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BRIEF NOTES AND DISCUSSIONS

Analysis of plates on Winkler foundation by wavelet collocation

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1 Introduction

Many engineering problems can be modeled as isotropic rectangular plates, such as bases of machines, pavement of roads or footing of buildings. One way to describe the behaviour of such plates is the Winkler model [1].

The analysis of plates on elastic foundations was conducted previously by several authors, using various approaches. Leissa [2] considered a thin-plate theory, Lam et al. [3] derived the exact solutions of bending, buckling and vibration of a Levy-plate, Xiang et al. [4] derived an analytical vibration solution, Omurtag et al. [5] used the finite element method, Matsunaga [6] developed a special higher-order plate theory, Shen et al. [7] used the Rayleigh-Ritz method, and Ayvaz et al. [8] used the modified Vlasov model to study the earthquake response of rectangular thin plates on elastic foundation.

In the recent years, some attempts have been made for the vibration analysis of rectangular thick plates on elastic foundations. Liew and Teo [9], and Liew et al. [10], Han and Liew [11] used the differential quadrature method to analyse the vibration characteristics of rectangular plates on elastic foundations. Also, Zhou et al. [12] used the Chebyshev polynomials as admissible functions to study the three-dimensional vibration of rectangular plates on elastic foundations by the Ritz method.

Meshless methods are not widely used for the analysis of Mindlin plates on elastic foundations. Civalek [13] used the singular convolution method for the bending analysis of Mindlin plates on elastic foundations. Also, a boundary element method was used by Chucheepsakul and Chinnaboon [14] to analyse plates by a two-parameter model. Recent papers by Chen et al. [15], Civalek [16], Liu [17], and Tomar et al. [18] have addressed the subject of vibration or bending of plates on elastic foundation via different numerical techniques.

This paper deals, for the first time, with the bending analysis of plates on Winkler foundations by a

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Table 1 Convergence study for deflections, moments and shear forces of uniformly loaded square SS plates on Winkler foundations ($K = 1$; $\nu = 0.3$)

h/a	Grid	w[a] $x = y = 0.5$	M_{xx}[b] $x = y = 0.5$	M_{yy}[b] $x = y = 0.5$	M_{xy}[b] $x = y = 0$	Q_x[c] $x = 0$; $y = 0.5$	Q_y[c] $x = 0.5$, $y = 0$
0.01	$(2^3 + 1)^2$ points	0.016	0.060	0.060	0.061	2.745	2.745
	$(2^4 + 1)^2$ points	4.025	4.745	4.745	3.197	0.334	−0.334
	$(2^5 + 1)^2$ points	4.053	4.774	4.774	3.236	0.337	−0.337
0.05	$(2^3 + 1)^2$ points	0.186	0.703	0.703	0.230	2.294	2.294
	$(2^4 + 1)^2$ points	4.102	4.773	4.773	3.228	0.337	−0.337
	$(2^5 + 1)^2$ points	4.104	4.775	4.775	3.238	0.337	−0.337
0.1	$(2^3 + 1)^2$ points	0.375	0.991	0.991	0.115	0.929	0.929
	$(2^4 + 1)^2$ points	4.261	4.774	4.774	3.229	0.337	−0.337
	$(2^5 + 1)^2$ points	4.261	4.774	4.774	3.237	0.337	−0.337
0.2	$(2^3 + 1)^2$ points	1.470	1.512	1.512	14.051	0.001	0.001
	$(2^4 + 1)^2$ points	4.888	4.772	4.772	3.227	0.337	−0.337
	$(2^5 + 1)^2$ points	4.888	4.772	4.772	3.236	0.337	−0.337

[a] $\times 10^{-3} qa^4/D$

[b] $\times 10^{-2} qa^2/D$

[c] $\times qa$

wavelet collocation method [19]. A first-order shear deformation theory is used to model the kinematics of the plates. Here we consider a three degree-of-freedom formulation, with one transverse displacement, and two rotations about the normal to the middle plane of the plate.

The method employed for the numerical solution is a collocation method based on Deslaurier-Dubuc interpolating basis in hierarchical form [20], which is the first necessary step towards the application to this class of problems of the adaptive wavelet collocation method introduced in [19]. This collocation algorithm can be viewed as a very effective meshless technique. The authors recently applied the method with success to composite structures [21]. The reader is welcomed to see details of the formulation in [21, 22].

