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A novel non-primitive Boundary Integral Equation Method for three-dimensional and axisymmetric Stokes flows

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Abstract A new Boundary Integral Equation (BIE) formulation for Stokes flow is presented for three-dimensional and axisymmetrical problems using non-primitive variables, assuming velocity field is prescribed on the boundary. The formulation involves the vector potential, instead of the classical stream function, and all three components of the vorticity are implied. Furthermore, following the Helmholtz decomposition, a scalar potential is added to represent the solenoidal velocity field. Firstly, the BIEs for three-dimensional flows are formulated for the vector potential and the vorticity by employing the fundamental solutions in free space of vector Laplace and biharmonic equations. The equations for axisymmetric flows are then derived from the three-dimensional formulation in a second step. The outcome is a domain integral free BIE formulation for both three-dimensional and axisymmetric Stokes flows with prescribed velocity boundary condition. Numerical results are included to validate and show the efficiency of the proposed axisymmetric formulation.

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1 Introduction

During the last few decades there has been an increasing interest in using BIE methods for incompressible Stokes flow. The benefits invoked are the reduction in dimensionality of the problem and a greater ease in dealing with unbounded domain boundary conditions. The theoretical foundation for handling Stokes flow by the Boundary Integral Equation (BIE) method was laid by Ladyzhenskaya [1] within the framework of hydrodynamic potentials. The first BIE formulation for simulating Stokes flow past particles was developed by Youngren and Acrivos [2]. Since then, there have been numerous investigations employing integral equation techniques in two and three dimensions. The implementation has mostly been performed using primitive variables, namely velocity and pressure. The book by Pozrikidis [3] offers an overview of the main developments.

In this work we focus on non-primitive variable (NPV) formulation in the context of BIE. Let us first consider the status of the NPV formulation, independently of BIE issues. The NPV formulation for two-dimensional problems (i.e. the well known stream function—vorticity formulation) involves two variables instead of three and has the important advantage that continuity condition is automatically fulfilled. On the other hand, for three-dimensional problems, the number of unknowns increases: six variables instead of four in primitive variables formulation. Furthermore, Hirasaki and Hellums [4] have shown that it is convenient to introduce one more harmonic scalar potential variable in order to simplify the boundary conditions when through-flow is allowed. In other words, the Helmholtz decomposition has to be employed to decompose the velocity field into a scalar and a vector potential. But even so, the NPV formulation remains attractive since it offers additional advantages: (1) the continuity condition keeps on being automatically enforced by the vector potential, (2) the pressure term is eliminated (the pressure can be calculated as a by-product).

Having summarized the features of the NPV formulation, we now focus on the BIE formulation based on non-primitive variables. For two-dimensional Stokes flows, one obtains a biharmonic problem in terms of stream function, which is usually transformed to an equivalent set of two coupled harmonic equations. This is carried out by introducing the vorticity, so that the two flow variables (vorticity ω and stream function ψ) are respectively the solutions of Laplace and Poisson type equations. The presence of the harmonic forcing term in the stream function equation gives rise to a domain integral. Although the domain integral can be computed, its direct evaluation is contrary to the spirit of BIEM and requires a significant amount of computational resources. Indeed the presence of a domain integral constitutes a bottleneck for the BIEM. However, when the source term is harmonic, the domain integral can be reformulated in terms of boundary integrals, following the method initiated by Fairweather et al. [5]. This domain integral free approach has been developed further for two-dimensional flows with steady fixed boundaries [6, 7] and steady and transient free surface problems [8–11].

In the same vein, the analysis of axisymmetrical flows can be performed in terms of the azimuthal components ω and ψ of the vorticity vector $\boldsymbol{\omega}$ and the stream vector $\boldsymbol{\psi}$ to represent the fluid motion in the meridian plane. These two flow variables are solutions of second order partial differential equations containing the elliptic differential operator E^2. Unfortunately, the procedure, valid for two-dimensional flows, leading to purely boundary integral equations cannot be extended in a straightforward manner to axisymmetric problems. Therefore, in previous works, the computation of the domain integral is carried out [12, 13].

Instead, it is shown in this paper that it is useful to address three-dimensional Stokes flows as a first step. It can be mentioned that, according to best of our knowledge, there does not exist any BIE method for three-dimensional Stokes flows using non-primitive variables. Obtaining such equations for three-dimensional problems is a goal in itself and at the same time useful for axisymmetrical flows. At the outset, the variables in the three-dimensional formulation due to the Helmholtz decomposition are two potentials (scalar ϕ and vector $\boldsymbol{\alpha}$) along with the vorticity vector $\boldsymbol{\omega}$. Due to the involvement of two potential this formulation is also referred as dual potential formulation. The equation for vorticity and vector potential are classical vector Laplace and Poisson equation. The boundary conditions associated to each dependent variable are extracted from the prescribed velocity field. The BIE for the scalar potential, governed by the Laplace equation, is straightforward [14, 15]. The integral equation formulation for the vector equations is directly carried out by employing the vector Green's theorem. The initial approach proposed by Fairweather et al. is then applied to get rid of the involved domain integral. The BIEs for axisymmetrical problems are derived in a second step from the three-dimensional formulation.

The outline of this paper is as follows. In Sect. 2, the procedure proposed by Fairweather et al. to obtain BIEs describing slow two-dimensional steady viscous flows is recalled. The difficulties that prevent obtaining a domain integral free BIE formulation for axisymmetric flows are also highlighted in Sect. 2. The governing equations for three-dimensional Stokes flow in dual potential formulation, along with the boundary conditions on each dependent variable, are presented in Sect. 3. Section 3 also recalls the necessary background on the fundamental solutions for vector Laplace and biharmonic equations. The BIEs are then formulated in Sect. 5 for three-dimensional problems and in Sect. 6 for axisymmetric problems. Section 7 comprises of numerical results for axisymmetric formulation. The last section summarizes the findings and includes some conclusive remarks.

2 BIE formulations for two-dimensional/planar flows

In this section we recall the BIE method for two-dimensional Stokes flows, originally proposed by Fairweather et al. [5], to provide the basis for the three-dimensional BIE formulation for Stokes flows.

For flows in two dimensions, lying in the xy plane, the scalar vorticity ω is often selected as the negative z-component of the vorticity vector normal to the plane of flow, namely

$$\omega = -\boldsymbol{\omega}.\mathbf{i}_z = -\nabla \times \mathbf{u}.\mathbf{i}_z \tag{1}$$

where $\mathbf{u} = (\mathrm{u}_x, \mathrm{u}_y)$ is the fluid velocity and $\mathbf{i}_z$ is the unit vector normal to the plane of flow. The condition of incompressibility $\nabla.\mathbf{u} = 0$ is satisfied by expressing $\mathbf{u}$ in terms of a stream function ψ according to

$$\mathbf{u} = \nabla \times \psi \mathbf{i}_z = \nabla\psi \times \mathbf{i}_z \tag{2}$$

This stream function [16] can be considered as the z-component of a stream vector $\boldsymbol{\psi}$ to be touched upon later for the study of genuinely three-dimensional flows. The equations of viscous incompressible flow in terms of scalar vorticity and stream function are

$$\nabla^2\omega = 0 \tag{3}$$

$$\nabla^2\psi = -\omega \tag{4}$$

where the vorticity transport equation (3) is obtained by taking the curl of the momentum equation and neglecting the inertia terms, while (4) is obtained by substituting (3) into the vorticity definition (1). The boundary conditions supplementing the two field equations result from separating the normal and tangential components of the velocity boundary condition $\mathbf{u}|_S = \mathbf{b}$. Here S represents the boundary of the domain Ω which is simply connected and is provided with an outward unit vector $\mathbf{n}$ and a tangential unit vector $\mathbf{t}$ in an anticlockwise orientation. Letting s be the curvilinear coordinate along the boundary, one obtains the following boundary conditions on the stream function

$$\mathbf{n}.\nabla\psi \times \mathbf{i}_z|_S = \mathbf{i}_z \times \mathbf{n}.\nabla\psi|_S = \mathbf{t}.\nabla\psi|_S = \left.\frac{\partial\psi}{\partial \mathrm{t}}\right|_S = \mathbf{n}.\mathbf{b} \tag{5}$$

$$\mathbf{t}.\nabla\psi \times \mathbf{i}_z|_S = \mathbf{i}_z \times \mathbf{t}.\nabla\psi|_S = -\mathbf{n}.\nabla\psi|_S = -\left.\frac{\partial\psi}{\partial \mathrm{n}}\right|_S = \mathbf{t}.\mathbf{b} \tag{6}$$

Integrating the right hand side of (5) to obtain the Dirichlet boundary conditions for the stream function

$$\psi|_S = \int_{s_1}^{s} \mathbf{n}.\mathbf{b}\, ds' + \text{constant} \tag{7}$$

gives a single valued function with an arbitrary additive constant, thanks to the global incompressibility condition

$$\int_S \mathbf{n}.\mathbf{b}\, ds = 0$$

Equations (6) and (7) provide the two required boundary conditions, deduced from the prescribed velocity field.

