Meccanica (2012) 47:257–268
DOI 10.1007/s11012-011-9431-3

BRIEF NOTES AND DISCUSSIONS

Comparison of Reissner, Mindlin and Reddy plate models with exact three dimensional solution for simply supported isotropic and transverse inextensible rectangular plate

Milan Batista

Received: 26 April 2010 / Accepted: 8 April 2011 / Published online: 3 May 2011

Abstract In the article, the exact solution of a sinusoidal loaded simply supported rectangular plate is given for the case of an isotropic plate and for the case of a transversally inextensible plate. Asymptotic and numerical comportment with Reissner, Mindlin and Reddy plate models is present.

Keywords Elasticity · Plate theory · Shear deformable plates

1 Introduction

The aim of various plate theories is to reduce three dimensional elasticity problems to two dimensional problems. When a theory is developed, one has the temptation to compare an exact solution of some simple three dimensional problems to results predicted by the plate theory and in this way obtain a direct difference between them. Lack of exact three-dimensional solutions often force authors to compare results only among plate theories [1] or/and among methods of solution, and by such comportment we can hardly decide which plate theory is better. A typical introductory problem in plate theory is the simply supported rectangular plate subject to sinusoidal distributed transverse load, which is treated by Navier's double series method [2, 3]. No comportment to an exact three-dimensional solution of this problem is provided in standard monographs and textbooks [2–9], though a three dimensional solution does exist.

Apparently, the three dimensional solution of the problem was first provided in 1931 by Galerkin [10], who expressed stress components through a single stress function which satisfies the biharmonic equation. In 1985 Levinson [11] gave a solution by solving Navier's equations by the semi-inverse method. Levinson's approach was later discussed in detail by Nicotra and Podio-Guidugli [12]. In 1988 Barret and Ellis [13] illustrated their exact general three-dimensional solution of elasticity equations by considering a sinusoidally loaded isotropic rectangular plate. However, they provided only formulas for displacements. In 1999 Werner [14] provided a solution of the problem by setting the transversal normal stress at zero so the load would become a characteristic volume force distribution across plate thickness. Recently Demasi [15] gave the exact three-dimensional for isotropic thick and thin rectangular plates using mixed form of Hooke's law and by solving the eigenvalue problem. This approach leads to algebraic field solutions for all the displacements and stresses which no longer require a new solution of the differential equation or eigenvalue problem. Future references for various other analytical and semi-analytical methods used to solve the problem may be found in Teo and Liew [16]. To that we add that various exact 3D elasticity solutions of orthotropic plates, sandwich plates,

M. Batista (✉)
Faculty of Maritime Studies and Transport, University of Ljubljana, Ljubljana, Slovenia
e-mail: milan.batista@fpp.uni-lj.si

piezoelectric plates, and variable-thickness inhomogeneous elastic plates may be found in [17–31].

In the article the three-dimensional problem of bending of a sinusoidally loaded simply supported rectangular plate will be solved once more by a method similar to one used by Levinson but in a more direct fashion by assuming displacements distribution which fulfils the boundary conditions. Aside from the isotropic plate, the solution for the transverse inextensible plate will be given since some plate theories are directly or tacitly based on assumptions that better suite such plates. All the expressions for displacements, stress components and stress resultants are present in their explicit form. The solutions are then compared to solutions given by three plate models: Reissner, Mindlin and Reddy.

2 Governing equations

We consider the equilibrium of a weightless homogeneous transversally isotropic elastic plate bounded by the planes

$$x = y = 0 \qquad x = a \qquad y = b \qquad z = \pm\frac{h}{2} \tag{1}$$

The stress-strain state of the plate is characterized by the normal stress components $\sigma_x, \sigma_y, \sigma_z$, the shear stresses components $\tau_{xy}, \tau_{xz}, \tau_{yz}$, and the displacement components u, v, w. The equations connecting these quantities are: equilibrium equations

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \\ &\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \end{aligned} \tag{2}$$

and constitutive equations. For an isotropic plate we have

$$\begin{aligned} \sigma_x &= \frac{2G}{(1-2\nu)}\left[(1-\nu)\frac{\partial u}{\partial x} + \nu\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] \\ \sigma_y &= \frac{2G}{(1-2\nu)}\left[(1-\nu)\frac{\partial v}{\partial y} + \nu\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right] \\ \sigma_z &= \frac{2G}{(1-2\nu)}\left[(1-\nu)\frac{\partial w}{\partial z} + \nu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] \end{aligned} \tag{3}$$

$$\begin{aligned} &\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ &\tau_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \end{aligned} \tag{4}$$

where $G \equiv \frac{E}{2(1+\nu)}$ is shear modulus, with E the modulus of elasticity and ν Poisson's ratio. For a transversally inextensible plate for which the shear properties are isotropic, the constitutive equations (3) are replaced by

$$\begin{aligned} &\sigma_x = \frac{2G}{1-\nu}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) \\ &\sigma_y = \frac{2G}{1-\nu}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right) \qquad \frac{\partial w}{\partial z} = 0 \end{aligned} \tag{5}$$

The boundary conditions of the problem are the following. Along the sides of the plate, we have

$$\begin{aligned} &\text{at} \quad x = 0 \quad \text{and} \quad x = a: \quad w = 0 \quad \sigma_x = 0 \\ &\text{at} \quad y = 0 \quad \text{and} \quad y = b: \quad w = 0 \quad \sigma_y = 0 \end{aligned} \tag{6}$$

and on the plate faces the boundary conditions we have are

$$\begin{aligned} &\text{on} \quad z = \frac{h}{2}: \quad \sigma_z = p \quad \tau_{xz} = \tau_{yz} = 0 \\ &\text{on} \quad z = -\frac{h}{2}: \quad \sigma_z = \tau_{xz} = \tau_{yz} = 0 \end{aligned} \tag{7}$$

where the load p is assumed to be in the form

$$p = p_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{8}$$

The shear stress components on the plate sides are not specified. However, it is required that the plate is in static equilibrium.