The solution of many problems is very smooth in a large part of the domain, while being globally not as smooth. In order to approximate it in an optimal way, the need arises of using high order methods, capable of taking advantage of the local smoothness of the solution by using a coarse grid in the greatest part of the domain, but capable, on the other hand of coping with singularities. Among the methods that display this potential one can count the methods based on wavelet bases and multiscale analysis. Due to their hierarchical nature, wavelet systems are well suited for the development of highly efficient adaptive procedures, since scaling functions capture the average information and wavelets are responsible for the representation of the details. Due to this fact, it is possible to enrich the approximation in the neighbourhood of singularities, simply by adding to the basis wavelets at higher levels of resolution. Such bases allow on one hand to design methods of arbitrarily high order, and on the other hand they display a local behaviour, allowing for the accurate treatment of singularities by locally refining the approximation space. It is important to stress the fact that when expressing a function in terms of the nodal basis, all the coefficients will be needed in order to get a good approximation. However, to express the same function in the wavelet basis, in order to get an approximation of the same order, only a subset of the coefficients is needed (essentially the ones corresponding to those basis functions whose centre is close to singularities). In other words, the nodal basis naturally corresponds to taking a uniform discretisation and the wavelet basis to a non-uniform one [22]. In the solution of computational mechanics problems, one would like to work

Table 2 Convergence study for deflections, moments and shear forces of uniformly loaded square SS plates on Winkler foundations ($K = 3$; $\nu = 0.3$)

h/a	Grid	w[a] $x = y = 0.5$	M_{xx}[b] $x = y = 0.5$	M_{yy}[b] $x = y = 0.5$	M_{xy}[b] $x = y = 0$	Q_x[c] $x = 0$; $y = 0.5$	Q_y[c] $x = 0.5$, $y = 0$
0.01	$(2^3+1)^2$ points	0.016	0.060	0.060	0.061	2.746	2.746
	$(2^4+1)^2$ points	3.325	3.850	3.850	2.710	0.290	−0.290
	$(2^5+1)^2$ points	3.348	3.875	3.875	2.746	0.293	−0.293
0.05	$(2^3+1)^2$ points	0.182	0.694	0.694	0.230	2.290	2.290
	$(2^4+1)^2$ points	3.380	3.864	3.864	2.733	0.292	−0.292
	$(2^5+1)^2$ points	3.381	3.865	3.865	2.743	0.292	−0.292
0.1	$(2^3+1)^2$ points	0.358	0.959	0.959	0.125	0.918	0.918
	$(2^4+1)^2$ points	3.482	3.834	3.834	2.717	0.291	−0.291
	$(2^5+1)^2$ points	3.483	3.834	3.834	2.725	0.291	−0.291
0.2	$(2^3+1)^2$ points	1.340	1.348	1.348	13.995	0.018	−0.018
	$(2^4+1)^2$ points	3.873	3.716	3.716	2.648	0.285	−0.285
	$(2^5+1)^2$ points	3.873	3.716	3.716	2.657	0.285	−0.285

[a] $\times 10^{-3} qa^4/D$

[b] $\times 10^{-2} qa^2/D$

[c] $\times qa$

Table 3 Convergence study for deflections, moments and shear forces of uniformly loaded square SS plates on Winkler foundations ($K = 5$; $\nu = 0.3$)

h/a	Grid	w[a] $x = y = 0.5$	M_{xx}[b] $x = y = 0.5$	M_{yy}[b] $x = y = 0.5$	M_{xy}[b] $x = y = 0$	Q_x[c] $x = 0$; $y = 0.5$	Q_y[c] $x = 0.5$, $y = 0$
0.01	$(2^3+1)^2$ points	0.016	0.059	0.059	0.061	2.751	2.751
	$(2^4+1)^2$ points	1.492	1.526	1.526	1.429	0.174	−0.174
	$(2^5+1)^2$ points	1.506	1.540	1.540	1.457	0.176	−0.176
0.05	$(2^3+1)^2$ points	0.159	0.646	0.646	0.224	2.286	2.286
	$(2^4+1)^2$ points	1.509	1.525	1.525	1.440	0.175	−0.175
	$(2^5+1)^2$ points	1.509	1.526	1.526	1.449	0.176	−0.176
0.1	$(2^3+1)^2$ points	0.257	0.787	0.787	0.256	0.884	0.884
	$(2^4+1)^2$ points	1.519	1.481	1.481	1.410	0.172	−0.172
	$(2^5+1)^2$ points	1.519	1.482	1.482	1.418	0.173	−0.173
0.2	$(2^3+1)^2$ points	0.827	0.713	0.713	13.059	0.091	−0.091
	$(2^4+1)^2$ points	1.551	1.328	1.328	1.299	0.162	−0.162
	$(2^5+1)^2$ points	1.551	1.328	1.328	1.308	0.162	−0.162