Following standard procedure to obtain the BIE, Eqs. (3) and (4) are transformed in their equivalent integral equation representations by employing the second Green's theorem, particularized by involving the fundamental solution of Laplace's equation G^H and the concerned variable: vorticity or stream function. The fundamental solution satisfies the singularly forced Laplace equation

$$\nabla^2 G^H(\mathbf{x}, \mathbf{x}') = -\delta_2(\mathbf{x} - \mathbf{x}') \tag{8}$$

where $\mathbf{x}$ is the observation point, $\mathbf{x}'$ is the fixed location of the singular source point and δ_2 is the Dirac delta function in two dimensions. Further, employing the singular behaviour of fundamental solution as the observation point approaches S, the integral equations for vorticity and stream function can be written as follows [14, 15]:

$$c(\mathbf{x})\omega(\mathbf{x}) = \int_S \omega(\mathbf{x}') \frac{\partial G^H}{\partial \mathrm{n}}(\mathbf{x}, \mathbf{x}')\, dS' - \int_S \frac{\partial\omega}{\partial \mathrm{n}}(\mathbf{x}') G^H(\mathbf{x}, \mathbf{x}')\, dS' \tag{9}$$

$$c(\mathbf{x})\psi(\mathbf{x}) = \int_S \psi(\mathbf{x}') \frac{\partial G^H}{\partial \mathrm{n}}(\mathbf{x}, \mathbf{x}')\, dS' - \int_S \frac{\partial\psi}{\partial \mathrm{n}}(\mathbf{x}') G^H(\mathbf{x}, \mathbf{x}')\, dS' + \int_\Omega \omega(\mathbf{x}') G^H(\mathbf{x}, \mathbf{x}') d\Omega' \tag{10}$$

where the coefficient $c(\mathbf{x})$ depends on the local geometry of the boundary at the evaluation point ($c = 1/2$ if the surface is regular at $\mathbf{x}$). As the stream function equation contains the vorticity as a forcing term, a domain integral appears in the corresponding integral equation (10).

The idea to transform the domain integral in (10) into a boundary integral is to introduce the free space fundamental solution of the biharmonic equation G^B defined by

$$\nabla^4 G^B(\mathbf{x},\mathbf{x}') = \nabla^2\nabla^2 G^B(\mathbf{x},\mathbf{x}') = -\delta_2(\mathbf{x}-\mathbf{x}') \quad (11)$$

Substituting in Green's theorem the fundamental solution G^B and ω one obtains

$$\int_\Omega \{\omega(\mathbf{x}')\nabla^2 G^B(\mathbf{x},\mathbf{x}') - G^B(\mathbf{x},\mathbf{x}')\nabla^2\omega(\mathbf{x}')\}\, d\Omega' = \int_S \omega(\mathbf{x}')\frac{\partial G^B}{\partial \mathrm{n}}(\mathbf{x},\mathbf{x}')\, dS' - \int_S \frac{\partial \omega}{\partial \mathrm{n}}(\mathbf{x}') G^B(\mathbf{x},\mathbf{x}')\, dS' \quad (12)$$

As vorticity is a harmonic function and due to the additional constraint that $\nabla^2 G^B = G^H$, the domain integral in (10) becomes a boundary integral, given by

$$\int_\Omega \omega(\mathbf{x}') G^H(\mathbf{x},\mathbf{x}')\, d\Omega' = \int_S \omega(\mathbf{x}')\frac{\partial G^B}{\partial \mathrm{n}}(\mathbf{x},\mathbf{x}')\, dS' - \int_S \frac{\partial \omega}{\partial \mathrm{n}}(\mathbf{x}') G^B(\mathbf{x},\mathbf{x}')\, dS' \quad (13)$$

which can be substituted in (10). Hence, the two-dimensional viscous incompressible flows are represented by a set of coupled boundary integral equations.

Let us now consider flows that are symmetrical about the z-axis. The velocity $\mathbf{u}$ is independent of the azimuthal angle ϕ and its azimuthal component $\mathbf{u}.\mathbf{i}_\phi$ is zero, $\mathbf{i}_\phi$ being the unit vector normal to every meridian plane $\phi =$ constant. The scalar vorticity ω and stream function ψ are the azimuthal components $\omega = \omega_\phi$ of the vorticity vector $\boldsymbol{\omega}$ and of the aforementioned stream vector $\boldsymbol{\psi}$, namely

$$\omega_\phi = \nabla \times \mathbf{u}.\mathbf{i}_\phi, \quad \mathbf{u} = \nabla\psi \times \mathbf{i}_\phi \quad (14)$$

The axisymmetric Stokes flow equations for the scalar vorticity and the stream function are obtained using the same procedure followed for two dimensions, given as

$$E^2\omega = 0 \quad (15)$$

$$E^2\psi = -\omega \quad (16)$$

where the second order elliptic operator for axisymmetrical problems E^2 is defined in cylindrical coordinates (ρ,ϕ,z) by $E^2 = \nabla^2 - 1/\rho^2$. The boundary conditions on S, which is the cross section of the three-dimensional fluid domain boundary with the meridian plane, can be obtained in the same way as for two-dimensional problems.

According to our literature survey, the only BIE implementation in NPV for axisymmetric Stokes flows has been proposed by Lu [12, 13]. The definition of the stream function 'ψ_L' used by Lu is the same as of Brenner [17] and is such that $\psi_L = -\rho\psi$. The resulting operator E_L^2 is not self-adjoint and the second Green's theorem requires defining the conjugate operator E_L^{*2}. It is thus needed to find G_L^*, the fundamental solution in free space of the conjugate equation $E_L^{*2}(G_L^*) = -\delta$.

However in our case E^2 is a self-adjoint operator, the second Green's theorem can be applied straightforwardly to transform Eqs. (15) and (16) into integral equations by involving the fundamental solution of E^2 and the concerned variable. The problem is simplified up to some extent but the vorticity always induces a forcing term in the differential equation for the stream function and the corresponding domain integral appears in the integral equation. Transforming this domain integral into a surface one, on the same lines as in the two dimensions proposed by Fairweather is no longer possible, since the fundamental solution in free space of $E^4 = -\delta$ which is necessary has not been yet obtained.

So, directly formulating the axisymmetrical problem cannot provide domain integral free BIE formulation. Instead, a two step approach can be followed. In a first step it is possible to obtain a domain integral free formulation in three dimensions by taking the key idea of Fairweather et al. for two-dimensional flows. In a second step, the formulation of axisymmetrical flows can be derived from the three-dimensional formulation through proper substitutions and azimuthal integration.

3 Three-dimensional flows: governing equations and boundary conditions

The three-dimensional formulations in non-primitive variables can be done by two different approaches. For each, the distinctive feature is the way in which the normal component of the prescribed boundary velocity is enforced. The first approach is based on the aforementioned stream vector $\mathbf{u} = \nabla \times \boldsymbol{\psi}$ (i.e. the three-dimensional counterpart of the stream function). An elliptic problem over the boundary, governing an additional surface scalar unknown must be introduced [18]. The second approach, based on the Helmholtz decomposition, introduces a vector potential $\boldsymbol{\alpha}$ and a harmonic scalar potential φ to respectively account for the rotational and irrotational parts of the velocity field. So in both cases an extra problem has to be solved, but the equation associated to the scalar potential is well known in the BIE literature and is much easier to solve than the elliptic problem arising in the first case. Therefore, the second approach is chosen for BIE formulations.

