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Cerruti's treatment of linear elastic trusses

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Abstract This paper presents the analysis of linear elastic trusses proposed by Valentino Cerruti in his graduation thesis. While Cerruti is famous among rational mechanicians, very little is known on this work of his. We will consider the work in some detail, putting into evidence the main subjects dealt with by the author: redundant trusses and uniform resistance. We will also provide a comparison with his contemporaries and some critical comments.

Keywords Trusses · Linear elasticity · Uniform resistance

1 Introduction

In 1873, two very interesting graduation theses were discussed in the Regia scuola d'applicazione per ingegneri (Royal technical school of applications for engineers) in Torino, one by Valentino Cerruti [1], the other by Carlo Alberto Castigliano [2]. Both dealt with linear elastic redundant trusses, a subject of great importance and up-to-date at the time, since the fast development of Europe and of recently unified Italy (1861–1870) saw the realization of huge structures in both civil and industrial fields (bridges, roofs, arches, cranes, decks, elevators, ...), of which trusses were important elements.

Cerruti strictly dealt with the subject of trusses composed of elements undergoing only tension or compression; on the other hand, Castigliano studied frames including also elements undergoing flexure and torsion. Their works got the same marks by the committee, yet Cerruti was given the first position among his colleagues, Castigliano the second, due to their preceding career. However, in the history of mechanics of structures Castigliano's work is given much importance and Cerruti's thesis has almost been forgotten.

To frame these two contributions in the development of mechanics of structures, we give some short hints on the matter. The early development of mechanics of structures in Europe has been dealt with in detail in some treatises [4–6], and we may roughly say that it ranged from 1820 to 1890, in two phases. The first phase was characterized by the original formulation of structural models for elastic beams, plates, and shells. Simple redundant structures were solved, without a precise and rigourous technique but rather by means of *ad hoc* procedures. The place of birth of these studies was France and the main founder was almost for sure Navier [7]. Almost contemporarily, Cauchy was one of the founders of mechanics of continua.

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The second phase, which we may say started in the 1840's, was characterized especially by works on frames and trusses; Germany emerged and we see a kind of dependance of England on the studies made in the Continent. A detailed and commented presentation of this aspect is found in the Introduction of [5].

During this phase, we may see two sub-phases: on one hand, attention was focused on the design of simple, statically determined, trusses. Efficient analytical and graphical techniques were developed to solve the kinematical and statical problems for these structures. We may quote the contributions by Möbius,[1] Rankine,[2] Clerk Maxwell [10–16], Fleeming Jenkin,[3] Culmann,[4] Cremona,[5] Lévy [8, 9], Williot,[6] Ritter.[7]

In the second sub-phase we see the search for the methods of solution for redundant systems, especially trusses. The method of displacements (stiffness approach) was anticipated by Navier [7] and developed by Poisson [17] and Clebsch [18, 19]. It is simple and undergoes implementation by algebraic calculations, since it requires the solution of a system of linear equations in the displacement components of all nodes. Trusses with more than a dozen nodes, however, required a huge amount of operations. These were practically impossible to perform at that time, when no calculating machines were at ease, so the method was soon abandoned. The method of forces (flexibility approach), also anticipated by Navier [7], saw three different developments. Menabrea [20] proposed a method based on the search for a minimum of what we now call elastic complementary energy with respect to redundant forces. Mohr [24] used the virtual work principle to calculate displacement components. Lévy [8] proposed an original technique to obtain global compatibility equations for the solution of statically indeterminate systems without the need to choose redundant actions.

Cerruti and Castigliano's theses find thus proper place in a time when there was still no universally accepted procedure for the problems of mechanics of structures. Cerruti tried to propose a general technique of solution for trusses, but was not able to define it properly and could not provide a precise algorithm of calculation. On the contrary, Castigliano found some results, perfected later [3] and now called after his name, that provided a means of evaluating redundant actions not only in trusses (strutures with elements subjected to tension and compression only) but in general frames.

Little is commonly known on the history of mechanics of structures in Italy. The main focus has been on Menabrea and Castigliano and on their dispute over the priority of the theorem of least elastic work,[8] in which Cerruti was also involved.[9]

In this paper we will consider Cerruti's thesis. We will give hints on the educational background of the technical school for engineers in Torino and some biographical notes. We will then expound and comment the graduation thesis trying to put into evidence the points of originality and the possible drawbacks. Indeed, Cerruti's treatment is elegant and written with intelligence and mastery of mathematical tools. On the

[1] Möbius AF, Lehrbuch der Statik, Göschen, Leipzig (1837). Möbius examines plane and spatial trusses, finds the necessary condition between the number of nodes and beams so that the truss is simple and shows that it is not sufficient when the matrix of constraints is singular. His work, however, appears almost unknown among the engineers of his time.

[2] Rankine WJM, A manual of applied mechanics, Griffin, London (1858); Principle of the equilibrium of polyhedral frames, Philosophical magazine s. 4, 27, 92 (1864).

[3] Fleeming Jenkin HC, On the practical application of reciprocal figures to the calculation of strains on framework, Transactions of the Royal Society of Edinburgh, 25, 441 ff. (1869).

[4] Culmann C, Die graphische Statik, Meyer & Zeller, Zürich (1866). Culmann is considered the first to have rationally placed statical graphics into the frame of projective geometry and to have developed a comprehensive treatise of how to apply graphical calculus to the various branches of engineering.

[5] Cremona L, Le figure reciproche nella statica grafica, Hoepli, Milano (1872). Here Cremona not only encompasses Culmann's statical graphics, but gives it a renewed geometrical rigour by means of the theory of reciprocal figures.

[6] Williot VJ, Notions pratiques sur la statique graphique, Lacroix, Paris (1878). Here a graphical technique for evaluating the displacements of the nodes of a truss is proposed.

[7] Ritter KW, Anwendungen der graphischen Statik nach Professor Dr. C. Culmann, Meyer & Zeller, Zürich (1888–1900). Here Ritter, pupil of Culmann, expounds the theories of graphical statics by his master and divulgates them.

[8] Benvenuto E, A brief outline of the scientific debate that preceded the works of Castigliano, Meccanica, 19-supplement, 19–32 (1984). Also in: Castigliano A, Selecta 1984, Levrotto e Bella, Torino (1984); Nascé V, Alberto Castigliano, railway engineer: his life and times, Meccanica, 19-supplement, 5–14 (1981). Also in: Castigliano A, Selecta 1984, cit.; Charlton TM, Least work theory according to Menabrea, Castigliano, and Fränkel, The structural engineer, 62A, 345–347 (1984); [27].

[9] Cerruti V, Sopra un teorema del Sig. Menabrea, Atti della Reale Accademia dei Lincei, s. 2, 2, 570–581 (1875); reprinted with the same title, Salviucci, Roma (1875).

other hand, some of the procedures he proposes are not original. Moreover, the presentation of the subject is not always precise and uniform (sometimes with conceptual jumps from one of the main subjects to the other) and a series of misprints are quite apparent. It looks like Cerruti did not spend too much time on a very precise preparation of this work, which seems reasonable since from his biography it turns out that at that time he was deeply busy with other research and activities.

2 Cerruti and the technical school in Torino

Valentino Cerruti was born in Crocemosso di Biella, near Torino, in 1850 and died there in 1909. When he was still a student, he published works on analytical geometry in the *Giornale di matematiche di Battaglini*.[10] He attended the Regia scuola di applicazione per ingegneri in Torino, one of the technical high schools established in the new born Kingdom of Italy after 1860 by the so-called "legge Casati" (Casati law).

