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PARALLEL MANIPULATORS

Forward displacement analysis of a quadratic 4-DOF 3T1R parallel manipulator

The Quadrupteron

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Abstract A quadratic parallel manipulator refers to a parallel manipulator with a quadratic characteristic polynomial. This paper revisits the forward displacement analysis (FDA) and the Type II singularity analysis of a quadratic 4-DOF 3T1R (SCARA) parallel manipulator: the Quadrupteron. It will be proved that there exists a one-to-one correspondence between the two formulas, each producing one solution to the FDA, and the two singularity-free regions. Therefore, a unique solution to the FDA can be obtained in a straightforward way for such a parallel manipulator if the singularity-free region in which it works is specified. The Type II singularity analysis in the joint space will also be investigated in order to identify the conditions on the inputs to keep the Quadrupteron working in the same singularity-free region in its Cartesian workspace.

Keywords Parallel manipulator · Forward displacement analysis · Singularity · Current solution · Input space

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1 Introduction

Theoretically, the workspace of a parallel manipulator is usually divided into several singularity-free regions [1, 2]. In practice, the parallel manipulator usually works in one singularity-free region, and one needs to find a unique solution (called the current solution in [3]) to its forward displacement analysis (FDA, also forward kinematics or direct kinematics) for a parallel manipulator in the singularity-free region (in its workspace) in which the parallel manipulator works.

Advances have been made from two perspectives in order to determine a unique solution for a parallel mechanism. On one hand, an approach has been proposed to the type synthesis of linear parallel manipulators [4]—parallel manipulators for which the solution to the FDA can be obtained by solving a set of linear equations. Linear parallel manipulators have been investigated thoroughly [4–11]. On the other hand, approaches have been proposed for determining a unique solution to a parallel manipulator, the FDA of which requires to solve a set of non-linear equations. The approaches proposed include numerical approaches [3] and algebraic approaches [12, 13]. In [3], the unique solution to its FDA has been determined for a 3-R$\underline{\text{P}}$R planar parallel manipulator in the singularity-free region in which the parallel manipulator works using a numerical approach. Here and throughout this paper, $\underline{\text{P}}$ stands for an actuated P (prismatic) joint, R (revolute) joints and P joints in a parallel manipulator denoted by

R̋ and P̋ have parallel axes,[1] and R joints denoted by Ŕ within a leg have parallel axes.

In [12, 13], it was proved that for the 3-R$\underline{\text{P}}$R parallel manipulator with similar platforms [12, 14, 15], there exists a one-to-one correspondence between the four formulas, each producing one solution to the FDA, and the four singularity-free regions. Therefore, one can determine the unique solution to its FDA in the singularity-free region in which it works. Inspired by the advances in [12, 13], the authors intend to investigate whether one can derive a formula that produces one solution to the FDA of other parallel manipulators. It is logical, from both the theoretical and practical points of view, to investigate quadratic parallel manipulators, which refers to parallel manipulators with a quadratic characteristic equation.

It has recently been revealed in [16] that the Quadrupteron is a quadratic 3T1R (SCARA) parallel manipulator. However, no formula has been proposed to determine its current solution. Considering the need for 3T1R parallel manipulators [17–34], we will revisit the singularity analysis and FDA of the Quadrupteron (Fig. 1)—a partially decoupled 4-DOF 3T1R parallel manipulator built at Laval University. Like in [12, 13], the singularity analysis is treated as an integral part of the FDA of the Quadrupteron in this paper.

This paper is organized as follows. In Sects. 2 and 3, we first recall the geometric description of the Quadrupteron and the inverse displacement analysis [16]. In Sect. 4, the Type II singularity analysis of the Quadrupteron is performed. It is revealed that the workspace of the Quadrupteron is divided into two singularity-free regions. In Sect. 5, the FDA of the Quadrupteron is then performed which shows that there exist two solutions to this problem. In Sect. 6, the distribution of the solutions to the FDA in the singularity-free regions for the Quadrupteron is discussed and the process of FDA is further simplified. In Sect. 8, the Type II singularity analysis in the joint space is investigated and the conditions on the inputs to keep the Quadrupteron working in the same singularity-free region in its Cartesian workspace are identified. Finally, conclusions are drawn.

