Meccanica (2011) 46:785–794
DOI 10.1007/s11012-010-9337-5

Two-temperature theory in generalized magneto-thermoelasticity with two relaxation times

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Received: 18 August 2009 / Accepted: 2 July 2010 / Published online: 5 August 2010

Abstract The model of one-dimensional equations of the two-temperature generalized magneto-thermoelasticity theory with two relaxation times in a perfect electric conducting medium is established. The state space approach developed in Ezzat (Can J. Phys. Rev. 86(11):1241–1250, 2008) is adopted for the solution of one-dimensional problems. The resulting formulation together with the Laplace transform techniques are applied to a specific problem of a half-space subjected to thermal shock and traction-free surface. The inversion of the Laplace transforms is carried out using a numerical approach. Some comparisons have been shown in figures to estimate the effects of the temperature discrepancy and the applied magnetic field.

Keywords Magneto-thermoelasticity · Two-temperature theory · Heat transfer · State space approach · Green-Lindsay theory of two relaxation times

Nomenclature

λ, μ	Lamè constants
t	Time
ρ	Density
C_E	Specific heat at constant strain
$\boldsymbol{H}$	Magnetic field intensity vector
$\boldsymbol{E}$	Electric field intensity vector
H_o	Constant component of magnetic field
J	Conduction current density vector
T	Thermodynamic temperature
φ	Conductive temperature
T_o	Reference temperature
α_T	Coefficient of linear thermal expansion
σ_{ij}	Components of stress tensor
e_{ij}	Components of strain tensor
u_i	Components of displacement vector
e	$=u_{i,i}$, dilatation
k	Thermal conductivity
κ	Diffusivity
μ_o	Magnetic permeability
ε_o	Electric permeability
τ, ν	Two relaxation times
β_o	The dimensionless temperature discrepancy
ε	$=\frac{\varphi_o \gamma^2}{\rho^2 c_o^2 C_E}$, the thermal coupling parameter
δ_{ij}	Kronecker's delta
γ	$=(3\lambda+2\mu)\alpha_T$
η_o	$=\frac{\rho C_E}{K}$
c_o	$=(\frac{\lambda+2\mu}{\rho})^{\frac{1}{2}}$, speed of propagation of isothermal elastic waves
α_o	$=(\frac{\mu_o H_o^2}{\rho})^{\frac{1}{2}}$, the Alfven velocity
c	$=\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, speed of light.

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1 Introduction

Linear elasticity is at the heart of almost all continuum-based constitutive models used in structural and geotechnical engineering, and therefore it is reasonable to concentrate efforts (initially at least) on improvement of solvers for these cases.

While in service, structural elements are frequently subjected to not only force loads but also nonuniform heating causing thermal stresses. These stress themselves or in combination with mechanical stresses due to external loads may cause the material to fracture. Therefore, to perform a complete strength analysis of structures, it is necessary to know the magnitude and distribution of thermal stresses. In this connection, issues associated with the determination of temperature fields and thermal stresses are of importance and draw the attention of experts of different professions.

The two temperatures theory of thermoelasticity was introduced by Gurtin and Williams [2, 3], Chen and Gurtin [4], and Chen et al. [5, 6], in which the classical Clausius-Duhem inequality was replaced by another one depending on two temperatures; the conductive temperature φ and the thermodynamic temperature T, the first is due to the thermal processes, and the second is due to the mechanical processes inherent between the particles and the layers of elastic material, this theory was also investigated by Ieşan [7].

The two-temperature model was underrated and unnoticed for many years thereafter. Only in the last decade has the theory been noticed, developed in many works, and find its applications mainly in the problems in which the discontinuities of stresses have no physical interpretations. Among the authors who contribute to develop this theory, Quintanilla [8] studied existence, structural stability, convergence and spatial behavior for this theory, Youssef [9] introduced the generalized Fourier law to the field equations of the two-temperature theory of thermoelasticity and proved the uniqueness of solution for homogeneous isotropic material, Puri and Jordan [10] studied the propagation of harmonic plane waves, recently, Magaña and Quintanilla [11] have studied the uniqueness and growth solutions for the model proposed by [9].

The heat conduction equations for the classical linear uncoupled and coupled thermoelasticity theories are of the diffusion type predicting infinite speed of propagation for heat wave contrary to physical observations. To eliminate the paradox inherent in the classical theories, the theories of generalized thermoelasticity were developed in attempt to amend the classical thermoelasticity in 1960's. Cattaneo [12] was the first to offer an explicit mathematical correction of the propagation speed defect inherent in Fourier's heat conduction law. Cattaneo's theory allows for the existence of thermal waves, which propagate at finite speeds. Starting from Maxwell's idea [13] and from the paper by Cattaneo, an extensive amount of literature [14–17] has contributed to the elimination of the paradox of instantaneous propagation of thermal disturbances. The approach used is known as extended irreversible thermodynamics, which introduces time derivative of the heat flux vector, Cauchy stress tensor and its trace into the classical Fourier law by preserving the entropy principle. Puri and Kythe [16] investigated the effects of using the (Maxwell-Cattaneo) model in Stoke's second problem for a viscous fluid. Joseph and Preziosi give a detail history of heat conduction theory in [15].

