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ORIGINAL ARTICLE

Vibration reduction of a nonlinear spring pendulum under multi external and parametric excitations via a longitudinal absorber

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Abstract The vibration of a ship pitch-roll motion described by a non-linear spring pendulum system (two degrees of freedom) subjected to multi external and parametric excitations can be reduced using a longitudinal absorber. The method of multiple scale perturbation technique (MSPT) is applied to analyze the response of this system near the simultaneous primary, sub-harmonic and internal resonance. The steady state solution near this resonance case is determined and studied applying Lyapunov's first method. The stability of the system is investigated using frequency response equations. Numerical simulations are extensive investigations to illustrate the effects of the absorber and some system parameters at selected values on the vibrating system. The simulation results are achieved using MATLAB 7.0 programs. Results are compared to previously published work.

Keywords Passive vibration control · Spring pendulum · Multi-excitation · Stability

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Nomenclature

c_j $(j = 1, 2, 3, 4)$	The damping coefficient of the spring pendulum degree-of-freedom and the absorber $(c_j = \varepsilon\hat{c}_j)$
ω_1, ω_2 and ω_3	The natural frequency of the spring pendulum and absorber
α, β	The non-linear parameters $(\beta = \varepsilon\hat{\beta})$
f_j	The forcing amplitude of the main system $(f_j = \varepsilon^2\hat{f}_j)$
Ω_j	The frequencies of the main system
ε	A small perturbation parameter
g	Gravity acceleration
M, m	Masses of spring pendulum and absorber respectively
l	Statically stretched length of the pendulum
l_1	Statically stretched length of the absorber
$x, \bar{x}$	Longitudinal response of the spring–pendulum $(x = \bar{x}/l)$
$u, \bar{u}$	Longitudinal response of the absorber $(u = \bar{u}/l)$
φ	Angular response of the pendulum
k_1, k_2	Linear stiffness of the spring–pendulum and the absorber
k_i $(i = 3, 4, 5, 6)$	Spring stiffness of non-linear parameters

1 Introduction

It is well known that both large amplitudes of vibration and dynamic chaos are undesired phenomenon in all engineering systems. It is necessary to get rid of them or to keep it to the minimum possible levels. The dynamic absorber represents a typical method for the suppression of vibrations through passive controls that can be added to the most engineering vibrating systems [1, 2]. Osama et al. [3] reported how the ship roll can be controlled using active and passive control. The spring pendulum system is one of the famous dynamically systems simulating many engineering applications and one of them is the ship roll motion [4, 5]. Lee et al. [6–8] studied the behavior of the spring–pendulum system subjected to single harmonic excitation force. They found that the system has a very complex behavior including jump phenomena and Hopf bifurcations. Moreover, they observed that the second-order approximation gives better agreement with the main system than the first-order approximation does. Eissa and El-Ganaini [9, 10] studied the control of both vibration and dynamic chaos of both internal combustion engines and mechanical structures having quadratic and cubic nonlinearties, subjected to multi excitation forces using single and multi-absorbers. Song et al. [11] investigated the vibration response of the spring–mass–damper system with a parametrically excited pendulum hinged to the mass applying the harmonic balance method. They showed that the area of unstable motion of the system obtained from the third order approximations to be fairly consistent with that obtained from numerical calculation. Eissa et al. [12–16] studied the effects of different controllers on both simple and spring pendulum describing ship roll motion. They studied the effects of the transverse and longitudinal tuned absorbers on both simple and spring pendulum system. It is worth to notice that their study is limited to single external excitation force only for each degree-of-freedom. Kamel [17] investigated the response and stability of two degree-of-freedom ship model under sinusoidal harmonic excitation. He obtained the bifurcation response equation near the combination resonance case in the presence of internal resonance of this system. El-Sayed et al. [18, 19] studied the nonlinear dynamics of multi-degree-of-freedom vibrating system using MSPT up to the third order approximation. They reported how effective is the passive vibration control at resonance. Eissa et al. [20] studied the vibration and stability of the non-linear spring pendulum simulating the ship roll motion. Furthermore, they illustrated the effects of the transverse absorber on the non-linear spring pendulum system. It is worth to notice that such study is limited to multi-parametric excitations for each degree-of-freedom.

In this paper, the vibration of a non-linear spring pendulum describing ship roll motion can be stabilized and controlled via a longitudinal absorber. This leads to a three-degrees-of-freedom system subjected to both multi external and parametric excitations. Multiple scale perturbation technique is applied to analyze the response of the system near the simultaneous primary, sub-harmonic and internal resonance to obtain semi-closed solution to the second order approximations. The steady state solution near the selected resonance case is studied using frequency response equations. The stability condition of the nonlinear solution is determined. The numerical solution and the effects of some system parameters on the vibrating system are investigated and reported. Optimum working conditions of the system are extracted when applying passive control methods. Comparison with the available published work is reported.

