Meccanica (2015) 50:1073–1087
DOI 10.1007/s11012-014-0053-4

Free vibrations of stepped axially functionally graded Timoshenko beams

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Received: 11 December 2013 / Accepted: 3 September 2014 / Published online: 14 October 2014

Abstract The paper studies free transverse vibrations of axially functionally graded beams with stepped changes in geometry and in material properties. The differential quadrature method with domain decomposition technique is used. The governing equations of motion are based on Timoshenko beam theory and are derived using Hamilton's principle. Material properties are assumed to vary along the beam in an abrupt or gradual way. General boundary conditions are considered by means of translatory and rotatory springs at both external ends of the beam. Results are presented for different combinations of boundary conditions, step locations and properties of axially functionally graded materials. The effect of dynamic stiffening of beams can be observed in various situations. There are no available previous results of axially functionally graded beams with stepped changes in material properties and in cross section. This study may be helpful for a variety of potential applications in characterizing the effect of stepped changes in material properties added to changes in geometry.

Keywords Free vibration · Timoshenko · Stepped beam · Axially functionally graded · Differential quadrature method · Discontinuity

1 Introduction

Dynamic behavior of stepped beam-like elements is of practical interest in many engineering applications, including civil, aerospace, shipbuilding and automobile engineering. Long span bridges, tall buildings, spacecraft antennae, rotor blades and robot arm manipulators can be modeled with beam-like elements.

In a dynamical environment, steps in cross-section and in material properties affect the natural frequencies. This situation may cause resonance if the changed frequency is close to the working frequency. It is crucial to predict the change in the frequency, as well as the mode shape.

The classical Bernoulli–Euler beam theory adequately predicts the frequencies of vibration of lower modes of slender beams. The governing characteristic differential equation of a non-uniform beam is a fourth order ordinary differential equation in the flexural displacement with variable coefficients. Many authors

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have performed analysis of vibration of stepped beams based on this theory. [1–10]. Among them, in 2010 the paper by Mao et al. [10] presents free vibrations of stepped homogeneous beams by Adomain decomposition method.

For Timoshenko beams, the governing characteristic differential equations are two differential equations coupled in terms of the flexural displacement and the angle of rotation which results from bending [11–15]. Free vibration of homogeneous stepped Timoshenko beam studies have been presented in [14, 15] among other papers. Various previous studies have been reported for beams made of AFG materials [11, 16–19] with a continuous variation of the cross-sectional area (tapered beams) [11, 16, 19] and with constant cross-sectional area [17, 18]. Exact solution for the behavior of vibrating Timoshenko beams with variable coefficients does not exist, then the problem must be analyzed by approximate procedures. Differential quadrature method, DQM, is a useful technique to solve the governing equations directly. Early references on the DQM can be found in Bellman and Casti [20], Bert and Malik [21], Laura and Gutiérrez [22] and more recent development and applications can be found in [6, 14, 15, 19, 23–26] among many others. In particular, Karami et al. [14] developed an accurate differential quadrature element method based on the theory of shear deformable beams. They employed it to analyze beams with non-uniform or discontinuous geometry and other complexities.

A recent literature survey on free vibration of stepped beams of functionally graded materials revealed that not many papers cover this topic. In particular to the authors' knowledge, there are no natural frequency data in the literature for axially functionally graded, AFG, Timoshenko beams with stepped changes in material properties and in cross-sectional area.

In the present paper a different point of view that adds the effect of the material inhomogeneity to the stepped change geometry is modeled for free vibration of Timoshenko beams with elastic boundary conditions. Functionally graded material properties are assumed to vary along the beam in a linear, quadratic or cubic fashion in each beam element, with an abrupt discontinuity at the stepped change geometry. The study considers the accuracy and convergence of the DQM as applied to the study of free transverse vibration of stepped beams of AFG materials.

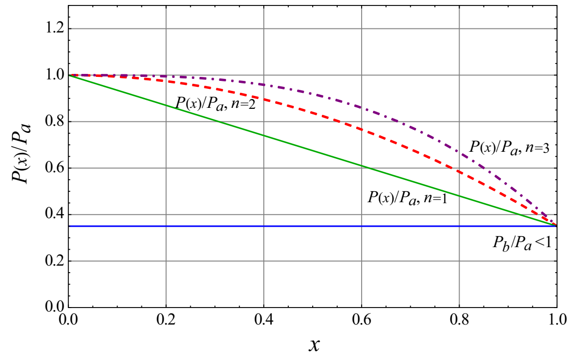

Fig. 1 Power law relation of AFG material properties. $x = \bar{x}/L$; $P_b/P_a < 1$

2 Theory

2.1 Axially functionally graded material properties

In the present paper the free vibration of stepped AFG Timoshenko beams with different combinations of boundary conditions is analyzed.

The beam could have stepped jumps in cross-sectional area and in material properties. In order to obtain the dynamic response, the beam is discretized into elements or subdomains depending on the geometrical and material discontinuities.

The inhomogeneous material [27], with gradient compositional variation of the constituents, varies in the longitudinal direction of the beam. Properties of AFG materials, like mass density ρ, Young's modulus E, shear modulus G, continuously vary in the axial direction.

A generic material property $P(\bar{x})$ [16] is assumed to vary along the beam axis $\bar{x}$ with a power law relation, Fig. 1:

$$P(\bar{x}) = P_a + (P_b - P_a)\left(\frac{\bar{x}}{L}\right)^n, \tag{1}$$

where P_a and P_b are properties of material "a" and material "b", respectively. They are the constituents of the inhomogeneous material of the beam; n is the material non-homogeneity parameter and $P(\bar{x})$ is a typical material property such as ρ, E or G. The percentage content of material "a" increases as n increases. When $n = 1$ the composition changes linearly through the length L, while $n = 1/2$ or $n = 2$

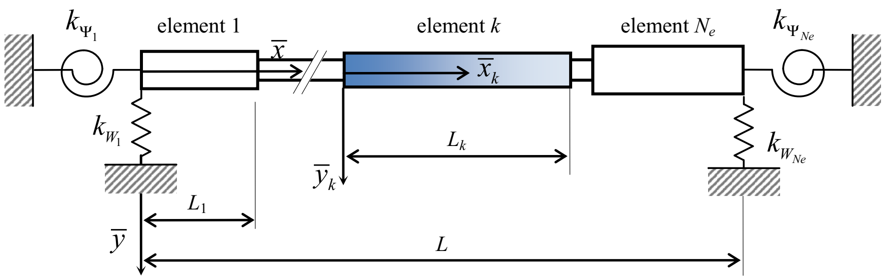

Fig. 2 Stepped beam with elastic restrains

corresponds to a quadratic distribution, and so on. In general, any value n outside the range (1/3, 3) is not desired [27] because such a functionally graded material would contain too much of one of the constituents. (When $n = 1/3$ or 3, one constituent has the 75 % of the total AFG material.)

2.2 The Timoshenko beam theory for AFG beams with stepped changes

Figure 2 presents the stepped Timoshenko beam of length L with elastic restrains at both ends. Cartesian global coordinate system $\bar{x}\bar{y}\bar{z}$ is adopted at the left hand end of the beam and the local coordinate systems are at the left hand end of each beam element k. The global $\bar{x}$ and local $\bar{x}_k$ axes are coincident; both are normal to the beam cross-section and pass through section's barycenter. The beam model is discretized in N_e subdomains or elements, depending on the geometrical and material discontinuities.