3 Isotropic plate

3.1 Solution

By examination of the problem, we may see that the boundary conditions (6) are satisfied with displacements of the form

$$\begin{aligned} u &= U(z)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \\ v &= V(z)\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \\ w &= W(z)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \end{aligned} \tag{9}$$

where the rigid body part of displacements is omitted. Substituting these expressions into constitutive equations (3)–(4) and then resulting expressions for stress components into equilibrium equations (2) we see that the functions $U(z)$, $V(z)$ and $W(z)$ must satisfy the system of second order ordinary differential equations

$$\begin{aligned}
&\frac{d^2U}{dz^2} - \left[\lambda^2 + \frac{\pi^2}{a^2(1-2\nu)}\right]U - \frac{\pi^2}{ab(1-2\nu)}V \\
&\quad + \frac{\pi}{a(1-2\nu)}\frac{dW}{dz} = 0 \\
&\frac{d^2V}{dz^2} - \left[\lambda^2 + \frac{\pi^2}{b^2(1-2\nu)}\right]V - \frac{\pi^2}{ab(1-2\nu)}U \\
&\quad + \frac{\pi}{b(1-2\nu)}\frac{dW}{dz} = 0 \\
&\frac{d^2W}{dz^2} - \frac{\lambda^2(1-2\nu)}{2(1-\nu)}W - \frac{\pi}{a(1-\nu)}\frac{dU}{dz} \\
&\quad - \frac{\pi}{b(1-\nu)}\frac{dV}{dz} = 0
\end{aligned} \tag{10}$$

where

$$\lambda \equiv \frac{\pi\sqrt{a^2+b^2}}{ab} \tag{11}$$

The solution of this system may be written in the form

$$\begin{aligned}
\lambda a U &= C_1\cosh\lambda z + C_2\sinh\lambda z \\
&\quad + C_3\lambda z\cosh\lambda z + C_4\lambda z\sinh\lambda z \\
\lambda b V &= C_1\cosh\lambda z + C_2\sinh\lambda z \\
&\quad + C_3\lambda z\cosh\lambda z + C_4\lambda z\sinh\lambda z \\
\pi W &= \left[C_2 - (3-4\nu)C_3\right]\cosh\lambda z \\
&\quad + \left[C_1 - (3-4\nu)C_4\right]\sinh\lambda z \\
&\quad + C_4\lambda z\cosh\lambda z + C_3\lambda z\sinh\lambda z
\end{aligned} \tag{12}$$

Note that the above solution contains only four integration constants—C_1, C_2, C_3, and C_4—instead of six. However, as will be see below, this is enough to satisfy all six face boundary conditions (7). Substituting expressions for displacements (9) into constitutive equations (3)–(4) and taking (12) into account yields expressions for in-plane stress components

$$\begin{aligned}
\sigma_x &= -p_0\left[\frac{\pi^2}{\lambda^2a^2}f(z) + 2\nu g(z)\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
\sigma_y &= -p_0\left[\frac{\pi^2}{\lambda^2b^2}f(z) + 2\nu g(z)\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
\tau_{xy} &= p_0\frac{\pi^2}{\lambda^2ab}f(z)\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}
\end{aligned} \tag{13}$$

and expressions for transverse stresses components

$$\begin{aligned}
\sigma_z &= p_0\left[f(z) - 2(1-\nu)g(z)\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
\tau_{xz} &= \frac{\pi p_0}{\lambda a}\left[f_1(z) - (1-2\nu)g_1(z)\right] \\
&\quad \times\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
\tau_{yz} &= \frac{\pi p_0}{\lambda b}\left[f_1(z) - (1-2\nu)g_1(z)\right] \\
&\quad \times\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
f(z) &= c_1\cosh\lambda z + c_2\sinh\lambda z \\
&\quad + c_3\lambda z\cosh\lambda z + c_4\lambda z\sinh\lambda z \\
g(z) &= c_3\sinh\lambda z + c_4\cosh\lambda z \\
f_1(z) &= \frac{1}{\lambda}\frac{df}{dz} = c_1\sinh\lambda z + c_2\cosh\lambda z \\
&\quad + c_3\lambda z\sinh\lambda z + c_4\lambda z\cosh\lambda z \\
g_1(z) &= \frac{1}{\lambda}\frac{dg}{dz} = c_3\cosh\lambda z + c_4\sinh\lambda z
\end{aligned} \tag{15}$$

and

$$C_k = \frac{p_0\pi}{2\lambda G}c_k \quad (k = 1, 2, 3, 4) \tag{16}$$

By means of transverse stress components (14), the plate faces boundary conditions (7) reduce to the following system of four linear equations for calculation of constants c_1, c_2, c_3 and c_4

$$\begin{aligned}
&f(h/2) - 2(1-\nu)g(h/2) = 1 \\
&f(-h/2) - 2(1-\nu)g(-h/2) = 0 \\
&f_1(\pm h/2) - (1-2\nu)g_1(\pm h/2) = 0
\end{aligned} \tag{17}$$

which has the solution

$$c_1 = \frac{\frac{\lambda h}{2}\cosh\frac{\lambda h}{2} - (1-2\nu)\sinh\frac{\lambda h}{2}}{\sinh\lambda h + \lambda h}$$

$$
\begin{aligned}
c_2 &= \frac{\frac{\lambda h}{2}\sinh\frac{\lambda h}{2} - (1-2\nu)\cosh\frac{\lambda h}{2}}{\sinh\lambda h - \lambda h} \\
c_3 &= -\frac{\cosh\frac{\lambda h}{2}}{\sinh\lambda h - \lambda h} \\
c_4 &= -\frac{\sinh\frac{\lambda h}{2}}{\sinh\lambda h + \lambda h}
\end{aligned}
\tag{18}
$$

In this way the problem is solved.

3.2 Stress resultants

By means of (13) the bending moments on a unit of length are by definition

$$
\begin{aligned}
M_x &\equiv \int_{-h/2}^{h/2} \sigma_x z dz \\
&= \frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 a^2} \\
&\quad \times \frac{\left(1-2\nu\frac{a^2}{b^2}\right)\sinh\lambda h + \nu\lambda h\frac{a^2}{b^2}\cosh\lambda h - \lambda h\left(1-\nu\frac{a^2}{b^2}\right)}{\sinh\lambda h - \lambda h} \\
&\quad \times \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
M_y &\equiv \int_{-h/2}^{h/2} \sigma_y z dz \\
&= \frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 b^2} \\
&\quad \times \frac{\left(1-2\nu\frac{b^2}{a^2}\right)\sinh\lambda h + \nu\lambda h\frac{b^2}{a^2}\cosh\lambda h - \lambda h\left(1-\nu\frac{b^2}{a^2}\right)}{\sinh\lambda h - \lambda h} \\
&\quad \times \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}
\end{aligned}
\tag{19}
$$

and the twisting moment is

$$
\begin{aligned}
M_{xy} &\equiv \int_{-h/2}^{h/2} \tau_{xy} z dz \\
&= \frac{p_0}{\lambda}\frac{\pi^2}{\lambda^2 ab} \\
&\quad \times \frac{(1+2\nu)\sinh\lambda h - \nu\lambda h\cosh\lambda h - \lambda h(1+\nu)}{\sinh\lambda h - \lambda h} \\
&\quad \times \cos\frac{\pi x}{a}\cos\frac{\pi y}{b}
\end{aligned}
\tag{20}
$$