The technical school in Torino was established in 1860, replacing an older institution of the Kingdom of Sardinia. It had among its founders Prospero Richelmy[11] and Quintino Sella.[12] In the teaching staff, we find Giovanni Curioni, of whom both Cerruti and Castigliano were pupils. Curioni deeply influenced the teaching of the subject in Italy by means of his massive handbook *L'arte di fabbricare*.[13] Curioni changed the name of his discipline into "Scienza delle costruzioni", which remained unchanged until nowadays, and promoted the passage from a technical to a scientific education for engineering students. As far as structures are concerned, Menabrea's influence[14] is apparent and the studies and discussions on redundant systems were for sure usual among teachers and students.

After his graduation, Cerruti moved to Roma, where he became private teacher of the children of Quintino Sella and became close friend with him. In 1873 he was appointed assistant professor in hydraulics at the technical school of applications for engineers in Roma. He was appointed professor of rational mechanics in 1877, full professor in 1881. In 1888 he became rector of the University of Roma, then dean of the Faculty of Sciences in 1892 and rector of the University again from 1900 to 1903. In 1901 he was elected at the Senate of the Kingdom of Italy and promoted the law which transformed two different technical schools in Torino into a polytechnical school, still operating under the name of Politecnico di Torino. In 1903 he became director of the school of applications for engineers in Roma.

Even if his education was in the field of engineering, the scientific work of Valentino Cerruti is mainly devoted to rational mechanics. This of course emphasizes his strong mathematical background, which should have been a main feature of the school in Torino, taking consideration also of the contributions by Castigliano. Leaving the details aside,[15] Cerruti's main contributions are two fundamental papers. The first[16] is on the extension of Betti's reciprocal theorem from statics to dynamics: Cerruti managed to find particular integrals of the equations of motion with characteristic singularities in space and time and provided

[10] This journal, founded in 1863 by the Italian mathematician Giuseppe Battaglini, was intended to spread mathematical knowledge and up-to-date research in new branches of mathematics in Italian universities, as its complete title says: *Giornale di matematiche ad uso degli studenti delle Università italiane, pubblicato per cura del professore G. Battaglini*.

[11] A very well known professional engineeer, Richelmy was also the first president of the school.

[12] Quintino Sella (1827–1884), mineralogist, professional engineer and important politician in unified Italy, studied in Torino and perfected in various European countries. He was professor of mineralogy, member of the Italian Parliament since 1860 and minister of finances in 1862.

[13] Curioni G, L'arte di fabbricare ossia corso completo di istituzioni pratiche per ingegneri, Negro, Torino (1864–1884). It consists of six volumes and five appendixes and had at least three different editions.

[14] General Luigi Federico Menabrea taught "Costruzioni" (strength of materials and mechanics of structures) at the University of Torino before the institution of the Royal technical school for engineers. He taught until 1865, when he devoted himself to political and military activities.

[15] More detailed biographies may be found in: Levi-Civita T, Commemorazione del Socio Valentino Cerruti, Rendiconti della Reale Accademia dei Lincei, s. 5, 18, 565–575 (1909); Silla L, Valentino Cerruti. Commemorazione letta alla Società Italiana di Fisica (sezione di Roma) nella seduta dell'8 gennaio, Il Nuovo Cimento, 19-1, 5–19 (1910); Lauricella G, Commemorazione di Valentino Cerruti, Giornale di matematiche di Battaglini, 50, 329–336 (1912).

[16] Cerruti V, Sulle vibrazioni dei corpi elastici isotropi, Memorie della Reale Accademia dei Lincei, s. 3, 8, 361–389 (1880).

resolving expressions. The second[17] solves the problem of finding the stress state in a half-space loaded by a tangential force, helping so in founding, together with Boussinesq, geomechanics.

3 Cerruti's "Sistemi elastici articolati"

In the following, we will summarize Cerruti's thesis, expounding the main points of interest.

After a short introduction in §2, Cerruti describes articulated systems as structures composed of bodies (understood to be rectilinear, even if this statement is never declared) connected by frictionless spherical hinges (nodes). Cerruti says that his results hold also for cylindrical hinges (i.e., plane trusses) and admits that the absence of friction is an idealization:

> I will suppose that at the contact surface of different elements friction does not exist, or is at least negligible: if this happens, the elements will bear only longitudinal stress. I must at once add, that frictionless articulated systems do not actually exist and are merely abstractions. In spite of this, their study leads to applications, at least in those cases in which the effects of bending may be neglected.[18]

Cerruti distinguishes among "simple" and "complex" articulated systems. The former are those in which nodes connect only two elements and, he says, reduce to chains and may easily be treated by means of graphical methods like that of funicular polygon. The latter are those in which nodes connect more than two elements.

3.1 The number of equations and constraints

In §3, Cerruti defines a truss as a system of points in space connected by rectilinear bars ("aste"). This view, focusing on the nodes seen as body-points rather than on bars, is the same as that of Menabrea [20–23], who for sure left his mark in Torino. Cerruti specifies the minimum condition number on the total $n(n-1)/2$ "distances" between the couples of points of the system to have a unique well-defined shape, resulting as $3n-6$. These conditions, as well as those on some terminal nodes (called vertexes) to fix the configuration may easily be interpreted, in contemporary language, as a definition of a statically determined truss.

The external constraints are the conditions imposed on the vertexes and must be at least in number of six to have a well constrained system. Their equations are given by Cerruti in terms of the coordinates $x, y, z; \xi, \eta, \zeta; \ldots$ of the points on which they act:[19]

$$f_1 = 0, \qquad f_2 = 0, \qquad f_3 = 0, \qquad \ldots \qquad f_6 = 0 \quad (1)$$

Constraint reactions are seen as the Lagrange multipliers of the first variation of condition equations. Indeed, if Π_i are proportionality constants, Cerruti states that the condition equations $f_i = 0$ imply the presence of the system of forces[20]

$$\begin{array}{lll} \Pi_i \dfrac{df_i}{dx}, & \Pi_i \dfrac{df_i}{dy}, & \Pi_i \dfrac{df_i}{dz}, \\ \Pi_i \dfrac{df_i}{d\xi}, & \Pi_i \dfrac{df_i}{d\eta}, & \Pi_i \dfrac{df_i}{d\zeta}, \\ \ldots & \ldots & \ldots \end{array} \quad (2)$$

applied to the constrained points.

Internal constraints are given by the presence of bars which impose a condition on the distance between couples of points i, j. They have the form:[21]

$$F_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - l_{ij}^2 = 0 \quad (3)$$

and forces in bars are again seen as constraint reactions:

> The conditions compelling the system because of bars connecting nodes are also expressed by equations: [...] in the same way as (1) imply the forces represented by the expressions (2), so (3) will imply other forces, that will be nothing else

[17] Cerruti V, Ricerche intorno all'equilibrio dei corpi elastici isotropi, Memorie della Reale Accademia dei Lincei, s. 3, 13, 81–122 (1882).

[18] Supporrò, che nella superficie di mutuo contatto dei diversi pezzi attrito non si sviluppi, o almeno sia trascurabile: se questo avviene, essi non sopporteranno che sforzi diretti nel senso della loro lunghezza. Ma debbo tosto soggiungere, che i sistemi articolati (senz'attrito) non esistono in realtà: esse sono mere astrazioni; contuttociò la loro teoria non è scevra di applicazioni pratiche, in tutti quei casi almeno, in cui gli effetti della flessione possono trascurarsi. Reference [1], p. 6.

[19] Reference [1], (1), p. 7. Cerruti's equation numbering is in square brackets, but here in order to avoid confusion with references we will enclose them in parentheses.

[20] Reference [1], (2), p. 8.

[21] Reference [1], (3), p. 8.

> than tensions in bars: among these forces there is [...] the fundamental difference between external and internal forces, in that if a force A is given by (3), then also $-A$ exists.[22]

The static problem of finding internal and external constraint reactions is not openly stated by Cerruti. Yet, he presents the correct number of balance equations and of unknown inner forces and displacement of nodes. However, he does not use a proper terminology that can avoid ambiguities.

