characterization of nonlinear shear behavior and the extension of classical elasticity, which is used for usual laminate composite theory, are studied especially. The simplicity of the final constitutive equations makes this nonlinear shear approach essentially useful from point of engineering applications.

The proposed material model is used for the study of experimental data. The analysis of test results based on fiberglass fabric composite specimens is carried out. Experiments with off-axis specimens are also studied and show that the proposed nonlinear approach, even with simplest introduction of nonlinear parameters, can approximate the results with quite satisfactory precision.

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References

1. Matzenmiller A, Lubliner J, Taylor RL (1995) A constitutive model for anisotropic damage in fiber-composites. Mech Mater 20:125–152
2. Puck A, Schürmann H (1998) Failure analysis of FRP laminates by means of physically based phenomenological models. Compos Sci Technol 58:1045–1067
3. Bogetti TA, Hoppel CPR, Harik VM, Newill JF, Burns BP (2004) Predicting the nonlinear response and progressive failure of composite laminates. Compos Sci Technol. doi:10.1016/S0266-3538(03)00217-3
4. Zinoviev PA, Grigoriev SV, Lebedeva OV, Tairova LP (1998) The strength of multilayered composites under a plane-stress state. Compos Sci Technol 58(7):1209–1223

5. Fedulov BN, Antonov FK, Safonov AA, Ushakov AE, Lomov SV (2014) Influence of fibre misalignment and voids on composite laminate strength. J Compos Mater. http://jcm.sagepub.com/content/early/2014/10/29/0021998314557533
6. Lubineau GA (2010) Pyramidal modeling scheme for laminates—identification of transverse cracking. Int J Damage Mech 19(4):499–518
7. Ladeveze P, Le Dantec E (1992) Damage modelling of the elementary ply for laminated composites. Compos Sci Technol 43:257–267
8. Smith EW, Pascoe KJ (1977) The role of shear deformation in the fatigue failure of a glass fiber-reinforced composite. Composites 8(4):237–243
9. Lagattu F, Lafarie-Frenot MC (2000) Variation of PEEK matrix crystallinity in APC-2 composite subjected to large shearing deformations. Compos Sci Technol 60:605–612
10. Makeev A, Ignatius C, He Y, Shonkwiler B (2009) A test method for assessment of shear properties of thick composites. J Compos Mater 43(25):3091–3105
11. Petit PH, Waddoups ME (1969) A method of predicting the nonlinear behavior of laminated composites. J Compos Mater 3(1):2–19
12. Schapery RA, Sun CT (2000) Time dependent and nonlinear effects in polymers and composites, Issue 1357. ASTM, Philadelphia
13. Chamis CC, Sinclar JH (1977) Ten-deg off-axis shear properties in fiber composites. Exp Mech 17:339–346
14. Hahn HT, Tsai SW (1973) Nonlinear elastic behavior of unidirectional composite laminae. J Compos Mater 7:102–118
15. Sims DF (1973) In-plane shear stress–strain response of unidirectional composite materials. J Compos Mater 7:124–128
16. Swanson SR, Messick M, Toombes GR (1985) Comparison of torsion tube and Iosipescu in-plane shear test results for a carbon fibre-reinforced epoxy composite. Composites 16(3):220–224
17. Sala G (2000) Mechanical characterization of angle–ply composite laminates by differential Moiré method: the influence of Specimens' length-to-width ratio. Meccanica 35(5):421–432
18. He Y, Makeev A (2014) Nonlinear shear behavior and interlaminar shear strength of unidirectional polymer matrix composites: a numerical study. Int J Solids Struct 51(6):1263–1273
19. Mantari JL, Bonilla EM, Guedes Soares C (2014) A new tangential-exponential higher order shear deformation theory for advanced composite plates. Compos B Eng 60:319–328
20. Weinberg M (1987) Shear testing of neat thermoplastic resins and their unidirectional graphite composites. Composites 18(5):386–392
21. Lomakin EV (2011) Constitutive models of mechanical behavior of media with stress state dependent material properties. Mechanics of generalized continua. Adv Struct Mater 7:339–350
22. Lomakin EV, Fedulov BN (2013) Plane strain extension of a strip made of a material with stress state type dependent material properties and weakened by cuts with circular base. Mech Solids 48(4):424–430
23. Lomakin EV, Rabotnov YuN (1978) A theory of elasticity for an isotropic body with different moduli in tension and compression. Mech Solids 13(6):25–30
24. Lomakin EV (2009) Plastic flow of dilatant solids with stress-state-dependent material properties. In: Gupta NK, Manzhirov AV (eds) Topical problems in solid mechanics. Elite Publishing House Pvt. Ltd., New Delhi, pp 122–132
25. Lomakin EV, Fedulov BN, Melnikov AM (2014) Constitutive models for anisotropic materials susceptible to loading conditions. In: Belyaev AK, Irschik H, Krommer M (eds) Mechanics and model-based control of advanced engineering systems. Springer, pp 209–216
26. Fedulov BN (2007) The limit plastic state of a holed strip made of a dilatant material. Moscow Univ Mech Bull 62(6):160–164
27. Alexandrov S, Jeng Y, Lomakin E (2014) An exact semi-analytic solution for residual stresses and strains within a thin hollow disc of pressure-sensitive material subject to thermal loading. Meccanica 49(4):775–794
28. Kratsch K, Schutzler J, Eitman D (1972) Carbon–carbon 3-D orthogonal material behavior. In: 13th Structures, structural dynamics and materials conference. AIAA paper, vol 72, 365 pp
29. Violeau D, Ladevèze P, Lubineau G (2009) Micromodel-based simulations for laminated composites. Compos Sci Technol 69:1364–1371
30. Lubineau G, Ladevèze P (2008) Construction of a micromechanics-based intralaminar mesomodel, and illustrations in ABAQUS/Standard. Comput Mater Sci 43:137–145
31. deBotton G, Hariton I (2002) High-rank nonlinear sequentially laminated composites and their possible tendency towards isotropic behavior. J Mech Phys Solids 50(12):2577–2595
32. Lenci S (2004) Elastic and damage longitudinal shear behavior of highly concentrated long fiber composites. Meccanica 39(5):415–439
33. Paggia M, Wriggers P (2012) Stiffness and strength of hierarchical polycrystalline materials with imperfect interfaces. J Mech Phys Solids 60(4):557–572
34. Talbot DRS (2000) Improved bounds for the effective properties of a nonlinear two-phase elastic composite. J Mech Phys Solids 48(6–7):1285–1294
35. Lekhnitskii SG (1963) Theory of the elasticity of an anisotropic elastic body. Holden Day, New York