Following the Timoshenko beam theory, [28, 29], the axial and shear strains could be expressed as:

$$\varepsilon = \varepsilon_x \cong -y\theta' + \frac{1}{2}(w')^2; \quad \gamma = \gamma_{xy} = w' - \theta, \tag{2}$$

where $w = w(\bar{x}, t)$ is the flexural displacement of the beam neutral axes in the $\bar{y}$ direction and $\theta = \theta(\bar{x}, t)$ is the cross-section rotation about the $\bar{z}$ axis. (Prime mark indicates derivative with respect to the spatial coordinate.)

The potential energy due to flexure stretching and shear:

$$U_1 = \int_A \int_0^L \frac{(E\varepsilon^2 + G\gamma^2)}{2} d\bar{x}\, dA = \frac{1}{2}\int_0^L EI(\theta')^2 d\bar{x} + \frac{1}{2}\int_0^L \kappa AG(w' - \theta)^2 d\bar{x} + C_1,$$

where C_1 is a constant [28], $E = E(\bar{x})$ is the Young modulus, $G = G(\bar{x})$ is the shear modulus. The area and the second moment of area of beam cross section are noted as $A = A(\bar{x})$ and $I = I(\bar{x})$. The coefficient κ is the shear correction factor.

The energy due to the elastic supports at beam's ends:

$$U_2 = \frac{1}{2}\left(k_{W_1} w_1^2 + k_{\Psi_1}\theta_1^2 + k_{W_{Ne}} w_{Ne}^2 + k_{\Psi_{Ne}}\theta_{Ne}^2\right),$$

where k_{W_1}, $k_{W_{Ne}}$ and k_{Ψ_1}, $k_{\Psi_{Ne}}$ are constants of the elastic restrains, w_1, w_{Ne} are the displacements in the $\bar{y}$ direction and θ_1, θ_{Ne} are the section rotation at beam ends.

Considering the energies for the k subdomains and summing them to the energies of the elastic supports, the total potential energy is given by:

$$U = \frac{1}{2}\sum_{k=1}^{N_e}\left\{\int_0^{L_k}\left[EI(\theta')^2 + \kappa AG(w' - \theta)^2\right]_k d\bar{x}_k\right\} + C_1 + \frac{1}{2}\left(k_{W_1} w_1^2 + k_{\Psi_1}\theta_1^2 + k_{W_{Ne}} w_{Ne}^2 + k_{\Psi_{Ne}}\theta_{Ne}^2\right). \tag{3}$$

The expression of the kinetic energy is derived from the velocity components of a point at a distance $\bar{y}$ from the neutral axis. The velocity components in the $\bar{x}$, $\bar{y}$ and $\bar{z}$ directions are expressed as:

$$V_x = -\bar{y}\dot{\theta}; \quad V_y = 0; \quad V_z = \dot{w} \tag{4}$$

and the kinetic energy T of the beam is given by:

$$T = \int_0^L \int_A \frac{\left(V_x^2 + V_y^2 + V_z^2\right)}{2} \rho\, dA\, d\bar{x} = \frac{1}{2}\sum_{k=1}^{N_e}\left\{\int_0^{L_k}\left[\rho I(\dot{\theta})^2 + \rho A(\dot{w})^2\right]_k d\bar{x}_k + C_2\right\}, \tag{5}$$

where C_2 is a constant and $\rho = \rho(\bar{x})$ is the material's density. Superimposed dot indicates differentiation with respect to time t.

The governing differential equations of motion are derived applying Hamilton's principle that states that $\int_{t_1}^{t_2}(T-U)dt$ taken between two specified times t_1 and t_2 is stationary for a dynamic trajectory, then:

$$\delta\int_{t_1}^{t_2}(T-U)dt=0. \tag{6}$$

When the system vibrates in one of its normal modes, the flexional displacement w and the section rotation θ can be written as:

$$w=\bar{W}\,e^{i\omega t};\quad \theta=\bar{\Psi}\,e^{i\omega t}, \tag{7}$$

where $\bar{W}=\bar{W}(\bar{x}_k)$ and $\bar{\Psi}=\bar{\Psi}(\bar{x}_k)$ are the spatial functions of the primary variables and ω is the circular natural frequency. Replacing Eqs. (7) in the energy expressions U and T of Eq. (6) and then integrating by parts, the governing element equations are obtained:

$$\left[\left(EI\bar{\Psi}'\right)'+\kappa AG\left(\bar{W}'-\bar{\Psi}\right)-\rho I\omega^2\bar{\Psi}\right]_k=0, \tag{8}$$

$$\left\{\kappa\left[AG\left(\bar{W}'-\bar{\Psi}\right)\right]'+\rho A\omega^2\bar{W}\right\}_k=0;\quad \text{for } k=1,2,\ldots N_e, \tag{9}$$

where $\kappa AG\left(\bar{W}'-\bar{\Psi}\right)=\bar{Q}$ the transverse shear force and $EI\bar{\Psi}'=\bar{M}$ the bending moment are the secondary variables.

The external boundary conditions at beam's ends are:

$$\begin{aligned}&\kappa A_jG_j\left(\bar{W}_j'-\bar{\Psi}_j\right)\pm k_{W_j}\bar{W}_j=0;\\ &E_jI_j\,\bar{\Psi}_j'\pm k_{\Psi_j}\bar{\Psi}_j=0\end{aligned} \tag{10}$$

for $j=1$ and $j=N_e$ at $\bar{x}_1=0$ and $\bar{x}_{Ne}=L_{Ne}$, respectively. Different combinations of classical boundary conditions can be obtained from Eqs. (10).

Geometrical compatibility conditions between two adjacent beam elements are:

$$\bar{W}_k(L_k)-\bar{W}_{k+1}(0)=0;\quad \bar{\Psi}_k(L_k)-\bar{\Psi}_{k+1}(0)=0;\quad \text{for } k=1,2,\ldots N_e-1, \tag{11}$$

and internal compatibility conditions of shear forces and bending moments are:

$$\begin{aligned}&\kappa A_kG_k\left(\bar{W}_k'(L_k)-\bar{\Psi}_k(L_k)\right)-\kappa A_{k+1}G_{k+1}\left(\bar{W}_{k+1}'(0)-\bar{\Psi}_{k+1}(0)\right)=0;\\ &E_kI_k\bar{\Psi}_k'(L_k)-E_{k+1}I_{k+1}\bar{\Psi}_{k+1}'(0)=0;\\ &\text{for } k=1,2,\ldots N_e-1.\end{aligned} \tag{12}$$

Notation of non-dimensional expressions is introduced as follows:

$$\begin{aligned}&x=\frac{\bar{x}_k}{L_k};\quad l_k=\frac{L_k}{L};\quad s_k=L\sqrt{\frac{(A_k)_{x=0}}{(I_k)_{x=0}}};\\ &W_k=\frac{\overline{W}_k}{L_k};\quad \Psi_k=\bar{\Psi}_k;\\ &\alpha_k=\frac{(A_k)_{x=0}}{A_0};\quad \beta_k=\frac{(I_k)_{x=0}}{I_0};\quad \delta_k=\frac{(E_k)_{x=0}}{E_0};\\ &\eta_k=\frac{(\rho_k)_{x=0}}{\rho_0};\, Q_k=\frac{\bar{Q}_k}{(E_kA_k)_{x=0}};\quad M_k=\frac{\bar{M}_k\,L_k}{(E_kI_k)_{x=0}},\end{aligned} \tag{13}$$

and the natural non dimensional frequency coefficient is expressed as:

$$\Omega=\sqrt{\rho_0A_0/(E_0I_0)}\,L^2\omega; \tag{14}$$

where $\rho_0=\rho_1(0)$; $A_0=A_1(0)$; $E_0=E_1(0)$; $I_0=I_1(0)$.