3.1.1 Evaluation of external constraint reactions

Cerruti examines the case when one or more nodes are fixed, so that unknown constraint reactions are present, and states the conditions for which statics of rigid bodies is sufficient to determine them. However, he does not consider the actual evaluation of constraint reactions, consequently no balance equations are written.

When there is only one fixed node, he says, the solution is always possible and unique. From an intuitive point of view, we may say that it is obvious. If there is a fixed point in a system and if the external active force is applied to it, the system remains in equilibrium. Cerruti, however, is not clear in these passages.

If the fixed nodes are two or three, Cerruti states that the applied forces must reduce either to a resultant force or to a resultant couple. This is a severe restriction, since in general a system of forces in a three-dimensional space can be reduced to both a force **f** and a couple **m** lying in a plane orthogonal to **f**.

Cerruti states that this requirement coincides with the vanishing of the trinomial invariant characteristic of the system of active forces:[23]

$$\sum X.\sum M_x + \sum Y.\sum M_y + \sum Z.\sum M_z = 0 \tag{4}$$

where X, Y, Z are the Cartesian components of the active forces and M_x, M_y, M_z are the Cartesian components of the active moments. The condition (4) is well known in statics.[24] Asking the system to reduce either to a force or to a couple equals to write (4).

However, the condition expressed by (4) is only necessary for the solution of the static problem by the ordinary laws of statics. Even if this is not stated by Cerruti, it might be inferred from the fact that he immediately provides sufficient conditions in the case of two and three fixed nodes: the resultant must be orthogonal either to the line joining the nodes, for two fixed nodes, or to the plane defined by the nodes, for three different nodes. Even if Cerruti does not explain why these are sufficient conditions, we may infer that it stems on the possibility of a unique decomposition of a force along two or three given directions. Note that Cerruti's reasoning is consistent only if the fixed nodes are interpreted as spherical hinges; however, he is ambiguous in this point.

3.2 A statically determinate system

In §4 Cerruti examines an example, shown in Fig. 1 with some slight changes with respect to the original. It is a statically determinate truss with a recursive sequence of both bars and loads. Cerruti thus finds, by means of balance equations, recursive formulæ[25] for constraint reactions and inner forces in all the elements of the truss. These formulæ are for sure compact and elegant from a mathematical point of view, but also have practical utility, and they will be a basic tool for Cerruti in order to solve redundant trusses.

Cerruti is able to present a complete and exhaustive description of the distribution of inner forces in bars: where they attain extrema, where they vanish, where they change sign so that some bars are in tension and others in compression, and which are the geometrical parameters influencing this change.[26]

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[22] Anche i legami da cui il sistema è astretto per causa delle aste, che ne collegano i vertici, si possono esprimere mediante equazioni: [...] come dalle equazioni (1) derivano le forze rappresentate dalle espressioni (2), così dalle equazioni (3) deriveranno altre forze, le quali non saranno altro, che le tensioni delle aste: ma tra queste due specie di forze corre [...] la differenza che intercede tra le forze esterne e le forze interne: epperò tra le forze provenienti dalle equazioni (3), se ve ne è una A, ve ne esiste ancora un'altra $-A$. Reference [1], p. 8.

[23] Reference [1], (4), p. 10.

[24] In general the resultant **f** and the resultant couple **m** of a system of forces are not orthogonal, except for the case in which forces are all contained in a plane. Hence, (4) implies the system to reduce either to **f** or to **m**. In the plane case, it is well known that forces reduce either to a force or to a couple, and the statement holds trivially true.

[25] Reference [1], (8)–(15), pp. 14–15. There are actually two equations (8) in [1], one on p. 12, the other on p. 14. This is an example of the apparent misprints in Cerruti's thesis, which we already talked of.

[26] Reference [1], (8)–(15), pp. 14–18.

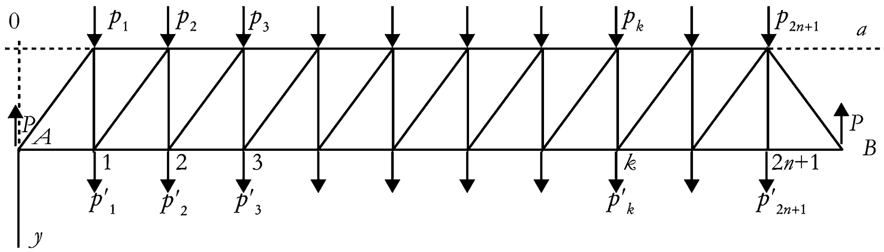

Fig. 1 A pattern truss according to Cerruti

In the last part of §4, Cerruti shows how, for the truss shown in Fig. 1 (designed so to have uniform resistance), it is easy to find the displacement of all nodes. The results[27] are obtained by elementary geometry and the linear elastic law of extension for the bars connecting each pair of nodes. Calculations are simplified since the design of uniform resistance lets the strain of the bars be the same, and Cerruti obtains new recursive and elegant formulæ, different from those obtained by Maxwell in [10–16], still showing his skills in what we nowadays would call "automatic implementation".

3.3 Redundant and uniform resistance trusses

At the end of his §4, Cerruti begins dealing with the core of his treatment, i.e. redundant trusses and the possibility to design them in order to have uniform resistance. These topics seem to be the ones to which Cerruti at first sight devotes his study. In particular, the study of structures of uniform resistance appears to be a pressing problem of engineering practice for him:

> The formulæ found here would provide [...] the shape of the deformed truss, [...] the variation of angles and the work spent by external forces during the deformation. But I will quit this subject and will discuss [...] the distribution of tensions and pressures in those cases where statics of rigid bodies throws us into indeterminacy and it is necessary to adopt the laws of elasticity.[28]

[27] Reference [1], (16)–(20), pp. 9–22.

[28] Le formole ora trovate ci fornirebbero [...] la figura della trave deformata, [...] la variazione degli angoli, ed il lavoro sviluppato nella deformazione dalle forze esterne. Ma io lascierò qui tale argomento e passerò a discorrere [...] della distribuzione delle tensioni e delle pressioni, nei casi, nei quali, la statica dei corpi rigidi gettandoci nell'indeterminazione, è mestieri aver ricorso alle leggi dell'elasticità. Reference [1], p. 22.

The treatment of these subjects will continue until §11. However, since there is not a unique thread in his work, we prefer to distinguish between the two subjects and to deal with them in the following Sects. 4 and 5, where we will not follow the order in which the subjects are exposed by Cerruti.

3.4 Open questions

The last two sections of Cerruti's thesis do not seem to have a direct link with articulated systems and trusses, but rather represent an outlook of Cerruti's interests in the open questions of rational mechanics, which would constitute the main field of his future research.

He so begins §12:

> One could ask which is the reason why the problem of the distribution of tensions and pressures in an articulated system can be solved by the preceding methods in a quite quick way and with all the rigour of the mathematical theory of elasticity, while many other problems remain unsolved because of the amount of difficulties they present. This is due to the fact, that in the considered case the laws of the displacement of the points of the system are known, and indeed this is the general problem of the theory of elasticity: "Provided the forces acting on a body, find the displacements parallel to three axes, which make any of its molecules undergo". When these displacements are known, as it was said in section 8, it will be very easy to find the expression of the elastic forces originated at each of its points. Yet all difficulties actually lie in finding the law of these displacements. The nature of systems sometimes indicates a priori this law: an example is in elastic articulated systems: in these cases nothing else remains, than to find their

magnitude, once external forces are given. However, hypotheses will never lead us to the true knowledge of displacements; and indeed, when a certain kind of deformation appears likely, it is easy to verify if it be possible or not; it is enough to prove, if by such an hypothesis the equations for internal equilibrium are verified. I state this, because the theories on the strength of materials, as usually exposed, rely on a particular hypothesis about the law of displacements; and this hypothesis is almost never verified, as I will now try to prove.[29]

Cerruti considers a linear elastic homogeneous cylindrical body and writes the balance, compatibility, and constitutive equations in terms of displacement components:[30]

$$
\begin{aligned}
&(\lambda+\mu)\frac{d\theta}{dx}+\mu\Delta^2 u=0,\\
&(\lambda+\mu)\frac{d\theta}{dy}+\mu\Delta^2 v=0,\\
&(\lambda+\mu)\frac{d\theta}{dz}+\mu\Delta^2 w=0,
\end{aligned}
\tag{5}
$$

with λ, μ the Lamé elastic moduli for the material, θ the volumetric strain and Δ the Laplace's operator.

