Meccanica (2012) 47:245–256
DOI 10.1007/s11012-011-9440-2

Kepler Conics S-code: golden ratio, Dandelin spheres, Fibonacci sequence

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Received: 24 February 2010 / Accepted: 24 May 2011 / Published online: 29 September 2011

Abstract We show how the unitary, genetic **S**-code description of the family of Kepler conic sections, not only enlightens the genesis of the so called Dandelin spheres but also naturally unfolds in the Kepler scenery the famous golden ratio, the golden rectangle and the Fibonacci sequence.

Keywords Kepler conic sections · Dandelin Spheres · Golden section · General Mechanics

1 Introduction

THE FAMILY OF THE KEPLER CONIC SECTIONS.

The three classical curves (ellipse, parabola and hyperbola) were described by the two great Greek geometers Menaechmus (c.375-325 B.C.) and Apollonius of Perga (c.262-190 B.C.) as *geometrical* objects, obtained by *intersecting* a cone with a plane: thus the expression *conic sections*.

The same curves were described geometrically by the Greek mathematician Pappus of Alexandria (c.290-350 AD) as the *locus* of points such that the *ratio* of the their distances from a given fixed point and from a straight fixed line is a *constant value*: the conic being a parabola, ellipse or hyperbola according as the constant ratio is equal to, less than or greater than 1.

We have to wait for Kepler and Newton to find a *physical application* for the conic sections in the dynamics of celestial bodies where it is shown that a mass point under the influence of a central, attractive, inverse square law field, describes an *orbit* which is a conic section. It was Kepler that introduced the term *focus* to denote the attractive, fixed point.

Actually, the Kepler conics, implemented by the peculiar perfect ellipse (the circle), are classified, both from a geometrical and physical point of view, in the following *four* types

circle, *ellipse*, *parabola*, *hyperbola*

corresponding, respectively, to the following constant values of

(1) the geometrical ratio e (the *eccentricity*):

$$e = 0, \qquad 0 < e < 1, \qquad e = 1, \qquad e > 1; \quad (1)$$

(2) the physical total mechanical *energy* E per unit of mass of the particle:

$$E = -K^4(2\Gamma^2)^{-1}, \qquad E < 0, \qquad E = 0, \qquad E > 0,$$

related to e by

$$e^2 - 1 = 2E\Gamma^2 K^{-4} \qquad (2)$$

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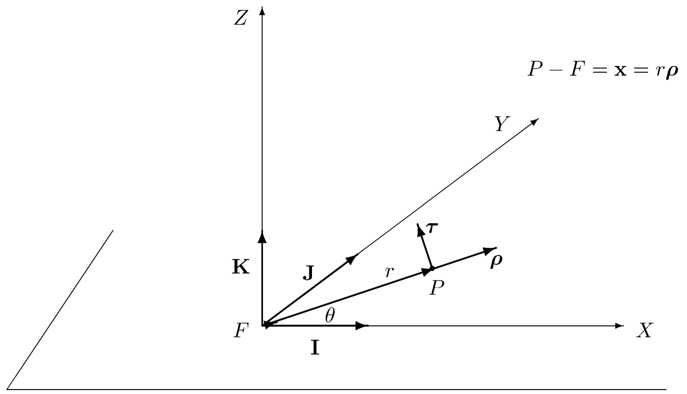

Fig. 1 The polar $\{F, \boldsymbol{\rho}, \boldsymbol{\tau}\}$ and the cartesian $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$ frames

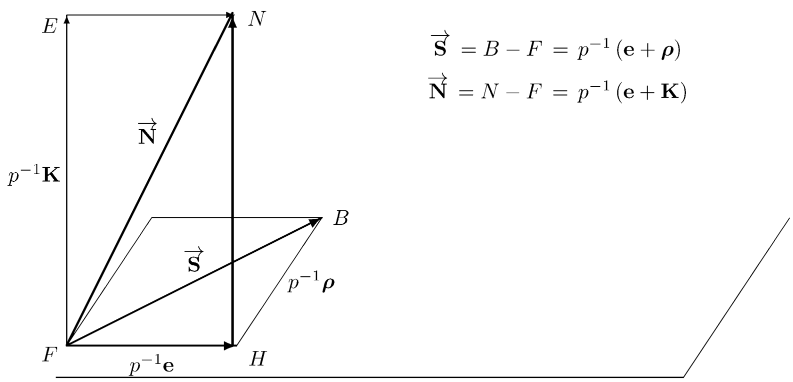

Fig. 2 The sum vector **S**. The orthogonal constant vector **N**

where Γ is the magnitude of the constant *angular momentum* vector per unit of mass and K^2 is the *universal constant of gravitation*.

The whole family of conic sections, characterized by the scalar couple (E, Γ) or by the equivalent couple of positive numbers

$$(e, p)$$

(giving both the shape e and dimension $p = \Gamma^2 K^{-2}$), is represented by the polar *scalar equation*

$$r = \frac{p}{1 + e\cos\theta} \tag{3}$$

in a standard plane polar coordinate system (r, θ), with the origin at the fixed focus F, and with $r = |\mathbf{x}|$, being $\mathbf{x}$ the point particle position vector (Fig. 1)

$$\mathbf{x} = P - F = r\boldsymbol{\rho}.$$

The axis $\theta = 0$ is characterized by the *eccentricity vector*

$$\mathbf{e} = e\boldsymbol{\rho}(0). \tag{4}$$

THE GENETIC **S**-CODE OF THE FAMILY. THE VECTOR **N** AND THE N-CONE.