Finally, the governing differential element equations become:

$$\frac{\kappa_ks_1^2\,s_k^2}{2(1+v)}\,E_kA_k\left(\Psi_k-W_k'\right)-\frac{s_1^2}{l_k^2}\left(E_k'I_k\Psi_k'+E_kI_k\Psi_k''\right)-\Omega^2\rho_kI_k\Psi_k=0; \tag{15}$$

$$\frac{\kappa_ks_1^2}{2(1+v_k)l_k^2}\left[E_k'A_k\left(W_k'-\Psi_k\right)+E_kA_k\left(W_k''-\Psi_k'\right)\right]+\Omega^2\rho_kA_kW_k=0; \tag{16}$$

for $k=1,2,\ldots N_e$.

3 The DQM

In order to obtain the DQM analog equations of the governing equations of the AFG stepped Timoshenko beam, each beam subdomain k is discretized in a grid of N points using the Chebyshev–Gauss–Lobato expression [20–22]

$$x_i=\{1-\cos[(i-1)\,\pi/\,(N-1)]\}/2\,,\quad i=1,2,\ldots,N,$$

where x_i is the coordinate of point i. Based on the quadrature rules [21], the qth order derivatives of flexural displacement W and section rotation Ψ at a point i of the grid are expressed as:

$$\left.\frac{d^{(q)}W_k}{dx^q}\right|_{x_i} = \sum_{j=1}^{N} C_{ij}^{(q)}\, W_{kj}; \left.\frac{d^{(q)}\Psi_k}{dx^q}\right|_{x_i} = \sum_{j=1}^{N} C_{ij}^{(q)}\, \Psi_{kj}, \quad (17)$$

where W_{kj} and Ψ_{kj} correspond to point j of subdomain k, and $C_{ij}^{(q)}$ are the weighting coefficients associated with the qth order derivative. They were obtained using Lagrange interpolating functions:

$$\Pi(x_i) = \prod_{j=1 j\neq i}^{N} \left(x_i - x_j\right) \quad \text{for} \quad i,j = 1,2,\ldots,N.$$

The off-diagonal terms of the weighting coefficient matrix of the first-order derivative are:

$$C_{ij}^{(1)} = \frac{\prod(x_i)}{(x_i - x_j)\ \prod(x_j)} \quad \text{for} \quad i,j = 1,2,\ldots,N \text{ and } i \neq j. \quad (18)$$

The off-diagonal terms of the weighting coefficient matrix of the second-order and higher-order derivatives are obtained through the recurrence expression:

$$C_{ij}^{(q)} = q\left[C_{ii}^{(q-1)} C_{ij}^{(1)} - \frac{C_{ij}^{(q-1)}}{(x_i - x_j)}\right] \quad \text{for} \quad i,j = 1,2,\ldots,N \text{ and } i \neq j;\ 2 \leq q \leq (N-1). \quad (19)$$

And the diagonal terms of the weighting coefficient matrix are given by:

$$C_{ii}^{(q)} = -\sum_{j=1 j\neq i}^{N} C_{ij}^{(r)}; \quad \text{for} \quad i = 1,2,\ldots,N; 1 \leq q \leq (N-1). \quad (20)$$

Using the quadrature rules, Eq. (18), the differential quadrature analogs of the governing Eqs. (16) and (17) of a node i are:

$$\frac{s_1^2 s_k^2 \kappa_k}{2(1+\nu_{ki})} A_{ki} E_{ki}\left(\Psi_{ki} - \sum_{j=1}^{N} C_{ij}^{(1)}\, W_{kj}\right) - \frac{s_1^2}{l_k^2} I_{ki} E_{ki}' \sum_{j=1}^{N} C_{ij}^{(1)}\, \Psi_{kj} - \frac{s_1^2}{l_k^2} I_{ki} E_{ki}\left(\sum_{j=1}^{N} C_{ij}^{(2)}\, \Psi_{kj}\right) - \Omega^2 \rho_{ki} I_{ki} \Psi_{ki} = 0, \quad (21)$$

$$\frac{\kappa_k}{2(1+\nu_k)} \frac{s_1^2}{l_k^2} E_{ki}' A_{ki}\left[\left(\sum_{j=1}^{N}\left(C_{ij}^{(1)}\right) W_{kj} - \Psi_{kj}\right) + E_{ki} A_{ki}\left(\sum_{j=1}^{N}\left(C_{ij}^{(2)}\right) W_{kj} - \sum_{j=1}^{N}\left(C_{ij}^{(1)}\right) \Psi_{kj}\right)\right] + \Omega^2 \rho_{ki} A_{ki}\, W_{ki} = 0. \quad (22)$$

The analog equations of internal forces at node i:

$$Q_{ki} = \frac{\kappa_{ki}}{2(1+\nu_{ki})} E_{ki} A_{ki}\left[\left(\sum_{j=1}^{N} C_{ij}^{(1)}\, W_{kj}\right) - \Psi_{ki}\right]; \quad M_{ki} = E_{ki} I_{ki}\left(\sum_{j=1}^{N} C_{ij}^{(1)}\, \Psi_{kj}\right); \quad (23)$$

the outer boundary conditions are given by:

$$Q_{1\,1} = k_{W_1} l_1^3 W_{1\,1}; M_{1\,1} = k_{\Psi_1} l_1 \Psi_{1\,1}; \quad Q_{Ne\,N} = -k_{W_{Ne}}\, l_{Ne}^3\, W_{Ne\,N}; \quad M_{Ne\,N} = -k_{\Psi_{Ne}} l_{Ne} \Psi_{Ne\,N}. \quad (24)$$

the analog continuity equations at adjacent beam elements become, for the geometrical compatibility conditions:

$$l_k W_{kN} - l_{k+1} W_{(k+1)1} = 0; \quad \Psi_{kN} - \Psi_{(k+1)1} = 0, \quad (25)$$

and for the internal compatibility conditions of shear forces and bending moment using secondary variables Eqs. (23):

$$\alpha_k \delta_k Q_{kN} - \alpha_{(k+1)} \delta_{(k+1)} Q_{(k+1)1} = 0; \quad \frac{\beta_k \delta_k}{l_k} M_{kN} - \frac{\beta_{(k+1)} \delta_{(k+1)}}{l_{k+1}} M_{(k+1)1} = 0, \quad (26)$$

where the constants α_k, β_k, δ_k, defined in Eqs. (13), take into account the discontinuities in material properties and in geometry. The set of analog Eqs. (21–26) constitute the linear system of equations that allows determining the natural frequencies of the stepped AFG Timoshenko beam.