[a] $\times 10^{-3} qa^4/D$

[b] $\times 10^{-2} qa^2/D$

[c] $\times qa$

Table 4 Deflections, moments and shear forces of uniformly loaded square SS plates on Winkler foundations ($\nu = 0.3$)

K	h/a	Method	w[a] $x = y = 0.5$	M_{xx}[b] $x = y = 0.5$	M_{yy}[b] $x = y = 0.5$	M_{xy}[b] $x = y = 0$	Q_x[c] $x = 0;\ y = 0.5$	Q_y[c] $x = 0.5,\ y = 0$
1	0.01	Present	4.053	4.774	4.774	3.236	0.337	−0.337
		Kobayashi and Sonoda [24]	4.054	4.775	4.775	3.241	0.337	−0.337
	0.05	Present	4.104	4.775	4.775	3.238	0.337	−0.337
		Kobayashi and Sonoda [24]	4.104	4.775	4.775	3.241	0.337	−0.337
	0.1	Present	4.261	4.774	4.774	3.237	0.337	−0.337
		Kobayashi and Sonoda [24]	4.261	4.774	4.774	3.240	0.337	−0.337
	0.2	Present	4.888	4.772	4.772	3.236	0.337	−0.337
		Kobayashi and Sonoda [24]	4.888	4.772	4.772	3.239	0.337	−0.337
3	0.01	Present	3.348	3.875	3.875	2.746	0.293	−0.293
		Kobayashi and Sonoda [24]	3.349	3.875	3.875	2.751	0.293	−0.293
	0.05	Present	3.381	3.865	3.865	2.743	0.292	−0.292
		Kobayashi and Sonoda [24]	3.381	3.865	3.865	2.746	0.292	−0.292
	0.1	Present	3.483	3.834	3.834	2.725	0.291	−0.291
		Kobayashi and Sonoda [24]	3.483	3.834	3.834	2.728	0.291	−0.291
	0.2	Present	3.873	3.716	3.716	2.657	0.285	−0.285
		Kobayashi and Sonoda [24]	3.873	3.716	3.716	2.660	0.284	−0.284
5	0.01	Present	1.506	1.540	1.540	1.457	0.176	−0.176
		Kobayashi and Sonoda [24]	1.506	1.540	1.540	1.462	0.176	−0.176
	0.05	Present	1.509	1.526	1.526	1.449	0.176	−0.176
		Kobayashi and Sonoda [24]	1.509	1.526	1.526	1.452	0.175	−0.175
	0.1	Present	1.519	1.482	1.482	1.418	0.173	−0.173
		Kobayashi and Sonoda [24]	1.519	1.482	1.482	1.421	0.172	−0.172
	0.2	Present	1.551	1.328	1.328	1.308	0.162	−0.162
		Kobayashi and Sonoda [24]	1.551	1.328	1.328	1.311	0.162	−0.162

[a] $\times 10^{-3} qa^4/D$

[b] $\times 10^{-2} qa^2/D$

[c] $\times qa$

with the second type of basis. Unfortunately, working with such a non-uniform basis presents some difficulties. If a Galerkin scheme is to be implemented, the need of evaluating integrals arises. The computation of such integrals by applying a classical quadrature rule is not convenient. It is known that wavelets are usually less regular than accurate. This implies the need of using a lower order quadrature rule, with a much finer grid of quadrature nodes, which raises the computational cost of the whole procedure [22]. This difficulty can be overcome by applying a technique developed by Dahmen and Micchelli [23]. Even if this technique is effective for linear problems, some difficulties may arise when dealing with nonlinearities. One way to overcome such problems is to use a collocation approach. In this paper it is used a collocation method based on the use of Deslaurier-Dubuc interpolating functions. This is a class of compactly supported scaling functions that satisfy a property of interpolation, rather than the usual property of orthonormality. However, most of the advantages associated to the use of wavelet systems holds, namely the property of smoothness characterisation through wavelet coefficients [22]. An interesting feature of this basis is the particular structure of the fast wavelet transform (which in this case takes the name of Interpolating Wavelet transform). Due to the particular properties of the scaling and wavelet functions, it exhibits some special features. First of all, due to the interpolation properties, the coefficients of the development