In our *previous works* [4–6, 8] we have shown that:

(A) a *vector equation* of the whole Kepler family of conic sections is

$$\mathbf{S} \cdot \mathbf{x} = 1 \tag{5}$$

where the *sum vector* **S**, defined by

$$\mathbf{S} \equiv B - F \equiv p^{-1}(\mathbf{e} + \boldsymbol{\rho}), \tag{6}$$

lies in the polar plane of the orbits (see Fig. 2);

(B) the vector **S**, by (5) and (6), not only encompasses the scalar equation (3), but also stores in a sort

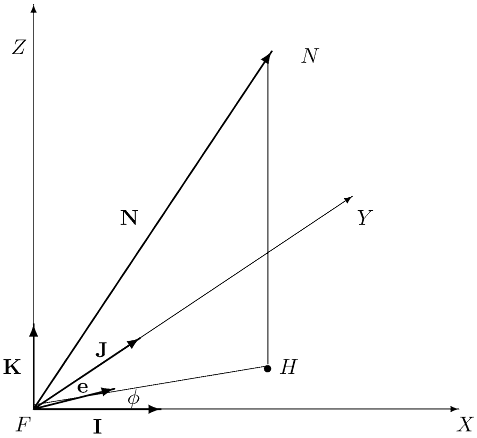

Fig. 3 The vectors **e** and **N** in the 3-space $\{F, X, Y, Z\}$

of *genetic code* all the characteristic geometrical and dynamical information about the Kepler family (for instance, Newton's gravitation law, Binet's equation, the energy E and the map which regularizes at collision the Newton's differential equations of motion);

(C) in particular, the vector **S** encodes a *constant vector* **N** which, in the inertial right-handed unit system $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$ (Figs. 1 and 2), is defined by

$$\mathbf{N} \equiv N - F \equiv p^{-1}(\mathbf{e} + \mathbf{K}), \tag{7}$$

is orthogonal to the $\{\mathbf{I}, \mathbf{J}\}$-plane of the conic orbits *and unravels naturally the N-cone which defines the Kepler orbits as conic sections*. Briefly, the **N**-breed of the N-cone was found by relaxing the restriction (4), so that

$$\mathbf{e} = e_X\mathbf{I} + e_Y\mathbf{J} = e\cos\phi\mathbf{I} + e\sin\phi\mathbf{J} = \mathbf{e}(\phi)$$

(the angle ϕ giving the direction of **e** with respect to **I** (Fig. 3)), whence

$$\mathbf{N} = \frac{e_X}{p}\mathbf{I} + \frac{e_Y}{p}\mathbf{J} + \frac{1}{p}\mathbf{K}$$

and the three coordinates of its *tip point* N, that is

$$X_N = p^{-1}e_X, \qquad Y_N = p^{-1}e_Y, \qquad Z_N = p^{-1} \tag{8}$$

satisfy

$$X_N^2 + Y_N^2 - Z_N^2 = (e^2 - 1)p^{-2} \tag{9}$$

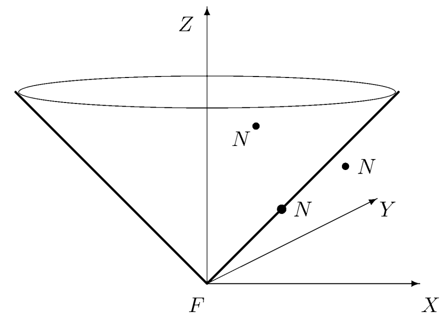

Fig. 4 The N-cone structure

(where $e = |\mathbf{e}| = \sqrt{e_X{}^2 + e_Y{}^2}$). Since the sign of $(e^2 - 1)$ depends on $e \lesseqgtr 1$, and $Z_N > 0$, we found several results, here summarized in the

Proposition 1.1 *The conics are sections of the N-cone.*

(1) *The N-cone with the vertex at $F(0,0,0)$ is the upper nappe of the circular right cone given by*

$$X^2 + Y^2 - Z^2 = 0. \tag{10}$$

(2) *The points N lying inside the N-cone correspond to elliptical orbits, those on the cone to parabolic ones and those outside the cone correspond to hyperbolic orbits (Fig. 4).*

(3) *Each point N defines a polar plane (with respect to the unit sphere $X^2 + Y^2 + Z^2 = 1$ with centre at F), that is the plane, orthogonal to the line FN, defined by the equation*

$$\frac{e_X}{p}X + \frac{e_Y}{p}Y + \frac{1}{p}Z - 1 = 0 \tag{11}$$

and which passes through the inverse point N^* of N (such that the magnitudes are $|N^* - F| = 1/|N - F|$).

(4) *The polar plane intersects the N-cone in a conic section, which, projected orthogonally onto the $\{\mathbf{I}, \mathbf{J}\}$-plane, gives exactly a Kepler orbit (Fig. 5).*

(5) *The polar plane makes an angle β with the X-axis such that*

$$\tan\beta = e \tag{12}$$

which is exactly the eccentricity of the Kepler orbit.

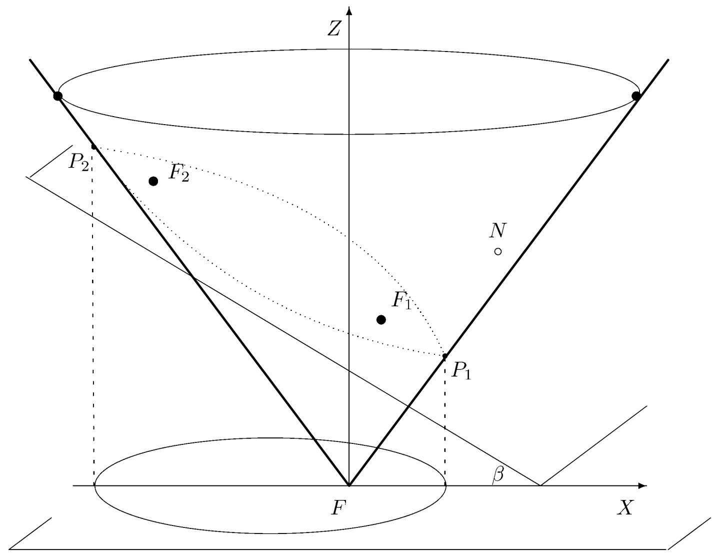

Fig. 5 The projection of an elliptic section onto the Kepler plane

Summarizing: each Kepler orbit is represented by a point N in the positive half-space $Z > 0$: the position of N with respect to the N-cone (inside, outside or on the cone) is determined by the type of the orbit, that is by e.