Then Cerruti remarks that in standard theories of strength of materials the small displacements are supposed to consist of a translation and a rigid rotation, generally varying from place to place in the body:[31]

$$
\begin{aligned}
&u=a+qz-ry,\\
&v=b+rx-pz,\\
&w=c+py-qx
\end{aligned}
\tag{6}
$$

where the components a, b, c of the infinitesimal translation and p, q, r of the infinitesimal rotation are actually fields defined on the cylinder.

By inserting (6) into (5), Cerruti obtains a set of equations[32] which

> [...] cannot hold in other case than when a, b, c are linear functions of z and p, q, r are uniform; a very special case, that for sure does not embrace all those occurring in practice. But [...] I will not consider further this question. It might perhaps be useful, at least in showing once more, that no remarkable perfections and advantages to the theories of the strength of materials will be brought without getting free from many (unjustified) hypotheses from which they start.[33]

In his §13, Cerruti shows that the methods of solution of redundant problems he presented in his §5 are not limited to the mechanics of structures but of more general application. He decides to show one of these applications, i.e., the well-known problem of the "pressures" (i.e., constraint reactions) of the support points of a heavy body over a plane. This problem had among its solvers Euler, Cournot, and Menabrea.[34] Cerruti replicates Euler's treatment with elegance and precision. However, he forgets to quote Cayley [25,

[29] Si potrebbe dimandare quale sia il motivo pel quale coi metodi precedenti il problema della distribuzione delle tensioni e delle pressioni in un sistema elastico articolato si sia potuto risolvere ed in modo abbastanza spedito con tutto il rigore della teoria matematica dell'elasticità, mentre tanti altri problemi rimangono ancora insoluti per le troppe difficoltà che presentano. Ciò è dovuto al fatto, che nel caso ora considerato sono conosciute le leggi degli spostamenti dei diversi punti del sistema; imperocchè questo è il problema generale della teoria dell'elasticità: "Date le forze, che sollecitano un corpo, trovare gli spostamenti paralleli a tre assi, che fanno subire ad una molecola qualunque di essa. Quando tali spostamenti sieno conosciuti, come si è accennato al n^o 8, sarà facilissimo trovare l'espressione delle forze elastiche provocate in ogni suo punto". Ma tutta la difficoltà versa appunto nel trovare la legge di questi spostamenti. La natura dei sistemi qualche volta indica a priori quale sia questa legge: un esempio l'abbiamo nei sistemi elastici articolati: in questi casi altro più non resta a fare, che a trovare la loro grandezza, conoscendo le forze estrinseche. Ma non saranno mai le ipotesi che ci guideranno alla verace conoscenza degli spostamenti; d'altronde quando sembra probabile un certo modo di deformazione, è agevole verificare se esso sia o no possibile; basta provare, se con una tale supposizione le equazioni dell'equilibrio interno restino soddisfatte. Io dico questo, perchè le teorie sulla resistenza dei materiali, come sono ordinariamente esposte, riposano su una ipotesi particolare intorno alla legge degli spostamenti; ipotesi che non è quasi mai verificata, come ora procurerò di dimostrare. [1], pp. 50–51.

[30] Reference [1], (66), p. 52.

[31] Reference [1], (67), p. 52.

[32] Reference [1], (68), p. 53.

[33] [...le quali] non possono sussistere altro che nel caso in cui a, b, c sieno funzioni lineari di z e p, q, r costanti; caso particolarissimo, e che non abbraccia certamente tutti quelli che si presentano in pratica. Ma [...] non mi fermerò ulteriormente su tal quistione. Essa può forse avere la sua utilità, quella almeno di far vedere una volta di più che non si arrecheranno mai perfezionamenti notevoli e di qualche vantaggio alle teorie della resistenza dei materiali senza svincolarsi da molte delle ipotesi (gratuite) da cui esse partono. Reference [1], p. 53.

[34] See, among others, [27].

26], who made a treatment very similar to Cerruti's.[35] Cerruti puts into evidence some remarkable features of such a relationship and, by the ordinary balance of force and moment, easily obtains the same results by Euler ([1], pp. 55–57).

4 Trusses with uniform resistance

Cerruti at the end of his §3 states that a truss will have uniform resistance if[36]

$$\frac{T_{ij}}{E_{ij}\sigma_{ij}} = \text{uniform} = T \tag{7}$$

where T_{ij} is the stress in the bar joining the nodes i, j, E_{ij} is its Young's modulus and σ_{ij} is the area of its cross-section. Remark that equation (7) defines a limit *strain*, hence Cerruti adopts a maximum strain (Navier) criterion. In §3 he limits himself to state that, if the truss is statically determinate, there is a unique distribution of the T_{ij} and hence of the required cross-sections of the bars.

The subject is considered again in §6, after Cerruti has somehow dealt with redundant trusses, and lets the author state some interesting theorems. Indeed, if there are m redundant external constraints and k redundant inner constraints, they must be described by constraint equations similar to (1), (3). By differentiating constraint equations with respect to the coordinates of the nodes and the length of the bars, respectively, Cerruti is able to insert the condition of uniform resistance (7) in both sets of equations.

For the external constraints he obtains[37]

$$\begin{aligned}
&\sum \frac{dF_1}{dl_{ij}} l_{ij} = 0,\\
&\sum \frac{dF_2}{dl_{ij}} l_{ij} = 0,\\
&\dots\\
&\sum \frac{dF_k}{dl_{ij}} l_{ij} = 0,
\end{aligned} \tag{8}$$

> which must be verified so that the problem be compatible: this depends on the functions F and we will also see later that depends on the forces transmitted by the bars. This is not enough: in equations (25)[38] let us replace the variations of the coordinates by their expressions via the elongations of the bars, hence by their forces. Let us introduce the condition of uniform resistance: after having eliminated the six variations still present, m conditions remain, independent of constraint reactions, to be satisfied in order that the problem be compatible: but it will in general be not so, the functions f_i being arbitrary. We conclude, then, that *a linear elastic truss cannot have uniform resistance, if the number of condition equations*[39] *exceeds six*.[40]

This result is very interesting, since, without solving the linear elastic problem for a redundant truss, Cerruti can provide a design suggestion: if the system has redundant external constraints, no structure with uniform resistance can be obtained. Still, the requirement that the simple external constraints do not exceed six is only a necessary condition:

> If the number of these conditions does not exceed six, it is necessary to check if equations (28) [our equations (8)] hold or not. In the latter case we can say that it is impossible to fix the cross-sections of the bars in order to compose a system with uniform resistance: on the contrary, in the former case this will be [...possible] in k infinite ways; indeed, by replacing T_{ij} in balance equations by his value $TE_{ij}\sigma_{ij}$, $3n-6$ equations in the cross-sections of the $3n-6+k$ bars of the system will result: yet choosing arbitrarily k of those cross-sections the above quoted

[35] Reference [1], (72), p. 55.

[36] Reference [1], (7), p. 12.

[37] Reference [1], (28), p. 28.

[38] The differentiated equations of external constraints, p. 27.

[39] That is, simple external constraints.