Using (14) the transverse shearing forces on a unit of length are by definition

$$
\begin{aligned}
Q_x &\equiv \int_{-h/2}^{h/2} \tau_{xz} dz = \frac{p_0}{\lambda}\frac{\pi}{\lambda a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
Q_y &\equiv \int_{-h/2}^{h/2} \tau_{yz} dz = \frac{p_0}{\lambda}\frac{\pi}{\lambda b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}
\end{aligned}
\tag{21}
$$

We may now check the static equilibrium of the plate. Using (21) the reaction forces along the plate sides are

$$
\begin{aligned}
F_x &= \int_0^b Q_x dy = -\frac{2b}{\lambda^2 a} p_0 \\
F_y &= \int_0^a Q_y dx = -\frac{2a}{\lambda^2 b} p_0
\end{aligned}
\tag{22}
$$

The sum of vertical forces is therefore

$$
2F_x + 2F_y = -\frac{4ab}{\pi^2} p_0
$$

and this equals total load

$$
\int_0^a \int_0^b p dx dy = \frac{4ab}{\pi^2} p_0
$$

The plate is thus in static equilibrium as expected.

3.3 Case of a thin plate

In order to compare the obtained solution with various plate theories we introduce dimensionless transverse coordinate ζ and a small dimensionless parameter η by writing

$$
\zeta \equiv \frac{z}{h/2} \in [-1, 1] \qquad \eta \equiv \lambda h \ll 1 \tag{23}
$$

Expansion of displacements (9) with respect to η and retaining only the first term in the expansion yields

$$
\begin{aligned}
u &= -\frac{p_0}{\lambda^4 D}\left[\frac{\pi\eta}{2\lambda a}\zeta + O\left(\eta^3\right)\right]\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
v &= -\frac{p_0}{\lambda^4 D}\left[\frac{\pi\eta}{2\lambda b}\zeta + O\left(\eta^3\right)\right]\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \\
w &= \frac{p_0}{\lambda^4 D}\left[1 + \frac{\eta^2}{40}\left(\frac{8-3\nu}{1-\nu} - \frac{5\nu}{1-\nu}\zeta^2\right) + O\left(\eta^4\right)\right] \\
&\quad \times \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}
\end{aligned}
\tag{24}
$$

Similarly the expansions of stress components (13)–(14) are

$$
\begin{aligned}
\sigma_x &= p_0\left[\frac{6\pi^2}{\lambda^2 b^2}\left(\nu+\frac{b^2}{a^2}\right)\frac{\zeta}{\eta^2}+O(\eta^0)\right]\\
&\quad\times\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\\
\sigma_y &= p_0\left[\frac{6\pi^2}{\lambda^2 a^2}\left(\nu+\frac{a^2}{b^2}\right)\frac{\zeta}{\eta^2}+O(\eta^0)\right]\\
&\quad\times\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\\
\tau_{xy} &= -p_0\left[\frac{6\pi^2(1-\nu)}{\lambda^2 ab}\frac{\zeta}{\eta^2}+O(\eta^0)\right]\\
&\quad\times\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}\\
\tau_{xz} &= p_0\left[\frac{3\pi}{2\lambda a}\frac{1-\zeta^2}{\eta}+O(\eta^0)\right]\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\\
\tau_{yz} &= p_0\left[\frac{3\pi}{2\lambda b}\frac{1-\zeta^2}{\eta}+O(\eta^0)\right]\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}\\
\sigma_z &= p_0\left[\frac{1}{4}(1+\zeta)^2(2-\zeta)+O(\eta^2)\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}
\end{aligned}
\tag{25}
$$

The above formulas illustrate that for a thin plate the in-plane displacements and in-plane stress components are in the first approximation linearly distributed across plate thickness while transverse shear stress components are distributed parabolically. Such distributions are customary initial assumptions of classical plate theories. Note that the order of magnitude of in-plane stress components is $O(\eta^{-2})$, the order of magnitude of transverse shear stress component is $O(\eta^{-1})$ and the order of magnitude of σ_z is $O(\eta^0)$. The expansion of moments (19) and (20) with respect to η are

$$
\begin{aligned}
M_x &= \frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 b^2}\left[\frac{b^2}{a^2}+\nu+\frac{\nu}{10}\eta^2+O(\eta^4)\right]\\
&\quad\times\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\\
M_y &= \frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 a^2}\left[\frac{a^2}{b^2}+\nu+\frac{\nu}{10}\eta^2+O(\eta^4)\right]\\
&\quad\times\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\\
M_{xy} &= -\frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 ab}\left[1-\nu-\frac{\nu}{10}\eta^2+O(\eta^4)\right]\\
&\quad\times\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}
\end{aligned}
\tag{26}
$$

This shows that the bending moments are of an order of magnitude $O(\eta^0)$. To that we note that the transverse shearing forces (21) do not depend on h so they have no expansion with respect to η.

4 Transversally inextensible plate

Consider now the transversally inextensible plate. The solution of the problem is sought in the form

$$
\begin{aligned}
u &= U(z)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\\
v &= V(z)\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}\\
w &= W_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}
\end{aligned}
\tag{27}
$$

where $U(z)$ and $V(z)$ are unknown functions and W_0 is a constant. Substituting these expressions into constitutive equations (4) and $(5)_{1,2}$ we obtain expressions for stress components. Introducing them into equilibrium equations $(2)_{1,2}$ we obtain a system of ordinary differential equations which may be written in the form

$$
\begin{aligned}
&\frac{d^2U}{dz^2}-\left[\lambda^2+\frac{\pi^2}{a^2(1-2\nu)}\right]U\\
&\quad-\frac{\pi^2(1+\nu)}{ab(1-\nu)}V=0\\
&\frac{d^2V}{dz^2}-\left[\lambda^2+\frac{\pi^2}{b^2(1-2\nu)}\right]V\\
&\quad-\frac{\pi^2(1+\nu)}{ab(1-\nu)}U=0
\end{aligned}
\tag{28}
$$

where

$$
\lambda\equiv\frac{\pi\sqrt{a^2+b^2}}{ab}\qquad \alpha^2\equiv\frac{2}{1-\nu}
\tag{29}
$$

The solution of this system may be expressed as

$$
\begin{aligned}
\lambda aU &= a^2(C_1\cosh\lambda z+C_2\sinh\lambda z)\\
&\quad+C_3\cosh\alpha\lambda z+C_4\sinh\alpha\lambda z\\
\lambda bV &= b^2(C_1\cosh\lambda z-C_2\sinh\lambda z)\\
&\quad+C_3\cosh\alpha\lambda z+C_4\sinh\alpha\lambda z
\end{aligned}
\tag{30}
$$

From the third equilibrium equation (2), one, after integration, obtains

$$\sigma_z = G\left[\frac{\pi}{ab}(bU + aV) + W_0\lambda^2 z + C\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{31}$$