The purpose of this paper The purpose of this paper is to show how, embodied in the **N**-vector and in the N-cone structure (that means deeply rooted into the sum **S**-vector) we not only find another aspect regarding the conic sections, that is the *Dandelin spheres*, but that we also unravel in an *unexpected* way (in the Kepler scenery) the famous entities: the *golden ratio*, the *golden rectangle* and the *Fibonacci sequence.*

2 Two fundamental N-elements. The N-encoding of the Dandelin spheres

The positive parameter p of the conics may be considered as a *dimension* parameter. We define *paradigmatic conics* those corresponding to the unit value $p = 1$ *and* with the eccentricity vector $\mathbf{e} = e\mathbf{I}$ (lying on the X-axis).

Let us restrict to the *paradigmatic circular, elliptic and parabolic orbits*, orbits characterized by the couple

$(0 \le e \le 1, p = 1)$.

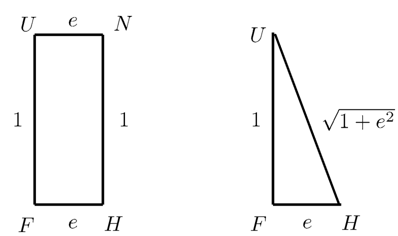

Fig. 6 The **N**-rectangle and the associated **N**-triangle

These particular ‘unit’ ($p = 1$) members of the conic family (with major axis on the **I**-axis) are ‘*paradigmatic*’, in the sense that, for each fixed value of e in $[0, 1]$, they represent all the other conceivable similar confocal orbits with $p \neq 1$ and with different inclination $\mathbf{e}(\phi)$ in the Kepler plane, being $\phi \in (0, 2\pi)$.

From the previous restriction and by (7), the vector $\mathbf{N} = e\mathbf{I} + \mathbf{K}$, having the tip pont N with coordinates

$$X_N = e, \qquad Y_N = 0, \qquad Z_N = 1, \tag{13}$$

defines in a natural way the two structures represented in Fig. 6, that is

the **N**-rectangle and the **N**-triangle

which show up *fundamental* **N**-*elements* in the whole paper.

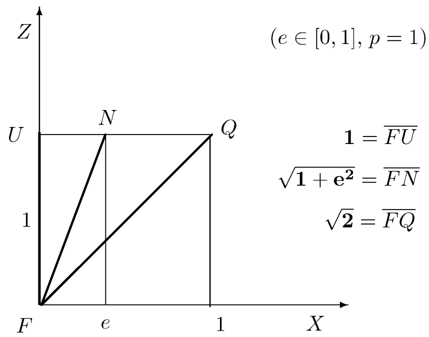

Fig. 7 The three **N**-embodied diagonal lengths 1, $\sqrt{1+e^2}$, $\sqrt{2}$

The first **N**-element, the **N**-rectangle, plays a key role in this section. Notice that, being $0 \le e \le 1$, the length of its *diagonal*

$$\overline{FN} = |\mathbf{N}| = |N - F| = \sqrt{1+e^2} \tag{14}$$

ranges through the *three particular lengths*:

$$1, \quad \sqrt{1+e^2}, \quad \sqrt{2}, \tag{15}$$

(from the unit segment FU, through the diagonals FN of the rectangles defined by N, up to diagonal of the unit square, as shown by the bold lines in Fig. 7).

As we shall show, *the ratios of the first unit length, respectively, to the sum and to the difference of the other two lengths gives exactly the radiuses of the so called Dandelin spheres*.

These spheres are named after their discoverer, the French/Belgium mathematician Germinal Pierre Dandelin (1794–1847), who demonstrated [1, 2] that if a cone is intersected by a plane, the intersection is a conic curve whose foci are exactly the points where the plane touches the *spheres* inscribed in the cone (see the Fig. 8, which depicts the two Dandelin spheres for an ellipse). A parabola has one Dandelin sphere; an hyperbola has two spheres, one for each nappe of the cone.

Proposition 2.1 *The two radius-vectors* $\mathbf{r}_1$, $\mathbf{r}_2$ *of the two Dandelin spheres associated to a paradigmatic elliptic orbit*, *that is* (*Fig*. 8)

$$\mathbf{r}_1 = F_1 - C_1 = r_1\mathbf{u}, \qquad \mathbf{r}_2 = F_2 - C_2 = -r_2\mathbf{u}$$

(*where* $\mathbf{u}$ *is the unit vector orthogonal to the common section plane*) *are related to the vector* $\mathbf{N}$ *and satisfy the intertwining relation* (*parallelism*)

$$\mathbf{r}_1 \wedge \mathbf{N} = \mathbf{r}_2 \wedge \mathbf{N} = \mathbf{0}$$

(*wedge product* $\wedge$).

Proof Recall that, by definition, also the vector $\mathbf{N}$ is orthogonal to the polar plane to which the elliptic section belongs, whence $\mathbf{N} = N - F = N\mathbf{u}$. □

We are now able to construct the Dandelin radiuses $\mathbf{r}_1$, $\mathbf{r}_2$ *directly* from $\mathbf{N}$.

THE **N**-BREED OF THE DANDELIN RADIUSES $\mathbf{r}_1$ AND $\mathbf{r}_2$.