2 Mathematical analysis

The diagrammatic sketch of the considered system is shown in Fig. 1. The absorber mass can move in the longitudinal direction to the pendulum axis. Here φ is the angular displacement of the pendulum which is assumed small (less than 5°, because if it is large one it will cause ship capsize) and x, u are the extensions of the spring and absorber respectively from their equilibrium positions.

The modified model of the non-linear spring pendulum that describes the ship roll motion with absorber in the longitudinal direction [15, 20] is considered as

$$\begin{aligned}&\ddot{x} + \varepsilon\hat{c}_1\dot{x} + \omega_1^2 x + \alpha_1 x^2 + \alpha_2 x^3 - (1+x)\dot{\varphi}^2 \\ &\quad + \omega_2^2(1-\cos\varphi) + \varepsilon\hat{\beta}\ddot{u} - \varepsilon\hat{\beta}(r+u)\dot{\varphi}^2 - \varepsilon\hat{c}_3\dot{u} \\ &\quad = \varepsilon^2\sum_{j=1}^{n}\hat{f}_{1j}\cos(\Omega_{1j}T_0) + \varepsilon^2 x\sum_{j=1}^{n}\hat{f}_{2j}\cos(\Omega_{2j}T_0)\end{aligned} \tag{1}$$

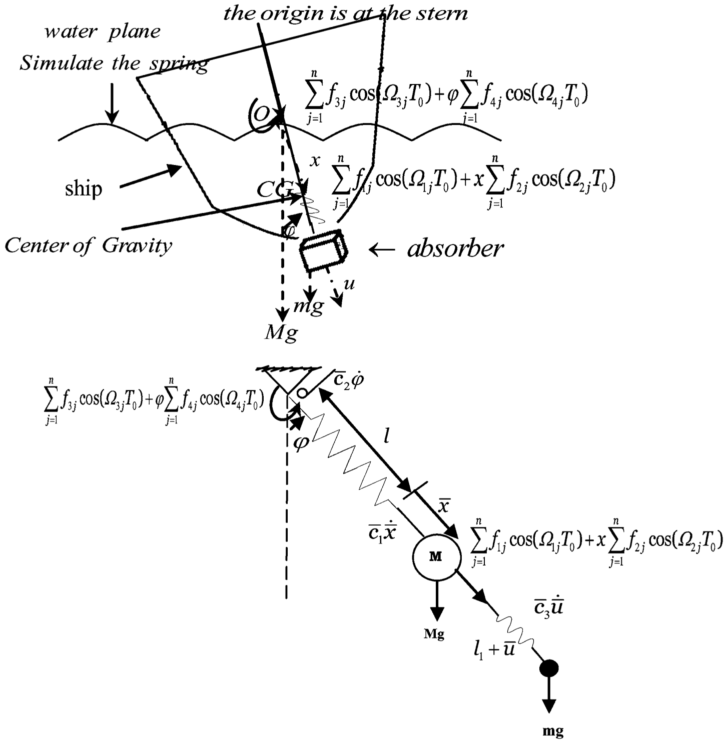

Fig. 1 Diagrammatic representation of the system

$$
\begin{aligned}
&(1+x)^2\ddot{\varphi} + \varepsilon\hat{c}_2\dot{\varphi} + 2(1+x)\dot{x}\dot{\varphi} + \omega_2^2(1+x)\sin\varphi \\
&\quad + \varepsilon\hat{\beta}[(2+2x+r+u)(r+u)\ddot{\varphi}] \\
&\quad + 2\varepsilon\hat{\beta}[(1+x+r+u)\dot{u}\dot{\varphi}] + 2\varepsilon\hat{\beta}[(r+u)\dot{x}\dot{\varphi}] \\
&\quad + \varepsilon\hat{\beta}\omega_2^2(r+u)\sin\varphi \\
&= \varepsilon^2\sum_{j=1}^{n}\hat{f}_{3j}\cos(\Omega_{3j}T_0) + \varepsilon^2\varphi\sum_{j=1}^{n}\hat{f}_{4j}\cos(\Omega_{4j}T_0)
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
&\ddot{u} + \omega_3^2 u + \varepsilon\hat{c}_4\dot{u} + \ddot{x} - (1+x+r+u)\dot{\varphi}^2 \\
&\quad + \omega_2^2(1-\cos\varphi) = 0
\end{aligned} \tag{3}
$$