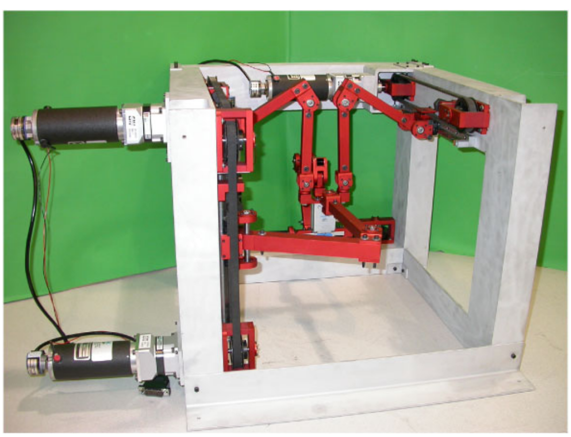
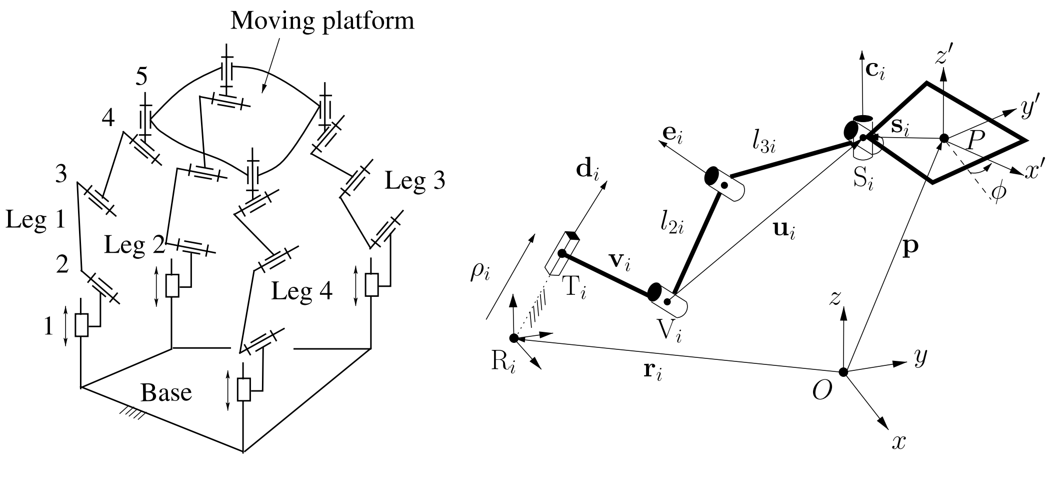


Fig. 1 Photograph of the Quadrupteron

[1] Strictly speaking, P joints do not have an axis but only a direction. The spatial location of the axis is arbitrary and does not affect the kinematic derivations.