On the same line of the vector $N - F = \mathbf{N} = \sqrt{1+e^2}\mathbf{u}$, construct:

(a) the two points N_+ and N_- which define, respectively, the vectors

$$\mathbf{N}_+ \equiv N_+ - F \equiv \sqrt{2}\mathbf{u} + \mathbf{N},$$
$$\mathbf{N}_- \equiv N_- - F \equiv \sqrt{2}\mathbf{u} - \mathbf{N}$$

with magnitudes (recall that for an ellipse $\sqrt{2} > \sqrt{1+e^2}$):

$$|\mathbf{N}_+| = |N_+ - F| = \sqrt{2} + \sqrt{1+e^2},$$
$$|\mathbf{N}_-| = |N_- - F| = \sqrt{2} - \sqrt{1+e^2};$$

(b) the *inverse* points N_+^* and N_-^* of the two previous points N_+, N_- (with respect to the unit sphere with centre at F), which define the vectors

$$\mathbf{N}_+^* \equiv N_+^* - F \equiv \frac{1}{\sqrt{2} + \sqrt{1+e^2}}\mathbf{u},$$
$$\mathbf{N}_-^* \equiv N_-^* - F \equiv \frac{1}{\sqrt{2} - \sqrt{1+e^2}}\mathbf{u}.$$

Proposition 2.2 *The magnitudes of the two vectors*

$$\mathbf{N}_+^*, \quad -\mathbf{N}_-^*$$

give exactly the two lengths r_1, r_2 *of the two Dandelin radius-vectors for the paradigmatic elliptic orbits.*

Proof The proof comes from simple geometrical facts related to the elliptical conic sections obtained by cutting the N-cone with the polar plane of N (with respect to the unit sphere with center at $F(0,0,0)$).

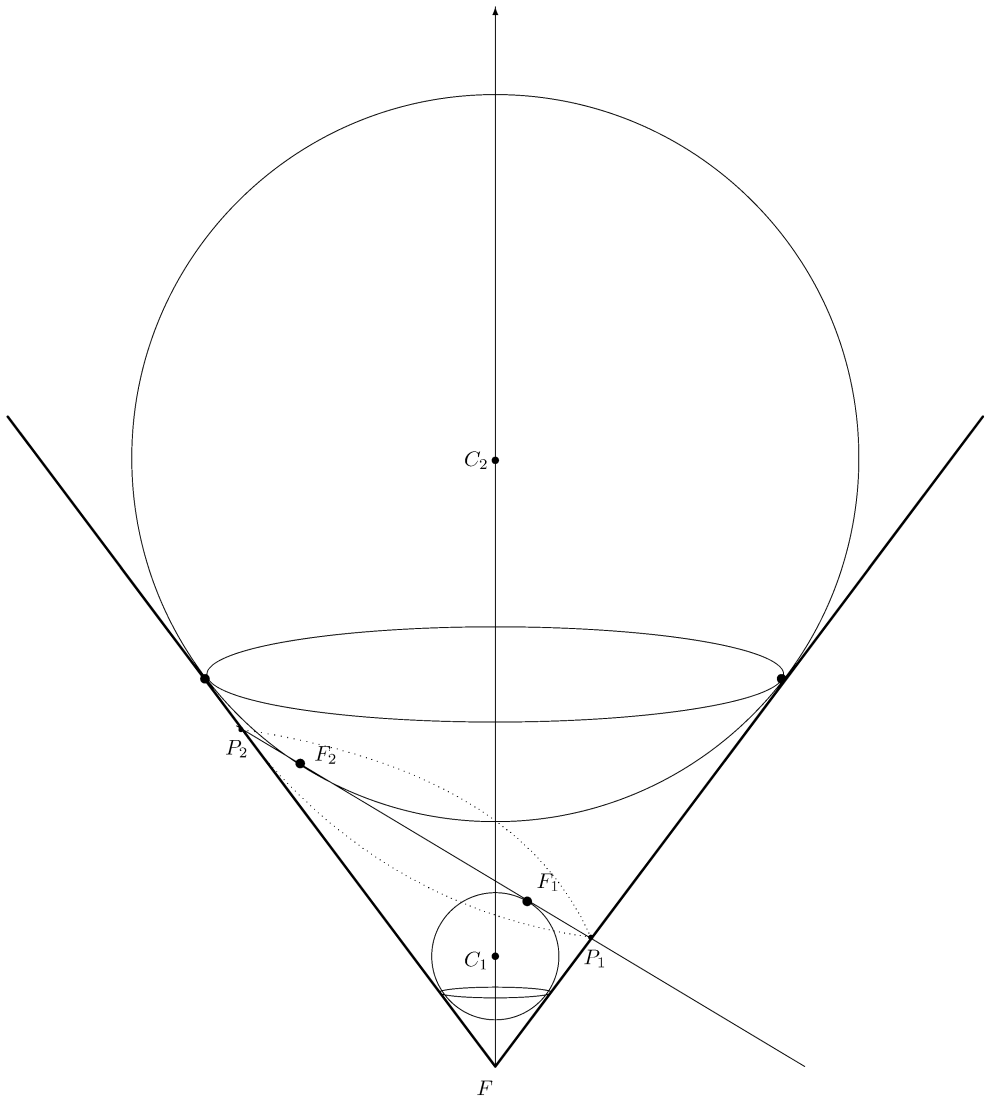

Fig. 8 The Dandelin spheres

In our hypothesis, (11) of the polar plane yields the polar line $z = -ex + 1$ in the $\{X, Z\}$-plane which:

(A) makes the angle β with the X-axis so that $\tan\beta = e$, intersects the X-axis in the point $(\frac{1}{e}, 0)$ and the Z-axis in the point (0, 1) (see Fig. 9);
(B) intersects the two lines $z = \pm x$ (of the cone) in the two points given by

$$\begin{cases} z = \pm x, \\ z = -ex + 1 \end{cases} \tag{16}$$

that is (see Fig. 10) in the two points P_1, P_2:

$$P_1 = \left(\frac{1}{1+e}, \frac{1}{1+e}\right), \quad P_2 = \left(\frac{1}{e-1}, \frac{1}{1-e}\right) \tag{17}$$