The values of the constants in (30) and (31) follow from boundary conditions (7)

$$\begin{aligned} &C_1 = C_2 = C_4 = 0 \\ &C_3 = \frac{\pi W_0}{\alpha\cosh\frac{\alpha\lambda h}{2}} \\ &C = \frac{p_0}{2} \end{aligned} \tag{32}$$

By means of (27), (30) and (32) the final expressions for displacement components are

$$\begin{aligned} u &= \frac{\pi p_0}{\lambda^2 aG}\frac{\sinh\alpha\lambda z}{\cosh\frac{\alpha\lambda h}{2}\left(\frac{\alpha\lambda h}{2} - \tanh\frac{\alpha\lambda h}{2}\right)}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \\ v &= \frac{\pi p_0}{\lambda^2 bG}\frac{\sinh\alpha\lambda z}{\cosh\frac{\alpha\lambda h}{2}\left(\frac{\alpha\lambda h}{2} - \tanh\frac{\alpha\lambda h}{2}\right)} \\ &\quad\times\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \\ w &= \frac{\alpha p_0}{2\lambda G}\frac{1}{\frac{\alpha\lambda h}{2} - \tanh\frac{\alpha\lambda h}{2}}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \end{aligned} \tag{33}$$

The stress components defined by above displacements follows from (4), (5) and (31). The in-plane stress components are

$$\begin{aligned} \sigma_x &= p_0\frac{\pi^2}{\lambda^2 b^2(1-\nu)}\frac{\left(\nu + \frac{b^2}{a^2}\right)\sinh\alpha\lambda z}{\frac{\alpha\lambda h}{2}\cosh\frac{\alpha\lambda h}{2} - \sinh\frac{\alpha\lambda h}{2}} \\ &\quad\times\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\ \sigma_y &= p_0\frac{\pi^2}{\lambda^2 a^2(1-\nu)}\frac{\left(\nu + \frac{a^2}{b^2}\right)\sinh\alpha\lambda z}{\frac{\alpha\lambda h}{2}\cosh\frac{\alpha\lambda h}{2} - \sinh\frac{\alpha\lambda h}{2}} \\ &\quad\times\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\ \tau_{xy} &= p_0\frac{\pi^2}{\lambda^2 ab}\frac{\sinh\alpha\lambda z}{\frac{\alpha\lambda h}{2}\cosh\frac{\alpha\lambda h}{2} - \sinh\frac{\alpha\lambda h}{2}} \\ &\quad\times\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} \end{aligned} \tag{34}$$

and the transverse stress components are

$$\begin{aligned} \sigma_z &= \frac{p_0}{2}\left(1 + \frac{\frac{\alpha\lambda z}{2}\cosh\frac{\alpha\lambda h}{2} - \sinh\alpha\lambda h}{\frac{\alpha\lambda h}{2}\cosh\frac{\alpha\lambda h}{2} - \sinh\frac{\alpha\lambda h}{2}}\right) \\ &\quad\times\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\ \tau_{xz} &= p_0\frac{\pi\alpha}{2\lambda a}\frac{\cosh\frac{\alpha\lambda h}{2} - \cosh\alpha\lambda z}{\frac{\alpha\lambda h}{2}\cosh\frac{\alpha\lambda h}{2} - \sinh\frac{\alpha\lambda h}{2}} \\ &\quad\times\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \\ \tau_{yz} &= p_0\frac{\pi\alpha}{2\lambda b}\frac{\cosh\frac{\alpha\lambda h}{2} - \cosh\alpha\lambda z}{\frac{\alpha\lambda h}{2}\cosh\frac{\alpha\lambda h}{2} - \sinh\frac{\alpha\lambda h}{2}} \\ &\quad\times\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \end{aligned} \tag{35}$$

4.1 Stress resultants

The bending stress resultants corresponding to the stress components (34)–(35) are

$$\begin{aligned} M_x &= \frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 b^2}\left(\nu + \frac{b^2}{a^2}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\ M_y &= \frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 a^2}\left(\nu + \frac{a^2}{b^2}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\ M_{xy} &= -\frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 ab}(1-\nu)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \end{aligned} \tag{36}$$

$$\begin{aligned} Q_x &= \frac{p_0}{\lambda}\frac{\pi}{\lambda a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \\ Q_y &= \frac{p_0}{\lambda}\frac{\pi}{\lambda b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \end{aligned} \tag{37}$$

From these we may see that unlike the isotropic plate for a transversally inextensible plate, besides shear forces, the moments are also independent of plate thickness h.

If we perform an expansion with respect to η similar to the isotropic plate we find that in the present case the expansions for displacement components differ from those of an isotropic plate (24) only for plate deflection, which is for present case given by

$$w_0 = \frac{p_0}{\lambda^4 D}\left[1 + \frac{\eta^2}{5}\frac{1}{1-\nu} + O\left(\eta^4\right)\right] \tag{38}$$

while the expansions for stress components are the same as for an isotropic plate (25).

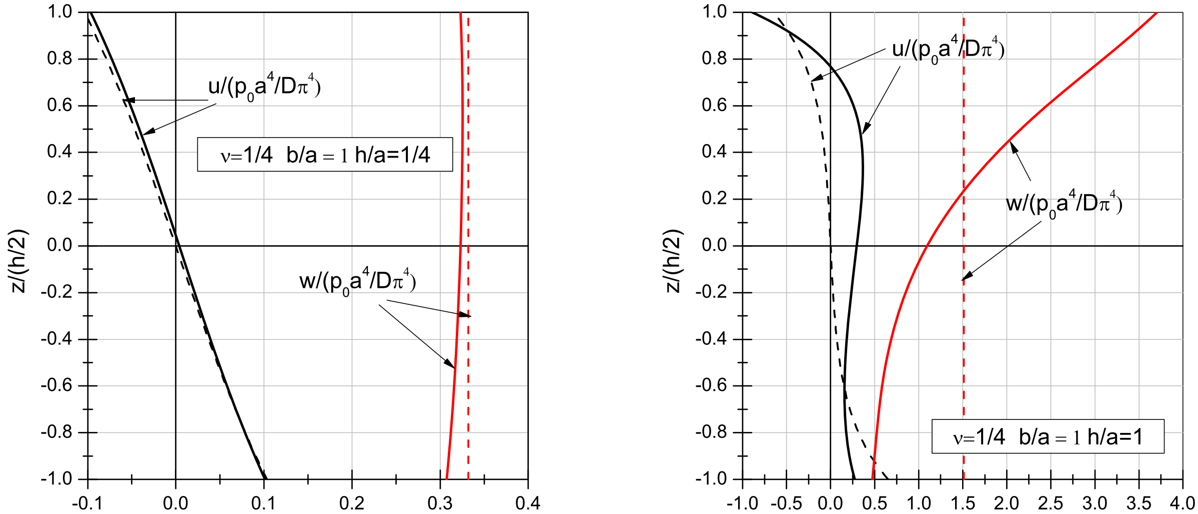

Fig. 1 Displacement distribution across the thickness of square plate. Distribution of u is at middle of the plate side and w in the center of the plate. *Full lines* give results for the isotropic plates, while the *dashed lines* are for the transversally inextensible plates