2 Description of the Quadrupteron

The Quadrupteron is an optimal design of the 4-$\underline{\text{P}}$ŔŔR̋R̋ parallel manipulator [22, 23, 32], which is shown in Fig. 2(a). The 4-$\underline{\text{P}}$ŔŔR̋R̋ parallel manipulator consists of a moving platform connected to a fixed base by four legs each having five joints namely, from base to moving platform, a P joint and four R joints which are labeled from 1 to 5 in sequence. For practical reasons, the axes of the last two R joints within a leg are arranged to have at least one common point. The P joints, mounted directly on the base, are the only actuated joints. In order for the architecture to produce the 3T1R motions and to be drivable by the four actuated P joints, the following geometric constraints must be satisfied: (i) The axes of the four R joints attached to the moving platform are all parallel. (ii) The axes of joints 2, 3 and 4 within a same leg are all parallel. (iii) The axes of joints 1 and 2 within a same leg are not orthogonal to each other. (iv) The axes of joints 2 of the four legs are not all parallel to a plane. This requires that the axes of joints 4 and 5 within a same leg are not orthogonal to each other in at least one of the four legs. (v) In at most one of the four legs, all the axes of joints 2–5 within a same leg are parallel. In this case, joints 4 and 5 degenerate to one joint. Such a leg, if it exists, is numbered as leg 1 for convenience. The moving platform of the 4-$\underline{\text{P}}$ŔŔR̋R̋ manipulator can translate along any direction and rotate about any axis that is parallel to the joint axes of the R joints on the moving platform. As pointed out in [22], the orientation of the axis of rotation of the moving platform does not change.



(a) Schematic representation. (b) Kinematic modeling of a leg.

Fig. 2 4-P̲ŔŔŔR̋ parallel manipulator [16]

As shown in Fig. 2(b), two unit vectors $\mathbf{c}_i$ and $\mathbf{e}_i$ are used to represent respectively the direction of joint 5 as well as the direction of joints 2, 3, 4 of the ith leg. Here and throughout this paper, it is assumed that $i = 1, \ldots, 4$ unless otherwise specified.

The 4-P̲ŔŔŔR̋ parallel manipulator belongs to the class 1-1-1-1[2] if $\mathbf{c}_1 \neq \mathbf{e}_1$ or to the class 2-1-1-1 if $\mathbf{c}_1 = \mathbf{e}_1$. The numbers within the name of the classes of a parallel manipulator represent the number of constraints exerted on the moving platform by its legs. In the Quadrupteron (Fig. 3), leg 1 is arranged such that $\mathbf{c}_1 = \mathbf{e}_1$. In this case, the axes of the R joints within leg 1 are parallel to the axes of rotation of the moving platform, and the translation along the direction parallel to the axes of the rotation of the moving platform is controlled only by the P joint within this leg. Moreover, the kinematics is further simplified by aligning the axes of the first two joints in each of the legs and by mounting the prismatic actuators in orthogonal directions. It is noted in Fig. 3 that, the axes of joints 4 and 5 in leg 1 are aligned, which makes one of them redundant. Therefore, leg 1 can be built as a P̲̋R̋R̋R̋ leg, and the notation for the specific architecture of the 4-P̲ŔŔŔR̋ parallel manipulator is changed for 1-P̲̋R̋R̋R̋+3-P̲ŔŔŔR̋.

It is noted that variations of the Quadrupteron can be obtained by (a) replacing the R̋R̋R̋ or ŔŔŔ in each leg with a serial chain, composed of R joints, P joints and planar parallelograms, that is equivalent to a planar kinematic chain, (b) replacing an ŔŔŔ with a planar-pseudo kinematic chain [11, 35] and/or (c) inserting redundant joints in the way proposed in [9].

[2]For a complete list of classes of 3T1R parallel manipulators, see [22].

3 Inverse displacement analysis

Referring to Fig. 2(b), a fixed reference frame O–xyz is defined on the base and a moving reference frame P–$x'y'z'$ is attached to the moving platform. The position of the moving platform is given by the vector connecting O to P, denoted by $\mathbf{p} = [x, y, z]^T$. Without loss of generality, both the z-axis and z'-axis are chosen to be parallel to $\mathbf{c}_i$. Hence we have $\mathbf{c}_i = [0, 0, 1]^T$. The orientation of the moving platform with respect to the base is denoted by orientation angle ϕ.