(recall that $e < 1$ and that $Z > 0$). □

Proposition 2.3 *The triangle with vertices F, P_1, P_2 has a right angle at F (Fig. 11). The radiuses of the two Dandelin spheres, that is the radius r_1 of the circle (incircle) inscribed in the right-angled triangle FP_1P_2*

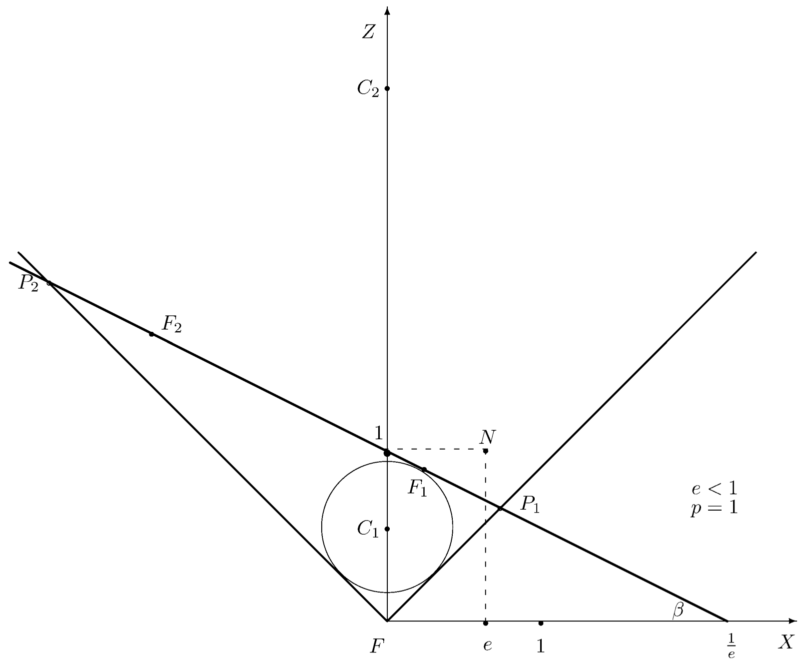

Fig. 9 The point N, the line P_1P_2 and the focuses F_1, F_2

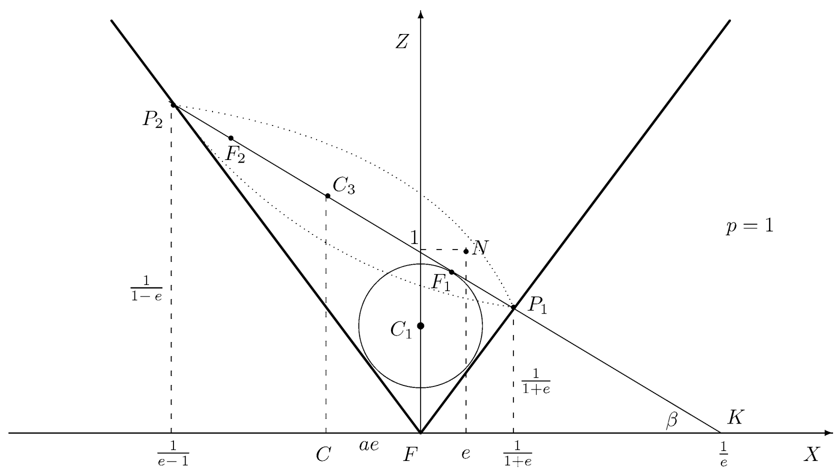

Fig. 10 The elliptic section

and the radius r_2 of the circle (excircle) ex-scribed to the same triangle, are given by

$$r_1 = \frac{1}{\sqrt{2}+\sqrt{1+e^2}}, \qquad r_2 = \frac{1}{\sqrt{2}-\sqrt{1+e^2}}. \quad (18)$$

Proof The well-known formula for a general triangle

$$S = \frac{P}{2}r$$

(where S, P and r are, respectively, the area, the perimeter of the triangle and the radius of the *inscribed circle*), yields for our right-angled triangle:

$$\overline{FP_1}\cdot\overline{FP_2} = Pr. \quad (19)$$

Being $\overline{FP_2}$ the hypothenuse of an isosceles triangle (since $|\frac{1}{e-1}| = |\frac{1}{1-e}|$, see Fig. 11) and $\overline{FP_1} = \overline{FP_2}\cdot\tan(\frac{\pi}{4}-\beta)$, with β satisfying (12), we find that

$$\overline{FP_2} = \sqrt{2}(1-e)^{-1}, \qquad \overline{FP_1} = \sqrt{2}(1+e)^{-1}$$

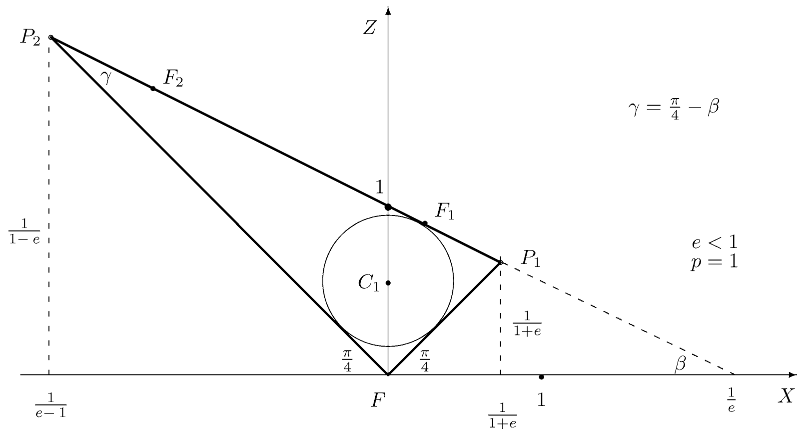

Fig. 11 The right-angled triangle FP_1P_2

whence $\overline{P_1P_2} = \sqrt{\overline{FP_1}^2 + \overline{FP_2}^2} = 2\sqrt{1+e^2} \times (1-e^2)^{-1}$. Thus the *whole* perimeter is

$$P = \frac{2\sqrt{1+e^2} + 2\sqrt{2}}{1-e^2} \tag{20}$$

whence, by (19), we obtain the value r_1 of (18).

