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On the unsteady rotational flow of fractional Oldroyd-B fluid in cylindrical domains

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Abstract This paper concerned with the unsteady rotational flow of fractional Oldroyd-B fluid, between two infinite coaxial circular cylinders. To solve the problem we used the finite Hankel and Laplace transforms. The motion is produced by the inner cylinder that, at time $t = 0^+$, is subject to a time-dependent rotational shear. The solutions that have been obtained, presented under series form in terms of the generalized G functions, satisfy all imposed initial and boundary conditions. The corresponding solutions for ordinary Oldroyd-B, fractional and ordinary Maxwell, fractional and ordinary second grade, and Newtonian fluids, performing the same motion, are obtained as limiting cases of general solutions.

The most important things regarding this paper to mention are that (1) we extracted the expressions for the velocity field and the shear stress corresponding to the motion of a fractional second grade fluid as limiting cases of general solutions corresponding to the fractional Oldroyd-B fluid, this is not previously done in the literature to the best of our knowledge, and (2) the expressions for the velocity field and the shear stress are in the most simplified form, and the point worth mentioning is that these expressions are free from convolution product and the integral of the product of the generalized G functions, in contrast with (Imran and Zamra in Commun. Nonlinear Sci. Numer. Simul. 16:226–238, 2011) in which the expression for the velocity field involving the convolution product as well as the integral of the product of the generalized G functions.

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1 Introduction

A wide range of commonly encountered fluids includes Newtonian fluids. But a large number of fluids appearing in industry differ greatly from Newtonian fluids in their rheology. Newtonian fluids are recognized by the linear relationship between stress and the rate of strain. In many existing fluids with complex molecular structure, the relation between stress and strain is found to be non-linear. Therefore, the Newtonian fluids model can not be used to predict, analyze and simulate the behavior of many viscoelastic fluids. Hence, in practical situations and applications in industry, it is necessary to study the flow behavior of non-Newtonian fluids.

Ordinary Navier-Stokes equations are a good tool to describe the flows of a large class of fluids (Newtonian fluids), but for the flow of fluids with complex microstructure (non-Newtonian fluids) which show non-linear viscoelastic behavior [2], a single governing equation is unable to describe the flow behavior and all its properties [3]. The study of fluid flows has a variety of applications in medicine, science and technology. During the past few decades, the flows of non-Newtonian fluids have been recognized more promising in industry and technology.

The inadequacy of the ordinary Navier-Stokes theory to describe the behavior of the rheologically complex fluids such as polymer solutions, heavy oils, blood and many emulsions, has led to the development of models of non-Newtonian fluids. In particular, many pastes, slurries, synovial, polymer solutions and suspensions exhibit shear thinning behavior. The study of non-Newtonian fluids [4] has become of increasing interest and importance in recent times. They are widely used in chemical engineering, food industry, biological analysis, petroleum industry and many other fields. The academic workers and engineers are very much interested in the geometry of flows of such types of fluids. As compared to Newtonian fluids, the analysis of the behavior of the motion of such fluids is much more complicated and not easy to handle because of non-linear relationship between stress and the rate of strain.

The flow analysis of non-Newtonian fluids is very important in the fields of fluid mechanics due to their several technological applications. Due to the complex stress-strain relationship, not many investigators have studied the flow behavior of non-Newtonian fluids in various flow fields. The study of non-Newtonian fluids [5–7] has got much attention because of their practical applications. A number of industrially important fluids including polymers, molten plastics, pulps, microfluids and food stuff display non-Newtonian characteristics. Exact analytic solutions for the flows of non-Newtonian fluids are important provided they correspond to the physically realistic problems and they can be used as checks against complicated numerical codes that have been developed for much more complex flows. In recent years many non-Newtonian models have been proposed. They include differential type, rate type and integral type fluids. Among them, the rate type fluid models have received special attention. The differential type fluids do not predict stress relaxation and they are not successful for describing the flows of some polymers.

The motion of a fluid in a rotating or translating cylinder is of interest to both theoretical and practical domains. It is very important to study the mechanism of viscoelastic fluids flow in many industry fields, such as oil exploitation, chemical and food industry and bio-engineering. For Newtonian fluids, the transient velocity distribution for the flow within a circular cylinder may be found in [8]. The first exact solutions for flows of non-Newtonian fluids in such a domain seem to be those of Ting [9] corresponding to second grade fluids and Srivastava [10] for Maxwell fluids. Later, Waters and King [11] studied the start-up Poiseuille flow of an Oldroyd-B fluid in a straight circular tube and its decay from the steady state condition when the pressure gradient is removed. During recent years quite many papers of this type have been published. The most general solutions corresponding to the helical flow of a second grade fluid seem to be those of Fetecau and Corina Fetecau [12], in which the cylinder is rotating around its axis and sliding along the same axis with time-dependent velocities. Other interesting solutions for different flows of the second grade and Maxwell fluids have been also obtained by Nadeem et al. [13]. Exact solutions for the helical flows of Oldroyd-B fluid in cylindrical domains have been obtained by Wood [14] and Fetecau et al. [15]. In the meantime a lot of papers regarding such motions have been published.