Table 5 Deflections, moments and shear forces of uniformly loaded square CC plates on Winkler foundations ($\nu = 0.3$)

K	h/a	Method	w[a] $x=y=0$	M_{xx}[b] $x=y=0$	M_{yy}[b] $x=y=0$	M_{xy}[b] $x=y=-1$	Q_x[c] $x=0;\ y=-1$	Q_y[c] $x=-1,\ y=0$
1	0.01	Present	1.917	2.437	3.319	0.026	0.244	−0.515
		Kobayashi and Sonoda [24]	1.918	2.437	3.320	0.101	0.244	−0.515
	0.05	Present	1.989	2.472	3.321	0.398	0.245	−0.509
		Kobayashi and Sonoda [24]	1.989	2.472	3.321	0.466	0.245	−0.509
	0.1	Present	2.206	2.575	3.321	0.833	0.248	−0.500
		Kobayashi and Sonoda [24]	2.206	2.575	3.321	0.850	0.245	−0.500
	0.2	Present	3.015	2.915	3.298	1.429	0.256	−0.474
		Kobayashi and Sonoda [24]	3.015	2.915	3.298	1.433	0.256	−0.474
3	0.01	Present	1.744	2.184	2.989	0.024	0.230	−0.481
		Kobayashi and Sonoda [24]	1.744	2.184	2.989	0.095	0.229	−0.480
	0.05	Present	1.802	2.207	2.978	0.373	0.230	−0.474
		Kobayashi and Sonoda [24]	1.802	2.207	2.978	0.437	0.230	−0.474
	0.1	Present	1.976	2.272	2.941	0.771	0.231	−0.462
		Kobayashi and Sonoda [24]	1.976	2.272	2.941	0.787	0.231	−0.462
	0.2	Present	2.590	2.463	2.790	1.274	0.232	−0.425
		Kobayashi and Sonoda [24]	2.590	2.463	2.790	1.278	0.231	−0.425
5	0.01	Present	1.069	1.212	1.709	0.019	0.172	−0.347
		Kobayashi and Sonoda [24]	1.069	1.212	1.709	0.073	0.172	−0.347
	0.05	Present	1.088	1.204	1.673	0.277	0.171	−0.340
		Kobayashi and Sonoda [24]	1.088	1.204	1.673	0.326	0.171	−0.340
	0.1	Present	1.141	1.181	1.569	0.544	0.167	−0.322
		Kobayashi and Sonoda [24]	1.141	1.181	1.569	0.556	0.167	−0.322
	0.2	Present	1.289	1.098	1.252	0.785	0.154	−0.269
		Kobayashi and Sonoda [24]	1.289	1.098	1.252	0.788	0.154	−0.269

[a] $\times 10^{-3} qa^4/D$

[b] $\times 10^{-2} qa^2/D$

[c] $\times qa$

of a given function with respect to the scaling function basis are its values at the Lagrange interpolation nodes. Moreover, with this basis it is particularly simple to work with non-uniform discretisation. In fact, it is possible to assign to each wavelet function a corresponding dyadic point and vice-versa. A detailed description of this wavelet system can be found in [22].

2 Numerical examples

In all following examples a regular grid was used and square domains have been considered. However, the collocation technique based on the use of interpolating wavelets can be used to solve non-rectangular structural domains. If this is the case, the domain must be mapped onto a unit square by means of a conformal mapping of class C2 with a C2 inverse [22].

2.1 Static results

Numerical results are presented for the uniformly loaded square plate ($a/b = 1, a \in [0, 1]$) for the various values of thickness-to-span ratio h/a and dimensionless foundation modulus $K = (k_f a^4/D)^{1/4}$. The shear correction factor is 5/6. The Poisson's ratio of 0.3 is taken for all cases. A convergence study for simply-supported plates in shown in Tables 1, 2, 3, for various values of K. In Tables 1,

Table 6 Comparison of frequency parameters Δ for the flexural modes of thin and moderately thick square plates on Winkler foundation