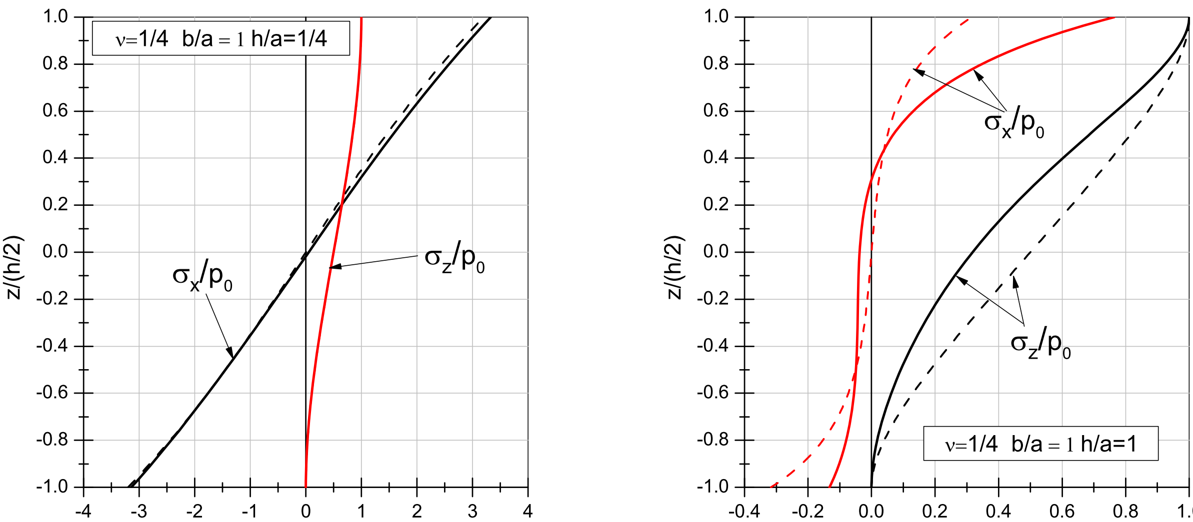

Fig. 2 Distribution of normal stress components across the thickness of a square plate at plate center. *Full lines* give results for isotropic plates, while *dashed lines* are for transversally inextensible plates

4.2 Comparison

A comparison between isotropic and transversally inextensible plates is shown in Figs. 1–4, where the displacements and stress components distribution across plate thickness for a square plate with ratios $h/a = 1/4$ and $h/a = 1$ are shown and Fig. 5 where the deflection and maximal stress are shown for various ratios of b/a and h/a. As my be seen from these figures there is a minor difference in distributions of displacement and stress components between isotropic and transversally inextensible plates for ratios $h/a < 0.25$. The difference becomes noticeable only for a higher ratio of h/a.

5 Comparison with plate theories

In this section, we compare the above exact solutions with three plate models: Reissner's, Mindlin's

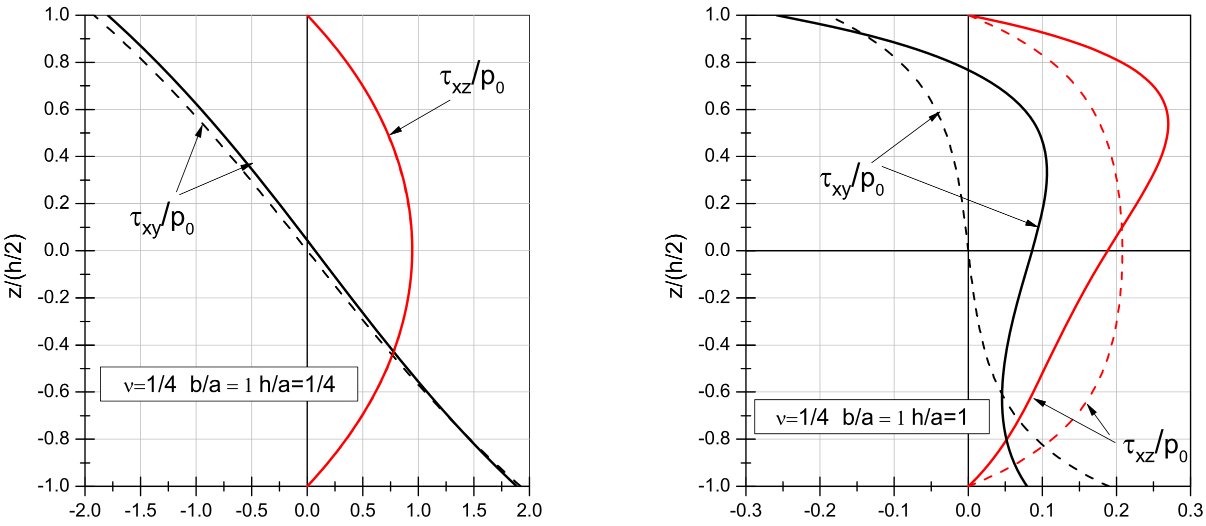

Fig. 3 Distribution of shear stress components across the thickness of a square plate: τ_{xy} at the middle of the plate, τ_{xz} at the middle of the plate side. Full lines give results for isotropic plates, while *dashed line* are for transversally inextensible plates

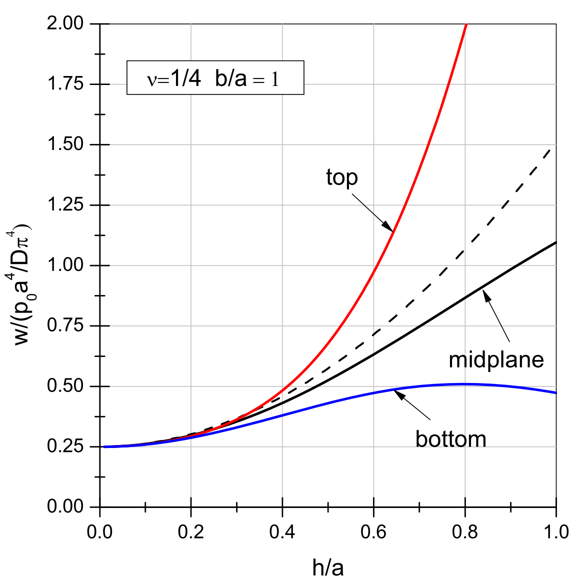

Fig. 4 Deflection of center of square plate. *Full lines* give results for isotropic plates, while *dashed line* are for transversally inextensible plates

and Reddy's. All three theories approximate $w \approx w_0(x, y)$, which is exact for the transversally inextensible plate; however, Reissner's model includes σ_z in constitutive equations for in-plane stress components while Mindlin' and Reddy' models do not so they may also be suitable for modelling transversally inextensible plates.