Three generalizations to the coupled theory were introduced. The first generalization to coupled thermoelasticity is due to Lord and Shulman [18], who introduced the theory of generalized thermoelasticity with one relaxation time. The heat equation of this theory is of the wave-type, it automatically ensures finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motion and constitutive relations, remain the same as those for the coupled and the uncoupled theories. The second generalization to the coupled theory of elasticity is what is known as the theory of thermoelasticity with two relaxation times or the theory of temperature-rate-dependent thermoelasticity. Müller [19], in a review of the thermodynamics of thermoelastic solids, proposed an entropy production inequality, with the help of which he considered restrictions on a class of constitutive equations. A generalization of this inequality was proposed by Green and Laws [20]. Green and Lindsay obtained an explicit version of the constitutive equations in [21]. These equations were also obtained independently by Şuhubi [22] and Ezzat [23] has obtained the fundamental solution for this theory. This theory contains two constants that act as relaxation times and modify all the equations of the coupled theory, not only the heat equation. Dhaliwal and Rokne [24] studied one dimensional thermal shook problem with two relaxation times. The third generalization to the coupled theory is known as the dual-phase-lag thermoelasticity, which were apparently developed by Maxwell, has been considered by Tzou [25], in which the Fourier law is replaced by an approximation to a modification of the Fourier law with two different translations for the heat flux and the temperature gradient.

Investigation of the interaction between magnetic field and stress and strain in a thermoelastic solid is very important due to its many applications in the fields of geophysics, plasma physics and related topics. Especially in nuclear fields, the extremely high temperature and temperature gradients, as well as the magnetic fields originating. Great attention has been devoted inside nuclear reactors, influence their design and operations to the study of electromagneto-thermoelastic coupled problems based on the generalized thermoelastic theories for non-rotating medium. In the context of Lord and Shulman's theory, Nayfeh and Nasser [26] studied the propagation of plane waves in a solid under the influence of an electromagnetic field. Choudhuri [27] extend these results to rotating media. Sharma and Chand [28] studied a one-dimensional transient magnetothermoelastic problem by introducing a potential function, Sherief and Ezzat [29] investigated a problem of an infinitely long annular cylinder in generalized magneto-thermoelasticity and Ezzat and Youssef [30] introduced a model of the equations of generalized magneto-thermoelasticity in a perfectly conducting medium. In the context of Green and Lindsay's theory, Ezzat et al. [31] solved an electromagneto-thermoelastic two-dimensional problem in generalized thermoelasticity with two relaxation times.

The solution is obtained using a state space approach. The first writers to introduce the state space formulation in thermoelastic problems were Bahar and Hetnarski [32]. Their work dealt with coupled thermoelasticity in the absence of heat sources. This work was followed by the work of Ezzat [33] in generalized thermoelasticity including heat sources.

The present work is an attempt to generalize these results to include the effects of a magnetic field in 2TT. The resulting formulation together with the Laplace transform is used to solve a one-dimensional thermal shock problem for a perfectly conducting half-space permeated by a primary uniform magnetic field whose surface is assumed to be perfect conductor and traction free. The inversion of the Laplace transform will be computed numerically by using a method based on Fourier expansion technique [34].

2 Formulation of the problem

We shall consider a thermoelastic medium of prefect conductivity permeated by an initial magnetic field $\boldsymbol{H}$. This produces an induced magnetic field $\boldsymbol{h}$ and induced electric field $\boldsymbol{E}$, which satisfy the linearized equations of electromagnetism and are valid for slowly moving media, Ezzat [33]:

The first set of equations constitutes the equations of electrodynamics of slowly moving bodies:

$$\operatorname{curl} \boldsymbol{h} = \boldsymbol{J} + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}, \tag{1}$$

$$\operatorname{curl} \boldsymbol{E} = -\mu_o \frac{\partial \boldsymbol{h}}{\partial t}, \tag{2}$$

$$\boldsymbol{E} = -\mu_o \left(\frac{\partial \boldsymbol{u}}{\partial t} \wedge \boldsymbol{H} \right), \tag{3}$$

$$\operatorname{div} \boldsymbol{h} = 0. \tag{4}$$

Here the vectors $\boldsymbol{h}$ and $\boldsymbol{E}$ denote perturbations of the magnetic and electric fields, respectively, $\boldsymbol{J}$ is the electric current density vector, $\boldsymbol{H}$ the initial constant magnetic field, $\boldsymbol{u}$ the displacement vector and μ_o and ε_o are the magnetic permeability in vacuum and electric permittivity in vacuum, respectively.