Recently, fractional calculus has encountered much success in the description of viscoelasticity [16–20]. The starting point of the fractional derivative model of a non-Newtonian fluid is usually a ordinary differential equation which is modified by replacing the time derivative of an integer order by the fractional calculus operators. This generalization allows one to define precisely non-integer order integrals or derivatives. Tan et al. [18] and Xu and Tan [21] examined the velocity field, stress field and vortex sheet of a generalized second-order fluid with fractional anomalous diffusion. Song and Jiang [22] achieved satisfactory result to apply the constitutive equation with fractional derivative to the experimental data of viscoelasticity. Tan et al. [23] and Tan and Xu [24] applied fractional derivative to the constitutive relationship models of Maxwell viscoelastic fluid and second grade fluid, and studied some unsteady flows.

The aim of this paper is to establish analytical solutions for the velocity field and the adequate shear stress corresponding to the unsteady flow of an incompressible fractional Oldroyd-B fluid between two infinite co-axial circular cylinders induced by a time-dependent shear. The motion of the fluid is produced by the inner cylinder, which at time $t = 0^+$, begins to rotate about its axis with a time-dependent shear stress. The solutions that have been obtained, presented under series form in terms of the generalized G functions, are established by means of the finite Hankel and Laplace transforms. The similar solutions for ordinary Oldroyd-B, fractional Maxwell, ordinary Maxwell fluids as well as those for the fractional and ordinary second grade fluids, can be obtained as limiting cases for $\gamma, \beta \to 1$; $\lambda_r \to 0$ and $\beta \to 1$; $\lambda_r \to 0$, γ and $\beta \to 1$; $\lambda \to 0$ and $\gamma \to 1$; $\lambda \to 0$, γ and $\beta \to 1$ respectively. The solutions for a Newtonian fluid performing the same motion, are obtained as limiting cases of the solutions for fractional Oldroyd-B fluid when $\lambda_r \to 0$, $\lambda \to 0$ and $\gamma, \beta \to 1$.

2 Governing equations

The flows to be considered here have the velocity $\mathbf{v}$ and the extra-stress $\mathbf{S}$ of the form [25]

$$\mathbf{v} = \mathbf{v}(r,t) = w(r,t)\mathbf{e}_\theta, \qquad \mathbf{S} = \mathbf{S}(r,t), \tag{1}$$

where $\mathbf{e}_\theta$ is the unit vector in the θ-direction of the cylindrical coordinates r, θ and z. For such flows, the constraint of incompressibility is automatically satisfied. Furthermore, if initially the fluid is at rest, then

$$\mathbf{v}(r,0) = \mathbf{0}, \qquad \mathbf{S}(r,0) = \mathbf{0}. \tag{2}$$

In the absence of body forces and a pressure gradient in the θ-direction, the governing equations corresponding to such motions of Oldroyd-B fluids are given by [25]

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial w(r,t)}{\partial t} = \nu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)w(r,t); \tag{3}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau(r,t) = \mu\left(1+\lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)w(r,t), \tag{4}$$

where $\tau(r,t) = S_{r\theta}(r,t)$ is the non-trivial shear stress, ρ is the constant density of the fluid, $\nu = \mu/\rho$ is the kinematic viscosity, λ is the relaxation time, and λ_r is the retardation time.

The governing equations corresponding to a fractional Oldroyd-B fluid, performing the same motion,

$$\left(1+\lambda D_t^\gamma\right)\frac{\partial w(r,t)}{\partial t} = \nu\left(1+\lambda_r D_t^\beta\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)w(r,t); \tag{5}$$

$$\left(1+\lambda D_t^\gamma\right)\tau(r,t) = \mu\left(1+\lambda_r D_t^\beta\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)w(r,t), \tag{6}$$

are obtained from (3) and (4) by replacing the inner time derivatives by the Riemann-Liouville fractional operators D_t^γ and D_t^β as given by [17, 26]

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^\beta}d\tau, \quad 0 \le \beta < 1, \tag{7}$$

where $\Gamma(.)$ is the Gamma function.

When $\gamma, \beta \to 1$, (5) and (6) reduce to (3) and (4) because $D_t^1 f = \frac{df}{dt}$. Furthermore, the new material constants λ and λ_r (although, for the simplicity, we keep the same notation) go to the initial λ and λ_r.

3 Rotational flow through an annulus

Let us consider an incompressible fractional Oldroyd-B fluid at rest in an annular region between two coaxial circular cylinders of radii R_1 and $R_2 (> R_1)$. At time $t = 0^+$, a time dependent rotational shear stress

$$\tau(R_1,t) = \frac{f}{\lambda}\left(\frac{R_1}{r}\right)^2 R_{\gamma,-1}\left(-\frac{1}{\lambda},t\right); \quad 0 \le \gamma < 1, \tag{8}$$

where f is a constant and the generalized functions R are defined by [27]

$$R_{a,b}(d,t) = \mathcal{L}^{-1}\left\{\frac{q^b}{q^a-d}\right\} = \sum_{n=0}^{\infty}\frac{d^n t^{(n+1)a-b-1}}{\Gamma[(n+1)a-b]}; \quad \mathrm{Re}(a-b)>0,\ \mathrm{Re}(q)>0,\ \left|\frac{d}{q^a}\right|<1, \tag{9}$$

is applied on the boundary of the inner cylinder. Due to the shear, the fluid is gradually moved and its velocity being of the form $(1)_1$. The governing equations are given by (5) and (6), while appropriate initial and boundary conditions are

$$w(r,0)=\frac{\partial w(r,0)}{\partial t}=0,\quad \tau(r,0)=0;\ r\in[R_1,R_2], \tag{10}$$

and

$$\begin{aligned}&\left(1+\lambda D_t^{\gamma}\right)\tau(r,t)|_{r=R_1}\\&\quad=\mu\left(1+\lambda_r D_t^{\beta}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)w(r,t)|_{r=R_1}=f,\\&\quad w(R_2,t)=0;\ t\geq 0.\end{aligned} \tag{11}$$

Of course, $\tau(R_1,t)$ given by (8) is just the solution of $(11)_1$, with the initial condition $(10)_3$.