5.1 Reissner and Mindlin plate

where

The basic equations of shear deformable plates established by Reissner [32, 33] and Mindlin [33, 34] may be written in the form

$$D\Delta^2 w_0 = p - \frac{h^2(2-\omega\nu)}{10(1-\nu)}\Delta p \tag{39}$$

The equation's shear forces equations are

$$\begin{aligned} &Q_x - \frac{h^2}{10}\Delta Q_x \\ &\quad = -D\frac{\partial \Delta w_0}{\partial x} - \frac{h^2}{10}\frac{1+\nu(1-\omega)}{1-\nu}\frac{\partial p}{\partial x} \\ &Q_y - \frac{h^2}{10}\Delta Q_y \\ &\quad = -D\frac{\partial \Delta w_0}{\partial y} - \frac{h^2}{10}\frac{1+\nu(1-\omega)}{1-\nu}\frac{\partial p}{\partial y} \end{aligned} \tag{40}$$

and the moments are determined by

$$M_x = -D\left(\frac{\partial^2 w_0}{\partial x^2} + \nu\frac{\partial^2 w_0}{\partial y^2}\right) + \frac{h^2}{5}\frac{\partial Q_x}{\partial x} - \frac{h^2}{10}\frac{\nu(2-\omega)}{1-\nu}p$$

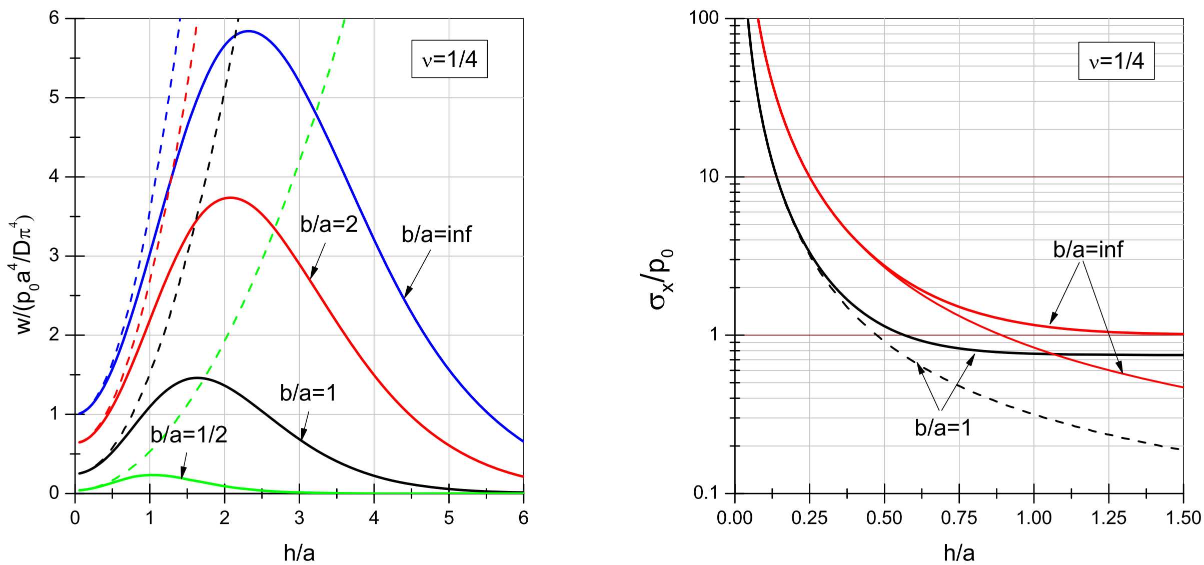

Fig. 5 Midplane deflection (*left*) and maximum stress (*right*) for various ratios of b/a and h/a. *Full lines* give results for isotropic plates, while *dashed line* are for transversally inextensible plates

$$M_x = -D\left(\frac{\partial^2 w_0}{\partial x^2} + \nu\frac{\partial^2 w_0}{\partial y^2}\right) + \frac{h^2}{5}\frac{\partial Q_x}{\partial x} - \frac{h^2}{10}\frac{\nu(2-\omega)}{1-\nu}p \tag{41}$$

$$M_{xy} = -(1-\nu)D\frac{\partial^2 w_0}{\partial x\partial y} + \frac{h^2}{10}\left(\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x}\right)$$

where

$$\omega = \begin{cases} 0 & \text{transversally inextensible plate (Mindlin)} \\ 1 & \text{isotropic plate (Reissner)} \end{cases} \tag{42}$$

and where for Mindlin's model the shear correction factor is taken to be $\kappa^2 = \frac{5}{6}$. Now, if we sought the solution of (39) and (40) in the form

$$w_0 = W\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$
$$Q_x = A\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{43}$$
$$Q_y = B\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

where W, A and B are constants, then by substituting (43) into (39)–(40) we obtain

$$w = \frac{p_0}{\lambda^4 D}\left(1 + \frac{\eta^2}{10}\frac{2-\omega\nu}{1-\nu}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{44}$$

$$Q_x = \frac{p_0}{\lambda}\frac{\pi}{\lambda a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}$$
$$Q_y = \frac{p_0}{\lambda}\frac{\pi}{\lambda b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} \tag{45}$$

From this it follows that the shear forces for both plate models are the same and match exact values (21). The magnitude of error for deflection in the case of an isotropic plate is for both theories of the order $O(\eta^2)$, as may be seen by comparing (44) and (24). In the case of a transversally inextensible plate the error of deflection for Mindlin's plate is of the order $O(\eta^4)$, as may be seen by comparing (38) and (44) with $\omega = 0$.

By means of (44) and (45) the moments (41) become

$$M_x = \frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 b^2}\left(\frac{b^2}{a^2} + \nu + \frac{\omega\nu}{10}\eta^2\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$
$$M_y = \frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 a^2}\left(\frac{a^2}{b^2} + \nu + \frac{\omega\nu}{10}\eta^2\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$
$$M_{xy} = -\frac{p_0}{\lambda^2}\frac{\pi^2}{\lambda^2 ab}\left(1 - \nu - \frac{\omega\nu}{10}\eta^2\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{46}$$

If we compare this with expansions (26) we find that for the case of an isotropic plate the difference between exact moments and moments of Reissner's model is of the order $O(\eta^4)$ while for Mindlin's model it is of the order $O(\eta^2)$. For a transversally inextensible plate the moments given by the Mindlin plate model match the exact values (36).