Boundary condition	h/b	K_1	Methods	$\Delta_{1,1}$	$\Delta_{1,2}$, $\Delta_{2,1}$	$\Delta_{2,2}$
SSSS ($\nu = 0.3$)	0.01	10^2	Zhou et al. [12]	2.2413	5.0973	8.0527
			Classical [2]	2.2420	5.1016	8.0639
			Xiang et al. [4]	2.2413	5.0971	8.0523
			Present ($(2^5+1)^2$ points)	2.2414	5.1016	8.0546
		5×10^2	Zhou et al. [12]	3.0214	5.4850	8.3035
			Classical [2]	3.0221	5.4894	8.3146
			Xiang et al. [4]	3.0215	5.4850	8.3032
			Present ($(2^5+1)^2$ points)	3.0215	5.4867	8.3054
	0.1	2×10^2	Zhou et al. [12]	2.3951	4.8262	7.2338
			Xiang et al. [4]	2.3989	4.8194	7.2093
			Present ($(2^5+1)^2$ points)	2.3989	4.8194	7.2093
		10^3	Zhou et al. [12]	3.7008	5.5661	7.7335
			Xiang et al. [4]	3.7212	5.5844	7.7353
			Present ($(2^5+1)^2$ points)	3.7213	5.5844	7.7353

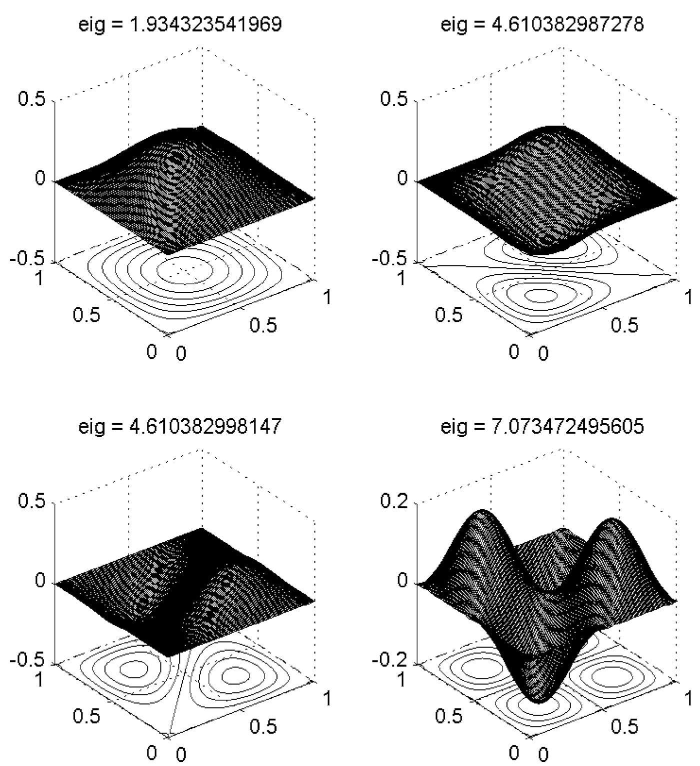

Fig. 1 First 4 vibrational modes: SSSS plate, with $h/b = 0.1$, $K_1 = 1$, using a $(2^4+1)^2$ points grid

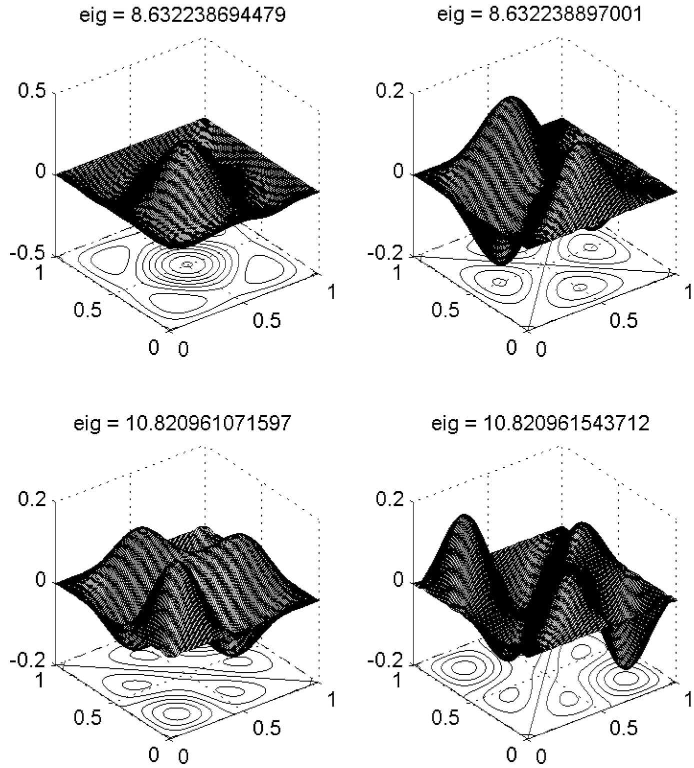

Fig. 2 Fifth to eighth vibrational modes: SSSS plate, with $h/b = 0.1$, $K_1 = 1$, using a $(2^4 + 1)^2$ points grid