The partial differential equations (5) and (6), also containing fractional derivatives, can be solved in principle by several methods, the integral transforms technique representing a systematic, efficient and powerful tool. In the following we shall use the Laplace transform to eliminate the time variable and the finite Hankel transform for the removal of spatial variable. However, in order to avoid the burdensome calculations of residues and contour integrals, we shall apply the discrete inverse Laplace transform method.

3.1 Calculation of the velocity field

Applying the Laplace transform to (5), and having in mind the initial and boundary conditions (10) and (11), we find that

$$\begin{aligned}&\left(q+\lambda q^{\gamma+1}\right)\overline{w}(r,q)\\&\quad=\nu\left(1+\lambda_r q^{\beta}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\overline{w}(r,q),\end{aligned} \tag{12}$$

where the image function $\overline{w}(r,q)=\mathcal{L}\{w(r,t)\}=\int_0^\infty e^{-qt}w(r,t)dt$ has to satisfy the conditions

$$\begin{aligned}&\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\overline{w}(r,q)|_{r=R_1}\\&\quad=\frac{f}{\mu q(1+\lambda_r q^{\beta})};\quad \overline{w}(R_2,q)=0.\end{aligned} \tag{13}$$

We shall denote by [28]

$$\overline{w}_H(r_n,q)=\int_{R_1}^{R_2} r\overline{w}(r,q)B(r,r_n)dr, \tag{14}$$

the finite Hankel transform of $\overline{w}(r,q)$, where

$$B(r,r_n)=J_1(rr_n)Y_2(R_1r_n)-J_2(R_1r_n)Y_1(rr_n), \tag{15}$$

r_n being the positive roots of the transcendental equation $B(R_2,r)=0$, while $J_p(\cdot)$ and $Y_p(\cdot)$ are the Bessel functions of the first and the second kind of order p.

The inverse Hankel transform of $\overline{w}_H(r_n,q)$ is given by [28]

$$\begin{aligned}&\overline{w}(r,q)\\&\quad=\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2J_1^2(R_2r_n)B(r,r_n)}{J_2^2(R_1r_n)-J_1^2(R_2r_n)}\overline{w}_H(r_n,q).\end{aligned} \tag{16}$$

Now multiplying both sides of (12) by $rB(r,r_n)$, then integrating with respect to r from R_1 to R_2 and taking into account the conditions (13) and the equality

$$\begin{aligned}&\int_{R_1}^{R_2} r\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right]\overline{w}(r,q)B(r,r_n)dr\\&\quad=-r_n^2\overline{w}_H(r_n,q)+\frac{2}{\pi r_n}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\overline{w}(r,q)|_{r=R_1}\\&\quad=-r_n^2\overline{w}_H(r_n,q)+\frac{2f}{\pi\mu r_n q(1+\lambda_r q^{\beta})},\end{aligned} \tag{17}$$

we find that

$$\begin{aligned}&\overline{w}_H(r_n,q)\\&\quad=\frac{2f}{\rho\pi r_n}\frac{1}{q(q+\lambda q^{\gamma+1}+\nu r_n^2+\nu r_n^2\lambda_r q^{\beta})}.\end{aligned} \tag{18}$$

Now, for a more suitable presentation of the velocity field $w(r,t)$, we shall rewrite (18) in the following equivalent form

$$\begin{aligned}&\overline{w}_H(r_n,q)\\&\quad=\frac{2f}{\pi\mu r_n^3}\frac{1}{q}\\&\qquad-\frac{2f(1+\lambda q^{\gamma}+\nu\lambda_r r_n^2q^{\beta-1})}{\pi\mu r_n^3(q+\lambda q^{\gamma+1}+\nu r_n^2+\nu\lambda_r r_n^2q^{\beta})}.\end{aligned} \tag{19}$$

Applying inverse Hankel transform to (19), and taking into account the following result

$$\int_{R_1}^{R_2}\left(r^2-R_2^2\right)B(r,r_n)dr=\frac{4}{\pi r_n^3}\left(\frac{R_2}{R_1}\right)^2, \tag{20}$$

we find that

$$\overline{w}(r,q) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right)\frac{1}{q} - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(r,r_n)}{r_n[J_2^2(R_1r_n) - J_1^2(R_2r_n)]} \times \frac{(1+\lambda q^{\gamma} + \nu\lambda_r r_n^2 q^{\beta-1})}{(q+\lambda q^{\gamma+1} + \nu r_n^2 + \nu\lambda_r r_n^2 q^{\beta})}. \tag{21}$$