5.2 Reddy's plate model

The basic equation of the Reddy plate model [3], Eqs. (6.4.4, 6.4.5, 6.4.6, 6.4.11a-6.4.11j) may be transformed into the following form which is convenient for analytical treatment

$$
\begin{aligned}
&D\Delta^2 w_0 = 5p - 4P - \frac{4h^2}{21(1-\nu)}\Delta P \\
&P \equiv -\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right)
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&Q_x - \frac{17h^2}{168}\Delta Q_x \\
&\quad = -D\frac{\partial \Delta w_0}{\partial x} - \frac{17h^2(1+\nu)}{168(1-\nu)}\frac{\partial P}{\partial x} \\
&Q_y - \frac{17h^2}{168}\Delta Q_y \\
&\quad = -D\frac{\partial \Delta w_0}{\partial y} - \frac{17h^2(1+\nu)}{168(1-\nu)}\frac{\partial P}{\partial y}
\end{aligned}
\tag{48}
$$

Observe that unlike Reissner' and Mindlin' plate models the above equations form the coupled system of differential equations for unknowns w_0, Q_x and Q_y. The moments are

$$
\begin{aligned}
M_x &= -D\left(\frac{\partial^2 w_0}{\partial x^2} + \nu\frac{\partial^2 w_0}{\partial y^2}\right) + \frac{h^2}{5}\frac{\partial Q_x}{\partial x} \\
&\quad - \frac{h^2\nu}{5(1-\nu)}P \\
M_y &= -D\left(\frac{\partial^2 w_0}{\partial y^2} + \nu\frac{\partial^2 w_0}{\partial x^2}\right) + \frac{h^2}{5}\frac{\partial Q_y}{\partial y} \\
&\quad - \frac{h^2\nu}{5(1-\nu)}P \\
M_{xy} &= -(1-\nu)D\frac{\partial^2 w_0}{\partial x \partial y} + \frac{h^2}{10}\left(\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x}\right)
\end{aligned}
\tag{49}
$$

The solution of the system (47)–(48) is again sought in the form (43). Omitting details, the final solution may be written in the form

$$
w = \frac{p_0}{\lambda^4 D}\frac{1 + \frac{17}{84}\frac{\eta^2}{1-\nu}}{1 + \frac{1}{420}\frac{\eta^2}{1-\nu}}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}
\tag{50}
$$

$$
\begin{aligned}
Q_x &= \frac{p_0}{\lambda}\frac{\pi}{\lambda a}\frac{1}{1 + \frac{\eta^2}{420}\frac{1}{1-\nu}}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
Q_y &= \frac{p_0}{\lambda}\frac{\pi}{\lambda b}\frac{1}{1 + \frac{\eta^2}{420}\frac{1}{1-\nu}}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}
\end{aligned}
\tag{51}
$$

It turns out that the moments (49) corresponding to this solution are the same as (46) with $\omega = 0$; i.e., as predicted by Mindlin's plate model. So, for the case of an isotropic plate the difference between exact moments and moments of Reddy's model is of the order $O(\eta^2)$ and for a transversally inextensible plate the moments match exact values (36). From (50) we see that the deflection error is of the order $O(\eta^2)$ even in the case of a transversally isotropic plate; so Mindlin's model in this case yields a smaller error. Also, from (51) it is seen that unlike Reissner's model and Mindlin's model the error in shear forces are of the order $O(\eta^2)$.

5.3 Numerical results

The quantitative comportment between models is given in Tables 1 and 2. The results in the tables confirm the results of qualitative asymptotic analyses. As may be seen from Table 1 the smallest deflection relative error is for a square isotropic plate obtained by Reissner's plate model, which gives a relative error below 0.5% even for the ratio $h/a = 0.3$. Comparing Reddy's and Mindlin's models, the first gives approximately two times the smallest relative error in deflection. However, for transversally inextensible plates the deflection error is smallest for Mindlin's plate model as was expected on the base of asymptotic analysis. Note that the relative error of deflection is for Mindlin's transversally inextensible plate for ratio $h/a = 0.3$ even below 0.2%. Table 2 illustrates the comportment of maximum normalized bending moment for a square plate. Note that in the table the values for the bending moment for the Mindlin and Reddy plate models are the same since the moments for both models are given by (46). It may be seen from the table that the relative error in the bending moment for Reissner's isotropic plate is in the case of deflection much smaller than that of Mindlin's and Reddy's. However, for a transversally inextensible plate Mindlin's and Reddy plate's give the exact value for the maximum bending moment.

Table 1 Comportment of maximum normalized plate deflection $w/(p_0 a^4/\pi^4 D)$ at middle of the square plate. A: isotropic plate, B: transversaly inextensible plate. *err* is relative error in %. $\nu = 0.3$

	Exact		Reissner		Mindlin			Reddy		
h/a	A	B		err_A		err_A	err_B		err_A	err_B
0.01	0.250127	0.250141	0.250120	0.00	0.250141	−0.01	0.00	0.250132	0.00	0.00
0.05	0.253126	0.253524	0.252996	0.05	0.253525	−0.16	0.00	0.253289	−0.06	0.09
0.10	0.262476	0.264090	0.261985	0.19	0.264099	−0.62	0.00	0.263151	−0.26	0.36
0.15	0.277966	0.281676	0.276965	0.36	0.281724	−1.35	−0.02	0.279567	−0.58	0.75
0.20	0.299456	0.306250	0.297938	0.51	0.306398	−2.32	−0.05	0.302506	−1.02	1.22
0.25	0.326748	0.337766	0.324903	0.56	0.338121	−3.48	−0.11	0.331926	−1.58	1.73
0.30	0.359588	0.376169	0.357861	0.48	0.376895	−4.81	−0.19	0.367771	−2.28	2.23

Table 2 Comportment of maximum normalized bending moment $M_x(\frac{a}{2}, \frac{b}{2})/(p_0 a^2/\pi^2)$ for square plate. A: isotropic plate, B: transversaly inextensible plate. *err* is relative error in %. $\nu = 0.3$

	Exact		Reissner		Mindlin and Reddy		
h/a	A	B		err_A		err_A	err_B
0.01	0.325014	0.325	0.325015	0.00	0.325	0.00	0.00
0.05	0.325370	0.325	0.325370	0.00	0.325	0.11	0.00
0.10	0.326480	0.325	0.326480	0.00	0.325	0.45	0.00
0.15	0.328327	0.325	0.328331	0.00	0.325	1.01	0.00
0.20	0.330910	0.325	0.330922	0.00	0.325	1.79	0.00
0.25	0.334224	0.325	0.334253	−0.01	0.325	2.76	0.00
0.30	0.338264	0.325	0.338324	−0.02	0.325	3.92	0.00

6 Conclusions

A comparison between exact 3D solutions of isotropic and transversally inextensible plates shows that the difference between calculated displacements and stress component become noticeable for approximately $h/a > 0.25$. More precisely, the expansions with respect to parameter η (23) for the stress components up to order $O(\eta^0)$ and in-plane displacement components up to order $O(\eta^3)$ are the same for both theories. The difference is only for the plate deflection which is of the order $O(\eta^2)$. In addition, the expansions show that the solution for small η is governed mostly by the plate bending, while the plate stretching is negligible.