The non-primitive variables formulation based on Helmholtz decomposition theorem is well known and is repeated here for convenience. The decomposed velocity field takes the following form:

$$\mathbf{u} = \nabla\varphi + \nabla \times \boldsymbol{\alpha} \tag{17}$$

Taking the curl of the velocity definition (17) and using the vorticity definition, one obtains the following vector equation for the vector potential

$$\nabla \times \nabla \times \boldsymbol{\alpha} = \boldsymbol{\omega} \tag{18}$$

along with the Laplace equation for the scalar potential

$$-\nabla^2\varphi = 0 \tag{19}$$

The prescribed velocity boundary conditions, to be imposed on the scalar and vector potentials, are derived by separating the normal and tangential components of the velocity on the boundary, according to Hirasaki and Hellums [4]. The normal velocity component is completely taken into account by the scalar potential through the following boundary conditions:

$$\mathbf{n}.\nabla\varphi|_S = \mathbf{n}.\mathbf{b} \tag{20}$$

$$\mathbf{n}.\nabla \times \boldsymbol{\alpha}|_S = 0 \tag{21}$$

Assuming that the scalar potential is calculated beforehand, the tangential component of the prescribed velocity is represented by a boundary condition on the vector potential

$$\mathbf{n} \times \nabla \times \boldsymbol{\alpha}|_S = \mathbf{n} \times (-\nabla\varphi + \mathbf{b}) \tag{22}$$

The scalar derivative boundary condition (21) can be transformed into a non-derivative homogenous boundary condition which fixes the tangential component of $\boldsymbol{\alpha}$

$$\mathbf{n} \times \boldsymbol{\alpha}|_S = 0 \tag{23}$$

However, this set of boundary conditions does not determine $\boldsymbol{\alpha}$ uniquely and a gauge condition $\nabla.\boldsymbol{\alpha} = \mathbf{0}$ is imposed. The vector potential equation (18) now becomes

$$\nabla^2\boldsymbol{\alpha} = -\boldsymbol{\omega} \tag{24}$$

It can be showed that in order to satisfy the gauge condition it is sufficient to impose the boundary condition $\nabla.\boldsymbol{\alpha}|_S = \mathbf{0}$. Taking the curl of the momentum equation result in the Laplace equation for the vorticity

$$\nabla^2\boldsymbol{\omega} = 0 \tag{25}$$

By the definition of the vorticity, $\nabla.\boldsymbol{\omega} = \mathbf{0}$. It is easy to show that the vorticity remains solenoidal throughout the domain if the homogenous boundary condition $\nabla.\boldsymbol{\omega}|_S = \mathbf{0}$ is satisfied. In conclusion, the set comprising the six scalar boundary conditions to be satisfied by the six components of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ is

$$\begin{aligned} &\mathbf{n} \times \boldsymbol{\alpha}|_S = 0, \qquad \mathbf{n} \times \nabla \times \boldsymbol{\alpha}|_S = \mathbf{n} \times (-\nabla\varphi + \mathbf{b}) \\ &\nabla.\boldsymbol{\alpha}|_S = 0, \qquad \nabla.\boldsymbol{\omega}|_S = 0 \end{aligned} \tag{26}$$

while $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ satisfy (24) and (25), or equivalently a biharmonic equation in $\boldsymbol{\alpha}$. The Stokes system given by (19), (24) and (25) with boundary conditions (20) and (26) is equivalent to the original system in primitive variables, see for instance [18] for more details.

The situation now resembles with that encountered in two-dimensional problems. It is indeed possible to obtain the free space fundamental solution of vector Laplace and biharmonic equations in three dimensions. The free space fundamental solution of vector Laplace equation satisfies, by definition, the singularly forced Laplace equation

$$\nabla^2\mathbf{G}^H(\mathbf{x}, \mathbf{x}') = -\mathbf{I}\delta_3(\mathbf{x} - \mathbf{x}') \tag{27}$$

where δ_3 is the three-dimensional Dirac delta function and $\mathbf{I}$ is the identity tensor. The fundamental solution $\mathbf{G}^H(\mathbf{x}, \mathbf{x}')$ satisfying (27) is [19]

$$\mathbf{G}^H(\mathbf{x}, \mathbf{x}') = \mathbf{I}\frac{1}{4\pi \mathrm{r}} \tag{28}$$

where $\mathrm{r} = |\mathbf{x} - \mathbf{x}'|$ is the distance between the source and the observation point. The free space fundamental solution for the biharmonic equation $\mathbf{G}^B$ which follows

$$\nabla^4\mathbf{G}^B(\mathbf{x}, \mathbf{x}') = -\mathbf{I}\delta_3(\mathbf{x} - \mathbf{x}') \tag{29}$$

is given by

$$\mathbf{G}^B(\mathbf{x}, \mathbf{x}') = \mathbf{I}\frac{\mathrm{r}}{8\pi} \tag{30}$$

and satisfies the following basic relation

$$\nabla^2\mathbf{G}^B = \mathbf{G}^H \tag{31}$$

4 Integral equation formulation in three dimensions

In this section the integral equations are formulated for Stokes flows, except for the scalar potential governed by the Laplace equation which is well known in the BIE literature (see e.g. [14, 15]). Here we focus on formulating the integral equation for vector Laplace equation (25) and extend it to the vector Poisson equation (24). Two points can be distinguished at this stage from the classical integral formulation of vector equations. First, during derivation, the operations and simplification must be carried out in a way that the components, known (26) and unknown, are preserved in the formulation. The second point is the use of the Fairweather approach in three dimensions, in order to transform the domain integral into a boundary integral.

The BIE formulation for the vector equations requires the Green's identity, which for the vector fields **E** and **F**, using the divergence theorem, is given by [19]

$$\int_\Omega (\mathbf{E}.\nabla^2\mathbf{F} - \mathbf{F}.\nabla^2\mathbf{E})\, d\Omega = \int_S \big[(\mathbf{n}'\times\mathbf{E}').(\nabla'\times\mathbf{F}') - (\mathbf{n}'\times\mathbf{F}').(\nabla'\times\mathbf{E}') + (\mathbf{n}'.\mathbf{E}')(\nabla'.\mathbf{F}') - (\mathbf{n}'.\mathbf{F}')(\nabla'.\mathbf{E}')\big]\, dS \qquad (32)$$

where Ω is the domain of interest enclosed by the boundary S, and $\mathbf{n}$ is the outward directed normal. On the right hand side of (32), the vectors with prime superscript lie on the boundary. For instance $\mathbf{n}' = \mathbf{n}(\mathbf{x}')$, where $\mathbf{x}'$ is the source point lying on the boundary S. Identifying in (32) the vector fields **E** and **F** as the vorticity vector and the fundamental solution $\mathbf{G}^H$ for the Laplace equation respectively, and using the definition (27), the integral equation for the vorticity is

$$c(\mathbf{x})\boldsymbol{\omega}(\mathbf{x}) = \int_S \big[(\mathbf{n}'\times\boldsymbol{\omega}').(\nabla'\times\mathbf{G}^H) - (\mathbf{n}'\times\mathbf{G}^H).(\nabla'\times\boldsymbol{\omega}') + (\mathbf{n}'.\boldsymbol{\omega}')(\nabla'.\mathbf{G}^H) - (\mathbf{n}'.\mathbf{G}^H)(\nabla'.\boldsymbol{\omega}')\big]\, dS \qquad (33)$$

Substituting for the fundamental solution $\mathbf{G}^H$ and simplifying by using identities provided in Appendix A, we obtain

$$c(\mathbf{x})\boldsymbol{\omega}(\mathbf{x}) = \frac{1}{4\pi}\int_S \Big[\frac{1}{\mathrm{r}^3}(\mathbf{n}'\times\boldsymbol{\omega}')\times(\mathbf{x}-\mathbf{x}') - \frac{1}{\mathrm{r}}\mathbf{n}'\times(\nabla'\times\boldsymbol{\omega}') + \frac{1}{\mathrm{r}^3}(\mathbf{n}'.\boldsymbol{\omega}')(\mathbf{x}-\mathbf{x}') - \frac{1}{\mathrm{r}}(\nabla'.\boldsymbol{\omega}')\mathbf{n}'\Big]\, dS \qquad (34)$$

The integral equation for the vector potential governed by the Poisson equation is formulated in the same way, except that the vorticity, acting as a source, gives rise to a domain integral. Identifying the vector fields **E** and **F** in (32) as the vector potential and the fundamental solution $\mathbf{G}^H$, we get after simplification

$$c(\mathbf{x})\boldsymbol{\alpha}(\mathbf{x}) = \int_S \big[(\mathbf{n}'\times\boldsymbol{\alpha}').(\nabla'\times\mathbf{G}^H) - (\mathbf{n}'\times\mathbf{G}^H).(\nabla'\times\boldsymbol{\alpha}') + (\mathbf{n}'.\boldsymbol{\alpha}')(\nabla'.\mathbf{G}^H) - (\mathbf{n}'.\mathbf{G}^H)(\nabla'.\boldsymbol{\alpha}')\big]\, dS - \int_\Omega (\mathbf{G}^H.\boldsymbol{\omega})\, d\Omega \qquad (35)$$