Introducing the expansion (48) (see Appendix) into (21), we get

$$\overline{w}(r,q) = \frac{f}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right)\frac{1}{q} - \frac{\pi f}{\mu\lambda}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(r,r_n)}{r_n[J_2^2(R_1r_n) - J_1^2(R_2r_n)]} \times \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \lambda_r^m \times \frac{(q^{\beta m-k-1} + \lambda q^{\gamma+\beta m-k-1} + \nu\lambda_r r_n^2 q^{\beta(m+1)-k-2})}{(q^{\gamma}+\lambda^{-1})^{k+1}}. \tag{22}$$

Applying the inverse Laplace transform to (22) and using (49) [27], we obtain

$$w(r,t) = \frac{1}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right)f - \frac{\pi f}{\mu\lambda}\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(r,r_n)}{r_n[J_2^2(R_1r_n) - J_1^2(R_2r_n)]} \times \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \lambda_r^m \times \big[G_{\gamma,\beta m-k-1,k+1}(-\lambda^{-1},t) + \lambda G_{\gamma,\gamma+\beta m-k-1,k+1}(-\lambda^{-1},t) + \nu\lambda_r r_n^2 G_{\gamma,\beta(m+1)-k-2,k+1}(-\lambda^{-1},t)\big]. \tag{23}$$

3.2 Calculation of the shear stress

Applying the Laplace transform to (6), we find that

$$\overline{\tau}(r,q) = \mu\frac{1+\lambda_r q^{\beta}}{1+\lambda q^{\gamma}}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\overline{w}(r,q). \tag{24}$$

In order to obtain a suitable form for the shear stress $\tau(r,t)$, we rewrite (18) into an equivalent form as

$$\overline{w}_H(r_n,q) = \frac{2f}{\mu\pi r_n^3(1+\lambda_r q^{\beta})}\frac{1}{q} - \frac{2f(1+\lambda q^{\gamma})}{\mu\pi r_n^3(1+\lambda_r q^{\beta})(q+\lambda q^{\gamma+1}+\nu r_n^2+\nu\lambda_r r_n^2 q^{\beta})}. \tag{25}$$

Applying the inverse Hankel transform to (25), we get

$$\overline{w}(r,q) = \frac{f}{2}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right)\frac{1}{\mu(1+\lambda_r q^{\beta})}\frac{1}{q} - \pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)B(r,r_n)}{r_n[J_2^2(R_1r_n) - J_1^2(R_2r_n)]} \times \frac{1+\lambda q^{\gamma}}{\mu(1+\lambda_r q^{\beta})(q+\lambda q^{\gamma+1}+\nu r_n^2+\nu\lambda_r r_n^2 q^{\beta})}, \tag{26}$$

where

$$\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\overline{w}(r,q) = f\left(\frac{R_1}{r}\right)^2\frac{1}{\mu(1+\lambda_r q^{\beta})}\frac{1}{q} + \pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)\widetilde{B}(r,r_n)}{[J_2^2(R_1r_n) - J_1^2(R_2r_n)]} \times \frac{1+\lambda q^{\gamma}}{\mu(1+\lambda_r q^{\beta})(q+\lambda q^{\gamma+1}+\nu r_n^2+\nu\lambda_r r_n^2 q^{\beta})}. \tag{27}$$

Introducing (27) into (24), we find that

$$\overline{\tau}(r,q) = f\left(\frac{R_1}{r}\right)^2\frac{1}{(1+\lambda q^{\gamma})}\frac{1}{q} + \pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2r_n)\widetilde{B}(r,r_n)}{[J_2^2(R_1r_n) - J_1^2(R_2r_n)]} \times \frac{1}{(q+\lambda q^{\gamma+1}+\nu r_n^2+\nu\lambda_r r_n^2 q^{\beta})}, \tag{28}$$

where

$$\widetilde{B}(r,r_n) = J_2(rr_n)Y_2(R_1r_n) - J_2(R_1r_n)Y_2(rr_n). \tag{29}$$

Introducing (48) into (28), we get

$$\overline{\tau}(r,q) = f\left(\frac{R_1}{r}\right)^2 \frac{1}{(1+\lambda q^{\gamma})}\frac{1}{q} + \frac{\pi f}{\lambda}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)\widetilde{B}(r,r_n)}{[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \times \lambda_r^m \frac{q^{\beta m-k-1}}{(q^{\gamma}+\lambda^{-1})^{k+1}}. \tag{30}$$

Applying the discrete inverse Laplace transform to (30), and using (9) and (49), we find that the shear stress $\tau(r,t)$ has the form

$$\tau(r,t) = \left(\frac{R_1}{r}\right)^2 \frac{f}{\lambda} R_{\gamma,-1}(-\lambda^{-1},t) + \frac{\pi f}{\lambda}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)\widetilde{B}(r,r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \times \lambda_r^m G_{\gamma,\beta m-k-1,k+1}(-\lambda^{-1},t). \tag{31}$$

4 Limiting cases

4.1 Ordinary Oldroyd-B fluid

Making $\gamma \to 1$ and $\beta \to 1$ into (23) and (31), we obtain the velocity field

$$w_{OB}(r,t) = \frac{1}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right) f - \frac{\pi f}{\mu\lambda}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r,r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!} \times \left(\frac{-\nu r_n^2}{\lambda}\right)^k \lambda_r^m\big[\lambda G_{1,m-k,k+1}(-\lambda^{-1},t) + (1+\nu\lambda_r r_n^2) G_{1,m-k-1,k+1}(-\lambda^{-1},t)\big], \tag{32}$$

and the associated tangential stress

$$\tau_{OB}(r,t) = \left(\frac{R_1}{r}\right)^2 f\left\{1 - \exp\left(\frac{-t}{\lambda}\right)\right\} + \frac{\pi f}{\lambda}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)\widetilde{B}(r,r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \times \lambda_r^m G_{1,m-k-1,k+1}(-\lambda^{-1},t), \tag{33}$$

corresponding to ordinary Oldroyd-B fluid performing the same motion.