[40] [...] che devono essere verificate, perchè il problema sia possibile: epperò tale possibilità dipende essenzialmente dalla forma delle funzioni F, vedremo poi che dipende altresì dalla natura degli sforzi subiti dalle diverse aste. Questo non basta: nelle equazioni (25) alle variazioni delle coordinate si sostituiscano le loro espressioni per mezzo degli allungamenti delle diverse aste; ed a questi poi le tensioni corrispondenti. Si introduca quindi la condizione di uniforme resistenza: dopo d'aver eliminato le sei variazioni che vi entrano ancora, rimarranno m condizioni indipendenti dalle reazioni dei vincoli, che dovrebbero essere soddisfatte, perchè il problema fosse possibile: ma non lo saranno generalmente potendo le funzioni f_i essere qualunque. Concludiamo dunque, che *un sistema elastico articolato non si può ridurre ad essere di ugual resistenza, se il numero delle equazioni di condizione sia superiore a sei*. Reference [1], p. 28.

> equations will uniquely provide the remaining $3n-6$, and each of these cross-section may be arbitrarily attributed infinitely different values.[41]

Thus, Cerruti concludes, a truss with n nodes may be designed to be with uniform resistance in a unique way only if it has not more than six simple external constraints and the number of bars does not exceed $3n-6$, i.e., when it is, in modern terms, statically determinate.

When there are k redundant bars and equations (8) hold, there are k different ways to design a truss with uniform resistance, and Cerruti easily proves that[42]

$$\delta l_{ij} = T l_{ij} \tag{9}$$

> which says that *the variations in length of the elements are independent of the way in which the arbitrary k cross-sections are chosen*.[43]

From this result it also easily follows a theorem on the work spent by the stress, clearly influenced by Menabrea's school:

> [...] *the work of external forces, and thus also that of molecular forces* [i.e., inner work] *during the deformation do not depend at all on the way the choice of those k cross-sections was made*.[44]

A very interesting theorem from the point of view of applications follows from the last result; from the expression of inner work L and the condition (7) of uniform resistance, Cerruti obtains:[45]

$$L = \frac{1}{2}\sum \frac{T_{ij}^2}{E_{ij}\sigma_{ij}} l_{ij} = \frac{T}{2}\sum E_{ij}\sigma_{ij} l_{ij}, \qquad \frac{2L}{T} = \sum E_{ij}\sigma_{ij} l_{ij} \tag{10}$$

> Now, L and T do not change with the choice of the k cross-sections, the right hand side [of the last equation of our (10)] shall then remain constant for any of those choices. Then we can say, that in this case *the sum of the products of the volumes of each bar times the relevant coefficient of elasticity is independent of the choice of the k arbitrary cross-sections*. If the coefficient E_{ij} is the same for each bar we may also add that, in any way we make this choice, *the weight of the employed material will always be the same*.[46]

The technological consequences of this result are apparent and any comment is straightforward.

It is interesting to compare this statement with the theorem proposed in 1870 by Clerk Maxwell [14–16], where the weigths of the compressed and extended bars of the truss are balanced. The two statements seem different, yet complementary, since Clerk Maxwell says that

> Theorem. In any system of points in equilibrium in a plane under the action of repulsions and attractions, the sum of the products of each attraction multiplied by the distance of the points between which it acts, is equal to the sum of the products of the repulsions multiplied each by the distance of the points between which it acts. [...]
>
> The importance of this theorem to the engineer arises from the circumstance that the strength of a piece is in general proportional to its section, so that if the strength of each piece is proportional to the stress which it has to bear, its

[41] Se il numero di tali equazioni non è superiore a sei, bisognerà vedere se le equazioni (28) reggano o non. Nel secondo caso potremo dire che è impossibile il determinare le sezioni delle diverse aste così da formare un sistema di egual resistenza: nel primo caso invece [...] questo sarà possibile [...] in un numero di modi k volte infinito; imperocchè nelle equazioni di equilibrio ponendo per T_{ij} il suo valore $TE_{ij}\sigma_{ij}$ ne seguiranno $3n-6$ equazioni tra le aree delle sezioni rette delle $3n-6+k$ aste del sistema: ma intanto scegliendo ad arbitrio k di tali sezioni le prefate equazioni ne daranno sempre in valor determinato per le $3n-6$ altre, e a ciascuna di tali sezioni si possono attribuire arbitrariamente infiniti valori diversi. Reference [1], pp. 28–29.

[42] Reference [1], (29), p. 29.

[43] [...] la quale ci dice, che *le variazioni di lunghezza dei diversi pezzi sono indipendenti dal modo, con cui si scelgono le k sezioni arbitrarie.* Reference [1], p. 29.

[44] [...] il *lavoro delle forze esterne, e quindi anche quello delle forze molecolari nella deformazione non dipendono per nulla dal modo con cui venne fatta la scelta di quelle k sezioni*. Reference [1], p. 29.

[45] Reference [1], (30) and the following unnumbered, p. 30.

[46] Ora L e T non cambiano col variare la scelta delle k sezioni, il secondo membro dovrà dunque rimanere costante, comunque tal scelta venga fatta. Epperò possiamo dire, che in questo caso *la somma dei prodotti dei volumi delle singole sbarre pel rispettivo coefficiente di elasticità è indipendente dal modo con cui venne fatta la scelta delle k sezioni rimaste arbitrarie*. Che se poi il coefficiente E_{ij} è lo stesso per ogni sbarra posiamo ancora soggiungere, che, comunque si faccia tal scelta, *il peso della materia impiegata sarà sempre lo stesso*. Reference [1], p. 30.

> weight will be proportional to the product of the stress multiplied by the length of the piece. Hence these sums of products give an estimate of the total quantity of material which must be used in sustaining tension and pressure respectively.[47]

That is, Clerk Maxwell provides an estimate for the amount of material employed in tension or in compression for the truss, while Cerruti states a theorem of equal quantity. It is apparent, however, that Cerruti did not know Clerk Maxwell's papers, published in a journal not very well spread in Italy and strongly influenced by projective geometry.

Cerruti in §7 is interested in the conditions for which (8) are satisfied. He then examines a particular case, the consequences of which are general enough: a spatial system with five nodes is geometrically determined by means of nine bars, and a tenth is necessarily dependent on the other nine distances. Cerruti gives credit to Cayley[48] for providing the condition to express this statement:[49]

$$C = \det\left(l_{ij}^2\right) = 0, \quad i, j = 0, 1, 2, \ldots, 5 \tag{11}$$

where $l_{00} = 0, l_{i0} = l_{0j} = 1, l_{ii} = 0, l_{ij} = -l_{ji}$. The first variation of equation (11) is the compatibility condition for the solution of the considered redundant system. After lengthy passages which show his mastery, Cerruti can state:

> [...] *the system may be reduced to have uniform resistance in a simply infinite way, when all bars undergo stresses of the same kind.*[50]

and, in general:

> [... when] the number of nodes is n, that of bars is $n(n-1)/2$, it is necessary to make some distinctions: either it is possible to select $(n^2-7n+12)/2$ groups of five nodes, through which ten bars liked with each other undergo stresses of the same kind (which in any case may vary from a group to the other), then the system may have uniform resistance in $(n^2-7n+12)/2$ infinite different ways; or this is impossible, and then, unless very special cases, it is impossible to design the system in order to have uniform resistance. In any case, if it is possible to select some groups of five nodes whiche satisfy the above mentioned conditions, the bars composing them may be designed to have uniform resistance in as many infinite ways as these groups are.[51]

Cerruti appears to be satisfied with these conclusions: he has indicated some design prescriptions and that seems enough, so that he skips to the other main subject of his thesis.

5 Statically indeterminate trusses

5.1 Poisson and Lévy's approaches

Cerruti begins to examine redundant trusses in his §5, where he declares his intentions in order to find the solution, in terms of inner forces, of the linear elastic problem.

The first approach presented by Cerruti looks for as many auxiliary unknowns as the balance equations, like in the problem of linear elastic continua, where the auxiliary unknowns are the displacement components:

> [...] the trick for the solution consists in letting the search for the unknown pressures and tensions depend on the search for $3n-6$ other quantities, as many as the independent balance equations, which is [...] always possible. This trick holds not only for the problem I consider,

[47] Reference [14–16], Scientific papers, pp. 172–173.