The direction of the ith fixed P joint is denoted by unit vector $\mathbf{d}_i$. A reference point R_i is defined on the axis of the ith fixed P joint, as shown in Fig. 2(b) and the motion of the P joint is measured with respect to the latter reference and denoted by the signed distance ρ_i between R_i and a point T_i defined on the sliding body. Point S_i is defined as the intersection of the axes of the last two joints in leg i. Furthermore, the position vector of point R_i (the vector connecting O to R_i) in the fixed frame is denoted by $\mathbf{r}_i = [r_{xi}, r_{yi}, r_{zi}]^T$, while the position vector of point S_i (the vector connecting P to S_i) is denoted by $\mathbf{s}_i^P = [s_{xi}, s_{yi}, s_{zi}]^T$ in the moving frame and $\mathbf{s}_i$ in the fixed frame.

Since the axes of joints 2, 3 and 4 of a given leg are parallel, it is possible to define a point V_i on the axis of joint 2 such that the vector connecting V_i to S_i, denoted by $\mathbf{u}_i$ is orthogonal to the unit vector $\mathbf{e}_i = [e_{xi}, e_{yi}, e_{zi}]^T$.

For the Quadrupteron (Fig. 3), we have

$$\mathbf{d}_1 = \mathbf{e}_1 = [0, 0, 1]^T$$

$$\mathbf{d}_2 = \mathbf{e}_2 = [0, 1, 0]^T$$

$$\mathbf{d}_3 = \mathbf{e}_3 = [1, 0, 0]^T$$

$$\mathbf{d}_4 = \mathbf{e}_4 = [0, 1, 0]^T$$

$$\mathbf{v}_i = [0, 0, 0]^T$$

$$\mathbf{c}_i = [0, 0, 1]^T$$

where $i = 1, 2, 3, 4$.

According to [16], the solution of the inverse displacement analysis for the 1-P̲̋R̋R̋R̋+3-Ṕ̲ŔŔŔR̋ parallel manipulator is

$$\begin{cases} \rho_1 = z + s_{z1} - r_{z1} \\ \rho_2 = y + s_{x2} \sin\phi + s_{y2} \cos\phi - r_{y2} \\ \rho_3 = x + s_{x3} \cos\phi - s_{y3} \sin\phi - r_{x3} \\ \rho_4 = y + s_{x4} \sin\phi + s_{y4} \cos\phi - r_{y4} \end{cases} \tag{1}$$

The solution for the displacement of the actuated P joint is unique and straightforward to compute. However, each leg usually has two assembly modes. These two assembly modes can be determined using an approach similar to the one given in [6] for the translational parallel manipulator.

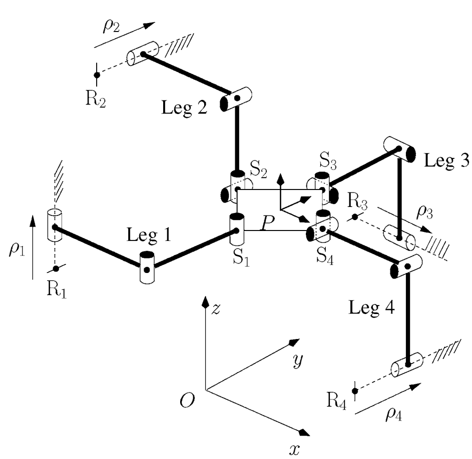


Fig. 3 Quadrupteron—a partially decoupled 1-C̲̋R̋R̋+3-Ć̲ŔŔR̋ parallel manipulator [16]

4 Type II singularity analysis in the Cartesian space

For a spatial parallel manipulator with than less than six-DOF, one may need to derive its velocity equation involving a 6 by 6 matrix [36] in order to deal with the constraint singularity [37]. However, since the Quadrupteron is constraint-singularity free, one may derive its velocity equation by simply differentiating the constraint equation with respect to time [16]. Both approaches lead to the same velocity equation.