Finally, r_2 comes from a well-known formula adapted to our case, or

$$r_2 = \frac{S}{\frac{P}{2} - \overline{P_1P_2}} \tag{21}$$

where $S = \frac{\overline{FP_1}\cdot\overline{FP_2}}{2} = \frac{1}{1-e^2}$ and P is given by (20). □

The Proposition 2.3 finally proves Proposition 2.2, showing that r_1 and r_2 are strictly related to the ratios of the three fundamental N-lengths (15).

LEADING IDEA. We have shown how the vector $\mathbf{N}$ (through its two offsprings under *inversion* $\mathbf{N}_+^*$, $\mathbf{N}_-^*$) encodes the radiuses r_1 and r_2 of the Dandelin spheres with centers C_1 and C_2.

The leading, key idea which has naturally brought us to attain this result is that the points C_1 and C_2 lie on the opposite sides of the polar line (and that *circle-inversion* changes inner points into outer points and vice-versa).

CIRCULAR, PARABOLIC, AND HYPERBOLIC ORBITS. The above results (obtained for the elliptic orbits) apply to the:

- paradigmatic *circular* orbits ($e = 0$), where $\mathbf{N} = \mathbf{K}$, the polar line is the horizontal line $z = 1$, the triangle F, P_1, P_2 is an isosceles one with unit height and $r_1 = (1+\sqrt{2})^{-1}$, $r_2 = (\sqrt{2}-1)^{-1}$;
- paradigmatic *parabolic* sections ($e = 1$), where, from (17) and (18), the triangle degenerates in a segment and we have accordingly only one Dandelin sphere with radius $r_1 = \frac{\sqrt{2}}{4}$;
- paradigmatic *hyperbolic* orbits ($e > 1$), where $r_1 \neq r_2$, the polar line passing through the points $(\frac{1}{e}, 0)$ and $(0, 1)$.

PROPERTY. It is worth noticing that: the perimeter of the right triangle FP_1P_2 is exactly twice the radius r_2 of the Dandelin sphere with centre C_2.

3 Physical interpretation of a Dandelin sphere

In recovering the Dandelin spheres we have considered the parameter p as a *dimension* parameter, that is as a geometrical quantity.

Now, we are going to give a physical flavor to our results, by recalling that the parameter p is related to the angular momentum magnitude Γ of the Kepler motion and to the universal constant of gravitation K^2 by the relation

$$p = \Gamma^2 K^{-2}.$$

For the *paradigmatic* orbits ($p = 1$), the relation (2) becomes $e^2 + 1 = 2 + 2EpK^{-2} = 2 + 2EK^{-2}$ which, substituted in the formula (18) for the Dandelin radius r_1, finally yields

$$r_1 = \frac{1}{\sqrt{2}}\left(\frac{1}{1+\sqrt{1+EK^{-2}}}\right) \tag{22}$$

which gives $r_1 = r_1(E)$.

Thus, we may state that *any sphere with a given radius* r_1 *in the Euclidean three-dimensional space represents a physical Kepler orbit with mechanical energy* E *related to* r_1 *by* (22). For the particular energy values $E = -K^4(2\Gamma^2)^{-1} = -\frac{1}{2}K^2$ (being $p = 1$) and $E = 0$, we recover the radiuses r_1 given explicitly in Sect. 2 for the circle and the parabola.

4 N-encoding: the golden ratio Φ and the golden rectangle

The second fundamental **N**-element, the **N**-triangle, plays a key role by encoding the famous golden number: the *golden section* or *golden ratio*.

Let us consider the perimeter of the **N**-triangle which is obviously

$$1 + e + \sqrt{1+e^2}.$$

Now, let us 'open' this perimeter (by rotating the hypotenuse $HU = \sqrt{1+e^2}$ about the vertex H till it becomes aligned to the cathetus FH, yielding the segment FH_{ext} (see Fig. 12, which shows this simple construction). The resulting whole length

$$1 + (e + \sqrt{1+e^2})$$

may now be considered as the semi-perimeter of a *rectangle* with

$$\text{height} = FU = 1, \qquad \text{base} = FH_{ext} = e + \sqrt{1+e^2}.$$

We call this rectangle the *extended* **N**-*rectangle* or, briefly, $\mathbf{N}_{ext}$-rectangle.

THE **golden ratio**. THE $\mathbf{N}_{ext}$-RECTANGLE FOR $e = \frac{1}{2}$ IS **a golden rectangle**.

When the eccentricity varies in the range [0, 1], the different extended rectangles vary from a degenerate unit segment, up to a rectangle of height 1 and base $1+\sqrt{2}$.

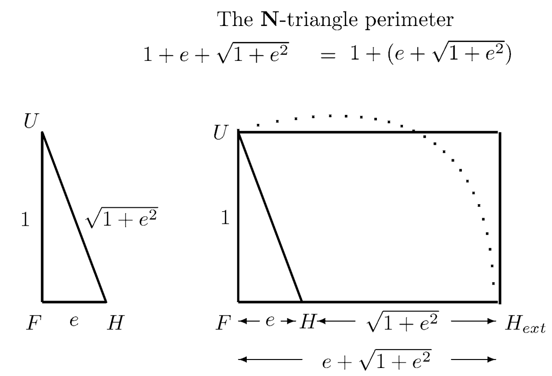

Fig. 12 Perimeter of the **N**-triangle = Semi-perimeter of its $\mathbf{N}_{ext}$-rectangle

When the eccentricity assumes the *medium* value $e = \frac{1}{2}$, the $\mathbf{N}_{ext}$-rectangle has

$$\text{height} = 1, \qquad \text{base} = \frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}.$$

This result:

(a) Unravels the *golden ratio*, for the length of the base is exactly

$$\Phi = \frac{1+\sqrt{5}}{2} = 1,618\ldots$$

that is *the famous irrational number, the golden ratio* or *golden section* (see [3]).