Since vorticity is a harmonic function, we can transform the domain integral in (35) into a boundary integral. This is carried out by incorporating the Green's theorem, this time by substituting the vector fields **E** and **F** as the vorticity vector and the fundamental solution $\mathbf{G}^B$ for biharmonic equation, to obtain the following integral relation:

$$\int_\Omega (\boldsymbol{\omega}.\nabla^2\mathbf{G}^B - \mathbf{G}^B.\nabla^2\boldsymbol{\omega})\, d\Omega = \int_S \big[(\mathbf{n}'\times\boldsymbol{\omega}').(\nabla'\times\mathbf{G}^B) - (\mathbf{n}'\times\mathbf{G}^B).(\nabla'\times\boldsymbol{\omega}') + (\mathbf{n}'.\boldsymbol{\omega}')(\nabla'.\mathbf{G}^B) - (\mathbf{n}'.\mathbf{G}^B)(\nabla'.\boldsymbol{\omega}')\big]\, dS \qquad (36)$$

Since $\mathbf{G}^H$ is a symmetric tensor $\boldsymbol{\omega}.\mathbf{G}^H = \mathbf{G}^H.\boldsymbol{\omega}$ and, using (31), (36) becomes

$$\int_\Omega \mathbf{G}^H.\boldsymbol{\omega}\, d\Omega = \int_S \big[(\mathbf{n}'\times\boldsymbol{\omega}').(\nabla'\times\mathbf{G}^B) - (\mathbf{n}'\times\mathbf{G}^B).(\nabla'\times\boldsymbol{\omega}') + (\mathbf{n}'.\boldsymbol{\omega}')(\nabla'.\mathbf{G}^B) - (\mathbf{n}'.\mathbf{G}^B)(\nabla'.\boldsymbol{\omega})\big]\, dS \qquad (37)$$

The domain integral has now been represented in boundary integral form. Substituting (37) into (35) gives the domain integral free formulation for the vector potential

$$c(\mathbf{x})\boldsymbol{\alpha}(\mathbf{x}) = \int_S \big[(\mathbf{n}'\times\boldsymbol{\alpha}').(\nabla'\times\mathbf{G}^H) - (\mathbf{n}'\times\mathbf{G}^H).(\nabla'\times\boldsymbol{\alpha}')$$

$$
\begin{aligned}
&+ (\mathbf{n}'.\boldsymbol{\alpha}')(\nabla'.\mathbf{G}^H) \\
&- (\mathbf{n}'.\mathbf{G}^H)(\nabla'.\boldsymbol{\alpha}')\big]\,dS \\
&- \int_S \big[(\mathbf{n}' \times \boldsymbol{\omega}').(\nabla' \times \mathbf{G}^B) \\
&- (\mathbf{n}' \times \mathbf{G}^B).(\nabla' \times \boldsymbol{\omega}') \\
&+ (\mathbf{n}'.\boldsymbol{\omega}')(\nabla'.\mathbf{G}^B) \\
&- (\mathbf{n}'.\mathbf{G}^B)(\nabla'.\boldsymbol{\omega}')\big]\,dS
\end{aligned} \tag{38}
$$

Substituting for the fundamental solutions and simplifying, by using the identities given in Appendix A, we get

$$
\begin{aligned}
c(\mathbf{x})\boldsymbol{\alpha}(\mathbf{x}) = {} & \frac{1}{4\pi}\int_S \bigg[\frac{1}{\mathrm{r}^3}(\mathbf{n}' \times \boldsymbol{\alpha}') \times (\mathbf{x}-\mathbf{x}') \\
& - \frac{1}{\mathrm{r}}\mathbf{n}' \times (\nabla' \times \boldsymbol{\alpha}') + \frac{1}{\mathrm{r}^3}(\mathbf{n}'.\boldsymbol{\alpha}')(\mathbf{x}-\mathbf{x}') \\
& - \frac{1}{\mathrm{r}}(\nabla'.\boldsymbol{\alpha}')\mathbf{n}'\bigg]\,dS \\
& + \frac{1}{8\pi}\int_S \bigg[\frac{1}{\mathrm{r}}(\mathbf{n}' \times \boldsymbol{\omega}') \times (\mathbf{x}-\mathbf{x}') \\
& + \mathrm{r}[\mathbf{n}' \times (\nabla' \times \boldsymbol{\omega}')] + \frac{1}{\mathrm{r}}(\mathbf{n}'.\boldsymbol{\omega}')(\mathbf{x}-\mathbf{x}') \\
& + \mathrm{r}(\nabla'.\boldsymbol{\omega}')\mathbf{n}'\bigg]\,dS
\end{aligned} \tag{39}
$$

The vector equations (34) and (39) consist in six scalar equations which can be projected on the orthonormal mobile basis. The mobile basis consisting of a unit normal vector **n** and two unit vectors in the tangential plane orthogonal to **n** is chosen. In this mobile basis the set of six equations consists of two normal components (one for each vorticity and vector potential) and four equations in the tangential plane (two for each vorticity and vector potential). These six equations can be obtained by operating the scalar and vector product with normal component on both sides of (34) and (39). Taking the scalar product of the vorticity equation with the normal component at the observation point, and by the vector identity $\mathbf{f}.(\mathbf{g} \times \mathbf{h}) = -(\mathbf{f} \times \mathbf{h}).\mathbf{g}$, the equation for the normal component of the vorticity is

$$
\begin{aligned}
c(\mathbf{x})(\mathbf{n}.\boldsymbol{\omega}) = {} & \frac{1}{4\pi}\int_S \bigg[-\frac{1}{\mathrm{r}^3}[\mathbf{n} \times (\mathbf{x}-\mathbf{x}')].(\mathbf{n}' \times \boldsymbol{\omega}') \\
& - \frac{1}{\mathrm{r}}\mathbf{n}.(\mathbf{n}' \times (\nabla' \times \boldsymbol{\omega}')) \\
& + \frac{1}{\mathrm{r}^3}[\mathbf{n}.(\mathbf{x}-\mathbf{x}')](\mathbf{n}'.\boldsymbol{\omega}') \\
& - \frac{1}{\mathrm{r}}[\mathbf{n}.\mathbf{n}'](\nabla'.\boldsymbol{\omega}')\bigg]\,dS
\end{aligned} \tag{40}
$$

Taking the vector product of the vorticity equation with the normal component at the observation point, and using the following identity:

$$
\begin{aligned}
& \frac{1}{\mathrm{r}^3}\mathbf{n} \times [(\mathbf{n}' \times \boldsymbol{\omega}') \times (\mathbf{x}-\mathbf{x}')] \\
& \quad = \frac{1}{\mathrm{r}^3}[\{\mathbf{n}.(\mathbf{x}-\mathbf{x}')\}(\mathbf{n}' \times \boldsymbol{\omega}') \\
& \qquad - \{\mathbf{n}.(\mathbf{n}' \times \boldsymbol{\omega}')\}(\mathbf{x}-\mathbf{x}')]
\end{aligned} \tag{41}
$$

in tensorial form

$$
\begin{aligned}
& \mathbf{n} \times [(\mathbf{n}' \times \boldsymbol{\omega}') \times (\mathbf{x}-\mathbf{x}')] \\
& \quad = \{\mathbf{n}.(\mathbf{x}-\mathbf{x}')\}(\mathbf{n}' \times \boldsymbol{\omega}') \\
& \qquad - \{\mathbf{n} \otimes (\mathbf{x}-\mathbf{x}').(\mathbf{n}' \times \boldsymbol{\omega}')\} \\
& \quad = \{\mathbf{n}.(\mathbf{x}-\mathbf{x}')\mathbf{I} - \mathbf{n} \otimes (\mathbf{x}-\mathbf{x}')\}.(\mathbf{n}' \times \boldsymbol{\omega}')
\end{aligned} \tag{42}
$$

the vector equation for the tangential components of the vorticity is

$$
\begin{aligned}
c(\mathbf{x})(\mathbf{n} \times \boldsymbol{\omega}) = {} & \frac{1}{4\pi}\int_S \frac{1}{\mathrm{r}^3}[\mathbf{n}.(\mathbf{x}-\mathbf{x}')\mathbf{I} \\
& - \mathbf{n} \otimes (\mathbf{x}-\mathbf{x}')].(\mathbf{n}' \times \boldsymbol{\omega}') \\
& - \frac{1}{\mathrm{r}}\mathbf{n} \times (\mathbf{n}' \times (\nabla' \times \boldsymbol{\omega}')) \\
& + \frac{1}{\mathrm{r}^3}[\mathbf{n} \times (\mathbf{x}-\mathbf{x}')](\mathbf{n}'.\boldsymbol{\omega}') \\
& - \frac{1}{\mathrm{r}}[\mathbf{n} \times \mathbf{n}'](\nabla.\boldsymbol{\omega})\,dS
\end{aligned} \tag{43}
$$