2, 3 we list the deflections and stress resultants for different edge conditions. The present method is compared with a Levy-type method by Kobayashi and Sonoda [24]. Due to the Levy approach, edges $y = constant$ are simply-supported. The boundary conditions denoted by SS represent simply-supported bords along the perimeter, etc. The results presented in Tables 4 and 5 show an excellent correlation with results by Kobayashi and Sonoda [24]. In most cases, results are identical. The larger differences are obtained for CC plates, and $h/a = 0.01$. As noted in the convergence study, for this aspect ratio, we should have used more points. However we decided to use the same number of nodes for all cases.

2.2 Free vibration results

Numerical results are presented for square plates ($a/b = 1$). The non-dimensional parameters are given as

$$\Delta = \frac{\omega b^2}{\pi^2}\sqrt{\rho t/D}, \qquad K_1 = \frac{k_f a^4}{D}, \tag{1}$$

In Table 6, we compare the present method with 3D results by Zhou et al. [12] who used a Ritz method to solve the three-dimensional problem of plates on elastic foundations. We show the frequency parameters Δ for the flexural modes of thin and moderately thick square plates on elastic foundation. In Figs. 1, 2 and 3, we illustrate the eigenmodes for a SSSS plate, with $h/b = 0.1$, $K_1 = 1$, using a $(2^4 + 1)^2$ points grid. The modes are quite stable. The results compare extremely well with the 3D Ritz method by Zhou et al. [12].

3 Conclusions

In this paper we used, for the first time, the wavelet collocation method to analyse static deformations and free vibrations of plates on Winkler foundations. The first-order shear deformation theory set of equations of motion define a static problem and an eigenproblem which can be solved by various algorithms.

The present results were compared with existing analytical solutions, and finite element schemes and are in very good agreement.

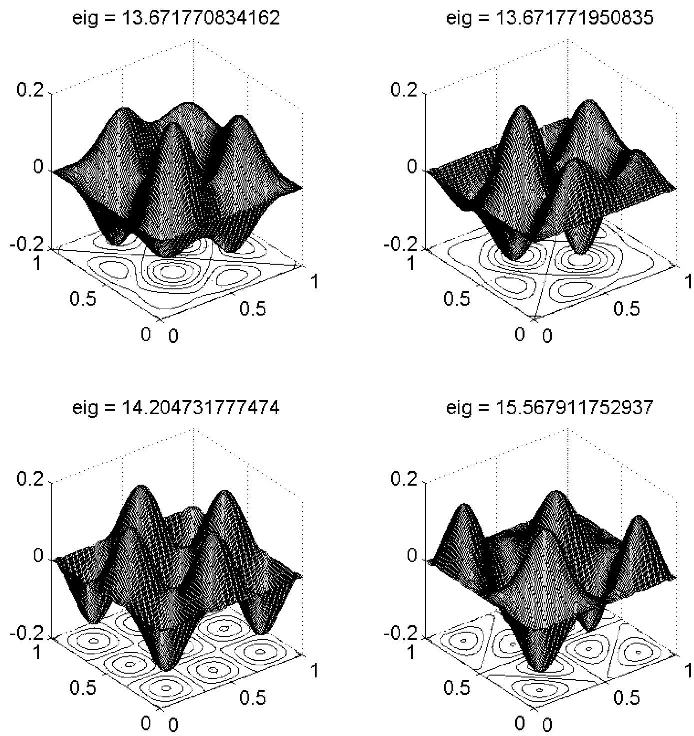

Fig. 3 Ninth to twelfth vibrational modes: SSSS plate, with $h/b = 0.1$, $K_1 = 1$, using a $(2^4 + 1)^2$ points grid

The present method is a simple yet powerful alternative to other finite element or meshless methods in the static deformation and free vibration analysis of plates on Winkler foundations.

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