The second group of equations is the equations of motion:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sigma_{ij,j} + T_{ij,j} + X_i, \tag{5}$$

where σ_{ij} is the stress tensor represents the constitutive equation:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma \left(T - T_0 + \nu \frac{\partial T}{\partial t} \right) \delta_{ij}, \tag{6}$$

and T_{ij} the Maxwell electromagnetic stress tensor related to the quantity h in the following manner [35]:

$$T_{ij} = \mu_o \left[H_i h_j + H_j h_i - \delta_{ij}(h_k H_k) \right], \tag{7}$$

so that the quantity $T_{ij,j} = \mu_o \in_{ijk} J_j H_k$ is the i-component of the Lorentz force, and X_i are the components of the body forces.

The above equations should be supplemented by the relations between strain and displacements

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{8}$$

and the heat conduction equation

$$k\varphi_{,ii} = \rho C_E\left(\frac{\partial T}{\partial t} + \tau\frac{\partial^2 T}{\partial t^2}\right) + \gamma\varphi_o\frac{\partial e}{\partial t}. \tag{9}$$

In the theory of two-temperature, the reference temperature is $\varphi_0 = T_0$, the Fourier law is $q_i = -k\varphi_{,i}$ and the relation between the thermodynamic absolute temperature T and conductive absolute temperature φ is given as [6]

$$\varphi - T = a\varphi_{,ii}, \tag{10}$$

where $a > 0$, and $[a] = m^2$ is the temperature discrepancy [4, 6].

In the above equations a comma denotes material derivatives and the summation convention are used.

The previous equations constitute a complete system of two-temperature generalized magneto-thermoelasticity with two relaxation times equations for a medium with a perfect electric conductivity.

Now, we shall consider a homogeneous isotropic thermoelastic conducting solid occupying half-space $x \geq 0$, which obey (1)–(9) when the body force is absent and the magnetization disregarded. The system is initially quiescent where all the state functions are depending only on the variable x and the time t.

The displacement vector has components

$$u_x = u(x,t), \qquad u_y = u_z = 0.$$

The strain component takes the form

$$e = e_{xx} = \frac{\partial u}{\partial x}. \tag{11}$$

The heat conduction equation is given by

$$k\frac{\partial^2\varphi}{\partial x^2} = \rho C_E\left(\frac{\partial T}{\partial t} + \tau\frac{\partial^2 T}{\partial t^2}\right) + \gamma T_o\frac{\partial e}{\partial t}. \tag{12a}$$

Substitution (10) into the heat conduction equation (12a) yields for the considered solid

$$\kappa\frac{\partial^2\varphi}{\partial x^2} + a\frac{\partial^3\varphi}{\partial x^2\partial t} + a\tau\frac{\partial^4\varphi}{\partial x^2\partial t^2} = \frac{\partial\varphi}{\partial t} + \tau\frac{\partial^2\varphi}{\partial t^2} + \varphi_o\frac{\partial e}{\partial t}, \tag{12b}$$

where $\kappa = \frac{k}{\rho C_E}$ is the diffusivity.

The constitutive equation will be

$$\sigma = \sigma_{xx} = (2\mu+\lambda)e - \gamma\left[(\varphi-\varphi_o) - a\frac{\partial^2\varphi}{\partial x^2} + \nu\frac{\partial\varphi}{\partial t} - \nu a\frac{\partial^3\varphi}{\partial x^2\partial t}\right]. \tag{13}$$

For the linear two-temperature thermoelasticity theory it is assumed that [4]:

$$\delta = \mathrm{Max}\left\{|\varphi-\varphi_o|, |\nabla\varphi|, |\nabla\nabla\varphi|, |\nabla\dot{\varphi}|, |\nabla\nabla\dot{\varphi}|\right\}$$

is small.

A constant magnetic field with components $(0, H_o, 0)$ is permeating the medium. The induced magnetic field h will have one component in the y-direction, while the induced electric field $\boldsymbol{E}$ will have one component in z-direction. Then, (1)–(3) yield.

$$J = \left(\frac{\partial h}{\partial x} - \varepsilon_0\frac{\partial E}{\partial t}\right), \tag{14}$$

$$h = -H_0 e, \tag{15}$$

$$E = -\mu_0 H_0\frac{\partial u}{\partial t}. \tag{16}$$

Expressing the components of the vector J in terms of displacement, by eliminating from (14) the quantities h and E and introducing them into the displacement equation (5), we get

$$\frac{\partial^2\sigma}{\partial x^2} + \mu_o H_o^2\frac{\partial^2 e}{\partial x^2} = \rho\alpha\frac{\partial^2 e}{\partial t^2}, \tag{17}$$

where $\alpha = 1 + \frac{\alpha_0^2}{c^2}$ and $\alpha_o = \sqrt{\frac{\mu_o H_o^2}{\rho}}$ is the Alfven velocity.