(b) Shows that the particular $\mathbf{N}_{ext}$-rectangle corresponding to $e = \frac{1}{2}$ (having height 1 and base Φ) is *a rectangle with sides in the ratio* 1 : Φ and thus is a *golden rectangle*, the famous rectangle [3] considered in art and in architecture to posses the most pleasing appearance (Fig. 13).

Thus, the introduction of the vector **N** unravels that:

Proposition 4.1 *The particular* **N**-*vector*

$$\mathbf{N} = \frac{1}{2}\mathbf{I} + \mathbf{K}$$

related to the particular couple $(p, e) = (1, \frac{1}{2})$, *exhibits, through the associated* $\mathbf{N}_{ext}$-*rectangle, the two famous golden elements*: *the golden ratio* Φ *and the golden rectangle.*

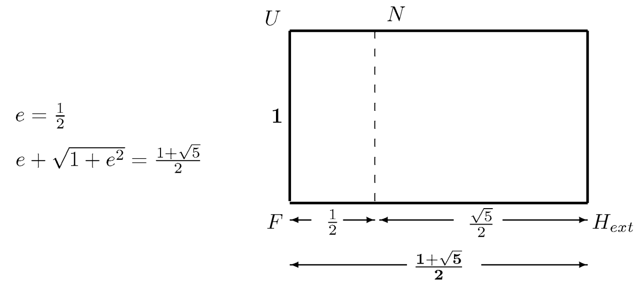

Fig. 13 The *golden* rectangle (the $\mathbf{N}_{ext}$-rectangle for $e = \frac{1}{2}$)

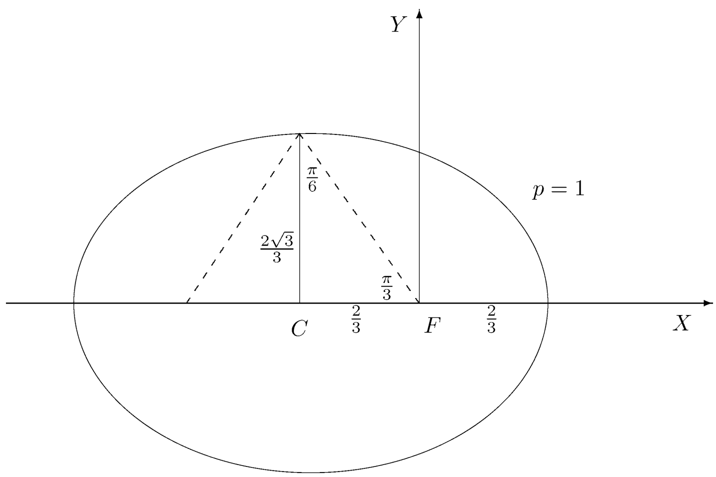

Fig. 14 The Kepler ellipse for the golden $\mathbf{N}_{ext}$-rectangle (for $e = \frac{1}{2}$)

THE GOLDEN RECTANGLE IN THE PHYSICAL KEPLER ARENA.

What is the *paradigmatic* Kepler orbit associated to the golden $\mathbf{N}_{ext}$-rectangle? By the well-known identities (written for $p = 1$)

$$a = \frac{1}{1-e^2}, \quad b = a\sqrt{1-e^2}, \quad c = ae,$$

the *physical Kepler orbit* $(p = 1, e = \frac{1}{2})$ *is an ellipse with* (Fig. 14):

$$a = \frac{4}{3} \quad \text{(semi-major axis);}$$

$$b = \frac{2\sqrt{3}}{3} \quad \text{(semi-minor axis);}$$

$$c = ae = \frac{2}{3} \quad \text{(focus } F \text{ in the middle point of the semi-major axis).}$$

such that $b = \sqrt{a}$ and which shows the particular angles $\frac{\pi}{3}$ and $\frac{\pi}{6}$ (*equilateral* triangle).

Comment The well-known golden rectangle is generally constructed in the literature by *starting* from a *whatsoever* unit square $ABCD$, by dividing it in two equal parts via a vertical segment TS (whence the bases $AT = TB = \frac{1}{2}$), by drawing the arc CE (centre at T and ray TC), thus obtaining the golden rectangle with base $AE = \Phi$ and the original height 1. Whereas we construct the golden rectangle by *starting* from a *well definite* figure, the **N**-triangle.

5 Golden N-cylinders. Pythagora's pentagrams

A plane $\mathbf{N}_{ext}$-rectangle corresponds to a given eccentricity vector $\mathbf{e} = e\mathbf{I}$.

Now, if (for a fixed value e) the eccentricity vector $\mathbf{e}$ is free to assume all the possible inclinations in the

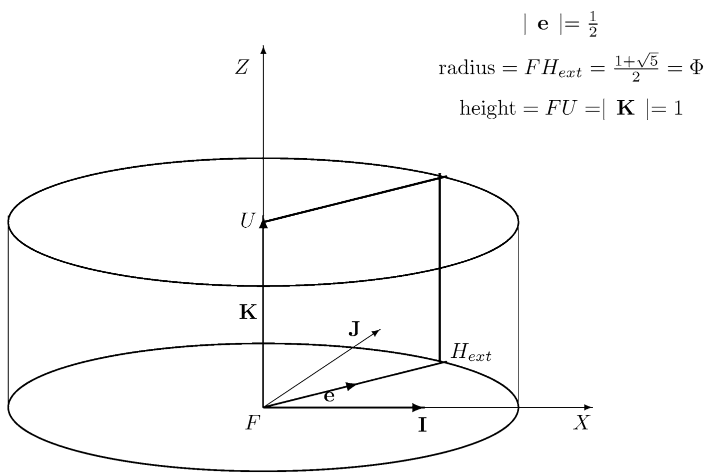

Fig. 15 The *golden* $\mathbf{N}_{ext}$-cylinder in the 3-space $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$