4 Numerical results

The natural frequency coefficients, Eq. (14), are obtained for a range of illustrative examples. Timoshenko beams with different material properties and different locations of the abrupt discontinuities are studied. The cross-section is of rectangular form and the geometrical relation between height and length can be expressed as:

$$\frac{h}{L} = \frac{\sqrt{12}}{s_1}\ ; \ \text{with} \quad h = h_1(0).$$

Table 1 Convergence analysis: first six natural frequency coefficients of uniform homogeneous Timoshenko beams: $h/L = 0.35$; $\kappa = 5/6$; $\nu = 0.30$

B.C.	N	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	
C–F	11	3.22713	14.4689	31.5016	47.8895	62.3557	68.0104	Present
	21	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
	31	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
	41	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
	51	3.22713	14.4689	31.5025	47.9090	62.3470	67.9901	
		3.23	14.47	21.50	47.91	62.35	–	[12]
		3.227128	14.468928	31.502540	47.911084	62.353342	–	[13]
		3.2272	14.4729	31.5425	48.0372	–	–	[16]
C–SS	41	11.082499	27.114378	44.843534	59.203032	63.339499	76.247312	Present
		11.08	27.11	44.84	59.20	63.34	–	[12]
		11.082499	27.114378	44.844585	59.203448	63.349869	–	[13]
C–C	41	13.834758	28.517925	45.665951	61.862050	68.283611	80.412094	Present
		13.84	28.52	45.67	61.86	68.28	–	[12]
		13.834758	28.517926	45.667237	61.867699	68.292529	–	[13]
F–F	41	16.791957[a]	33.814869	51.521440	58.991998	73.739689	75.304144	Present
		16.79	33.82	51.52	58.99	73.74	–	[12]
		16.791957	33.814869	51.526943	58.993336	73.763812	–	[13]

[a] The repeated null eigenvalues for rigid translation and rotation for the F–F case are omitted in the table

In all the numerical examples the shear correction factor is assumed as: $\kappa = 5/6$.

Table 1 contains a convergence analysis. The DQM results for the first six frequency coefficients of a uniform homogeneous Timoshenko beam under various classical boundary conditions are listed. The rate of convergence and accuracy of the proposed differential quadrature procedure can be observed as the number of grid points, N, increases. The obtained values are compared with results available in the literature. The agreement between those results is excellent, and it can be concluded that the proposed procedure has adequate accuracy with $N = 41$ grid points.

Table 2 presents the first six frequency coefficients of a tapered AFG Timoshenko beam under three different combinations of boundary conditions. The grid is obtained taking $N = 41$ points. To make a comparison with published results, material properties are assumed to vary according to Eq. (1), with $n = 1$, 2, 3 and 4.

$$E_k = E_k(x) = E_a\left[1 + \left(\chi_{E_k} - 1\right)x^n\right];$$
$$\rho_k = \rho_k(x) = \rho_a\left[1 + \left(\chi_{\rho_k} - 1\right)x^n\right], \qquad (27)$$

with $\chi_{E_k} = E_b/E_a$ and $\chi_{\rho_k} = \rho_b/\rho_a$.

The beam cross section has variable height $h(x)$ and constant wide $b(x) = b$.

In the calculations, the constituents of the inhomogeneous material are assumed to be aluminum Al and zirconia ZrO_2. Their Young modulus and density are:

$$E_{Al} = 70\,\text{GPa}; \quad \rho_{Al} = 2{,}700\,\text{kg/m}^3;$$
$$E_{ZrO_2} = 200\,\text{GPa}; \quad \rho_{ZrO_2} = 5{,}700\,\text{kg/m}^3; \qquad (28)$$
$$\nu_{Al} = \nu_{ZrO_2} = 0.30$$

It can be seen that the agreement with previous published results is excellent.

Tables 1 and 2 demonstrate the rate of convergence and accuracy of the approach proposed.

The results on Table 3 show the effect of an AFG material on the frequency coefficients of a uniform Timoshenko beam. $s_1 = 12.5$, is equivalent to $h_0/L \cong 0.28$; $L_1 = L$. Eight different combinations of classical boundary conditions are adopted. The material properties vary according to Eqs. (27), with $n = 1, 2$ and 3. The domain is discretized in a grid of $N = 41$ points.

Next, free vibrations of stepped AFG Timoshenko beams are studied. Different boundary conditions, step locations and material properties are considered. Cases A–C described the geometric variation as follows:

Case A	$L = L_1 + L_2$	$h_2 = h_1$;	$b_2 = \xi_b\, b_1$;	$A_2 = \xi_b\, A_1$;	$I_2 = \xi_b\, I_1$.
Case B	$L = L_1 + L_2$	$h_2 = \xi_h\, h_1$;	$b_2 = b_1$;	$A_2 = \xi_h\, A_1$;	$I_2 = \xi_h^3\, A_1$.
Case C	$L = L_1 + L_2$	$h_2 = \xi_h\, h_1$;	$b_2 = \xi_b\, b_1$;	$A_2 = \xi_h\, \xi_b\, A_1$;	$I_2 = \xi_b\, \xi_h^3\, A_1$.

with ξ_b and ξ_h constants. The three geometrical cases, Fig. 3, are assumed introducing stepped variations of the area and the second moment of area, [10]. One of the elements of the stepped beam has constant material properties while the other has AFG properties.

The material properties of the portion of the beam of length L_1 are supposed to have AFG characteristics, Eqs. (27–28): $n_1 = 1$, 2 and 3; $\chi_{E_1} = 70/200 = 0.35$; $\chi_{\rho_1} = 2{,}700/5{,}700 = 0.474$ with constant cross-section A_1.While the other part of the stepped beam of length L_2, has homogeneous material: $\chi_{E_2} = 200/200 = 1$; $\chi_{\rho_2} = 5{,}700/5{,}700 = 1 = 1$, and the cross-sectional area being constant and equal to A_2.

Tables 4, 5 and 6 present the first six natural frequency coefficients of cantilever beams of Fig. 3 with a step located at $l_1 = L_1/L = 0.250$, 0.370, 0.620 and 0.750.

In Tables 4 and 5 a comparison is made with Mao et al. [10] when the material properties are assumed to be constant in both beam elements. Mao et al. [10] have based their results on Euler–Bernoulli beam theory. In the present paper, $h_0/L = 0.0017$ is used to

Table 2 First six natural frequency coefficients of AFG Timoshenko beams, with a small taper in height: $h(x) = h_0(1 - 0.1x)$; $h_0/L = 0.35$; $\kappa = 5/6$; $\nu = 0.30$; $\chi_E = 0.35$; $\chi_\rho = 0.47$; $N = 41$

B.C.	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	
C–F	1	3.944636	14.93640	30.57274	46.40688	60.9420	65.7584	Present
		3.944636	14.93640	30.57274	46.40888	–	–	[13]
	2	3.935789	15.15333	31.22390	47.58364	62.7344	66.9431	Present
		3.935789	15.15333	31.22390	47.58572	–	–	[13]
		3.9359	15.1577	31.2638	47.7164	–	–	[16]
	3	3.849497	15.19867	31.59328	48.24423	63.7301	67.5523	Present
		3.849497	15.19869	31.59328	48.24669	–	–	[13]
	4	3.77127	15.1970	31.8164	48.6325	64.3432	67.9315	Present
		3.771269	15.19695	31.81639	48.63501	–	–	[13]
C–SS	1	10.88465	25.56609	42.18263	58.13438	60.9556	74.1197	Present
		10.88465	25.56609	42.18907	58.14309	–	–	[13]
	2	10.80070	25.61789	42.64742	58.85281	62.7800	75.2574	Present
		10.80070	25.61789	42.64780	58.85946	–	–	[13]
	3	10.73937	25.63540	42.85451	59.08722	63.7788	75.7602	Present
		10.73937	25.63540	42.85506	59.09377	–	–	[13]
	4	10.71567	25.66653	42.95520	59.14091	64.391	75.9913	Present
		10.71567	25.66653	42.95581	59.14709	–	–	[13]
C–C	1	12.68158	26.49101	42.64171	58.65182	66.816	75.9159	Present
		12.68158	26.49101	42.64203	58.66849	–	–	[13]
	2	12.46329	26.38044	42.96071	59.39162	68.058	77.0951	Present
		12.46329	26.38044	42.92108	59.40234	–	–	[13]
		12.4689	26.4153	43.0904	59.6829	–	–	[16]
	3	12.37525	26.31883	43.08334	59.68936	68.5813	77.5992	Present
		12.37525	26.31883	43.08388	59.69942	–	–	[13]
	4	12.36220	26.31154	43.13433	59.80551	68.8795	77.8249	Present
		12.36220	26.31158	43.13499	59.81504	–	–	[13]