where:

$$
c_1 = \frac{\bar{c}_1}{M+m}, \qquad c_2 = \frac{\bar{c}_2}{(M+m)l^2},
$$

$$
c_3 = \frac{\bar{c}_3}{M+m}, \qquad c_4 = \frac{\bar{c}_3}{m}, \qquad \omega_1^2 = \frac{k_1}{M+m},
$$

$$
\omega_2^2 = \frac{g}{l}, \qquad \omega_3^2 = \frac{k_4}{m}, \qquad \alpha_1 = \frac{k_2 l}{M+m},
$$

$$
\alpha_2 = \frac{k_3 l^2}{M+m}, \qquad \beta = \frac{m}{M+m}, \qquad r = \frac{l_1}{l}
$$

The method of perturbation (MSPT) is used to obtain a uniformly valid, asymptotic expansion of the solutions for (1)–(3), taking into account the resonance condition $\Omega_{11} \cong \omega_1, \Omega_{21} \cong 2\omega_1, \Omega_{31} \cong \omega_2, \Omega_{41} \cong 2\omega_2$ and $\omega_3 \cong \omega_1$. Introducing the detuning parameters $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, and σ_5 ($\sigma_z = \varepsilon\hat{\sigma}_z, z = 1, 2, \ldots, 5$) according to

$$
\begin{aligned}
&\Omega_{11} \cong \omega_1 + \varepsilon\hat{\sigma}_1, \qquad \Omega_{21} \cong 2\omega_1 + \varepsilon\hat{\sigma}_2, \\
&\Omega_{31} \cong \omega_2 + \varepsilon\hat{\sigma}_3, \qquad \Omega_{41} \cong 2\omega_2 + \varepsilon\hat{\sigma}_4 \quad \text{and} \\
&\omega_3 \cong \omega_1 + \varepsilon\hat{\sigma}_5
\end{aligned} \tag{4}
$$

The asymptotic approximate solution of (1)–(3) will be in the form:

$$
x(t;\varepsilon) = \sum_{m=1}^{3}\varepsilon^m x_m(T_0, T_1, T_2) + O(\varepsilon^4) \tag{5a}
$$

$$
\varphi(t;\varepsilon) = \sum_{m=1}^{3}\varepsilon^m \varphi_m(T_0, T_1, T_2) + O(\varepsilon^4) \tag{5b}
$$

$$
u(t;\varepsilon) = \sum_{m=1}^{3}\varepsilon^m u_m(T_0, T_1, T_2) + O(\varepsilon^4), \tag{5c}
$$

where $T_p = \varepsilon^p t$ and $D_p = \partial/\partial T_p$ $(p = 0, 1, 2)$.

Substituting (5) and its derivatives into (1)–(3) and equating the coefficients of like power of ε, we get the following

$o(\varepsilon)$:

$$(D_0^2 + \omega_1^2)x_1 = 0 \tag{6a}$$

$$(D_0^2 + \omega_2^2)\varphi_1 = 0 \tag{6b}$$

$$(D_0^2 + \omega_3^2)u_1 = -D_0^2 x_1 \tag{6c}$$

$o(\varepsilon^2)$:

$$\begin{aligned}(D_0^2 + \omega_1^2)x_2 &= -2D_0D_1x_1 - \hat{c}_1 D_0 x_1 - \alpha_1 x_1^2 + (D_0\varphi_1)^2 \\ &\quad - \omega_2^2\frac{\varphi_1^2}{2} - \hat{\beta} D_0^2 u_1 + \hat{c}_3 D_0 u_1 \\ &\quad + \sum_{j=1}^{n} \hat{f}_{1j}\cos(\Omega_{1j}T_0)\end{aligned} \tag{7a}$$

$$\begin{aligned}(D_0^2 + \omega_2^2)\varphi_2 &= -2D_0D_1\varphi_1 - 2x_1 D_0^2\varphi_1 - \hat{c}_2(D_0\varphi_1) \\ &\quad - 2D_0x_1D_0\varphi_1 - \omega_2^2 x_1\varphi_1 - (\hat{\beta}r^2 + 2\hat{\beta}r)D_0^2\varphi_1 \\ &\quad - \hat{\beta}r\omega_2^2\varphi_1 + \sum_{j=1}^{n} \hat{f}_{3j}\cos(\Omega_{3j}T_0)\end{aligned} \tag{7b}$$

$$\begin{aligned}(D_0^2 + \omega_3^2)u_2 &= -2D_0D_1u_1 - \hat{c}_4 D_0 u_1 - (D_0^2 x_2 + 2D_0D_1x_1) \\ &\quad + (r+1)(D_0\varphi_1)^2 - \omega_2^2\frac{\varphi_1^2}{2}\end{aligned} \tag{7c}$$