Equation (43) is a vector equation containing one scalar equation along each basis component in the tangential plane. Using the same operations as on the vorticity equation, the equations for the normal component of the vector potential is

$$
\begin{aligned}
& c(\mathbf{x})(\mathbf{n}.\boldsymbol{\alpha}) \\
& \quad = \frac{1}{4\pi}\int_S \bigg[-\frac{1}{\mathrm{r}^3}[\mathbf{n} \times (\mathbf{x}-\mathbf{x}')].(\mathbf{n}' \times \boldsymbol{\alpha}') \\
& \qquad - \frac{1}{\mathrm{r}}\mathbf{n}.(\mathbf{n}' \times (\nabla' \times \boldsymbol{\alpha}')) + \frac{1}{\mathrm{r}^3}[\mathbf{n}.(\mathbf{x}-\mathbf{x}')](\mathbf{n}'.\boldsymbol{\alpha}') \\
& \qquad - \frac{1}{\mathrm{r}}[\mathbf{n}.\mathbf{n}'](\nabla'.\boldsymbol{\alpha}')\bigg]\,dS \\
& \qquad + \frac{1}{8\pi}\int_S \bigg[-\frac{1}{\mathrm{r}}[\mathbf{n} \times (\mathbf{x}-\mathbf{x}')].(\mathbf{n}' \times \boldsymbol{\omega}') \\
& \qquad + \mathrm{r}\mathbf{n}.(\mathbf{n}' \times (\nabla' \times \boldsymbol{\omega}')) + \frac{1}{\mathrm{r}}[\mathbf{n}.(\mathbf{x}-\mathbf{x}')](\mathbf{n}'.\boldsymbol{\omega}') \\
& \qquad + \mathrm{r}[\mathbf{n}.\mathbf{n}'](\nabla'.\boldsymbol{\omega}')\bigg]\,dS
\end{aligned} \tag{44}
$$

and the vector equation for the tangential components of the vector potential is

$$
\begin{aligned}
&c(\mathbf{x})(\mathbf{n}\times\boldsymbol{\alpha})\\
&\quad=\frac{1}{4\pi}\int_S \frac{1}{\mathrm{r}^3}\big[\mathbf{n}.(\mathbf{x}-\mathbf{x}')\mathbf{I}-\mathbf{n}\otimes(\mathbf{x}-\mathbf{x}')\big].(\mathbf{n}'\times\boldsymbol{\alpha}')-\frac{1}{\mathrm{r}}\mathbf{n}\times\big(\mathbf{n}'\times(\nabla'\times\boldsymbol{\alpha}')\big)+\frac{1}{\mathrm{r}^3}\big[\mathbf{n}\times(\mathbf{x}-\mathbf{x}')\big](\mathbf{n}'.\boldsymbol{\alpha}')\\
&\qquad-\frac{1}{\mathrm{r}}\big[\mathbf{n}\times\mathbf{n}'\big](\nabla'.\boldsymbol{\alpha}')\,dS+\frac{1}{8\pi}\int_S\bigg[\frac{1}{\mathrm{r}}\big[\mathbf{n}.(\mathbf{x}-\mathbf{x}')\mathbf{I}-\mathbf{n}\otimes(\mathbf{x}-\mathbf{x}')\big].(\mathbf{n}'\times\boldsymbol{\omega}')+\mathrm{r}\mathbf{n}\times\big(\mathbf{n}'\times(\nabla'\times\boldsymbol{\omega}')\big)\\
&\qquad+\frac{1}{\mathrm{r}}\big[\mathbf{n}\times(\mathbf{x}-\mathbf{x}')\big](\mathbf{n}'.\boldsymbol{\omega}')+\mathrm{r}\big[\mathbf{n}\times\mathbf{n}'\big](\nabla'.\boldsymbol{\omega}')\bigg]dS
\end{aligned}
\tag{45}
$$

The set of equations composed of (40) and (43)–(45) contains ten variables along with the divergence free condition on the boundary for vorticity and the vector potential. For clarity, these variables are listed: one normal component for each $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ (46), two tangential components for $\boldsymbol{\omega}$ and two tangential components of the rotational of $\boldsymbol{\omega}$ (47). These six variables constitute the unknowns. The remaining four components, namely the two tangential components of the vector potential and the two tangential components of its rotational (48) are fixed by the boundary conditions, as shown in the next section.

$$\mathbf{n}'.\boldsymbol{\omega}',\ \mathbf{n}'.\boldsymbol{\alpha}' \tag{46}$$

$$\mathbf{n}'\times\boldsymbol{\omega}',\ \mathbf{n}'\times(\nabla'\times\boldsymbol{\omega}') \tag{47}$$

$$\mathbf{n}'\times\boldsymbol{\alpha}',\ \mathbf{n}'\times(\nabla'\times\boldsymbol{\alpha}') \tag{48}$$

4.1 Substitution of boundary conditions

Assuming that the Neumann problem is solved beforehand for the scalar potential, let us form the six scalar equations by inserting in (40) and (43)–(45) the prescribed boundary conditions given by (26). In order to represent these equations in a convenient form, the scalar, vector and tensorial kernel functions are respectively denoted by k, $\mathbf{k}$ and $\mathbf{K}$. The set of equations becomes

$$
\begin{aligned}
c(\mathbf{x})(\mathbf{n}.\boldsymbol{\omega})=\frac{1}{4\pi}\int_S\frac{1}{\mathrm{r}^2}\big[&-\mathbf{k}_1(\mathbf{x},\mathbf{x}').(\mathbf{n}'\times\boldsymbol{\omega}')\\
&-\mathbf{k}_2(\mathbf{x},\mathbf{x}').\big(\mathbf{n}'\times(\nabla'\times\boldsymbol{\omega}')\big)\\
&+\mathrm{k}(\mathbf{x},\mathbf{x}')(\mathbf{n}'.\boldsymbol{\omega}')\big]\,dS
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
c(\mathbf{x})(\mathbf{n}\times\boldsymbol{\omega})=\frac{1}{4\pi}\int_S\frac{1}{\mathrm{r}^2}\big[&\mathbf{K}(\mathbf{x},\mathbf{x}').(\mathbf{n}'\times\boldsymbol{\omega}')\\
&-\mathbf{k}_2(\mathbf{x},\mathbf{x}')\times\big(\mathbf{n}'\times(\nabla'\times\boldsymbol{\omega}')\big)\\
&+\mathbf{k}_1(\mathbf{x},\mathbf{x}')(\mathbf{n}'.\boldsymbol{\omega}')\big]\,dS
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
c(\mathbf{x})(\mathbf{n}.\boldsymbol{\alpha})=\frac{1}{4\pi}\int_S\frac{1}{\mathrm{r}^2}\big[&-\mathbf{k}_2(\mathbf{x},\mathbf{x}').\big(\mathbf{n}'\times(-\nabla\varphi|_S+\mathbf{b})\big)\\
&+\mathrm{k}(\mathbf{x},\mathbf{x}')(\mathbf{n}'.\boldsymbol{\alpha}')\big]\,dS\\
+\frac{1}{8\pi}\int_S\big[&-\mathbf{k}_1(\mathbf{x},\mathbf{x}').(\mathbf{n}'\times\boldsymbol{\omega}')\\
&+\mathbf{k}_2(\mathbf{x},\mathbf{x}').\big(\mathbf{n}'\times(\nabla'\times\boldsymbol{\omega}')\big)\\
&+\mathrm{k}(\mathbf{x},\mathbf{x}')(\mathbf{n}'.\boldsymbol{\omega}')\big]\,dS
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
0=\frac{1}{4\pi}\int_S\frac{1}{\mathrm{r}^2}\big[&-\mathbf{k}_2(\mathbf{x},\mathbf{x}')\times\big(\mathbf{n}'\times(-\nabla\varphi|_S+\mathbf{b})\big)\\
&+\mathbf{k}_1(\mathbf{x},\mathbf{x}')(\mathbf{n}'.\boldsymbol{\alpha}')\big]\,dS\\
+\frac{1}{8\pi}\int_S\big[&\mathbf{K}(\mathbf{x},\mathbf{x}').(\mathbf{n}'\times\boldsymbol{\omega}')\\
&+\mathbf{k}_2(\mathbf{x},\mathbf{x}')\times\big(\mathbf{n}'\times(\nabla'\times\boldsymbol{\omega}')\big)\\
&+\mathbf{k}_1(\mathbf{x},\mathbf{x}')(\mathbf{n}'.\boldsymbol{\omega}')\big]\,dS
\end{aligned}
\tag{52}
$$

where

$$
\begin{aligned}
&\mathrm{k}(\mathbf{x},\mathbf{x}')=\frac{1}{\mathrm{r}}\big[\mathbf{n}.(\mathbf{x}-\mathbf{x}')\big]\\
&\mathbf{k}_1(\mathbf{x},\mathbf{x}')=\frac{1}{\mathrm{r}}\big[\mathbf{n}\times(\mathbf{x}-\mathbf{x}')\big],\qquad \mathbf{k}_2(\mathbf{x},\mathbf{x}')=\mathrm{r}\mathbf{n}\\
&\mathbf{K}(\mathbf{x},\mathbf{x}')=\frac{1}{\mathrm{r}}\big[\mathbf{n}.(\mathbf{x}-\mathbf{x}')\mathbf{I}-\mathbf{n}\otimes(\mathbf{x}-\mathbf{x}')\big]
\end{aligned}
\tag{53}
$$

The number of equations is consistent with the number of unknown variables (46)–(48). The BIEs (49)–(52) and an additional BIE for the scalar potential completely characterizes the three-dimensional Stokes flows problem with prescribed velocity boundary condition.