Let us introduce the following non-dimensional variables:

$$x^* = c_o\eta_o x, \qquad u^* = c_o\eta_o u, \qquad t^* = c_o^2\eta_o t,$$

$$\tau^* = c_o^2\eta_o\tau, \qquad \nu^* = c_o^2\eta_o\nu, \qquad \theta^* = \frac{\gamma(T-T_o)}{\rho c_o^2},$$

$$\varphi^* = \frac{\gamma(\varphi-\varphi_o)}{\rho c_o^2}, \qquad \eta_o = \frac{\rho C_E}{k_o} = \frac{1}{\kappa},$$

$$\varepsilon = \frac{\delta_o\gamma}{\rho C_E}, \qquad \sigma^* = \frac{\sigma}{\rho c_o^2},$$

$$E^* = \frac{E}{\mu_o H_o c_o}, \qquad c_o^2 = \frac{\lambda+2\mu}{\rho},$$

$$h^* = \frac{h}{H_o}, \qquad T^*_{ij} = \frac{T_{ij}}{\mu_o H_o^2}.$$

Using the above values, then we have (dropping the asterisks for convenience):

$$h = -e, \tag{18}$$

$$E = -\frac{\partial u}{\partial t}, \tag{19}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \left(\frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2}\right)\theta + \varepsilon \frac{\partial e}{\partial t}, \tag{20a}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \left(1 - \beta_o \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2}\right)\varphi + \varepsilon \frac{\partial e}{\partial t}, \tag{20b}$$

$$\sigma = e - \left(1 + \nu \frac{\partial}{\partial t}\right)\theta, \tag{21}$$

$$\frac{\partial^2 \sigma}{\partial x^2} + \beta \frac{\partial^2 e}{\partial x^2} = \alpha \frac{\partial^2 e}{\partial t^2}, \tag{22}$$

$$\varphi - \theta = \beta_o \frac{\partial^2 \varphi}{\partial x^2}, \tag{23}$$

where $\beta = \frac{\alpha_o^2}{c_o^2}$ and $\beta_o = a c_o^2 \eta_o^2$.

Taking the Laplace transforms defined by the relation

$$\overline{f}(s) = \int_0^\infty e^{-st} f(t) dt,$$

of both sides of (18)–(23), we obtain:

$$\bar{h} = -\bar{e}, \tag{24}$$

$$\bar{E} = -s\bar{u}, \tag{25}$$

$$\frac{\partial^2 \bar{\varphi}}{\partial x^2} = (s + \tau s^2)\bar{\theta} + s\varepsilon\bar{e}, \tag{26}$$

$$\bar{\sigma} = \bar{e} - (1 + \nu s)\bar{\theta}, \tag{27}$$

$$\frac{\partial^2 \bar{\sigma}}{\partial x^2} + \beta \frac{\partial^2 \bar{e}}{\partial x^2} = \alpha s^2 \bar{e}, \tag{28}$$

$$\bar{\varphi} - \bar{\theta} = \beta_o \frac{\partial^2 \bar{\varphi}}{\partial x^2}, \tag{29}$$

where all the initial state functions are equal to zero.

The conductive temperature attains its minimum value at $\beta_o^c = \varepsilon(1 + \varepsilon)^{-3}$ and equal to the thermodynamic temperature at $\beta_o = 0$ [10].

Determining the conductive temperature φ it is easy to obtain the thermodynamic temperature T by simple operations using (23), while knowing T it needs to solve second order differential equation to get φ from (23). Therefore, usually the problem is solving in φ rather than in T.

Eliminating $\bar{e}$ and $\bar{\theta}$ from (26)–(29), we obtain

$$\frac{\partial^2 \bar{\varphi}}{\partial x^2} = L_1 \bar{\varphi} + L_2 \bar{\sigma}, \tag{30}$$

where

$$L_1 = \frac{s + \tau s^2 + \varepsilon s(1 + \nu s)}{1 + \beta_o[s + \tau s^2 + s\varepsilon(1 + \nu s)]},$$

$$L_2 = \frac{\varepsilon s}{1 + \beta_o[s + \tau s^2 + s\varepsilon(1 + \nu s)]},$$

and

$$\frac{\partial^2 \bar{\sigma}}{\partial x^2} = M_1 \bar{\varphi} + M_2 \bar{\sigma}, \tag{31}$$

where

$$M_1 = \frac{m(\alpha s^2 - \beta L_1)}{1 + n\beta}, \qquad M_2 = \frac{n\alpha s^2 - m\beta L_2}{1 + n\beta}$$