4.2 Fractional Maxwell fluid

Making $\lambda_r \to 0$ into (32) and (33), we obtain the velocity field

$$w_{FM}(r,t) = \frac{1}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right) f - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r,r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \big[G_{\gamma,\gamma-k-1,k+1}(-\lambda^{-1},t) + \lambda^{-1} G_{\gamma,-k-1,k+1}(-\lambda^{-1},t)\big], \tag{34}$$

and the shear stress

$$\tau_{FM}(r,t) = \left(\frac{R_1}{r}\right)^2 \frac{f}{\lambda} R_{\gamma,-1}(-\lambda^{-1},t) + \frac{\pi f}{\lambda}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)\widetilde{B}(r,r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\gamma,-k-1,k+1}(-\lambda^{-1},t), \tag{35}$$

corresponding to the fractional Maxwell fluid performing the same motion.

4.3 Ordinary Maxwell fluid

Making $\lambda_r \to 0$ and $\gamma \to 1$ into (23) and (31), we obtain the velocity field

$$w_M(r,t) = \frac{1}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right)f - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r,r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k\left[G_{1,-k,k+1}\left(-\lambda^{-1},t\right) + \lambda^{-1}G_{1,-k-1,k+1}\left(-\lambda^{-1},t\right)\right], \tag{36}$$

and its associated tangential stress

$$\tau_M(r,t) = \left(\frac{R_1}{r}\right)^2 f\left\{1 - \exp\left(\frac{-t}{\lambda}\right)\right\} + \frac{\pi f}{\lambda}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)\widetilde{B}(r,r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{1,-k-1,k+1}\left(-\lambda^{-1},t\right), \tag{37}$$

corresponding to ordinary Maxwell fluid performing the same motion.

4.4 Fractional second grade fluid

Making $\lambda \to 0$ and $\gamma \to 1$ into (23) and (31) and using (50), (51) and (52), we obtain the velocity field

$$w_{FSG}(r,t) = \frac{1}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right)f - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r,r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k\left[G_{1-\beta,-\beta k-\beta,k+1}\left(-\nu\lambda_r r_n^2,t\right) + \nu\lambda_r r_n^2 G_{1-\beta,-\beta k-1,k+1}\left(-\nu\lambda_r r_n^2,t\right)\right], \tag{38}$$

and the associated tangential stress

$$\tau_{FSG}(r,t) = \left(\frac{R_1}{r}\right)^2 f + \pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)\widetilde{B}(r,r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k G_{1-\beta,-\beta k-\beta,k+1}\left(-\nu\lambda_r r_n^2,t\right), \tag{39}$$

corresponding to fractional second grade fluid performing the same motion.

By taking $\nu\lambda_r = \alpha$ and $\alpha_1 = \alpha\rho$ (the material constants for second grade fluid) into (38) and (39), we shall get the similar solutions for fractional second grade fluid as we get directly from the governing equations of fractional second garde fluid as under

$$w_{FSG}(r,t) = \frac{1}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right)f - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r,r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k\left[G_{1-\beta,-\beta k-\beta,k+1}\left(-\alpha r_n^2,t\right) + \alpha r_n^2 G_{1-\beta,-\beta k-1,k+1}\left(-\alpha r_n^2,t\right)\right], \tag{40}$$

and

$$\tau_{FSG}(r,t) = \left(\frac{R_1}{r}\right)^2 f + \pi f\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)\widetilde{B}(r,r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k G_{1-\beta,-\beta k-\beta,k+1}\left(-\alpha r_n^2,t\right). \tag{41}$$

4.5 Ordinary second grade fluid

Making $\beta \to 1$ into (40) and (41) and using (53), the expressions for velocity field

$$w_{SG}(r,t) = \frac{1}{2\mu}\left(\frac{R_1}{R_2}\right)^2\left(r - \frac{R_2^2}{r}\right)f - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_1^2(R_2 r_n)B(r,r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \exp\left(-\frac{\nu r_n^2 t}{1 + \alpha r_n^2}\right), \tag{42}$$

and the associated tangential stress

$$\tau_{SG}(r,t) = \left(\frac{R_1}{r}\right)^2 f + \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n)\widetilde{B}(r,r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \frac{1}{1+\alpha r_n^2} \times \exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right), \quad (43)$$

corresponding to an ordinary second grade fluid performing the same motion are recovered. The velocity field (42) is identical to (5.17) from [29], obtained by a different technique.