5 Axisymmetric formulation

Consider Eqs. (49)–(52) and assuming that the solution domain Ω and the solution itself are both rotationally symmetric about the z-axis, as illustrated in Fig. 1. For sake of simplicity, let the evaluation point $\mathbf{x}$ lie in the origin meridian plane $\phi=0$. At this point, two Euclidean mobile frames can be defined. The first one is

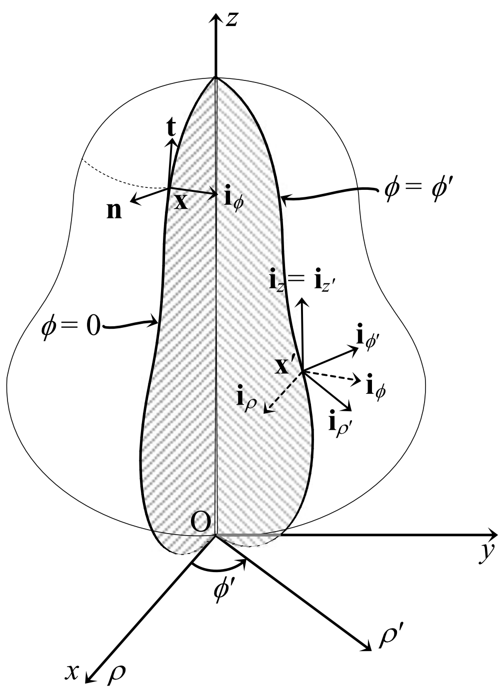

Fig. 1 Sketch of the axisymmetric domain and meridian plane ($\phi = 0$ and $\phi = \phi'$)

a Darboux frame $(\mathbf{x}, \mathbf{n}, \mathbf{i}_\phi, \mathbf{t})$ where $\mathbf{n}$ is the unit outer normal to the surface S, $\mathbf{i}_\phi$ is the azimuthal unit vector tangent to the surface and $\mathbf{t}$ is the unit vector tangent to the meridian curve. The tangent vector $\mathbf{t}$ is such that the basis $(\mathbf{n}, \mathbf{i}_\phi, \mathbf{t})$ is right handed in that order. The second one is a cylindrical coordinates moving frame constituted by $(\mathbf{x}, \mathbf{i}_\rho, \mathbf{i}_\phi, \mathbf{i}_z)$.

Due to the axial symmetry, the two basis are related by

$$\mathbf{n} = \mathrm{n}_\rho \mathbf{i}_\rho + \mathrm{n}_z \mathbf{i}_z, \qquad \mathbf{t} = \mathbf{n} \times \mathbf{i}_\phi = -\mathrm{n}_z \mathbf{i}_\rho + \mathrm{n}_\rho \mathbf{i}_z \tag{54}$$

while the vorticity vector and the vector potential have a single azimuthal component

$$\boldsymbol{\omega} = \omega \mathbf{i}_\phi, \qquad \boldsymbol{\alpha} = \alpha \mathbf{i}_\phi \tag{55}$$

which implies

$$\mathbf{n} \times \boldsymbol{\omega} = -\omega \mathbf{t}, \qquad \mathbf{n}.\boldsymbol{\omega} = 0, \qquad \mathbf{n}.\boldsymbol{\alpha} = 0 \tag{56}$$

Similarly two frames $(\mathbf{x}', \mathbf{n}', \mathbf{i}'_\phi, \mathbf{t}')$ and $(\mathbf{x}', \mathbf{i}'_\rho, \mathbf{i}'_\phi, \mathbf{i}'_z)$ can be defined at the source point $\mathbf{x}'$. Because of the axial symmetry, the azimuthal component of the gradient of any function vanishes

$$\nabla' = \frac{\partial}{\partial \mathrm{n}'} \mathbf{n}' + \frac{\partial}{\partial \mathrm{t}'} \mathbf{t}' \tag{57}$$

$$\begin{aligned} \nabla \times \boldsymbol{\omega} &= \frac{\partial \omega}{\partial \mathrm{t}} \mathbf{n} + \left(-\frac{\partial \omega}{\partial \mathrm{n}} \right) \mathbf{t} \\ \mathbf{n} \times \nabla \times \boldsymbol{\omega} &= -\frac{\partial \omega}{\partial \mathrm{n}} \mathbf{i}_\phi = -\kappa_\mathrm{n} \mathbf{i}_\phi \end{aligned} \tag{58}$$

In the latter expression, κ_n is a shorthand notation used for convenience. Let $\boldsymbol{\beta} = (-\nabla\varphi|_S + \mathbf{b})$ be the corrected prescribed velocity, then

$$\mathbf{n}' \times (-\nabla\varphi|_S + \mathbf{b}) = \beta_{\mathrm{t}'} \mathbf{i}_{\phi'} \tag{59}$$

where β_t is the tangential component of the corrected prescribed velocity. Taking advantage of these simplifications (49)–(52) reduce to

$$\begin{aligned} 0 = \frac{1}{4\pi} \int_S \frac{1}{\mathrm{r}^2} \big[&\mathbf{k}_1(\mathbf{x}, \mathbf{x}').(\omega' \mathbf{t}') \\ &+ \mathbf{k}_2(\mathbf{x}, \mathbf{x}').(\kappa'_\mathrm{n} \mathbf{i}_{\phi'}) \big] \, dS \end{aligned} \tag{60}$$

$$\begin{aligned} c(\mathbf{x})(\omega \mathbf{t}) = \frac{1}{4\pi} \int_S \frac{1}{\mathrm{r}^2} \big[&\mathbf{K}(\mathbf{x}, \mathbf{x}').(\omega' \mathbf{t}') \\ &- \mathbf{k}_2\left(\mathbf{x}, \mathbf{x}'\right) \times \left(\kappa'_\mathrm{n} \mathbf{i}_{\phi'}\right) \big] \, dS \end{aligned} \tag{61}$$

$$\begin{aligned} 0 = \frac{1}{8\pi} \int_S \big[&\mathbf{k}_1(\mathbf{x}, \mathbf{x}').(\omega' \mathbf{t}') \\ &- \mathbf{k}_2(\mathbf{x}, \mathbf{x}').(\kappa'_\mathrm{n} \mathbf{i}_{\phi'}) \big] \, dS \\ &- \frac{1}{4\pi} \int_S \frac{1}{\mathrm{r}^2} \big[\mathbf{k}_2(\mathbf{x}, \mathbf{x}').(\beta'_\mathrm{t} \mathbf{i}_{\phi'}) \big] \, dS \end{aligned} \tag{62}$$

$$\begin{aligned} 0 = \frac{1}{8\pi} \int_S \big[&\mathbf{K}(\mathbf{x}, \mathbf{x}').(\omega' \mathbf{t}') \\ &+ \mathbf{k}_2(\mathbf{x}, \mathbf{x}') \times (\kappa'_\mathrm{n} \mathbf{i}_{\phi'}) \big] \, dS \\ &+ \frac{1}{4\pi} \int_S \frac{1}{\mathrm{r}^2} \big[\mathbf{k}_2(\mathbf{x}, \mathbf{x}') \times (\beta'_\mathrm{t} \mathbf{i}_{\phi'}) \big] \, dS \end{aligned} \tag{63}$$