$o(\varepsilon^3)$:

$$\begin{aligned}(D_0^2 + \omega_1^2)x_3 &= -2D_0D_1x_2 - (D_1^2 + 2D_0D_2)x_1 \\ &\quad - \hat{c}_1(D_1x_1 + D_0x_2) - 2\alpha_1 x_1 x_2 - \alpha_2 x_1^3 \\ &\quad + 2D_0\varphi_1 D_0\varphi_2 + 2D_0\varphi_1 D_1\varphi_1 + x_1(D_0\varphi_1)^2 \\ &\quad - \omega_2^2\varphi_1\varphi_2 - \hat{\beta}(D_0^2 u_2 + 2D_0D_1u_1) \\ &\quad + \hat{\beta}r(D_0\varphi_1)^2 + \hat{c}_3(D_1u_1 + D_0u_2) \\ &\quad + x_1\sum_{j=1}^{n} \hat{f}_{2j}\cos(\Omega_{2j}T_0)\end{aligned} \tag{8a}$$

$$\begin{aligned}(D_0^2 + \omega_2^2)\varphi_3 &= -2D_0D_1\varphi_2 - 4x_1D_0D_1\varphi_1 - (D_1^2 + 2D_0D_2)\varphi_1 \\ &\quad - x_1^2 D_0^2\varphi_1 - 2x_2 D_0^2\varphi_1 - 2x_1 D_0^2\varphi_2 \\ &\quad - \hat{c}_2(D_1\varphi_1 + D_0\varphi_2) - 2D_0x_1D_0\varphi_2 \\ &\quad - 2D_0x_1D_1\varphi_1 - 2D_1x_1D_0\varphi_1 - 2D_0x_2D_0\varphi_1 \\ &\quad - 2x_1D_0x_1D_0\varphi_1 - \omega_2^2 x_2\varphi_1 + \frac{\omega_2^2\varphi_1^3}{6} - \omega_2^2 x_1\varphi_2 \\ &\quad - 2\hat{\beta}r(u_1D_0^2\varphi_1 + x_1D_0^2\varphi_1 + D_0^2\varphi_2 + 2D_0D_1\varphi_1) \\ &\quad - \hat{\beta}r^2(D_0^2\varphi_2 + 2D_0D_1\varphi_1) - 2\hat{\beta}u_1D_0^2\varphi_1 \\ &\quad - (2\hat{\beta}r + 2\hat{\beta})D_0u_1D_0\varphi_1 - (2\hat{\beta}rD_0x_1D_0\varphi_1) \\ &\quad - \hat{\beta}\omega_2^2(u_1\varphi_1 + r\varphi_2) + \varphi_1\sum_{j=1}^{n} \hat{f}_{4j}\cos(\Omega_{4j}T_0)\end{aligned} \tag{8b}$$

$$\begin{aligned}(D_0^2 + \omega_3^2)u_3 &= -2D_0D_1u_2 - (D_1^2 + 2D_0D_2)u_1 \\ &\quad - \hat{c}_4(D_1u_1 + D_0u_2) - D_0^2x_3 - 2D_0D_1x_2 \\ &\quad - (D_1^2 + 2D_0D_2)x_1 + (u_1 + x_1)(D_0\varphi_1)^2 \\ &\quad + 2(r+1)(D_0\varphi_1)(D_0\varphi_2 + D_1\varphi_1) - \omega_2^2\varphi_1\varphi_2\end{aligned} \tag{8c}$$

The general solutions of (6) can be written in the form

$$x_1 = A_1(T_1, T_2)\exp(i\omega_1 T_0) + \text{c.c.} \tag{9a}$$

$$\varphi_1 = A_2(T_1, T_2)\exp(i\omega_2 T_0) + \text{c.c.} \tag{9b}$$

$$\begin{aligned}u_1 &= A_3(T_1, T_2)\exp(i\omega_3 T_0) \\ &\quad + \Gamma A_1(T_1, T_2)\exp(i\omega_1 T_0) + \text{c.c.}\end{aligned} \tag{9c}$$

where A_m are complex function in T_1 and T_2, c.c. represents the complex conjugate of the previous terms and $\Gamma = \frac{\omega_1^2}{\omega_3^2 - \omega_1^2}$ is a real constant.