Table 3 First six natural frequency coefficients of uniform cross-section AFG Timoshenko beams with various boundary conditions

n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
	SS–C						*C–SS*					
(*)	12.1785	31.2031	52.8839	75.5682	91.1848	98.6071	12.1785	31.2031	52.8839	75.5682	91.1848	98.6071
1	10.6527	28.775	49.4047	71.0515	87.9797	93.3051	12.1651	29.9625	50.4151	71.7136	84.8549	93.2976
2	10.6989	28.9102	49.9738	72.0967	89.5464	94.6851	12.0899	30.0762	50.9878	72.6812	87.2899	94.7715
3	10.7643	28.9303	50.1384	72.4798	90.0939	95.3002	12.0341	30.1315	51.2538	73.0823	88.5729	95.4611
	C–C						*SS–SS*					
(*)	15.6659	33.6285	54.4651	76.0997	98.6071	98.8973	8.82664	28.3570	51.2257	74.8810	88.4591	98.5858
1	14.6202	31.5549	51.3116	71.8519	93.0976	94.0592	8.28580	26.8203	48.4458	70.8139	83.4161	93.2972
2	14.3771	31.4512	51.6786	72.7317	94.5697	95.9212	8.45284	27.2194	49.237	71.9953	85.4714	94.5548
3	14.2839	31.3979	51.8204	73.097	95.176	96.7421	8.53962	27.3644	49.5312	72.4364	86.4428	95.0597
	C–F						*F–C*					
(*)	3.32139	16.2331	36.5346	57.9414	79.6803	93.6481	3.32139	16.2331	36.5346	57.9414	79.6803	93.6481
1	4.02882	16.8325	35.8482	56.0353	76.3509	88.5229	2.39704	13.9273	33.4166	53.8750	74.8178	89.1143
2	4.01239	17.0959	36.6134	57.3718	78.3942	90.6695	2.47269	14.2114	33.9864	54.8942	76.3463	90.7864
3	3.91997	17.1605	37.0517	58.1304	79.5352	91.8588	2.56071	14.4004	34.1414	55.1771	76.8348	91.3604
	SS–F						*F–SS*					
(*)	0	13.1082	33.8752	56.692	78.8321	90.6865	0	13.1082	33.8752	56.692	78.8321	90.6865
1	0	13.3545	32.9559	54.5518	75.286	87.1940	0	11.6817	31.5971	53.1957	74.4974	84.2197
2	0	13.8453	34.0090	56.1060	77.5417	88.6983	0	12.1240	32.4552	54.417	76.2008	86.1989
3	0	13.9978	34.5127	56.9025	78.7123	89.4351	0	12.3723	32.7588	54.811	76.7682	87.1194

$h_0/L = 0.28$; $\chi_E = 0.35$; $\chi_\rho = 0.47$; $N = 41$

(*) Homogeneous material $\chi_E = 200/200 = 1$; $\chi_\rho = 5{,}700/5{,}700 = 1$

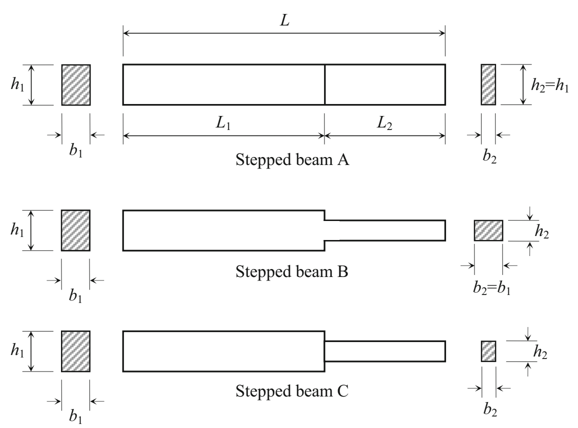

Fig. 3 Stepped AFG Timoshenko beams

propose a limiting situation. The mentioned value is small enough to neglect the effects of shear force and rotary inertia in the application of the Timoshenko beam theory. For that reason, results calculated with 0.0017, become comparable to Euler–Bernoulli theory's results (as shown in Tables 4, 5). It can be seen that the agreement with [10] is excellent.

Figure 4 shows the fundamental frequency coefficients for cantilever beams, with different locations of the step. l_1 is equal to 0.25, 0.375, 0.625, 0.75 and AFG material properties for the part of the beam of length L_1 are $\chi_{E_1} = 70/200 = 0.35$; $\chi_{\rho_1} = 2{,}700/5{,}700 = 0.474$; and for the element of length L_2: $\chi_{E_2} = 200/200 = 1$; $\chi_{\rho_2} = 5{,}700/5{,}700 = 1$. $h_0/L = 0.0017$ (Beam A, color solid line; Beam B, color dotted line; Beam C, color dashed line). The frequency coefficients of the stepped beams can be compared with the coefficients of the uniform beam of similar material properties, which is indicated by a solid black line. It can be seen that it is possible to have lighter structures with higher coefficients of fundamental frequency when the beams are of AFG materials. [1].

Hereafter there are several numerical examples of frequency coefficients of stepped Timoshenko beams with different AFG materials and combinations of classical or elastic boundary conditions.

Table 6 is similar to 4, with $h_0/L = 0.28$. Table 7 presents natural frequency coefficients of stepped

Table 4 First six natural frequency coefficients of clamped-free AFG beams with a step

l_1	n_1	Ω_1	[10]	Ω_2	[10]	Ω_3	[10]	Ω_4	[10]	Ω_5	Ω_6
0.250	(*)	4.3468	4.3468	24.1602	24.1602	62.4786	62.4811	120.3563	120.365	200.7861	299.6742
	1	3.8637		23.6956		64.1197		121.8507		199.6484	298.4878
	2	4.0457		24.1105		64.2078		121.4757		199.4563	298.747
	3	4.1295		24.2318		63.9806		121.055		199.3913	299.0133
0.375	(*)	4.6338	4.6337	22.9914	22.992	61.3733	61.3763	121.9037	121.9125	198.2126	299.2762
	1	4.0354		23.6227		62.0314		120.8812		197.9171	293.4266
	2	4.2798		23.7724		61.7598		120.9824		198.3788	294.4107
	3	4.3884		23.6998		61.5064		121.1276		198.5901	294.9978
0.625	(*)	4.6338	4.6337	22.9916	22.992	61.3755	61.3763	121.9103	121.9125	198.2126	299.2554
	1	4.3126		22.2184		61.1649		117.8723		193.6996	290.8491
	2	4.5714		22.2770		61.3557		118.5899		195.4600	293.7625
	3	4.6547		22.2811		61.4775		119.0521		196.4258	295.3792
0.750	(*)	4.3469	4.3468	24.1607	24.1602	62.4806	62.4811	120.3628	120.365	200.8059	299.7157
	1	4.3801		22.0700		59.4769		117.2304		193.6328	287.011
	2	4.5772		22.4126		59.9505		118.4397		195.8656	290.6275
	3	4.6072		22.6572		60.2381		119.1656		197.1076	292.4955