Comparison between asymptotic expansion of the results of an exact solution with respect to η and the results obtained by Reissner's, Mindlin's and Reddy's plate theories shows that for the case of an isotropic plate the difference between exact moments and moments given by Reissner's model is of the order $O(\eta^4)$, while for Mindlin's and Reddy's model the difference is of the order $O(\eta^2)$. For transversally inextensible plates both the Mindlin and Reddy plate models give moments which match expressions for the exact moments. Further, the Reissner and Mindlin models also match the exact expressions for shear forces (which are the same for isotropic and transversally inextensible plate) while for the Reddy model the difference is of the order $O(\eta^2)$. The difference for deflection is in the case of an isotropic plate and transversally inextensible plate for all three plate models of the order $O(\eta^2)$ except for the transversally inextensible Mindlin plate, for which the difference is of the order $O(\eta^4)$.

The present results show that for the case of sinusoidally loaded simply supported rectangular plates the Reissner plate model is superior to Mindlin's and Reddy's plate models. For the transversally inextensible plate it is a bit surprising that Mindlin's first order plate model gives better results than Reddy's third order model.

Appendix

The present solution may be easily generalized to a more general load case that may be represented by

$$p = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{52}$$

simply by replacing a with $\frac{a}{m}$ and b with $\frac{b}{n}$ so (11) becomes

$$\lambda_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \tag{53}$$

If (53) is used in the above solution instead of λ then all displacement components and stress components may be expressed as double series. For example the deflection given by (9) becomes

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{54}$$

References

1. Reddy JN, Wang CM (1998) Deflection relationships between classical and third-order plate theories. Acta Mech 130:199–208
2. Timoshenko S, Woinowsky-Krieger S (1959) Theory of plates and shells. McGraw-Hill, New York
3. Wang CM, Reddy JN, Lee KH (2000) Shear deformable beams and plates. Amsterdam, Elsevier
4. Girkmann K (1963) Flächentragwerke. Springer, Berlin
5. Gould PL (1999) Analysis of plates and shells. Prentice Hall, New York
6. Reddy JN (2007) Theory and analysis of elastic plates and shells, 2nd edn. CRC Press, Boca Raton
7. Szilard R (2004) Theories and applications of plate analysis: classical numerical and engineering methods. Wiley, New York
8. Ventsel E, Krauthammer T (2001) Thin plates and shells: theory, analysis, and application. CRC Press, Boca Raton
9. Vinson JV (1974) Structural mechanics: the behavior of plates and shells. Willey, New York
10. Galerkin BG (1951) Collected works, vol I. Academy of Science, Moscow (In Russian)
11. Levinson M (1985) The simply supported rectangular plate: An exact, three dimensional, linear elasticity solution. J Elast 15(3):283–291
12. Nicotra V, Podio-Guidugli P, Tiero A (1999) Exact equilibrium solutions for linearly elastic plate-like bodies. J Elast 56(3):231–245
13. Barrett KE, Ellis S (1988) An exact theory of elastic plates. Int J Solids Struct 24:859–880
14. Werner H (1999) A three dimensional solution for rectangular plate bending free of transversal normal stresses. Commun Numer Methods Eng 15:295–302
15. Demasi L (2007) Three-dimensional closed form solutions and exact thin plate theories for isotropic plates. Compos Struct 80:183–195
16. Teo TM, Liew KM (1999) Three-dimensional elasticity solutions to some orthotropic plate problems. Int J Solids Struct 36:5301–5326
17. Demasi L (2008) 2D, quasi 3D and 3D exact solutions for bending of thick and thin sandwich plates. J Sandw Struct Mater 10:271–310
18. Demasi L (2009) Three-dimensional closed form solutions and ∞^3 theories for orthotropic plates. Mech Adv Mater Struct
19. Pagano NJ (1969) Exact solutions for composite laminates in cylindrical bending. J Compos Mater 3:398–411
20. Pagano NJ (1970) Exact solutions for rectangular bidirectional composite and sandwich plates. J Compos Mater 4:20–34
21. Kardomateas G (2009) Three dimensional elasticity solution for sandwich plates with orthotropic phases: the positive discriminant case. J Appl Mech 76:041505
22. Zenkour AM (2007) Three dimensional elasticity solution for uniformly loaded cross-ply laminates and sandwich plates. J Sandw Struct Mater 9:213
23. Pan E (2001) Exact solution for simply supported and multilayered magneto-electro-elastic plates. J Appl Mech 68:608–618
24. Heyliger P (1997) Exact solutions for simply supported laminated piezoelectric plates. J Appl Mech 63:299–306
25. Vel SS, Batra RC (1997) Three-dimensional analytical solution for hybrid multilayered piezoelectric plates. J Appl Mech 67:558–567
26. Meyer-Piening HR (2004) Application of the elasticity solution to linear sandwich beam plate and shell analyses. J Sandwich Struct 6(4):295–312
27. Savoia M, Reddy JN (1992) A variational approach to three-dimensional elasticity solutions of laminated composite plates. J Appl Mech 59:S166–75
28. Noor AK, Burton WS (1990) Three-dimensional solutions for antisymmetrically laminated anisotropic plates. J Appl Mech 57:182–188
29. Carrera E, Giunta G, Brischetto S (2007) Hierarchical closed form solutions for plates bent by localized transverse loadings. J Zhejiang Univ Sci A 8:1–12
30. Demasi L (2009) ∞^6 mixed plate theories based on the generalized unified formulation. Part V: Results. Compos Struct 88:1–16
31. Zenkour AM, Mashat DS (2009) Exact solutions for variable-thickness inhomogeneous elastic plates under various boundary conditions. Meccanica 44:433–447
32. Reissner E (1945) The effect of transverse shear deformation on the bending of elastic plates. J Appl Mech 12:A-69–A-77 (Trans ASME 67)
33. Nardinocchi P, Podio-Guidugli P (1994) The equations of Reissner-Mindlin plates obtained by the method of internal constraints. Meccanica 29(2):143–157
34. Mindlin RD (1951) Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. Trans ASME J Appl Mech 18:31–38