[48] Cerruti does not make a precise quotation. We may infer that he refers to the paper On a theorem relating to five points in a plane, in: Cayley's collected papers, vol. 5, pp. 480–483, Cambridge University Press, 1892.

[49] Reference [1], (31), p. 31.

[50] [...] *il sistema si può rendere di egual resistenza ed in un numero di maniere semplicemente infinito, se tutte e dieci le aste sopportano sforzi della stessa natura*. Reference [1], p. 34.

[51] [...] il numero dei punti essendo n, quello delle aste $n(n-1)/2$, bisogna fare alcune distinzioni: o si possono formare $(n^2-7n+12)/2$ gruppi di cinque punti, pei quali dieci aste che li collegano sopportano sforzi della stessa natura (la quale peraltro può cambiare da un gruppo all'altro) e allora il sistema si può ancora ridurre ad essere di egual resistenza, e questo in $(n^2-7n+12)/2$ infinite maniere differenti; o ciò è impossibile, ed allora, meno casi specialissimi, non si potrà ridurre il sistema ad essere di egual resistenza. Tuttavia, se sia possibile il formare alcuni gruppi di cinque punti, che soddisfino alle summenzionate condizioni, si possono foggiare le aste che li formano, cosicchè costituiscono un complesso di egual resistenza, e ciò in tante infinite maniere differenti, quanti sono questi gruppi. Reference [1], pp. 35–36.

> but for many other general questions [. . .]: it is known indeed that the knowledge of molecular forces in a body depends on that of six functions [the stress components] related by three partial differential equations, that are not sufficient to determine them, if the mentioned functions could not be expressed by means of other three only. The nature of these three functions is determined by the model we make of molecular forces: for elastic forces, these three functions are the orthogonal components of the displacement of any molecule of the body.[52]

As an example of this approach for an articulated system, Cerruti assumes the node displacement components as auxiliary unknowns. First he writes the length l_{ij} of a bar in terms of the differences among the coordinates of its terminal nodes i, j and differentiates it:[53]

$$(x_j - x_i)(\delta x_j - \delta x_i) + (y_j - y_i)(\delta y_j - \delta y_i) + (z_j - z_i)(\delta z_j - \delta z_i) - l_{ij}\delta l_{ij} = 0 \tag{12}$$

then, by using the linear elastic constitutive relation for the force borne by the bar, he obtains:[54]

$$T_{ij} = \frac{E_{ij}\sigma_{ij}}{l_{ij}^2}[(x_j - x_i)(\delta x_j - \delta x_i) + (y_j - y_i)(\delta y_j - \delta y_i) + (z_j - z_i)(\delta z_j - \delta z_i)]. \tag{13}$$

> Inserting the expressions for stresses given by (22) [our (12)] into the balance equations, these will contain the variation of the coordinates only, that can so be determined: once known their values, by equations (22) the forces can be calculated.[55]

Remark that in the whole of his thesis Cerruti does not explicitly write the balance equations for the nodes; the same had been done by Menabrea in [20]. On the contrary, Castigliano in [2] does actually write them.

Cerruti's second approach avoids the use of auxiliary unknowns. It is based on the choice of k independent relations among the $3n - 6 + k$ distances. If the displacements, as supposed, are small, all variations may be written in the reference configuration. Thus, the k relations among distances may be differentiated and the expressions of forces in terms of displacement variations may be inserted in them. These conditions, in addition to the $3n - 6$ independent balance equations for the first, will determine the forces in the $3n - 6 + k$ bars.

Cerruti gives credit to Poisson [17][56] for the first approach and to Lévy [8] for the second. Poisson actually studied the motion of a material point P subjected to a given active force and constrained to fixed points A_i by means of elastic threads. The strain ζ_i of each thread i with initial length l_i is given by

$$\zeta_i = \frac{1}{l_i}[(\alpha - a_i)u + (\beta - b_i)v + (\gamma - c_i)w], \tag{14}$$

where u, v, w are the displacement components of P, and the balance equations are

$$\sum_i \frac{(\alpha - a_i)\zeta_i}{l_i\epsilon_i} = X, \quad \sum_i \frac{(\beta - b_i)\zeta_i}{l_i\epsilon_i} = Y, \quad \sum_i \frac{(\gamma - c_i)\zeta_i}{l_i\epsilon_i} = Z, \tag{15}$$

where $\alpha, \beta, \gamma; a_i, b_i, c_i$ are the cartesian coordinates of P and A_i, respectively, and ϵ_i are the extensibilities of the threads. By inserting equations (14) into

[52] [. . .] l'artifizio della soluzione consiste nel far dipendere la ricerca delle pressioni e delle tensioni incognite dalla ricerca di $3n - 6$ altre quantità tante quante sono le equazioni di equilibrio fra loro indipendenti: cosa [. . .] sempre possibile. Nè questo artifizio è applicabile soltanto al mio problema, ma sì a ben altre quistioni più generali [. . .]: è noto infatti che la conoscenza delle forze molecolari destate in un corpo dipende da quella di sei funzioni legate fra loro da tre equazioni alle derivate parziali, equazioni che non sarebbero sufficienti a determinarle, se le sei funzioni in discorso non si potessero esprimere mercè tre altre soltanto. La natura poi di queste tre funzioni resta sempre determinata dal concetto che altri si fa sull'origine delle forze molecolari: nel caso delle forze elastiche queste tre funzioni sono gli spostamenti paralleli a tre assi di una molecola qualunque del corpo. Reference [1], pp. 22–23.

[53] Reference [1], (21), p. 24.

[54] Reference [1], (22), p. 24.

[55] Mettendo poi nelle equazioni di equilibrio al posto delle tensioni le loro espressioni forniteci dalle (22) esse verranno a non contenere più che le variazioni delle coordinate, le quali potrannosi in tal modo determinare: una volta conosciuti i loro valori mercè le equazioni (22) si calcoleranno quelli delle tensioni. Reference [1], pp. 24–25.

[56] The quoted passage is contained in vol. 2, pp. 402–404.

(15) Poisson obtained three independent equations of motion in the unknown coordinates α, β, γ, and then provided the conditions for equilibrium. In contemporary language, such an approach is a version of the displacement method, in which the unknowns are the components of the displacement of the nodes and the equations to be solved are those of equilibrium. For an engineer of the 1800's it was straightforward to interpret P as a node of a truss and the linear elastic threads as the bars connecting the node to the others of the truss.

Lévy presented and improved ([8, 9]) a method to solve redundant trusses which is a version of what we now call the force method. The method is based on the possibility, in a truss with k redundant bars, to write k compatibility equations linking the lengths of the redundant bars to those of the remaining m:

$$F_j(l_1, l_2, \dots, l_n) = 0, \quad j = 1, 2, \dots, k, \; n = m + k. \tag{16}$$

Equations (16) hold in the reference configuration and for small deformations in its neighbourhood, so the first variation of (16) provides

$$\frac{\partial F_j}{\partial l_1} dl_1 + \frac{\partial F_j}{\partial l_2} dl_2 + \cdots + \frac{\partial F_j}{\partial l_n} dl_n = 0. \tag{17}$$

By inserting the forces f_j in the bars in terms of the variation of length dl_j into (17), one obtains k independent compatibility equations which, together with the $3n - 6$ independent balance equations for the nodes, completely determine the forces in all the $3n - 6 + k$ bars.

The reference to Lévy's paper, contemporary to Cerruti's thesis, puts into evidence how the school of engineering in Torino was up-to-date and well documented on the most important researches in France.