Substituting (9) into (7) and using the response condition of (4) leads to secular terms. Eliminating these secular terms leads to solvability conditions for the first-order approximation:

$$\begin{aligned}2i\omega_1 D_1 A_1 &= [-\hat{c}_1 i\omega_1 A_1] + [\hat{\beta}\omega_3^2 A_3 + i\hat{c}_3\omega_3 A_3] \\ &\quad \times \exp(i\hat{\sigma}_5 T_1) + \frac{\hat{f}_{11}}{2}\exp(i\hat{\sigma}_1 T_1)\end{aligned} \tag{10a}$$

$$2i\omega_2 D_1 A_2 = [-i\hat{c}_2\omega_2 A_2 + (\hat{\beta}r^2 + \hat{\beta}r)\omega_2^2 A_2] + \frac{\hat{f}_{31}}{2}\exp(i\hat{\sigma}_3 T_1) \tag{10b}$$

$$2i\omega_3 D_1 A_3 = [-\hat{c}_4 i\omega_3 A_3] + [-2i\omega_1 D_1 A_1] \times \exp(-i\hat{\sigma}_5 T_1) \tag{10c}$$

After eliminating the secular terms, the particular solutions of (7) will be in the form:

$$x_2 = \left[\frac{\alpha_1 A_1^2}{3\omega_1^2}\right]\exp(2i\omega_1 T_0) - \left[\frac{3\omega_2^2}{2(\omega_1^2 - 4\omega_2^2)}A_2^2\right] \times \exp(2i\omega_2 T_0) + \frac{[-\alpha_1 A_1\bar{A}_1 + \frac{\omega_2^2}{2}A_2\bar{A}_2]}{\omega_1^2} + \text{c.c.} \tag{11a}$$

$$\varphi_2 = \frac{[\omega_2^2 A_1 A_2 + 2\omega_1\omega_2 A_1 A_2]}{\omega_2^2 - (\omega_1+\omega_2)^2}\exp(i(\omega_1+\omega_2)T_0) + \frac{[\omega_2^2 A_1\bar{A}_2 - 2\omega_1\omega_2 A_1\bar{A}_2]}{\omega_2^2 - (\omega_1-\omega_2)^2}\exp(i(\omega_1-\omega_2)T_0) + \text{c.c.} \tag{11b}$$

$$u_2 = \frac{[4\alpha_1 A_1^2]}{3(\omega_3^2 - 4\omega_1^2)}\exp(2i\omega_1 T_0) - \frac{[(r+\frac{3}{2}) + \frac{6\omega_2^2}{(\omega_1^2 - 4\omega_2^2)}]\omega_2^2 A_2^2}{\omega_3^2 - 4\omega_2^2}\exp(2i\omega_2 T_0) + \frac{[(r+\frac{1}{2})\omega_2^2 A_2\bar{A}_2]}{\omega_3^2} + \text{c.c.} \tag{11c}$$

Substituting (9), (11) into (8) and using the response condition, then (4) and eliminating the secular terms leads to solvability condition for the second-order approximation:

$$2i\omega_1 D_2 A_1 = [-D_1^2 A_1 - \hat{c}_1 D_1 A_1 + \eta_1 A_1 A_2\bar{A}_2 + \eta_2 A_1^2\bar{A}_1] + [\hat{c}_3 D_1 A_3 - 2i\hat{\beta}\omega_3 D_1 A_3] \times \exp(i\hat{\sigma}_5 T_1) + \left[\frac{\hat{f}_{21}}{2}\bar{A}_1\right]\exp(i\hat{\sigma}_2 T_1) \tag{12a}$$

$$2i\omega_2 D_2 A_2 = [-D_1^2 A_2 - \hat{c}_2 D_1 A_2 - 2i\omega_2\hat{\beta}(r^2 + 2r)D_1 A_2 + \eta_1 A_1\bar{A}_1 A_2 + \eta_3 A_2^2\bar{A}_2] - \left[\frac{\hat{\beta}r\hat{f}_{31}}{2}\right]\exp(i\hat{\sigma}_3 T_1) + \left[\frac{\hat{f}_{41}}{2}\bar{A}_2\right]\exp(i\hat{\sigma}_4 T_1) \tag{12b}$$

$$2i\omega_3 D_2 A_3 = [2\omega_2^2 A_3 A_2\bar{A}_2 - D_1^2 A_3 - \hat{c}_4 D_1 A_3] + [\eta_4 A_1 A_2\bar{A}_2 - 2i\omega_1 D_2 A_1] \times \exp(-i\hat{\sigma}_5 T_1) \tag{12c}$$

where $\eta_q\{q = 1, 2, 3, 4\}$ are constants. (See Appendix.)