Four scalar equations (60)–(63) still remain for two variables (ω' and κ'_n), implying that two equations should be redundant. To detect which are these equations, integration over the azimuthal angle must be carried out. In this manner we specify that the contribution of the ring source distribution is calculated at the evaluation point placed on the meridian curve. Some preliminary transformations have to be introduced. First the differential surface area dS is expressed as $\rho d\phi d\Gamma$, where $d\Gamma$ is the differential arc length along the meridian curve. Second, as all quantities appearing either in the left or right hand side integrals are expressed in distinct moving frames, it is necessary to recast all of them in a common fixed frame which is chosen to be $(\mathrm{O}, \mathbf{i}_\rho, \mathbf{i}_\phi, \mathbf{i}_z)$. We can write

$$\mathbf{i}_{\rho'} = \cos\phi'\mathbf{i}_\rho + \sin\phi'\mathbf{i}_\phi$$
$$\mathbf{i}_{\phi'} = -\sin\phi'\mathbf{i}_\rho + \cos\phi'\mathbf{i}_\phi, \qquad \mathbf{i}_{z'} = \mathbf{i}_z$$
$$\mathbf{n}' = \mathrm{n}_{\rho'}\mathbf{i}_{\rho'} + \mathrm{n}_{z'}\mathbf{i}_{z'} = \mathrm{n}_{\rho'}\cos\phi'\mathbf{i}_\rho + \mathrm{n}_{\rho'}\sin\phi'\mathbf{i}_\phi + \mathrm{n}_{z'}\mathbf{i}_z$$
$$\mathbf{t}' = \mathbf{n}' \times \mathbf{i}_{\phi'} = -\mathrm{n}'_z\mathbf{i}_{\rho'} + \mathrm{n}'_\rho\mathbf{i}_{z'}$$
$$= -\mathrm{n}'_z\cos\phi'\mathbf{i}_\rho - \mathrm{n}'_z\sin\phi'\mathbf{i}_\phi + \mathrm{n}'_\rho\mathbf{i}_z$$
$$\mathbf{x}' = \rho'\mathbf{i}_{\rho'} + z'\mathbf{i}_{z'} = \rho'\cos\phi'\mathbf{i}_\rho + \rho'\sin\phi'\mathbf{i}_\phi + z'\mathbf{i}_z$$
$$(\mathbf{x} - \mathbf{x}') = (\rho - \rho'\cos\phi')\mathbf{i}_\rho - \rho'\sin\phi'\mathbf{i}_\phi + (z - z')\mathbf{i}_z$$

and substitute these expressions into (60)–(63). After integration in the azimuthal direction and simplifications we find that (60) and (62) are identically zero and the two remaining equations are thus

$$c(\mathbf{x})\omega(\mathbf{x}) = \frac{1}{4\pi}\int_\Gamma \big[\big(\mathrm{n}'_\rho\rho G_{03}^{AX} - \mathrm{n}'_\rho\rho' G_{13}^{AX} + (z - z')\mathrm{n}'_z G_{13}^{AX}\big)\omega' - G_{11}^{AX}\kappa'_n\big]\rho' d\Gamma \tag{64}$$

$$\int_\Gamma G_{11}^{AX}\beta'_t\rho' d\Gamma = -\frac{1}{8\pi}\int_\Gamma \big[\big(\mathrm{n}'_\rho\rho G_{01}^{AX} - \mathrm{n}'_\rho\rho' G_{11}^{AX} + (z - z')\mathrm{n}'_z G_{11}^{AX}\big)\omega' + G_{1-1}^{AX}\kappa'_n\big]\rho' d\Gamma \tag{65}$$

In the above equations, the axisymmetric kernel functions are represented using the general notation

$$G_{ij}^{AX} = \int_0^{2\pi}\frac{\cos^i\phi'}{\mathrm{r}^j}d\phi' \tag{66}$$

where r is the distance between the ring source and the evaluation points.

The formulation composed of BIEs (64) and (65) and an additional BIE for the scalar potential fully characterizes the axisymmetric Stokes flows problem with prescribed velocity boundary condition. The axisymmetric kernel functions involved in these equations can be represented in terms of complete elliptic integrals (Appendix B).

6 Numerical results

The BIE formulation for axisymmetric problems is implemented and validated for exterior and interior flow problems. The BIEs are solved by the standard collocation approach. The integral equation for the scalar potential is solved independently, whereas (64) and (65) are solved in a coupled manner, since the unknowns appear in both equations. The boundary Γ is discretized into N elements represented by cubic splines. Over each element the variables (specified and unknowns) are approximated by linear shape functions (hat functions). In this manner we obtain a balanced matrix system, denoting $2N$ equations in discretized form for $2N$ unknowns, which is solved by a direct Gauss elimination solver. The quadrature formula used to compute the kernels functions is adaptive: for regular integrals the order of quadrature varies from 4 to 20, depending on the distance between the evaluation and source points, whereas the logarithmic singular integrals are computed using 20 integration points.

6.1 Exterior Stokes flow: Translation of a sphere

A well known example of an axisymmetrical flow is that arising from the motion of a solid sphere of radius a moving with a constant velocity U through an unbounded fluid otherwise at rest [17]. The velocity components in spherical coordinates (r, θ, ϕ) are

$$\begin{aligned}\mathrm{u}_r &= -\frac{1}{2}U\cos\theta\left(\frac{a}{r}\right)^2\left[\frac{a}{r} - 3\frac{r}{a}\right],\\ \mathrm{u}_\theta &= -\frac{1}{4}U\sin\theta\left(\frac{a}{r}\right)\left[\left(\frac{a}{r}\right)^2 + 3\right]\end{aligned} \tag{67}$$

The boundary condition for the scalar potential is the normal component of the prescribed velocity

$$\mathbf{n}.\mathbf{b} = \mathbf{n}.\nabla\varphi|_\Gamma = U\cos\theta \tag{68}$$

The analytical expression for vorticity and its normal derivative are given by

$$\begin{aligned}\boldsymbol{\omega} &= \nabla \times \mathbf{u}\mathbf{i}_\phi = \frac{3U}{2}\frac{a\sin\theta}{r^2}\mathbf{i}_\phi,\\ \mathbf{n} \times \nabla \times \boldsymbol{\omega} &= -\frac{\partial\omega}{\partial \mathrm{n}}\mathbf{i}_\phi = -\frac{3U}{2}\frac{\mathrm{n}_r a\sin\theta}{r^3}\mathbf{i}_\phi\end{aligned} \tag{69}$$

We consider the radius of the sphere and the constant velocity U as unity. First the Laplace equation is solved for the scalar potential to obtain the corrected tangential velocity β_t (59) which is then substituted in (65). The relative errors for computed variables on the boundary: vorticity and its normal derivative for varying number of nodes in the boundary discretization are shown in Table 1.

6.2 Interior Stokes flow

In these examples the flow inside a sphere of unit radius with two slightly different boundary conditions

Table 1 Relative errors for the vorticity and the normal derivative for translation of a sphere

Number of nodes	Relative errors	
	ω	$\frac{\partial\omega}{\partial n}$
21	1.1×10^{-3}	1.9×10^{-3}
51	1.9×10^{-4}	1.2×10^{-3}
101	4.9×10^{-5}	8.9×10^{-4}
201	1.2×10^{-5}	6.4×10^{-4}

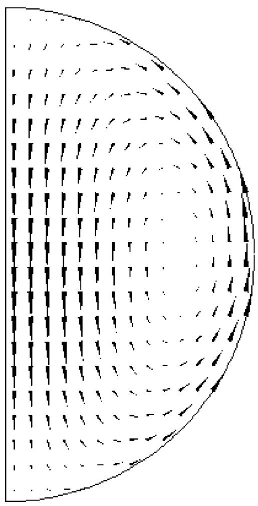

Fig. 2 Velocity vector field for interior Stokes flow example in test case 1

is considered. We impose only the tangential velocity condition such that

in test case 1:

$$\mathbf{n.u}|_\Gamma = 0, \qquad \mathbf{t.u}|_\Gamma = \rho = \sin\theta$$

and in test case 2:

$$\mathbf{n.u}|_\Gamma = 0, \qquad \mathbf{t.u}|_\Gamma = \rho^2 = \sin^2\theta$$

Both examples exhibit a re-circulating profile such as the one presented in Fig. 2. Since the analytical solution is not readily available, the boundary values of the vorticity for both test cases are obtained through commercial finite element software COMSOL Multiphysics (R) for a quantitative comparison with BIEM. The vorticity profiles computed through BIEM and COMSOL are shown in Figs. 3 and 4, respectively. The BIEM discretization involves 51 nodes on the boundary.

7 Summary

A new BIE formulation has been presented for three-dimensional and axisymmetric Stokes flows using Helmholtz decomposition based non-primitive variables. The dependent variables governing the flow field are classically a scalar potential, a vector potential and the vorticity vector. For prescribed boundary velocity the boundary conditions applied to each dependent variable are chosen according to the approach of Hirasaki and Hellums [4]. In a first step, the BIE formulation for three-dimensional flows is obtained. The domain integral appearing in the integral equations for the vector Poisson equation is transformed into boundary integrals by invoking the fundamental solution for the biharmonic equation, using the initial framework provided by Fairweather et al. [5] for two-dimensional problems. The BIE formulation for axisymmetric problems is derived in a second step from the three-dimensional one. This formulation is validated for exterior and interior flows.