4.6 Newtonian fluid

Making $\lambda \to 0$ into (36) and (37) and using (52), we obtain the corresponding solutions for the Newtonian fluid, as follows

$$w_N(r,t) = \frac{1}{2\mu}\left(\frac{R_1}{R_2}\right)^2 \left(r - \frac{R_2^2}{r}\right) f - \frac{\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r,r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=0}^{\infty} (-\nu r_n^2)^k \frac{t^k}{k!}, \quad (44)$$

and

$$\tau_N(r,t) = \left(\frac{R_1}{r}\right)^2 f + \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n)\widetilde{B}(r,r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \sum_{k=0}^{\infty} (-\nu r_n^2)^k \frac{t^k}{k!}. \quad (45)$$

These solutions can also be written in a more simpler form as

$$w_N(r,t) = \frac{1}{2\mu}\left(\frac{R_1}{R_2}\right)^2 \left(r - \frac{R_2^2}{r}\right) f - \frac{\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r,r_n)}{r_n[J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \exp(-\nu r_n^2 t), \quad (46)$$

$$\tau_N(r,t) = \left(\frac{R_1}{r}\right)^2 f + \pi f \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n)\widetilde{B}(r,r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \exp(-\nu r_n^2 t). \quad (47)$$

5 Conclusions and numerical results

The purpose of this paper is to establish exact solutions for the velocity field and the adequate shear stress corresponding to the unsteady flow of an incompressible fractional Oldroyd-B fluid between two infinite coaxial circular cylinders induced by a time-dependent shear. The motion of the fluid is produced by the inner cylinder, which at time $t = 0^+$, begins to rotate about its axis with a time-dependent shear stress. The solutions that have been obtained by means of the finite Hankel and Laplace transforms, are presented under series form in terms of the generalized G functions. The similar solutions for ordinary Oldroyd-B, fractional Maxwell, ordinary Maxwell, fractional second grade and ordinary second grade fluids as well as those for the Newtonian fluid, performing the same motion, are obtained as limiting cases of the solutions for fractional Oldroyd-B fluid.

In order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity $w(r,t)$ and the shear stress $\tau(r,t)$ given by (23) and (31), have been drawn against r for different values of the time t and of the material parameters. Figures 1a and 1b show the influence of time on the fluid motion. From these figures it is clearly seen that the velocity as well as the shear stress (in absolute value) is an increasing function of t. In Figs. 2a and 2b, it is shown the influence of the kinematic viscosity ν on the fluid motion. It is clearly seen from these figures that the velocity is an increasing function of ν, while the shear stress (in absolute value) is a decreasing function of ν. The influence of the relaxation and the retardation times on the fluid motion is shown in Figs. 3 and 4. The two parameters, as it was to be expected, have opposite effects on the fluid motion. Both the velocity and the shear stress (in absolute value) are decreasing functions with respect to λ and increasing ones with regard to λ_r. Figures 5a and 5b show the influence of the fractional parameter γ on the fluid motion. It is clearly seen from these figures that both the velocity and the shear stress (in absolute value) are increasing functions of γ. In Figs. 6a and 6b, it is shown the influence of the fractional parameter β on

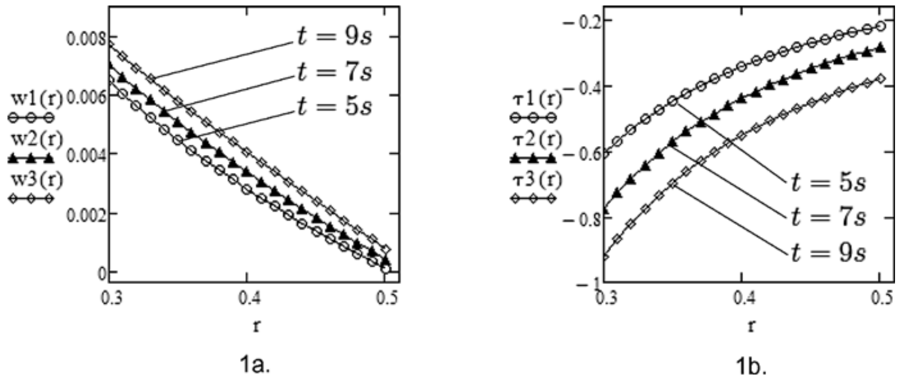


Fig. 1 Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by (23) and (31) for $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0.035$, $\mu = 30$, $\lambda = 12$, $\lambda_r = 2.2$, $\gamma = 0.9$, $\beta = 0.6$ and different values of t

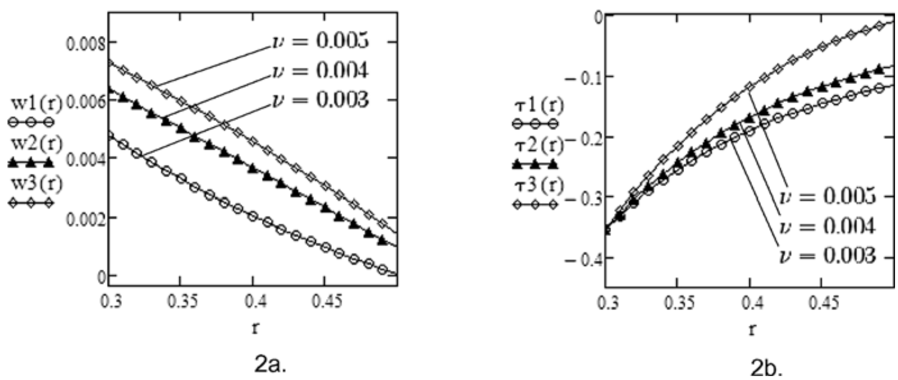


Fig. 2 Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by (23) and (31) for $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $t = 6$ s, $\mu = 40$, $\lambda = 9$, $\lambda_r = 3$, $\gamma = 0.3$, $\beta = 0.3$ and different values of ν