3 Periodic solution

The simultaneous primary, sub-harmonic and internal resonance case ($\Omega_{11} \cong \omega_1$, $\Omega_{21} \cong 2\omega_1$, $\Omega_{31} \cong \omega_2$, $\Omega_{41} \cong 2\omega_2$, $\omega_3 \cong \omega_1$) which is the worst resonance case, has been chosen to study the stability from the second-order approximation solution.

From the definition of the derivative and multiplying both sides by $2i\omega_m$ we get:

$$2i\omega_m\frac{dA_m}{dt} = \varepsilon 2i\omega_m D_1 A_m + \varepsilon^2 2i\omega_m D_2 A_m + O(\varepsilon^3) \tag{13}$$

To analyze the solution of (10)–(12), it is convenient to express $A_m(T_1, T_2)$ in the form:

$$A_m = \left(\frac{\hat{a}_m}{2}\right)\exp(i\gamma_m) \qquad a_m = \varepsilon\hat{a}_m \tag{14}$$

where a_m and γ_m are real and represent the steady state amplitudes and phases of the motions respectively. Inserting (14) and (10)–(12) into (13) and equating the real and imaginary parts yields,

$$\dot{a}_1 = \zeta_1 a_1 + \frac{\zeta_2 f_{11}}{2\omega_1}\cos(\theta_1) - \frac{\zeta_3 f_{11}}{2\omega_1}\sin(\theta_1) + \frac{f_{21}a_1}{4\omega_1}\sin(\theta_2) + \zeta_4 a_3\cos(\theta_5) - \zeta_5 a_3\sin(\theta_5) \tag{15a}$$

$$a_1\dot{\gamma}_1 = \zeta_6 a_1 - \frac{\eta_2 a_1^3}{8\omega_1} - \frac{\eta_1 a_1 a_2^2}{8\omega_1} + \frac{\zeta_2 f_{11}}{2\omega_1}\sin(\theta_1) + \frac{\zeta_3 f_{11}}{2\omega_1}\cos(\theta_1) - \frac{f_{21}a_1}{4\omega_1}\cos(\theta_2) + \zeta_4 a_3\sin(\theta_5) + \zeta_5 a_3\cos(\theta_5) \tag{15b}$$

$$\dot{a}_2 = \zeta_7 a_2 + \frac{c_2 f_{31}}{8\omega_2^2}\cos(\theta_3) - \frac{\zeta_8 f_{31}}{4\omega_2}\sin(\theta_3) + \frac{f_{41} a_2}{4\omega_2}\sin(\theta_4) \tag{16a}$$

$$a_2\dot{\gamma}_2 = \zeta_9 a_2 - \frac{\eta_3}{8\omega_2}a_2^3 - \frac{\eta_1}{8\omega_2}a_1^2 a_2 + \frac{c_2 f_{31}}{8\omega_2^2}\sin(\theta_3) + \frac{\zeta_8 f_{31}}{4\omega_2}\cos(\theta_3) - \frac{f_{41} a_2}{4\omega_2}\cos(\theta_4) \tag{16b}$$

$$\begin{aligned}\dot{a}_3 = \zeta_{10} a_3 &+ \left[\zeta_{11} a_1 - \frac{(\eta_4 - \eta_1)}{8\omega_3}a_1 a_2^2 + \frac{\eta_2}{8\omega_3}a_1^3\right] \\ &\times \sin(\theta_5) + \zeta_{12} a_1 \cos(\theta_5) \\ &+ \zeta_{13}\frac{f_{11}}{4\omega_3}\cos(\theta_1 - \theta_5) - \zeta_{14}\frac{f_{11}}{4\omega_3}\sin(\theta_1 - \theta_5) \\ &- \frac{f_{21}}{4\omega_3}a_1 \sin(\theta_2 - \theta_5)\end{aligned} \tag{17a}$$

$$\begin{aligned}a_3\dot{\gamma}_3 = \zeta_{15} a_3 &- \frac{\omega_2^2}{4\omega_3}a_3 a_2^2 + \left[\zeta_{11} a_1 - \frac{(\eta_4 - \eta_1)}{8\omega_3}a_1 a_2^2 \right.\\ &\left. + \frac{\eta_2}{8\omega_3}a_1^3\right]\cos(\theta_5) - \zeta_{12} a_1 \sin(\theta_5) \\ &+ \zeta_{13}\frac{f_{11}}{4\omega_3}\sin(\theta_1 - \theta_5) + \zeta_{14}\frac{f_{11}}{4\omega_3}\cos(\theta_1 - \theta_5) \\ &+ \frac{f_{21}}{4\omega_3}a_1 \cos(\theta_2 - \theta_5)\end{aligned} \tag{17b}$$