Differentiating (1) with respect to time, we obtain the velocity equation of the Quadrupteron

$$\begin{bmatrix} \dot\rho_1 \\ \dot\rho_2 \\ \dot\rho_3 \\ \dot\rho_4 \end{bmatrix} = \mathbf{J} \begin{bmatrix} \dot x \\ \dot y \\ \dot z \\ \dot\phi \end{bmatrix} \tag{2}$$

where the Jacobian matrix is

$$[\mathbf{J}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & (s_{x2}\cos\phi - s_{y2}\sin\phi) \\ 1 & 0 & 0 & (-s_{x3}\sin\phi - s_{y3}\cos\phi) \\ 0 & 1 & 0 & (s_{x4}\cos\phi - s_{y4}\sin\phi) \end{bmatrix} \tag{3}$$

The Type II singularity occurs when matrix $\mathbf{J}$ is singular, i.e.,

$$\det(\mathbf{J}) = 0 \tag{4}$$

Equation (4) leads to

$$-C\cos\phi + D\sin\phi = 0. \tag{5}$$

$$C = s_{x2} - s_{x4}$$

$$D = s_{y2} - s_{y4}$$

It is noted that in the Cartesian space, the singular locus is only related to the orientation angle, ϕ, of the moving platform.

Let us define α as

$$\begin{cases} \sin\alpha = C/S_{12} \\ \cos\alpha = D/S_{12} \end{cases}$$

where

$$S_{12} = (C^2 + D^2)^{1/2} \tag{6}$$

Equation (5) can be rewritten as

$$\sin(\phi - \alpha) = 0 \quad (7)$$

Equation (7) leads to two solutions, α and $\alpha + \pi$, for ϕ. These two singular orientations divide the four-dimensional workspace of a 3T1R parallel manipulator into two singularity-free regions:

(a) Region I ($\alpha < \phi < \alpha + \pi$) in which

$$\sin(\phi - \alpha) > 0 \quad (8)$$

(b) Region II ($\alpha + \pi < \phi < \alpha + 2\pi$) in which

$$\sin(\phi - \alpha) < 0 \quad (9)$$

As long as the orientation of the moving platform is kept away from the two singular orientations ($\phi = \alpha$ and $\phi = \alpha + \pi$), no Type II singularities will be encountered.

5 Forward displacement analysis

The FDA can be performed by solving (1) for variables x, y, z and ϕ.

First, we obtain z by solving the first equation in (1) as

$$z = \rho_1 + r_{z1} - s_{z1} \quad (10)$$

Subtracting the fourth equation from the second equation of (1), we obtain

$$C \sin\phi + D \cos\phi + E = 0 \quad (11)$$

where

$$E = -r_{y2} + r_{y4} - \rho_2 + \rho_4 \quad (12)$$

Equation (11) can be rewritten as

$$\cos(\phi - \alpha) + E/S_{12} = 0 \quad (13)$$

Solving (13), we obtain

(a) two solutions for ϕ if $|E/S_{12}| < 1$

$$\begin{cases} \cos(\phi - \alpha) = -E/S_{12} \\ \sin(\phi - \alpha) = (1 - \cos^2(\phi - \alpha))^{1/2} \end{cases} \quad (14)$$

$$\begin{cases} \cos(\phi - \alpha) = -E/S_{12} \\ \sin(\phi - \alpha) = -(1 - \cos^2(\phi - \alpha))^{1/2} \end{cases} \quad (15)$$

(b) one solution for ϕ if $|E/S_{12}| = 1$

$$\cos(\phi - \alpha) = -E/S_{12} \quad (16)$$

or (c) no real solution for ϕ if $|E/S_{12}| > 1$

For each value of ϕ, one solution for y and x can be obtained, respectively, using the second and the third equations of (1) as

$$y = \rho_2 - s_{x2} \sin\phi - s_{y2} \cos\phi + r_{y2} \quad (17)$$

$$x = \rho_3 - s_{x3} \cos\phi + s_{y3} \sin\phi + r_{x3}. \quad (18)$$

The above analysis shows that there are two sets of poses of the moving platform for a given set of inputs in a non-singular (regular) configuration and that the FDA is easily solved in closed form.