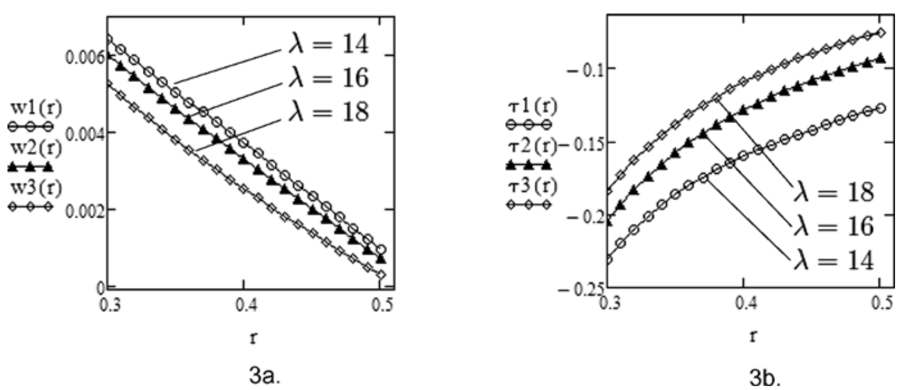
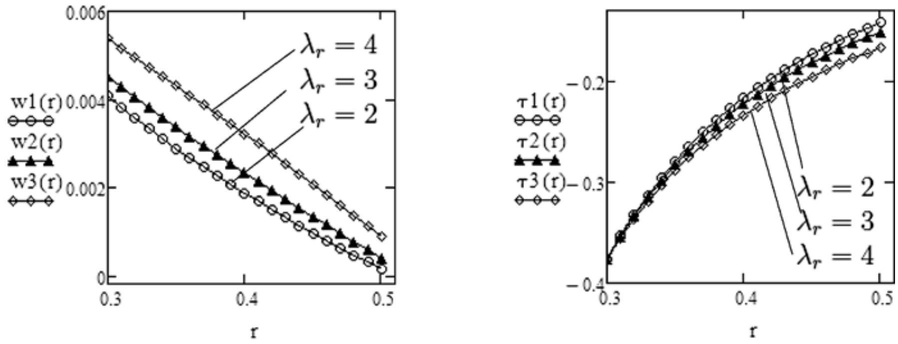


Fig. 3 Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by (23) and (31) for $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $t = 5$ s, $\nu = 0.04$, $\mu = 40$, $\lambda_r = 7$, $\gamma = 0.3$, $\beta = 0.3$ and different values of λ

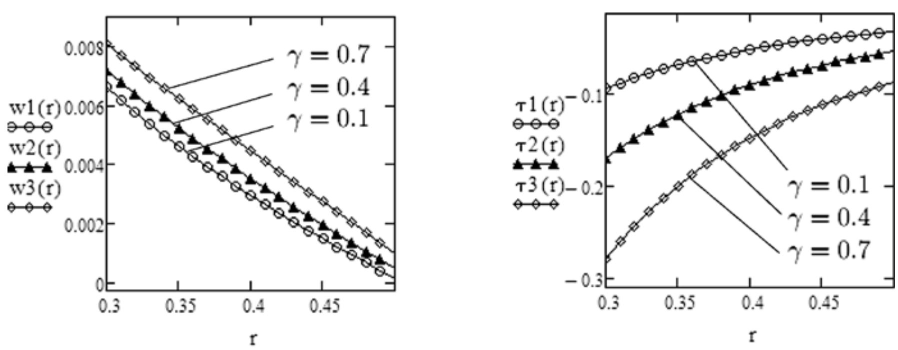
the fluid motion. It is clearly seen from these figures that the velocity is increasing function of β, while the shear stress (in absolute value) is a decreasing function of β.

Finally, for comparison, the diagrams of $w(r,t)$ and $\tau(r,t)$ corresponding to the seven models (fractional Oldroyd-B, ordinary Oldroyd-B, fractional Maxwell, ordinary Maxwell, fractional second grade, ordinary



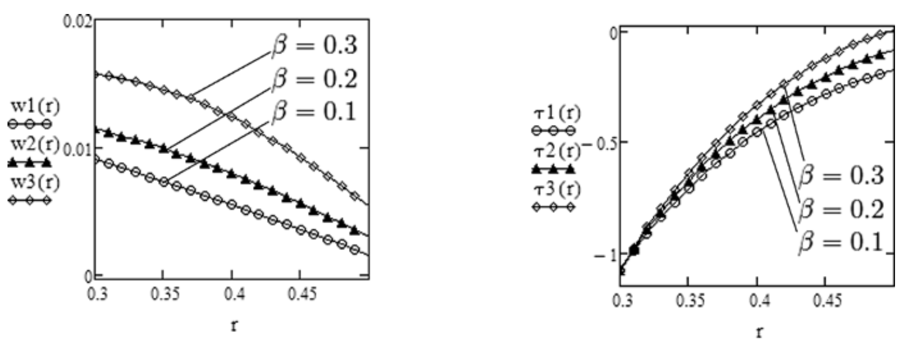
4a. 4b.

Fig. 4 Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by (23) and (31) for $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $t = 5$ s, $\nu = 0.04$, $\mu = 50$, $\lambda = 8$, $\gamma = 0.3$, $\beta = 0.9$ and different values of λ_r



5a. 5b.