$h_0/L = 0.0017$. $b_2 = 0.5b_1$; $h_2 = h_1$. Beam A; $N = 41$

(*) Homogeneous material $\chi_{E_1} = \chi_{E_2} = 200/200 = 1$; $\chi_{\rho_1} = \chi_{\rho_2} = 5{,}700/5{,}700 = 1$

Table 5 First six natural frequency coefficients of clamped-free AFG beams with a step

l_1	n_1	Ω_1	[10]	Ω_2	[10]	Ω_3	[10]	Ω_4	[10]	Ω_5	Ω_6
0.250	(*)	2.7846	2.7846	15.6249	15.6246	37.9882	37.9887	68.9783	68.9797	115.7087	177.3212
	1	2.6467		14.9821		38.2198		71.5726		118.2374	177.1747
	2	2.7016		15.3217		38.6780		71.4441		117.7290	176.9778
	3	2.7255		15.4488		38.7363		71.0993		117.2629	176.8448
0.375	(*)	3.4955	3.4954	15.5133	15.5134	37.6981	37.6984	78.7317	78.7355	127.8980	182.6741
	1	3.2168		15.4993		39.7631		78.6336		125.6686	185.0525
	2	3.3340		15.8160		39.5096		78.4952		126.3286	185.3328
	3	3.3840		15.8727		39.1949		78.4009		126.7853	185.3927
0.625	(*)	4.4912	4.4914	16.7903	16.7903	46.8926	46.8937	89.9449	89.9482	148.9669	226.3189
	1	4.1806		17.6457		45.6956		90.2874		144.5670	221.5364
	2	4.4176		17.4948		46.0733		90.4663		146.0521	223.2275
	3	4.4952		17.3331		46.3366		90.4909		147.0350	224.0949
0.750	(*)	4.3318	4.3318	21.8649	21.8650	48.1350	48.1358	99.8838	99.8900	168.7895	238.3905
	1	4.3597		20.6312		49.2036		95.5215		161.9129	236.5752
	2	4.5553		20.8023		49.3366		96.7237		163.6992	238.9683
	3	4.5855		20.9209		49.3557		97.5346		164.6782	240.1911

$h_0/L = 0.0017$. $b_2 = b_1$; $h_2 = 0.5h_1$. Beam-B; $N = 41$

(*) Homogeneous material $\chi_{E_1} = \chi_{E_2} = 200/200 = 1$; $\chi_{\rho_1} = \chi_{\rho_2} = 5{,}700/5{,}700 = 1 = 1$

Table 6 First six natural frequency coefficients of clamped-free AFG beams with a step

l_1	n_1	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.250	(*)	4.06449	17.8630	37.1720	57.3490	79.4140	92.6848
	1	3.63133	17.2152	37.8868	58.8358	79.3544	94.5491
	2	3.79451	17.5841	38.1841	58.6410	79.0353	94.5874
	3	3.86955	17.7206	38.1507	58.3178	78.8752	94.4485
0.375	(*)	4.32412	17.2855	35.7138	58.0554	78.0655	92.5543
	1	3.78354	17.2768	37.0641	57.7466	78.9122	91.8550
	2	4.00354	17.5597	36.7862	57.7511	79.1242	91.3234
	3	4.10153	17.6018	36.4833	57.7650	79.1120	91.0414
0.625	(*)	4.32733	16.9958	36.7192	57.6770	76.8641	93.2328
	1	4.02863	16.6640	35.8378	56.0373	78.1264	91.7424
	2	4.26506	16.7748	36.0281	55.9663	78.6646	93.2880
	3	4.34232	16.7660	36.1687	55.9462	78.9014	94.1504
0.750	(*)	4.06743	17.5799	37.1502	59.1402	81.0677	89.7291
	1	4.08903	16.5377	35.4589	55.3383	75.7887	90.5304
	2	4.27107	16.7656	35.8330	56.0908	76.3512	91.5395
	3	4.30051	16.8773	36.0521	56.7025	76.6374	91.8365

$h_0/L = 0.28$. $b_2 = 0.5b_1$; $h_2 = h_1$. Beam A; $N = 41$

(*) Homogeneous material $\chi_{E_1} = \chi_{E_2} = 200/200 = 1$; $\chi_{\rho_1} = \chi_{\rho_2} = 5{,}700/5{,}700 = 1$

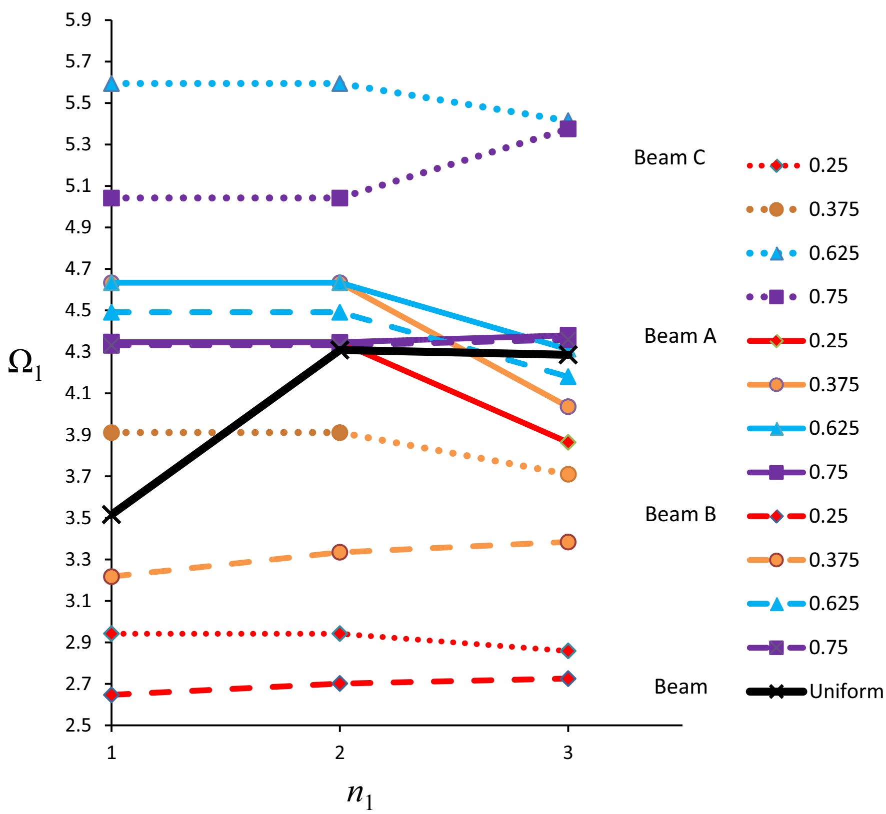

Fig. 4 Fundamental natural frequency coefficient for cantilever AFG stepped beams ($N = 41$)

Table 7 First six natural frequency coefficients of beams of AFG materials with a step