where $\theta_1 = \hat{\sigma}_1 T_1 - \gamma_1, \theta_2 = \hat{\sigma}_2 T_1 - 2\gamma_1$, $\theta_3 = \hat{\sigma}_3 T_1 - \gamma_2, \theta_4 = \hat{\sigma}_4 T_1 - 2\gamma_2, \theta_5 = \hat{\sigma}_5 T_1 + \gamma_3 - \gamma_1$ and $\zeta_\upsilon\{\upsilon = 1, 2, \ldots, 15\}$ are constants. (See Appendix.)

Thus, the first approximation periodic solution can be written in the form

$$x_1 = a_1 \cos(\Omega_{11} T_0 - \theta_1) \tag{18a}$$

$$\varphi_1 = a_2 \cos(\Omega_{31} T_0 - \theta_3) \tag{18b}$$

$$u_1 = a_3 \cos(\Omega_{11} T_0 - (\theta_1 - \theta_5)) + \Gamma a_1 \cos(\Omega_{11} T_0 - \theta_1) \tag{18c}$$

where $a_1, a_2, a_3, \theta_1, \theta_3$ and θ_5 are obtained by the solution of (15)–(17).

3.1 Stability of the fixed points

The steady state solution of our dynamical system corresponding to the fixed point of (15)–(17) are obtained when $\dot{a}_n = 0$ and $\dot{\theta}_n = 0$, then we get the frequency response equations (FRE) for practical case (where, $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0$) as follows:

$$\begin{aligned}\{\sigma_1^2 &+ H_1\sigma_1 + H_2 = 0, \sigma_2^2 + H_3\sigma_2 + H_4 = 0, \\ \sigma_3^2 &+ H_5\sigma_3 + H_6 = 0, \sigma_4^2 + H_7\sigma_4 + H_8 = 0 \text{ and} \\ \sigma_5^2 &+ H_9\sigma_5 + H_{10} = 0\}\end{aligned} \tag{19}$$

where $H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9$ and H_{10} are defined in the Appendix.

To determine the stability of the steady-state solution, one lets

$$a_m = a_{m0} + a_{m1}, \qquad \theta_m = \theta_{m0} + \theta_{m1} \tag{20}$$

where a_{m0} and θ_{m0} are the solutions of (15)–(17) and a_{m1}, θ_{m1} are perturbations which are assumed to be small compared to a_{m0} and θ_{m0}. Substituting (20) into (15)–(17) and keeping only the linear terms in a_{m1} and θ_{m1}, we obtain

$$\begin{aligned}\dot{a}_{11} = &\left[\zeta_1 + \frac{f_{21}}{4\omega_1}\sin(\theta_{20})\right]a_{11} - \left[\frac{\zeta_2 f_{11}}{2\omega_1}\sin(\theta_{10}) \right.\\ &\left. + \frac{\zeta_3 f_{11}}{2\omega_1}\cos(\theta_{10}) - \frac{f_{21} a_{10}\theta_{21}}{4\omega_1\theta_{11}}\cos(\theta_{20})\right]\theta_{11} \\ &+ [\zeta_4 \cos(\theta_{50}) - \zeta_5 \sin(\theta_{50})]a_{31} \\ &- [\zeta_4 a_{30} \sin(\theta_{50}) + \zeta_5 a_{30} \cos(\theta_{50})]\theta_{51}\end{aligned} \tag{21a}$$

$$\begin{aligned}\dot{\theta}_{11} = &\left[\frac{(\sigma_1 - \sigma_2)}{a_{10}} + \frac{\zeta_6}{a_{10}} - \frac{3\eta_2 a_{10}}{8\omega_1} - \frac{\eta_1 a_{20}^2}{8a_{10}\omega_1} \right.\\ &\left. - \frac{f_{21}}{4\omega_1 a_{10}}\cos(\theta_{20})\right]a_{11} - \left[\frac{\eta_1 a_{20}}{4\omega_1}\right]a_{21} \\ &+ \left[\frac{\zeta_2 f_{11}}{2\omega_1 a_{10}}\cos(\theta_{10}) - \frac{\zeta_3 f_{11}}{2\omega_1 a_{10}}\sin(\theta_{10}) \right.\\ &\left. + \frac{f_{21}\theta_{21}}{4\omega_1\theta_{11}}\sin(\theta_{20})\right]\theta_{11} + \left[\frac{\zeta_4}{a_{10}}\sin(\theta_{50}) \right.\\ &\left. + \frac{\zeta_5}{a_{10}}\cos(\theta_{50})\right]a_{31} + \left[\frac{\zeta_4 a_{30}}{a_{10}}\cos(\theta_{50}) \right.\\ &\left. - \frac{\zeta_5 a_{30}}{a_{10}}\sin(\theta_{50})\right]\theta_{51}\end{aligned} \tag{21b}$$