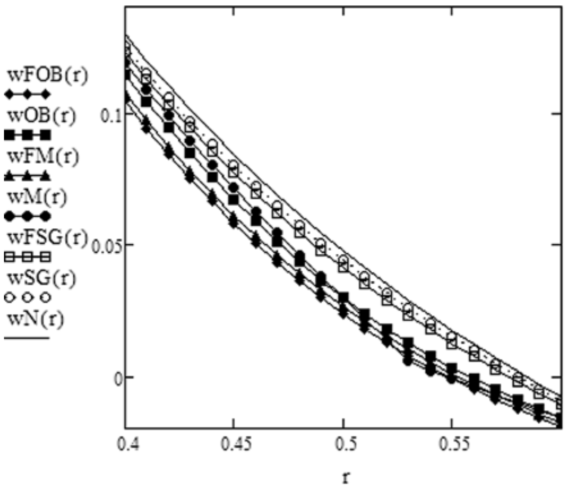
Fig. 5 Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by (23) and (31) for $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $t = 6$ s, $\nu = 0.045$, $\mu = 30$, $\lambda = 25$, $\lambda_r = 8$, $\beta = 1$, and different values of γ



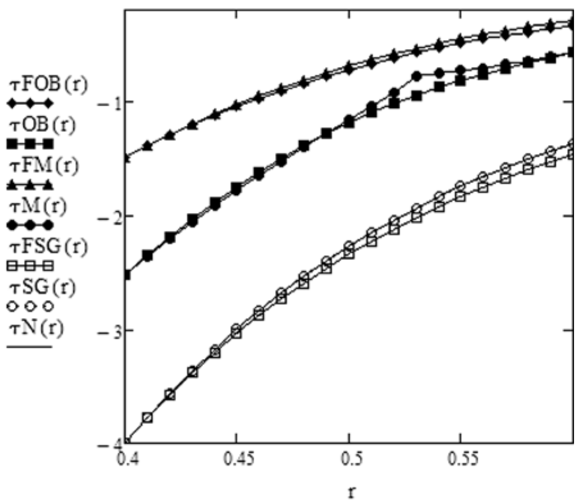
6a. 6b.

Fig. 6 Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by (23) and (31) for $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $t = 6$ s, $\nu = 0.04$, $\mu = 30$, $\lambda = 8$, $\lambda_r = 5.5$, $\gamma = 1$, and different values of β

second grade and Newtonian) are together depicted in Fig. 7 for the same values of the common material constants and time t. In all cases the velocity of the fluid is a decreasing function with respect to r and the Newtonian fluid is the swiftest while the fractional Oldroyd-B fluid has the smallest velocity on the whole flow domain. One thing is of worth mentioning that units of the material constants are SI units in all



7a.



7b.

Fig. 7 Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ corresponding to the Newtonian, fractional second grade, second grade, Maxwell, fractional Maxwell, Oldroyd-B, and fractional Oldroyd-B fluids, for $R_1 = 0.4$, $R_2 = 0.6$, $f = -4$, $t = 5$ s, $\nu = 0.035$, $\mu = 2.96$, $\lambda = 5$, $\lambda_r = 1$, $\gamma = 0.5$ and $\beta = 0.5$

figures, and the roots r_n have been approximated by $(2n-1)\pi/[2(R_2 - R_1)]$.

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Appendix

$$\frac{1}{(q+\lambda q^{\gamma+1}+\nu r_n^2+\nu\lambda_r r_n^2 q^{\beta})} = \frac{1}{\lambda}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \times \lambda_r^m\frac{q^{\beta m-k-1}}{(q^{\gamma}+\lambda^{-1})^{k+1}}, \tag{48}$$

$$G_{a,b,c}(d,t) = \mathcal{L}^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\} = \sum_{j=0}^{\infty}\frac{d^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]};$$

$$\mathrm{Re}(ac-b)>0,\ \left|\frac{d}{q^a}\right|<1, \tag{49}$$

$$\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(-\nu r_n^2\right)^k\lambda_r^m\frac{t^{-\beta m+k}}{\Gamma(-\beta m+k+1)} = \sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k G_{1-\beta,-\beta k-\beta,k+1}\left(-\nu\lambda_r r_n^2,t\right), \tag{50}$$

$$\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{k!}{m!(k-m)!}\left(-\nu r_n^2\right)^k \times \lambda_r^m\frac{t^{-\beta m-\beta+k+1}}{\Gamma(-\beta m-\beta+k+2)} = \sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k G_{1-\beta,-\beta k-1,k+1}\left(-\nu\lambda_r r_n^2,t\right), \tag{51}$$

$$\lim_{\lambda\to 0}\frac{1}{\lambda^k}G_{1,b,k}\left(-\lambda^{-1},t\right) = \frac{t^{-b-1}}{\Gamma(-b)};$$
$$\lim_{\lambda\to 0}\frac{1}{\lambda}R_{1,b}\left(-\lambda^{-1},t\right) = \frac{t^{-b-1}}{\Gamma(-b)};\quad b<0, \tag{52}$$

$$\sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k G_{0,-k-1,k+1}\left(-\alpha r_n^2,t\right) = \frac{1}{1+\alpha r_n^2}\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right). \tag{53}$$

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