On the other hand, it is strange and worth remarking that no credits are given neither to Clebsch [18, 19], who for sure perfected the displacement method, nor to Navier [7], who quite likely introduced it. The first omission, which may at first glance seem the most serious, is in part justified by the fact that Clebsch' treatise was translated in French, a kind of second mother language for scholars in Torino, only in 1883. As a confirmation of this hypothesis, reading the early works of Castigliano ([2, 3]) puts into evidence the same omission, which lets us think that Clebsch was not at all known in the school of engineering in Torino.

The omission of the reference to Navier has no easy interpretation. However, even if Poisson's treatise was for sure well known, Navier's should have been better known, since it had successive editions until the most famous one, commented by Saint-Venant.[57]

Maybe an explanation for the reference to Poisson only is due to the fact that Cerruti in his thesis often refers to the links between nodes seen as material points connected by elastic forces and the general view of continua seen as molecules interacting by mean of central forces. This view, present also in Navier's treatise, is for sure perfected and better explained by Poisson, one of the fathers of the molecular theory of elasticity.

5.2 Cerruti's contribution to the solution of redundant trusses

After having presented these examples in literature, Cerruti advances a method of his own, which seems to be a version of Lévy's method:

> Let us consider the case in which the system satisfies certain geometrical conditions, i.e., the case in which a certain number of *surface equations* exist, to which the coordinates of the vertexes of the system shall obey (we will suppose, however, that no fixed points exist, or, if they exist, the conditions indicated in §3 are also verified). Let these conditions be $m + 6$: if $m = 0$ no difficulty exists and this subject was already dealt with in §3; if $m > 0$ the rules expressed there are no more sufficient. But on this purpose we note that the surface equations will hold for any value that the coordinates attain during deformation, hence, if differentiated, will also be satisfied when the variations of the coordinates will be replaced by the actual values they have attained under the action of external forces. This posed, let us find by one of the above quoted methods the forces in function of the external forces and of the $m + 6$ constraint reactions: let us express the variations of the coordinates by means of these forces and let us insert these expressions in the differentiated $m + 6$ equations of condition: we will thus have $m + 6$ equations

[57] Navier CLMH, Résumé des leçons données a l'École des Ponts et Chaussées …, avec des notes et des appendices par M. Barré de Saint-Venant, Dunod, Paris (1864).

among the constraint reactions and six variations of the coordinates, in fact by means of forces we can express but the values of $3n-6$ variations, and in our case all the variations are determined and no one remains arbitrary. Yet by combining balance equations one obtains six of them relating external forces and constraint reactions that, in conjunction with the first $m+6$ makes $m+12$ among $m+6$ reactions and six variations of coordinates, that is, as many as the unknowns of the problem. In any case, one can have $m+6$ equations among the constraint reactions only by eliminating among the first $m+6$ the six variations of the coordinates.[58]

Cerruti's approach is applied to the truss of Fig. 2, composed by six bars along the sides and the diagonals of a plane four-sided polygon. There are four external forces applied to the nodes, fulfilling the conditions of global balance. The eight independent scalar components of the balance equations for the four nodes in terms of the six unknown bar forces reduce to five,[59] since three equations are needed for global balance.

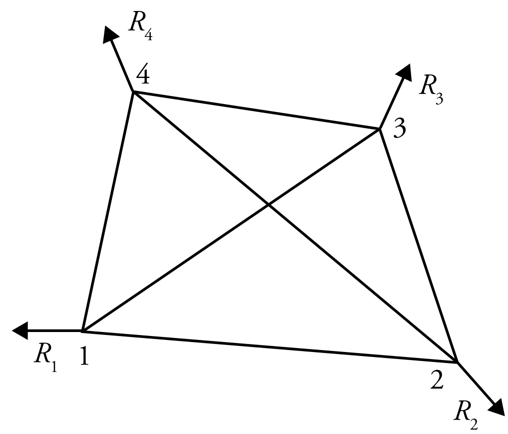

Fig. 2 A redundant truss with no fixed points

Another independent equation is needed to close the problem and Cerruti indicates that such equation has the form given by Cayley's condition (11) on the distances among a given number of points in space:

$$C = \det\left(l_{ij}^2\right) = 0, \quad i, j = 0, 1, 2, \ldots, 4 \tag{18}$$

where this time the indexes i, j range from 0 to 4.[60]

Indeed, by differentiating this condition and replacing the variations δl_{ij} by their expressions in terms of the elastic forces in the bars, Cerruti obtains a compatibility equation in the form:[61]

$$\begin{aligned}
&\frac{\sin(3\hat{1}4)\sin(3\hat{2}4)}{l_{34}}\frac{T_{12}}{\epsilon_{12}} - \frac{\sin(2\hat{1}4)\sin(2\hat{3}4)}{l_{24}}\frac{T_{13}}{\epsilon_{13}} \\
&\quad + \frac{\sin(2\hat{1}3)\sin(2\hat{4}3)}{l_{23}}\frac{T_{14}}{\epsilon_{14}} + \frac{\sin(1\hat{2}4)\sin(1\hat{3}4)}{l_{14}}\frac{T_{23}}{\epsilon_{23}} \\
&\quad - \frac{\sin(1\hat{2}3)\sin(1\hat{4}3)}{l_{13}}\frac{T_{24}}{\epsilon_{24}} \\
&\quad + \frac{\sin(1\hat{3}2)\sin(1\hat{4}2)}{l_{12}}\frac{T_{34}}{\epsilon_{34}} = 0.
\end{aligned} \tag{19}$$

The use of (18), in our opinion, represents Cerruti's major contribution to the solution of redundant trusses. This equation represents indeed an easy and well established algorithm to obtain Lévy's equations (16). Applying (18) to a truss with only six bars, as depicted in Fig. 2, is not a serious limitation because

[58] Consideriamo il caso, in cui il sistema debba soddisfare a certe condizioni geometriche, il caso cioè in cui esista un certo numero di *equazioni alla superficie*, alle quali sieno obbligate le coordinate dei vertici del sistema (supporremo però che o non vi siano dei punti fissi, o, quando ve ne sono, si verifichino altresì le condizioni indicate nel n^o 3). Sieno $m+6$ queste condizioni: quando $m=0$ non si ha alcuna difficoltà e questo argomento venne già discusso nel citato n^o 3; se $m>0$ le regole ivi enunziate non sono più sufficienti. Ma intorno a ciò osserveremo che le equazioni alla superficie, dovendo sempre sussistere qualunque sia il valore, che le coordinate vengano ad ottenere durante la deformazione, differenziate saranno pur soddisfatte sostituendo alle variazioni delle coordinate quelle effettive, che esse han subito sotto l'azione delle forze esterne. Ciò posto si ricavino con uno dei metodi precedenti le tensioni in funzione delle forze esterne e delle $m+6$ reazioni dei vincoli: si esprimano le variazioni delle coordinate per mezzo di queste tensioni e si sostituiscano tali espressioni nelle $m+6$ equazioni di condizione differenziate: avremo così $m+6$ equazioni tra le reazioni dei vincoli e sei variazioni delle coordinate, imperocchè per mezzo delle tensioni non si possono esprimere, che i valori di $3n-6$ delle variazioni, e nel nostro caso tutte quante le variazioni sono determinate e niuna arbitraria. Ma combinando le equazioni di equilibrio se ne ricavano sei tra le forze esterne e le reazioni dei vincoli che congiunte colle prime $m+6$ fanno $m+12$ equazioni tra $m+6$ reazioni e sei variazioni di coordinate, tante cioè quante sono le incognite del problema. Però si possono avere $m+6$ equazioni tra le sole reazioni dei vincoli eliminando tra le prime $m+6$ le sei variazioni delle coordinate. Reference [1], pp. 26–27.

[59] Reference [1], (40), p. 37.

[60] Reference [1], (41), p. 37.

[61] Reference [1], (44), p. 38.