B.C.	n_1	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
SS–C	(*)	7.19564	27.5087	48.6301	67.294	90.7471	94.8681
	1	8.10075	27.1250	47.9073	67.3493	87.9421	91.0087
	2	7.88432	27.7650	48.2833	68.4196	89.4862	92.1438
	3	7.74909	28.0319	48.3773	68.8784	90.3165	92.5564
	(*)	*(8.51776 **)*	*(37.4279)*	*(80.3227)*	*(129.222)*	*(211.882)*	*(277.673)*
C–SS	(*)	9.07035	28.0166	44.0097	67.7693	91.2451	97.3789
	1	10.3578	27.4970	45.4545	66.7572	88.6132	96.2024
	2	10.2765	27.7143	45.2915	67.5657	89.9042	97.5837
	3	10.1043	27.8085	45.1025	68.0746	90.5237	98.1914
	(*)	*(10.4072 **)*	*(42.1479)*	*(71.4341)*	*(141.237)*	*(195.943)*	*(295.073)*
C–C	(*)	11.0143	30.3741	49.5723	69.2024	92.5762	99.0514
	1	12.8290	30.4488	49.6334	68.9892	89.9604	96.9636
	2	12.5740	30.7034	49.6123	69.7247	91.5159	98.1565
	3	12.3046	30.7872	49.5514	70.1695	92.2960	98.6610
	(*)	*(13.6160**)*	*(49.4501)*	*(87.6013)*	*(151.384)*	*(224.476)*	*(308.164)*
C–F	(*)	5.20353	13.5788	30.9211	51.5550	70.5594	93.5004
	1	5.05507	14.4658	31.1255	51.3059	71.2779	91.3077
	2	5.28648	14.3217	31.3292	51.2914	71.9772	93.3965
	3	5.34654	14.1618	31.3901	51.2428	72.3584	94.5246
	(*)	*(5.59372**)*	*(15.9079)*	*(49.0616)*	*(87.1239)*	*(151.407)*	*(224.520)*
F–C	(*)	0.929072	9.20821	34.9496	51.9305	71.4098	94.814
	1	1.012010	10.1953	32.1938	51.6411	70.3562	90.5051
	2	0.966524	10.0134	33.4188	52.4965	72.1808	91.6169
	3	0.950797	9.89641	34.0806	52.7623	72.8943	91.9021
	(*)	*(0.947401**)*	*(11.4575)*	*(50.0108)*	*(87.6521)*	*(151.347)*	*(224.476)*
SS–SS	(*)	4.98837	25.1872	42.901	65.7235	90.6115	92.4199
	1	5.63974	24.0047	43.5573	64.9097	87.9153	89.0929
	2	5.48827	24.6502	43.7541	66.1185	89.469	90.1089
	3	5.39267	24.9659	43.7127	66.6739	90.1937	90.5524
	(*)	*(5.34078**)*	*(32.6569)*	*(63.1071)*	*(120.781)*	*(184.236)*	*(263.781)*
SS–F	(*)	0	11.7121	28.0948	50.5545	68.684	90.7496
	1	0	11.4088	27.8168	49.5263	69.6462	87.9270
	2	0	11.5026	28.4034	49.9353	70.6627	89.4768
	3	0	11.5485	28.6469	50.0470	71.0366	90.3489
	(*)	*(0)*	*(12.9282 **)*	*(37.2397)*	*(79.7088)*	*(129.215)*	*(211.932)*
F–SS	(*)	0	6.72053	31.5573	47.1766	70.2838	92.4005
	1	0	7.37897	28.6308	47.6707	68.3732	89.0052
	2	0	7.25495	29.6362	48.521	70.3443	90.0524
	3	0	7.17966	30.1991	48.7932	71.1526	90.255
	(*)	*(0)*	*(7.52668 **)*	*(42.5766)*	*(71.6266)*	*(141.200)*	*(195.939)*
F–F	(*)	0	0	13.5386	35.5016	54.0474	72.4241
	1	0	0	13.4427	32.8224	53.5280	72.2935
	2	0	0	13.5085	34.0342	54.4367	74.0526
	3	0	0	13.5416	34.6933	54.7046	74.6769
	(*)	*(0)*	*(0)*	*(15.2543 **)*	*(49.6032)*	*(87.1795)*	*(151.369)*

$l_1 = 0.625$; $b_2 = 0.5b_1$; $h_2 = 0.5h_1$; $h_0/L = 0.28$. Beam-C; $N = 41$

(*) Homogeneous material; $\chi_{E_1} = \chi_{E_2} = 200/200 = 1$; $\chi_{\rho_1} = \chi_{\rho_2} = 5{,}700/5{,}700 = 1$

(**) Slender beam is assumed with $h_0/L = 0.0017$

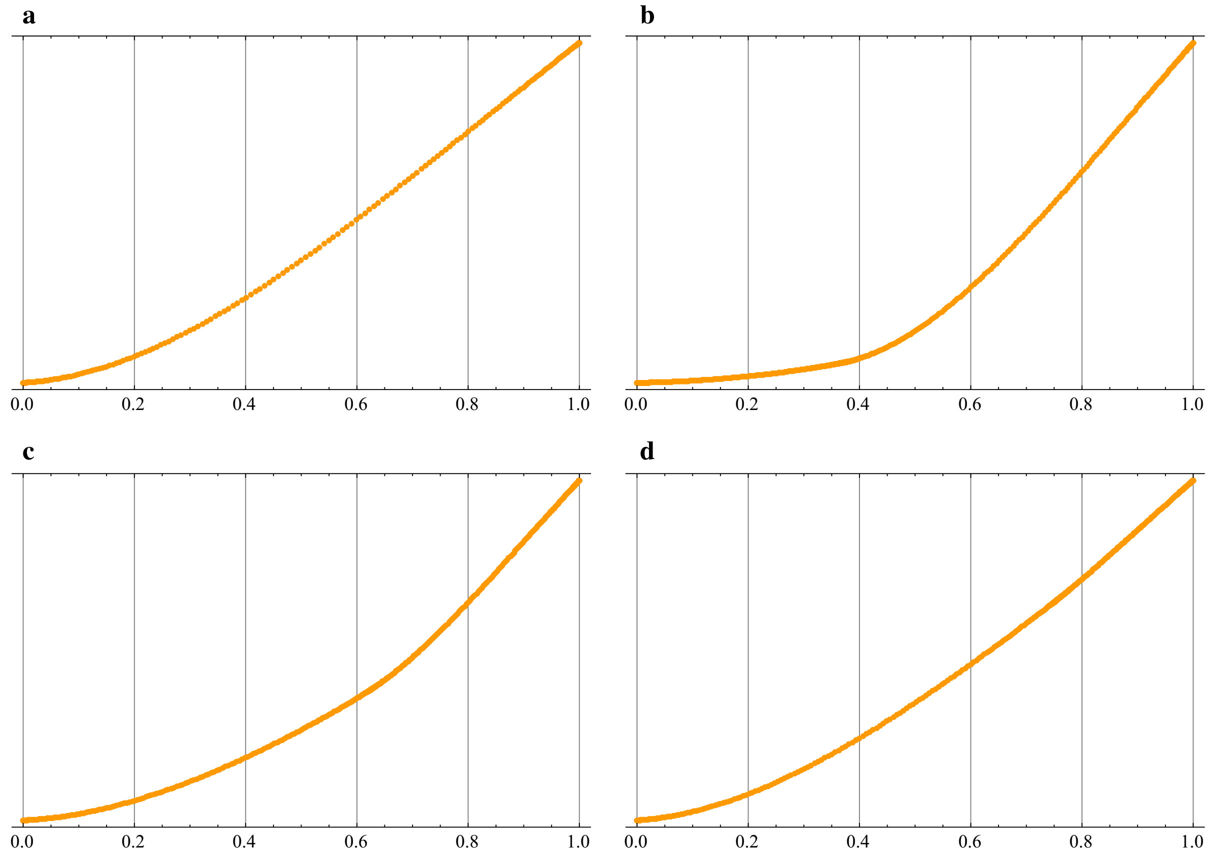

Fig. 5 Fundamental mode shapes of cantilever Timoshenko beams. Beam C; $\kappa = 0.833333$; $h_0/L = 0.28$; $h_0 = h_1$; $h_2 = 0.5h_1$; $b_2 = 0.5b_1$. $N = 41$. **a** Uniform beam $l_1 = 1$, homogeneous material. **b** Stepped beam $l_1 = 0.375$, AFG material, $n = 3$. **c** Stepped beam $l_1 = 0.625$, AFG material, $n = 3$. **d** Stepped beam $l_1 = 0.750$, AFG material, $n = 3$