$$m = (1 + \nu s)(1 - \beta_o L_1), \text{ and}$$

$$n = 1 - \beta_o L_2(1 + \nu s)$$

Choosing as state variables the temperature of heat conduction $\bar{\varphi}$ and the stress component $\bar{\sigma}$ in the x-direction, (30) and (31) can be written in the matrix form as:

$$\frac{d^2 \bar{v}(x, s)}{dx^2} = A(s)\bar{v}(x, s), \tag{32}$$

where

$$\bar{v}(x, s) = \begin{bmatrix} \bar{\varphi}(x, s) \\ \bar{\sigma}(x, s) \end{bmatrix} \quad \text{and} \quad A(s) = \begin{bmatrix} L_1 & L_2 \\ M_1 & M_2 \end{bmatrix}.$$

The formal solution of system (32) can be written in the form

$$\bar{v}(x, s) = \exp\left[-\sqrt{A(s)}x\right]\bar{v}(0, s), \tag{33}$$

where

$$\bar{v}(0, s) = \begin{bmatrix} \bar{\varphi}(0, s) \\ \bar{\sigma}(0, s) \end{bmatrix} = \begin{bmatrix} \bar{\varphi}_o \\ \bar{\sigma}_o \end{bmatrix}$$

where for bounded solution with large x, we have canceled the part of exponential that has a positive power.

We shall use the well-known Cayley-Hamilton theorem to find the form of the matrix $\exp[-\sqrt{A(s)}x]$.

The characteristic equation of the matrix $A(s)$ can be written as follows:

$$k^2 - k(L_1 + M_2) + (L_1 M_2 - L_2 M_1) = 0. \tag{34}$$

The roots of this equation, namely, k_1 and k_2, satisfy the following relations:

$$k_1 + k_2 = L_1 + M_2, \tag{35a}$$

$$k_1 k_2 = L_1 M_2 - L_2 M_1. \tag{35b}$$

The Taylor series expansion of the matrix exponential in (33) has the form

$$\exp\left[-\sqrt{A(s)}x\right] = \sum_{n=0}^{\infty} \frac{[-\sqrt{A(s)}x]^n}{n!}. \tag{36}$$

Using the Cayley-Hamilton theorem, we can express A^2 and higher orders of the matrix A in terms of I and A, where I is the unit matrix of second order.

Thus, the infinite series in (36) can be reduced to

$$\exp\left[-\sqrt{A(s)}x\right] = a_o(x,s)I + a_1(x,s)A(s), \tag{37}$$

where a_o and a_1 are coefficients depending on x and s.

By the Cayley-Hamilton theorem, the characteristic roots k_1 and k_2 of the matrix A must satisfy (37), thus

$$\exp\left[-\sqrt{k_1}x\right] = a_o + a_1 k_1, \tag{38}$$

and

$$\exp\left[-\sqrt{k_2}x\right] = a_o + a_1 k_2. \tag{39}$$

The solution of the above system is given by

$$a_o = \frac{k_1 e^{-\sqrt{k_2}x} - k_2 e^{-\sqrt{k_1}x}}{k_1 - k_2}, \quad \text{and}$$

$$a_1 = \frac{e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x}}{k_1 - k_2}.$$

Hence, we have

$$\exp\left[-\sqrt{k_1}x\right] = L_{ij}(x,s), \quad i,j = 1,2, \quad \text{where}$$

$$L_{11} = \frac{e^{-\sqrt{k_2}x}(k_1 - L_1) - e^{-\sqrt{k_1}x}(k_2 - L_1)}{k_1 - k_2},$$

$$L_{12} = \frac{L_2(e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x})}{k_1 - k_2},$$

$$L_{22} = \frac{e^{-\sqrt{k_1}x}(k_2 - M_2) - e^{-\sqrt{k_2}x}(k_1 - M_2)}{k_1 - k_2},$$

$$L_{21} = \frac{M_1(e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x})}{k_1 - k_2}. \tag{40}$$

The solution in (33) can be written in the form

$$\bar{v}(x,s) = L_{ij}\bar{v}(0,s). \tag{41}$$

Hence, we obtain

$$\bar{\varphi}(x,s) = \frac{(k_1\bar{\varphi}_o - L_1\bar{\varphi}_o - L_2\bar{\sigma}_o)e^{-\sqrt{k_2}x} - (k_2\bar{\varphi}_o - L_1\bar{\varphi}_o - L_2\bar{\sigma}_o)e^{-\sqrt{k_1}x}}{k_1 - k_2}, \tag{42}$$

$$\bar{\sigma}(x,s) = \frac{(k_1\bar{\sigma}_o - M_1\bar{\varphi}_o - M_2\bar{\sigma}_o)e^{-\sqrt{k_2}x} - (k_2\bar{\sigma}_o - M_1\bar{\varphi}_o - M_2\bar{\sigma}_o)e^{-\sqrt{k_1}x}}{k_1 - k_2}. \tag{43}$$