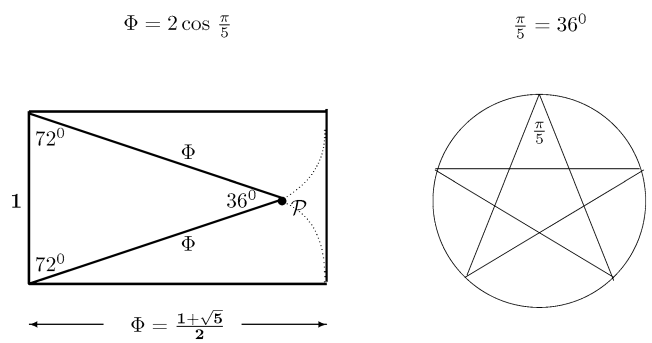

Fig. 16 The golden **N**-rectangle and the starry pentagram

$\mathbf{I}, \mathbf{J}$ plane, that is $\mathbf{e} = \mathbf{e}(\phi)$, the corresponding extended $\mathbf{N}_{ext}$-rectangles (generated by a ϕ-rotation about the Z-axis) define in the three-space a **N**-*cylinder* with unit height.

If also the magnitude $|\mathbf{e}| = e \geq 0$ is free to assume all the values, so that $\mathbf{e} = \mathbf{e}(\phi) = e_X \mathbf{I} + e_Y \mathbf{J}$, we have infinite concentric cylinders, one for each e.

Among these cylinders, the particular one with height 1 and radius $\Phi = \frac{1+\sqrt{5}}{2}$ (Fig. 15) which we call *golden* $\mathbf{N}_{ext}$-*cylinder*, represents all the Kepler ellipses ($p = 1, e = \frac{1}{2}$) obtained by rotating the peculiar paradigmatic one of Fig. 14 in its plane.

An intriguing aspect: by the simple construction of Fig. 16 and by the trigonometric relation $\Phi = 2\cos\frac{\pi}{5} = 2\cos 36°$, a golden rectangular section of our golden cylinder generates an isosceles triangle which is a 'corner' of the beautiful starry *pentagram of Pythagoras*.

Summarizing:

$$\textit{ellipse}\left(1, \frac{1}{2}\right) \rightarrow \textit{golden rectangle}\ \mathbf{N}_{ext}$$
$$\rightarrow \textit{golden}\ \Phi \rightarrow \textit{Phythagora'sstar}$$

(one may say that a peculiar *celestial* elliptical Kepler orbit in the sky finds a *starry* representation).

6 The Fibonacci sequence

Let us dwell on a fascinating result regarding particular parabolic orbits ($p = 1, e = 1$).

The sides $(1, e, \sqrt{1+e^2})$ of the associated right-angled **N**-triangles are

$$1,\ 1,\ \sqrt{2}.$$

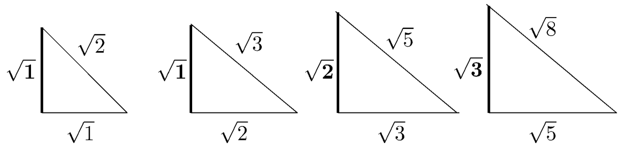

Fig. 17 The $\sqrt{\text{triangles}}$ related to the Fibonacci sequence

Now, let us treat all these three values in a *uniform* way (*by regarding all of them as square roots*). Then we may state that the **N**-triangle has sides

$$\sqrt{1},\ \sqrt{1},\ \sqrt{2}$$

(the first triangle in Fig. 17).

Now, let us adopt a sort of *shift-back clockwise-operation*, by shifting the base side to the height position, by shifting the hypothenuse to the base side and by evaluating the new hypothenuse, whence generating a new triangle with sides

$$\sqrt{1},\ \sqrt{2},\ \sqrt{3}$$

(second triangle, Fig. 17). Similarly we construct the third triangle ($\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$), and so on.

In so doing we have generated the following *sequence of heights*:

$$\sqrt{1},\ \sqrt{1},\ \sqrt{2},\ \sqrt{3},\ \sqrt{5},\ \sqrt{8},\ \sqrt{13}, \ldots$$

which, once squared, gives the sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

which is exactly the famous, fascinating *Fibonacci sequence*.

Let us notice (in this **N**-triangle approach) the *correct and natural repetition* of the first two terms of the Fibonacci sequence.

7 Conclusions and outlook

It is well-known that the Kepler orbits are *plane* curves. In [4, 7] we have shown that the whole Kepler family (and several of its geometrical and physical characters) are generated by the *plane* vector **S**. Moreover [8], when this vector *pops up* in the three-dimensional space and generates both the vector **N** and the *N-cone* structure, the curves show up undoubtedly as *conic sections*.

In this paper, we have naturally embedded in this *unitary and physical* **N**-*description* not only the so called Dandelin spheres, but also several other renowned (and different) geometrical quantities (such as the famous golden ratio Φ, the celebrated golden rectangle and the famous Fibonacci sequence).

Thus, as an outlook, since the (**S** → **N**)-approach has not only provided by birth a 'primigenial unity' among the different features strictly belonging to the Kepler motion, but also, as shown in this paper, has revealed a fundamental role in bringing together different traditional aspects of the literature (which, at first sight, do not strictly belong to the Kepler scenery but are shown to be deeply intertwined to it), let us state that this approach, suitably developed, may help in understanding the intimate structure of other geometrical and dynamical research areas.

Acknowledgement Work supported by the Italian Ministry for University and Scientific, Technological Research MIUR.

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