Table 8 First six natural frequency coefficients of beams of AFG material with a step and elastic boundary conditions

$K_{W_1} = K_{W_{Ne}}$	$K_{\Psi_1} = K_{\Psi_{Ne}}$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
1	0	5.2293	21.0805	34.846	49.7056	67.014	80.1735
10	0	5.37586	24.5944	42.8232	64.784	84.576	89.4175
100	0	5.39099	24.9293	43.627	66.5091	89.8933	90.2136
1	1	6.61123	21.3832	35.2881	50.4601	68.2869	81.9322
10	1	6.89815	25.4599	43.534	65.0541	85.6113	90.6153
100	1	6.92807	25.8747	44.4565	66.9906	90.3376	91.5538
1	2	7.41451	21.5809	35.587	50.9588	69.1919	83.3195
10	2	7.82332	26.0726	44.0646	65.2568	86.1678	91.6842
100	2	7.86625	26.5516	45.0818	67.3511	90.5234	92.6861
100	3	8.51358	27.0604	45.5686	67.6306	90.6665	93.5384
100	5	9.35946	27.7752	46.2747	68.0353	90.8759	94.707
100	10	10.4037	28.7337	47.2588	68.599	91.1743	96.1681

$l_1 = 0.625$; $b_2 = 0.5b_1$; $h_2 = 0.5h_1$; $h_0/L = 0.28$. Beam-C; AFG material with $n = 3$; $N = 41$

Table 9 First six natural frequency coefficients of beams of AFG materials with a step and elastic boundary conditions

$k_{\Psi_1} = k_{\Psi_{Ne}}$	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0	(*)	3.84424	9.47446	20.5316	37.1839	55.3907	72.7467
	1	4.38649	10.0526	20.053	34.855	54.6429	72.7562
	2	4.23327	9.7874	20.1879	35.9804	55.5704	74.476
	3	4.1482	9.70539	20.2749	36.6016	55.8557	75.0765
1	(*)	4.38549	10.2496	21.3403	39.0375	57.8117	74.7584
	1	5.06205	10.8117	21.1405	37.021	57.0023	75.0265
	2	4.84748	10.6166	21.1735	38.0236	57.8369	76.62
	3	4.73719	10.5483	21.2125	38.6008	58.1074	77.2004
10	(*)	5.00368	11.8244	23.2906	43.6574	64.3114	81.5589
	1	5.75503	12.0887	23.7343	42.1856	63.4559	82.492
	2	5.49418	12.1407	23.6681	43.0553	64.0902	83.8209
	3	5.36764	12.1508	23.6043	43.5384	64.2884	84.3639
100	(*)	5.15025	12.3819	24.1386	45.6905	67.2619	85.0984
	1	5.90898	12.464	24.7852	44.307	66.331	86.1555
	2	5.63961	12.6228	24.7585	45.2416	66.9631	87.5202
	3	5.51084	12.6748	24.672	45.7045	67.1246	88.061

$l_1 = 0.625$; $b_2 = 0.5b_1$; $h_2 = 0.5h_1$; $h_0/L = 0.28$. Beam-C; $k_{W_1} = k_{W_{Ne}} = 0.10$; $N = 41$

(*) Homogeneous material; $\chi_{E_1} = \chi_{E_2} = 200/200 = 1$; $\chi_{\rho_1} = \chi_{\rho_2} = 5{,}700/5{,}700 = 1$

AFG Timoshenko beams. For element $k = 1$: $l_1 = 0.625$; $\chi_{E_1} = 70/200 = 0.35$; $\chi_{\rho_1} = 2{,}700/5{,}700 = 0.474$; and for element $k = 2$: $l_2 = 0.375$; $\chi_{E_2} = 200/200$; $\chi_{\rho_2} = 5{,}700/5{,}700$ (Case C).

Figure 5 shows the fundamental mode shapes of cantilever Timoshenko beams. Figure 5a corresponds to a uniform beam of homogeneous material. Figure 5b–d correspond to stepped beams, (case Beam C; $b_2 = 0.5b_1$; $h_2 = 0.5h_1$), with the step located at $l_1 = 0.375$, 0.625 and 0.750, respectively. Again the portion of the beam of length L_1 is made of AFG material with $n = 3$ Eqs. (27–28). The span of length L_2 has homogeneous material. In general, the effect of the step on the dynamic behavior of the beam can be observed in the magnitude of the fundamental frequency coefficient and in the shape associated to this mode.

Tables 8 and 9 present frequency coefficients for stepped Timoshenko beams with elastic restrains at external ends. Both Tables are related with the case of stepped Beam C with $b_2 = 0.5b_1$; $h_2 = 0.5h_1$. The beam element of length $L_1 = 0.625L$ is made of AFG material with cubic variation, $n = 3$: $\chi_{E_1} = 70/200 = 0.35$; $\chi_{\rho_1} = 2{,}700/5{,}700 = 0.474$; the beam element of length L_2 is of homogeneous material: $\chi_{E_2} = 200/200 = 1$; $\chi_{\rho_2} = 5{,}700/5{,}700 = 1$. Boundary conditions are assumed elastic. The translational restrain conditions indicated by K_{W_j} with $j = 1$ and $j = N_e$ and the rotational restrains by K_{Ψ_j} with $j = 1$ and $j = N_e$. are varied.

In Table 8 variable boundary conditions are presented, while in Table 9 constant boundary conditions for translational displacements are assumed: $k_{W_1} = k_{W_{Ne}} = 0.10$.

In both tables it can be observed that frequency coefficients increase as the boundary conditions stiffen.

5 Conclusions

This paper examines the case of vibrations of stepped inhomogeneous beams on the basis of the Timoshenko beam theory. Different combinations of classical and elastic boundary conditions are considered. The equations of motion for the AFG stepped beams are obtained applying Hamilton's principle.

The DQM directly solves the ordinary differential equations and it is applied for any type of inhomogeneity in the axial direction (stepped change in geometry and/or material properties).

The variation of the material properties and stepped changes play an important role on the variations of the natural frequency coefficients. It is possible to have lighter structures with higher coefficients of fundamental frequency when the beams are of AFG materials and have stepped variations of the cross-sectional area, second moment of area and material properties.

Additionally, since to the authors' knowledge this technological situation has not been previously studied in the literature, the present results may be used as a means of comparison for future studies.

Acknowledgments The authors acknowledge the Universidad Nacional del Sur and the Consejo Nacional de Investigaciones Científicas y Técnicas for the financial support which has enabled the present research. The authors are indebted to the unknown reviewers for the valuable suggestions.

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