By using (42) and (43) with (29) we get

$$\bar{\theta}(x,s) = \frac{(k_1\bar{\varphi}_o - L_1\bar{\varphi}_o - L_2\bar{\sigma}_o)(1 - \beta_o k_2)e^{-\sqrt{k_2}x} - (k_2\bar{\varphi}_o - L_1\bar{\varphi}_o - L_2\bar{\sigma}_o))(1 - \beta_o k_1)e^{-\sqrt{k_1}x}}{k_1 - k_2}. \tag{44}$$

It should be noted that the corresponding expressions for generalized thermoelasticity of two-temperature with relaxation time in the absence of magnetic field can be deduced by setting $\alpha = 1.0$ and $\beta = 0$ in (40).

We consider a semi-space homogeneous elastic medium of perfect conductivity occupying the region $x \geq 0$ with quiescent initial state and boundary conditions in the following form:

(i) Thermal boundary condition:

A thermal shock is applied to the boundary plane $x = 0$ in the form

$$\varphi(0,t) = \Phi_o H(t) \quad \text{or} \quad \bar{\varphi}(0,s) = \bar{\varphi}_o = \frac{\Phi_o}{s}, \tag{45}$$

where Φ_o is a constant and $H(t)$ is the Heaviside unit step function.

(ii) Mechanical boundary condition:

The bounding plane $x = 0$ is taken to be traction-free, i.e.

$$\sigma(0,t) + T_{11}(0,t) - T_{11}^o(0,t) = 0, \tag{46}$$

where T_{11}^o is the Maxwell stress tensor in a vacuum.

Since the transverse components of the vectors E and h are continuous across the bounding plane, i.e. $E(0,t) = E^0(0,t)$ and $h(0,t) = h^0(0,t)$, $t > 0$, where E^0 and h^0 are the components of the induced electric and magnetic field in free space and the relative permeability is very nearly unity, it follows that $T_{11}(0,t) = T_{11}^o(0,t)$ and (46) reduces to [33]:

$$\sigma(0,t) = 0, \quad \text{or} \quad \bar{\sigma}(0,s) = \bar{\sigma}_0 = 0. \tag{47}$$

Hence, we can use the conditions on (45) and (46) into (42) and (43) to get the exact solution in the Laplace transform domain in the following forms:

$$\bar{\varphi}(x,s) = \frac{\Phi_o[(k_1 - L_1)e^{-\sqrt{k_2}x} - (k_2 - L_1)e^{-\sqrt{k_1}x}]}{s(k_1 - k_2)}, \tag{48}$$

$$\bar{\sigma}(x,s) = \frac{\Phi_o M_1(e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x})}{s(k_1 - k_2)}, \tag{49}$$

$$\bar{\theta}(x,s) = \frac{\Phi_o[Be^{-\sqrt{k_2}x} - Ae^{-\sqrt{k_1}x}]}{s(k_1 - k_2)}, \tag{50}$$

$$\bar{e}(x,s) = \frac{\Phi_o[[m(k_1 - L_1) - nM_1]e^{-\sqrt{k_2}x} - [m(k_2 - L_1) - nM_1]e^{-\sqrt{k_1}x}]}{s(k_1 - k_2)}, \tag{51}$$

where $A = (k_2 - L_1)(1 - \beta_o k_1)$, $B = (k_1 - L_1)(1 - \beta_o k_2)$.

From (11), the displacement takes the form:

$$\bar{u}(x,s) = \frac{\Phi_o(Ce^{-\sqrt{k_1}x} - De^{-\sqrt{k_2}x})}{s(k_1 - k_2)},$$

where $C = \dfrac{m(k_2 - L_1) - nM_1}{\sqrt{k_1}}$,

$$D = \frac{m(k_1 - L_1) - nM_1}{\sqrt{k_2}}. \tag{52}$$

The induced magnetic and electric field takes the following forms

$$\bar{h}(x,s) = -\frac{\Phi_o[[m(k_1 - L_1) - nM_1]e^{-\sqrt{k_2}x} - [m(k_2 - L_1) - nM_1]e^{-\sqrt{k_1}x}]}{s(k_1 - k_2)}, \tag{53}$$

$$\bar{E}(x,s) = -\frac{\Phi_o(Ce^{-\sqrt{k_1}x} - De^{-\sqrt{k_2}x})}{s^2(k_1 - k_2)}. \tag{54}$$

Those complete the solution in the Laplace transform domain.