It should be pointed out that for a set of Cartesian coordinates (vector **p** and angle ϕ) of the moving platform obtained using (10)–(18) associated with a set of valid inputs, there usually exist two assembly modes for each leg.

6 Distribution of the solutions to the FDA into the singularity-free regions

In Sect. 4, it has been shown that the workspace of the Quadrupteron is divided into two singularity-free regions (see (8) and (9)). Moreover, it has been shown in Sect. 5 that the number of the solutions to the FDA of the Quadrupteron in a non-singular configuration within its workspace is two (see (14) and (15)). That is to say that the number of singularity-free regions is equal to the number of solutions to its FDA for the Quadrupteron.

From (8), (9), (14) and (15), we can conclude that there exists a one-to-one correspondence—shown in Table 1—between the two formulas, each producing one solution to the FDA, and the two singularity-free regions of the Quadrupteron in non-singular configurations. In other words, singularities correspond exactly with the change of branches of the solutions of the FDA.

Once the singularity-free region in which the Quadrupteron works is given, a unique solution to the FDA can be computed directly according to the corresponding formula listed in Table 1. There is no need to calculate both solutions to the FDA first and then chose the appropriate one from these two solutions. This further simplifies the FDA of the Quadrupteron.

Table 1 Distribution of the solutions to FDA into the singularity-free regions of the Quadrupteron

Singularity-free region	Solution to FDA
I	Equations (14), (17), (18) and (10)
II	Equations (15), (17), (18) and (10)

Table 2 Geometric parameters (in mm) of the $1\text{-}\underline{\hat{P}}\hat{R}\hat{R}\hat{R} + 3\text{-}\underline{\hat{P}}\hat{R}\hat{R}\hat{R}\hat{R}$ parallel manipulator

i	$\mathbf{r}_i$	$\mathbf{s}_i^P$	l_{2i}	l_{3i}
1	$[-180, -225, 0]^T$	$[0, -40, 0]^T$	203	223
2	$[-180, 0, 135]^T$	$[-40, 0, 0]^T$	154	161
3	$[0, 140, -135]^T$	$[0, 40, 0]^T$	167	167
4	$[180, 0, 135]^T$	$[40, 0, 0]^T$	154	161

7 Numerical example

A numerical example is given below to verify the results obtained in the previous sections.

The geometric parameters of the Quadrupteron are given in Table 2.

The two singularity-free regions of this Quadrupteron are: (a) Singularity-free region I ($-90° < \phi < 90°$) and (b) Singularity-free region II ($90° < \phi < 270°$).

The Quadrupteron, as shown in Fig. 1, always works in the singularity-free region I. Therefore, the unique current solution to its FDA can be obtained directly using the associated equations listed in Table 1 ((14), (17), (18) and (10)). If the inputs of the manipulator are $\rho_1 = 0$, $\rho_2 = 0$, $\rho_3 = 0$ and $\rho_4 = 0$, the solution to the FDA we obtained is $x = 0$, $y = 0$, $z = 0$ and $\phi = 0°$. If the inputs of the manipulator are $\rho_1 = 10$, $\rho_2 = 20$, $\rho_3 = 30$ and $\rho_4 = 40$, the solution to the FDA we obtained is $x = 40$, $y = 30$, $z = 10$ and $\phi = 14.48°$.

8 Type II singularity analysis in the joint space

In order to guarantee that the Quadrupteron works in a singularity-free region in its Cartesian workspace, we will discuss the Type II singularity analysis in the joint space in this section.