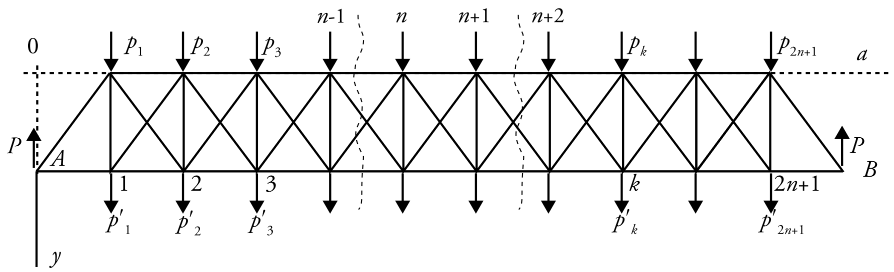

Fig. 3 A redundant truss obtained starting from a simple one

most trusses of civil and industrial architecture can be decomposed into meshes similar to that of Fig. 2.

When the system reduces to three bars joined by the node 4 and fixed at their other extremities 1, 2, 3, which is a standard problem in the study of redundant structures, Cerruti remarks that he may obtain:[62]

$$\sin(2\hat{4}3)\frac{T_{14}}{\epsilon_{14}} - \sin(1\hat{4}3)\frac{T_{24}}{\epsilon_{24}} + \sin(1\hat{4}2)\frac{T_{34}}{\epsilon_{34}} = 0, \quad (20)$$

which, he observes, by taking into consideration the geometrical properties of the triangles, is the solution for the same problem obtained by Menabrea in [21].[63] Another tribute to Menabrea in a work where almost no hint on work and energy is given, is a meaningful sign of how Menabrea's teaching was deeply rooted in the school of engineering in Torino.

Always with the aim of providing an iterative procedure, Cerruti reduces the linear algebraic system composed by the five balance equations for the nodes and the compatibility condition (19) to a linear algebraic system of two equations in the two unknowns T_{23}, T_{14},[64] so that he can examine a series of particular cases of interest in the applications.[65]

He begins by examining the case where the external forces are directed along the diagonals of the polygon, then goes on to study the case when the polygon reduces to a trapeze and eventually to a parallelogram. In this last case, he obtains a universal formula for the forces in the bars in terms of the external ones.[66] This formula is simplified again when the parallelogram becomes a rectangle or a square.[67]

Cerruti remarks that this result is now independent of possible symmetries both in geometry and external load which could otherwise provide an answer for the problem of redundant trusses. Indeed, he remarks, these symmetry considerations fail as soon as deformation begins and may lead to errors, while the formulation of additional compatibility conditions based only on the geometry of distances among points is not affected by such errors.

In §9, Cerruti applies his recursive formulæ obtained for the truss of Fig. 1 for the redundant truss in Fig. 3, which is obtained from the simple one in Fig. 1 by adding bars along the other diagonal of each rectangular element composing the truss. He first examines the rectangle between the nodes n and $n+1$, which is but a particular case of the polygon he studied in §8.

By means of ordinary balance equations on the portion of the truss comprised by the transverse imaginary sections between the nodes $n+1, n+2$ and $n-1, n$, he obtains the external forces applied to the vertexes of the considered rectangle.[68] He then writes the balance equations for the considered rectangle as another portion of the truss,[69] as well as the balance equations for the nodes of the rectangle, even if, he remarks, not all of these equations are linearly dependent.[70] He then provides the compatibility equation corresponding to (19)[71] and remarks that one has nine equations in ten unknowns, which

[62] Reference [1], (45), p. 39.

[63] Menabrea at pp. 152–156 of [21] studies a five bar system similar to that considered by Cerruti and at p. 155 refers an equation connected to Cerruti's one. See also [27].

[64] Reference [1], (46), p. 39 and (47), p. 40.

[65] Reference [1], pp. 41–44.

[66] Reference [1], (51), p. 43.

[67] Reference [1], (52) and (53), p. 44.

[68] Reference [1], (54)–(57), p. 45.

[69] Reference [1], (58)–(59), p. 46.

[70] Reference [1], (60), p. 46.

[71] Reference [1], (61), p. 46.

> [...] will let us determine nine of the forces in terms of the tenth. After that, by using the already obtained results, one finds the forces of the different elements of the preceding rectangle, and so on until one reaches the [external] support. All forces will be expressed in terms of that tenth, which had been left undetermined in the calculation of the first rectangle, the value of which will be found at the end of the procedure. By replacing its value found in this way into the preceding expressions, all the forces in the elements will be known.[72]

In §10 Cerruti leaves the applications aside for a while and considers again Poisson's study of body-points motion, to which he referred as inspiring one of the methods of solution for redundant systems. He does not add anything to Poisson's original treatment, but frames the method directly into a structural environment by considering the points as hinges and the threads as elastic bars, and studying the standard case of three bars hinged at fixed points and joined at a common node to which an external force is applied. Cerruti puts into evidence the meaning of the constraint equations in terms of length of the bars and applies the rules of determinants to solve the linear system of balance equations in terms of displacements. Cerruti suggests that a matrix approach to a displacement method is fruitful:

> This example is useful to show how simple is to solve the problem of the stress distribution by letting it depend on the search of as many quantities as balance equations.[73]

In §11 Cerruti spends some more words on the possible cases in which the *k* redundant bars cannot be described by the additional compatibility condition (18). Indeed, this is effective when considering five nodes (only one redundant distance which can be expressed by means of other nine), but even in the case of six nodes the situation is more complicated. Indeed, twelve bars are sufficient for a statically determined truss, but when considering a single redundant bar, it is not possible to express a single compatibility condition for it in terms of other distances. Cerruti remarks that this possibility depends much on how the truss is actually built.

6 Final remarks

The study of Cerruti's thesis lets us know for sure that the level of education at the school of engineering in Torino was very high. The quotations are up-to-date, like the problems dealt with by Castigliano and Cerruti. Still, as it was already stated above, Cerruti's thesis is somehow not fully appreciable and it is not strange that, even if Cerruti's importance in the field of rational mechanics is undoubtable, his graduation work is for sure second to that of Castigliano, in spite of the judgement of the commitee.

Indeed, it is apparent that the thesis has some interesting features: the search for an iterative procedure, partially fulfilled, and the search for the conditions of uniform resistance, which gets some satisfactory results. On the other hand, a lot of the proposed procedures are not original, and the technique for solving redundant trusses, even if recursive and interesting from a contemporary point of view because of the possibility of automatic implementation, is of limited applications.

There are hints of originality and apparent signs of mastery and some new results in the thesis, however. The study of the conditions of uniform resistance is well done and, in particular, some results are obtained without the need to solve the linear elastic static problem, which is of course very important. The idea that it is possible to provide a recursive formulation for the resolution of redundant problems is for sure modern and absolutely convincing, but it is put forth for a very limited set of applications.

In conclusion, we wish to put forth a last comment, that is, as already mentioned, reading Cerruti's thesis with a contemporary eye lets us see that Cerruti was a bright and very well educated fellow, ready for a brilliant career in mechanics, but who had no time for fulfilling a very accurate work.

[72] Esse ci permetteranno dunque di ricavare nove delle tensioni in funzione della decima. Dopo ciò, facendo uso dei risultati già ottenuti, si passerà a trovar le tensioni dei diversi pezzi del rettangolo precedente, e così via di mano in mano sino a che si sia giunto all'appoggio. Tutte le tensioni si potranno esprimere mediante quella decima, che era rimasta indeterminata nel calcolo relativo al primo rettangolo, ma il cui valore si potrà trovare poi al termine dell'operazione. Sostituendo quindi il suo valore così trovato nelle espressioni precedenti, tutte le tensioni dei diversi pezzi diverranno conosciute. Reference [1], p. 47.

[73] Questo esempio serve a far vedere con quanta semplicità si possa sciogliere il problema della distribuzione delle tensioni, facendolo dipendere dalla ricerca di tante quantità soltanto quante sono le equazioni di equilibrio. Reference [1], p. 49.

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