The main advantage of both formulations is that they are free from domain integral, thus preserving the 'boundary only' character of the method. From a comparative point of view, the NPV based axisymmetric BIE formulation contains logarithmic singular kernel functions, instead of the first order singularity occurring in conventional primitive variable formulation. The first order singularity is evaluated only in Cauchy principal value sense and, as a consequence, the integrals involved in domain value computation are known to behave as nearly singular. Accordingly, this issue may require special treatment for such computations [20]. The drawback associated with the formulation is the extra equations: seven equations for three-dimensional and three equations for axisymmetric problems, whereas in primitive variable there are only four and three equations respectively. To somewhat compensate this issue, it is possible to couple the formulation with fast solution methods, for instance based on multipole techniques [21, 22].

Appendix A: Vector operations on tensors

The necessary formulas involving the divergence and curl operation of a function, irrespective of the coordinate system are as follows:

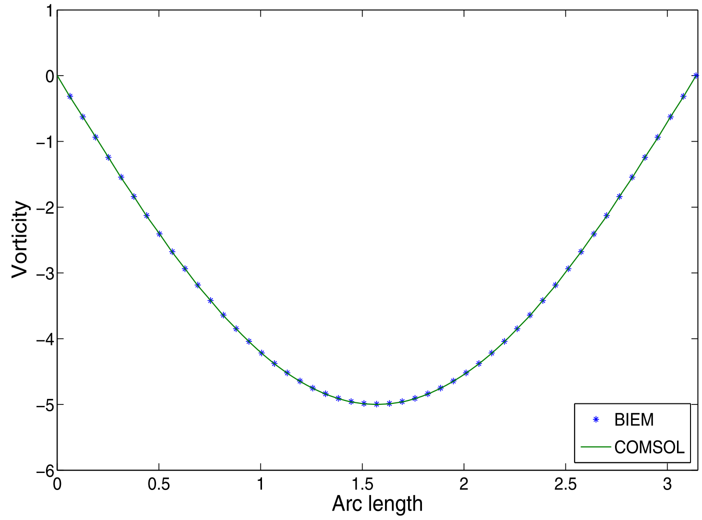

Fig. 3 Vorticity profile on the meridian curve for test case 1

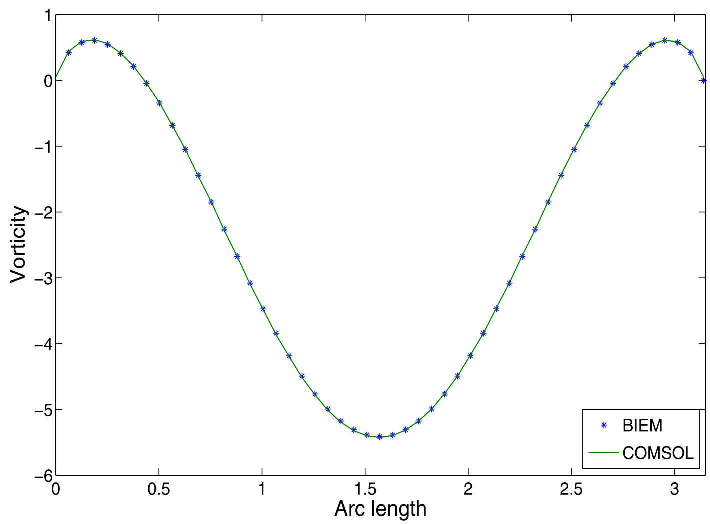

Fig. 4 Vorticity profile on the meridian curve for test case 2

$$(\mathbf{n}.f(r)\mathbf{I})\nabla.\boldsymbol{\omega} = \big(f(r)\nabla.\boldsymbol{\omega}\big)\mathbf{n} \tag{A.1}$$

$$(\mathbf{n}.\boldsymbol{\omega})\big(\nabla.f(r)\mathbf{I}\big) = \mathbf{n}.\boldsymbol{\omega}\big(\nabla f(r)\big) \tag{A.2}$$

$$\big(\mathbf{n}\times f(r)\mathbf{I}\big).(\nabla\times\boldsymbol{\omega}) = \mathbf{n}\times\big(f(r)\mathbf{I}.\nabla\times\boldsymbol{\omega}\big)$$

$$= f(r)\big[\mathbf{n}\times(\nabla\times\boldsymbol{\omega})\big] \tag{A.3}$$

$$(\mathbf{n}\times\boldsymbol{\omega}).(\nabla\times f(r)\mathbf{I}) = (\mathbf{n}\times\boldsymbol{\omega})\times\nabla f(r) \tag{A.4}$$

Appendix B: Axisymmetric kernel functions in terms of Complete Elliptic Integrals

The complete elliptic integral of the first and second kind, $K(\mathrm{m})$ and $E(\mathrm{m})$, are defined as

$$K(\mathrm{m}) = \int_0^{\pi/2} \frac{d\phi'}{\sqrt{1-\mathrm{m}\cos^2\phi'}} \tag{B.1}$$

$$E(\mathrm{m}) = \int_0^{\pi/2} \sqrt{1-\mathrm{m}\cos^2\phi'}\,d\phi' \tag{B.2}$$

where m is the elliptic integral parameter ($0 \le \mathrm{m} \le 1$). Some functions which can be represented in the form of complete elliptic integrals required later to express the kernel functions are

$$\int_0^{\pi/2} \frac{1}{[1-\mathrm{m}\cos^2\phi']^{3/2}}\,d\phi' = \frac{1}{1-\mathrm{m}}E(\mathrm{m}) \tag{B.3}$$

$$\int_0^{\pi/2} \frac{\cos 2\phi'}{[1-\mathrm{m}\cos^2\phi']^{3/2}}\,d\phi' = \frac{(2-\mathrm{m})E(\mathrm{m})-2(1-\mathrm{m})K(\mathrm{m})}{\mathrm{m}(1-\mathrm{m})} \tag{B.4}$$

$$\int_0^{\pi/2} \frac{\cos 2\phi'}{[1-\mathrm{m}\cos^2\phi']^{1/2}}\,d\phi' = \frac{-2E(\mathrm{m})-(2-\mathrm{m})K(\mathrm{m})}{\mathrm{m}} \tag{B.5}$$

$$\int_0^{\pi/2} \cos 2\phi'\big(1-\mathrm{m}\cos^2\phi'\big)^{1/2}\,d\phi' = \frac{(-2+\mathrm{m})E(\mathrm{m})+2(1-\mathrm{m})K(\mathrm{m})}{3\mathrm{m}} \tag{B.6}$$

The axisymmetric integration is carried out in cylindrical coordinates (ρ,ϕ,z), where the distance between the source and evaluation points is

$$\mathrm{r} = \big[\big(z-z'\big)^2+\rho^2+\rho'^2-2\rho\rho'\cos\big(\phi-\phi'\big)\big]^{1/2} \tag{B.7}$$

and the elliptic integral parameter is defined as

$$\mathrm{m} = \frac{4\rho\rho'}{(z-z')^2+(\rho+\rho')^2} \tag{B.8}$$

Taking into account that the plane of computation is the origin plane ($\phi = 0$), we obtain

$$\mathrm{r} = \big[a-b\cos\phi'\big]^{1/2}, \qquad \mathrm{m} = \frac{2b}{a+b} \tag{B.9}$$

where

$$a = \big(z-z'\big)^2+\rho^2+\rho'^2, \qquad b = 2\rho\rho' \tag{B.10}$$

For specific values of i and j, the kernel functions appearing in the axisymmetric BIE formulation are

For $i=0$ and $j=1$:

$$G_{01}^{AX} = \frac{4K(\mathrm{m})}{(a+b)^{1/2}} \tag{B.11}$$

For $i=0$ and $j=3$:

$$G_{03}^{AX} = \frac{4}{(a+b)^{3/2}}\int_0^{\pi/2} \frac{1}{[1-\mathrm{m}\cos^2\phi']^{3/2}}\,d\phi' \tag{B.12}$$

For $i=1$ and $j=1$:

$$G_{11}^{AX} = \frac{4}{(a+b)^{1/2}}\int_0^{\pi/2} \frac{\cos 2\phi'}{[1-\mathrm{m}\cos^2\phi']^{1/2}}\,d\phi' \tag{B.13}$$

For $i=1$ and $j=3$:

$$G_{13}^{AX} = \frac{4}{(a+b)^{3/2}}\int_0^{\pi/2} \frac{\cos 2\phi'}{[1-\mathrm{m}\cos^2\phi']^{3/2}}\,d\phi' \tag{B.14}$$

For $i=1$ and $j=-1$:

$$G_{1-1}^{AX} = 4(a+b)^{1/2} \times \int_0^{\pi/2} \cos 2\phi'\big(1-\mathrm{m}\cos^2\phi'\big)^{1/2}\,d\phi' \tag{B.15}$$

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