When singularity happens, (7) is satisfied. We obtain

$$\cos(\phi - \alpha) = \pm 1 \tag{19}$$

The substitution of (13) into the above equation yields

$$-E/S_{12} = \pm 1 \tag{20}$$

i.e.,

$$-r_{y2} + r_{y4} - \rho_2 + \rho_4 \pm S_{12} = 0 \tag{21}$$

From (21), we learn that the singular locus of the Quadrupteron in the joint space, is composed of two straight lines in the ρ_2–ρ_4 plane. The inputs must lie between these two lines, which can be expressed as

$$\begin{aligned} &\rho_2 + r_{y2} - r_{y4} - S_{12} \\ &\quad < \rho_4 < \rho_2 + r_{y2} - r_{y4} + S_{12} \end{aligned} \tag{22}$$

Otherwise, there is no solution to its FDA.

It is also noted that (a) there is no singularity inside the joint space between the above two straight lines, and (b) the singularity locus, $-r_{y2} + r_{y4} - \rho_2 + \rho_4 + S_{12} = 0$ and $-r_{y2} + r_{y4} - \rho_2 + \rho_4 - S_{12} = 0$, in the joint space correspond, respectively, with the singularity locus, $\phi = \alpha$ and $\phi = \alpha + \pi$. Therefore, we can guarantee that the Quadrupteron works in the same singularity-free region in the Cartesian workspace by letting the inputs satisfy (22).

For the Quadrupteron shown in Fig. 1, we have $S_{12} = 80$, $-r_{y2} + r_{y4} = 0$. Equation (21) becomes (Fig. 4)

$$-\rho_2 + \rho_4 \pm 80 = 0 \tag{23}$$

In order to guarantee that the manipulator works in singularity-free region I, the inputs must satisfy

$$\rho_2 - 80 < \rho_4 < \rho_2 + 80 \tag{24}$$

In order to avoid the interference between the machine elements of top joints of legs 2 and 4, the inputs must satisfy

$$\rho_2 - (80 - \Delta) < \rho_4 < \rho_2 + (80 - \Delta) \tag{25}$$

where Δ is a small positive number.

Considering the actual strokes of the actuators, which are 220 mm for both actuators in legs 2 and 4, the inputs falls into the part of the square in dashed-line that lies between the lines $-\rho_2 + \rho_4 + (80 - \Delta) = 0$, $-\rho_2 + \rho_4 - (80 - \Delta) = 0$ (Fig. 4).

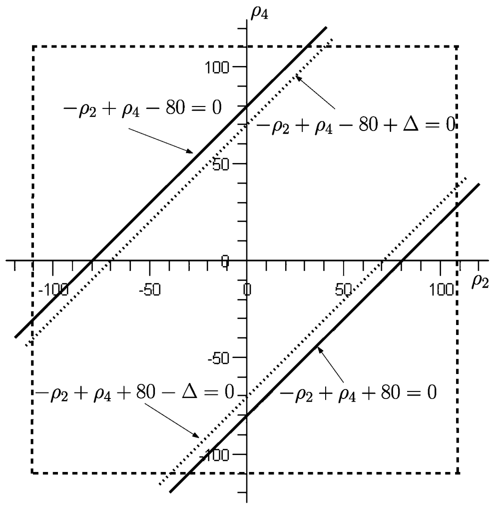


Fig. 4 Type II singularity locus in the joint space

9 Conclusion

It has been proved that there exists a one-to-one correspondence between the two formulas, each producing one solution to the FDA, and the two singularity-free regions for the Quadrupteron. Once the singularity-free region in which the parallel manipulator works is given, one can obtain a unique solution to its FDA using the associated formula. This simplifies the FDA of the Quadrupteron. Unlike most of parallel manipulators, there is no need to calculate multiple solutions to the FDA first and then select the appropriate one, if algebraic closed-form solutions are used.

The Type II singularity analysis in the joint space has been investigated and the conditions on the inputs to keep the Quadrupteron working in the same singularity-free region in its Cartesian workspace have been identified. This is useful to the control of the Quadrupteron.

The approach presented in this paper can be readily extended to the investigation on the general 4-P̲ŔŔŔ̋ 3T1R parallel manipulators.

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