3 Inversion of the Laplace transforms

In order to invert the Laplace transform in the above equations, we adopt a numerical inversion method based on a Fourier series expansion [34]. In this method, the inverse $g(t)$ of the Laplace transform $\bar{g}(s)$

is approximated by the relation

$$g(t) = \frac{e^{ct}}{t_1}\left[\frac{1}{2}\bar{g}(c) + \mathrm{Re}\left(\sum_{k=1}^{\infty} e^{ik\pi t/t_1}\bar{g}(c+ik\pi/t_1)\right)\right],$$

$$0 \le t \le 2t_1, \tag{55}$$

where N is a sufficiently large integer representing the number of terms in the truncated infinite Fourier series and it must chosen such that

$$e^{ct}\mathrm{Re}\left[e^{iN\pi t/t_1}\bar{g}(c+iN\pi/t_1)\right] \le \varepsilon_1,$$

where ε_1 is a persecuted small positive number that corresponds to the degree of accuracy to be achieved. The parameter c is a positive free parameter that must be greater than the real parts of all singularities of $\bar{g}(s)$. The optimal choice of c was obtained according to the criteria described in [34].

4 Numerical results

The copper material was chosen for purposes of numerical evaluations. The constants of the problem were taken as following [36] (see Table 1).

The computations were carried out for $t = 0.1$ and $t = 0.2$. Formula (55) was used to invert the Laplace transforms in (48)–(54) and giving the conductive temperature, the thermodynamic temperature, the stress, the displacement, the strain, induced magnetic field and the induced electric field distributions. The results are represented graphically at different positions of x.

In Figs. 1–7, we noticed the difference in all functions for the value of the non-dimensional temperature discrepancy β_o where the case of $\beta_o = 0.0$ indicates the old situation (one type temperature) and the case $\beta_o = 0.075$ [10], indicates the new case (two-temperature).

In all figures we notice that the curves are smoother in the case $\beta_o = 0.075$.

In Figs. 1–3, we observe that at time $t = 0.1$, the conductive temperature, the thermo-dynamical tem-

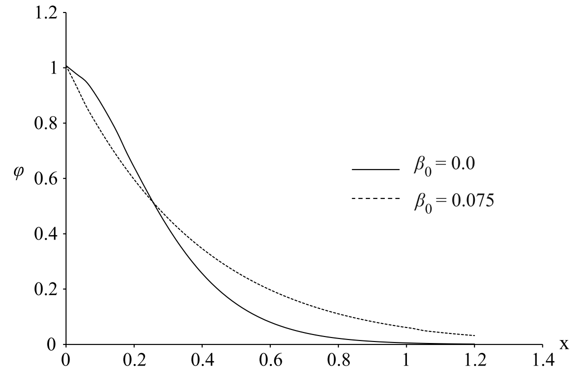

Fig. 1 The conductive temperature distribution

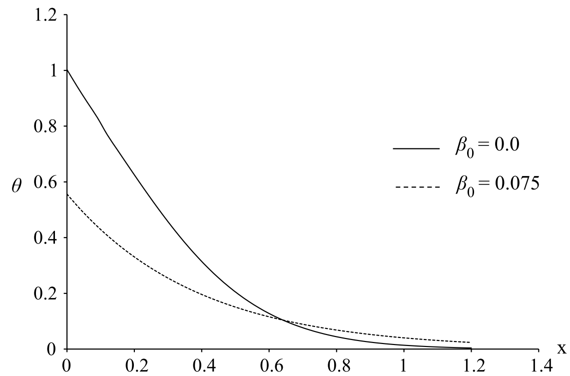

Fig. 2 The thermodynamic temperature distribution

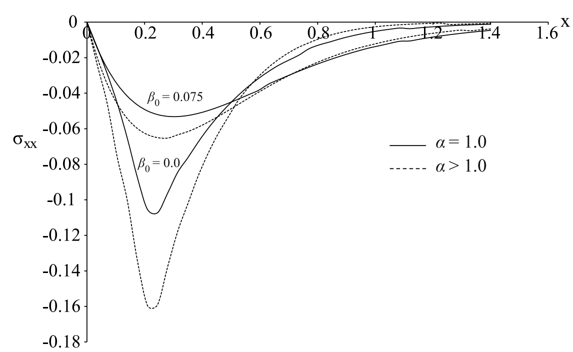

Fig. 3 The stress distribution

Table 1 Values of the constants

$k = 386$ N/Ks, $\alpha_T = 1.78 \times 10^{-5}$ K^{-1}, $C_E = 383.1$ m^2/K, $\eta_o = 8886.73$ s/m^2, $\mu = 3.86 \times 10^{10}$ N/m^2, $\lambda = 7.76 \times 10^{10}$ N/m^2,
$\rho = 8954$ kg/m^3, $T_o = 293$ K, $c_o = 415$ m/s, $\varepsilon = 0.0168$, $\tau = 0.002$ s, $\varepsilon_o = 8.854 \times 10^{-12}$ C^2/Nm^2, $\mu_o = 1.256 \times 10^{-6}$ Ns^2/C^2,
$\nu = 0.003$ s, $B_o = \mu_o H_o = 1$ Tesla, $\alpha = 1.001$, $\beta = 3.8 \times 10^{12}$.

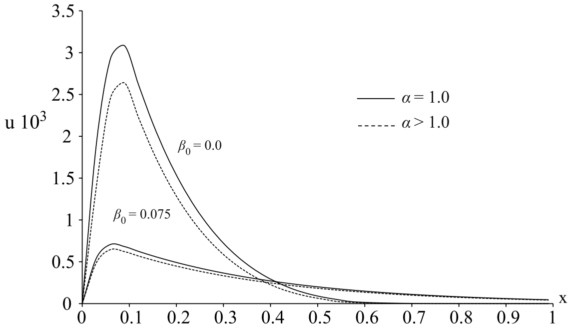

Fig. 4 The displacement distribution

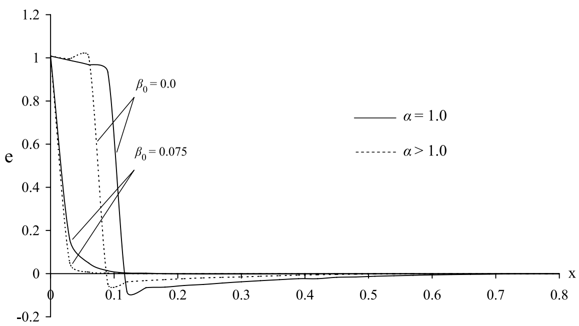

Fig. 5 The strain distribution

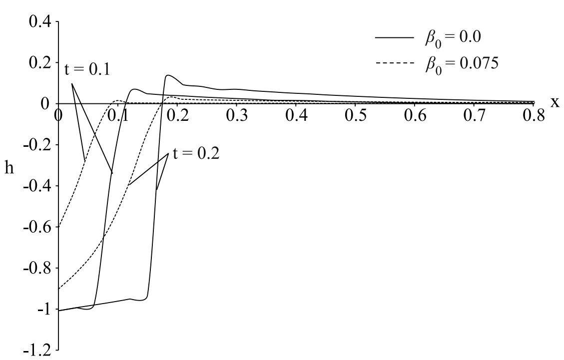

Fig. 6 The induced magnetic field distribution

perature and the stress waves cut the x-axis rapidly when $\beta_o = 0.0$ than when $\beta_o = 0.075$.

In Figs. 3–7 exhibiting the space variation of the displacement, the strain and the stress as well as the induced magnetic and electric field, we observe the following:

(i) Significant differences in the stress are noticed for different value of the non-dimensional temperature discrepancy in presence ($\alpha > 1.0$) or absence ($\alpha = 1.0$) of magnetic field.

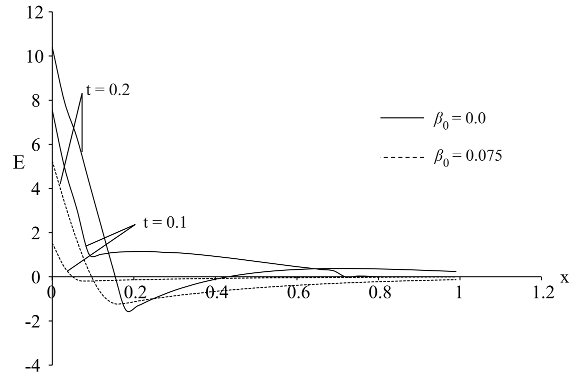

Fig. 7 The induced electric field distribution

(ii) The absolute value of the maximum stress decreases when $\beta_o = 0.075$ and increases when $\beta_o = 0.0$.

(iii) The magnetic field acts to decrease the displacement, the strain and the magnitude of the stress component. This is mainly due to the fact that the magnetic field corresponds to term signifying positive force that tends to accelerate the charge carriers.

(iv) The absolute value of the induced magnetic and electric field decreases when $\beta_o = 0.075$ and increases when $\beta_o = 0.0$.

5 Conclusions

Previously, the discontinuity of the stress distribution was a critical situation and no one has explained the reason physically, while in the context of the two-temperature theory of thermoelasticity, the stress function is continuous. This paper indicates that, the two-temperature generalized theory of magneto-thermo-viscoelasticity describes the behavior of the particles of an elastic body more realistically than the one-temperature theory of generalized magneto-thermoelasticity with two relaxation times.

In this work, the method of direct integration by means of the matrix exponential, which is standard approach in modern control theory and developed in detail in many texts [1], is easier than in the classical situation (with one thermodynamic temperature).

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