$$\begin{aligned}\dot{a}_{21} = &\left[\zeta_7 + \frac{f_{41}}{4\omega_2}\sin(\theta_{40})\right]a_{21} - \left[\frac{c_2 f_{31}}{8\omega_2^2}\sin(\theta_{30}) \right.\\ &\left. + \frac{\zeta_8 f_{31}}{4\omega_2}\cos(\theta_{30}) + \frac{f_{41} a_{20}\theta_{41}}{4\omega_2\theta_{31}}\cos(\theta_{40})\right]\theta_{31}\end{aligned} \tag{22a}$$

$$\dot{\theta}_{31} = \left[\frac{(\sigma_3-\sigma_4)}{a_{20}} + \frac{\zeta_9}{a_{20}} - \frac{3\eta_3}{8\omega_2}a_{20} - \frac{\eta_1}{8\omega_2 a_{20}}a_{10}^2 - \frac{f_{41}}{4\omega_2 a_{20}}\cos(\theta_{40})\right]a_{21} - \left[\frac{\eta_1}{4\omega_2}a_{10}\right]a_{11} + \left[\frac{c_2 f_{31}}{8\omega_2^2 a_{20}}\cos(\theta_{30}) - \frac{\zeta_8 f_{31}}{4\omega_2 a_{20}}\sin(\theta_{30}) + \frac{f_{41}\theta_{41}}{4\omega_2\theta_{31}}\sin(\theta_{40})\right]\theta_{31} \tag{22b}$$

$$\begin{aligned}\dot{a}_{31} = \zeta_{10}a_{31} &+ \left[\zeta_{11}\sin(\theta_{50}) - \frac{(\eta_4-\eta_1)}{8\omega_3}a_{20}^2\sin(\theta_{50}) + \frac{3\eta_2}{8\omega_3}a_{10}^2\sin(\theta_{50}) + \zeta_{12}\cos(\theta_{50}) - \frac{f_{21}}{4\omega_3}\sin(\theta_{20}-\theta_{50})\right]a_{11} \\ &+ \left[-\zeta_{13}\frac{f_{11}}{4\omega_3}\sin(\theta_{10}-\theta_{50}) - \zeta_{14}\frac{f_{11}}{4\omega_3}\cos(\theta_{10}-\theta_{50}) - \frac{f_{21}\theta_{21}}{4\omega_3\theta_{11}}a_{10}\cos(\theta_{20}-\theta_{50})\right]\theta_{11} \\ &- \left[\frac{(\eta_4-\eta_1)}{4\omega_3}a_{10}a_{20}\sin(\theta_{50})\right]a_{21} \\ &+ \left[\zeta_{11}a_{10}\cos(\theta_{50}) - \frac{(\eta_4-\eta_1)}{8\omega_3}a_{10}a_{20}^2\cos(\theta_{50}) + \frac{\eta_2}{8\omega_3}a_{10}^3\cos(\theta_{50}) - \zeta_{12}a_{10}\sin(\theta_{50}) \right. \\ &\left. + \zeta_{13}\frac{f_{11}}{4\omega_3}\sin(\theta_{10}-\theta_{50}) + \zeta_{14}\frac{f_{11}}{4\omega_3}\cos(\theta_{10}-\theta_{50}) + \frac{f_{21}}{4\omega_3}a_{10}\cos(\theta_{20}-\theta_{50})\right]\theta_{51}\end{aligned} \tag{23a}$$

$$\begin{aligned}\dot{\theta}_{51} = &\left[\frac{(\sigma_1-\sigma_2+\sigma_5)}{a_{30}} + \frac{\zeta_{15}}{a_{30}} - \frac{\omega_2^2}{4\omega_3 a_{30}}a_{20}^2\right]a_{31} - \left[\frac{\omega_2^2}{2\omega_3}a_{20} + \frac{(\eta_4-\eta_1)}{4\omega_3 a_{30}}a_{10}a_{20}\cos(\theta_{50})\right]a_{21} \\ &+ \left[\frac{\zeta_{11}}{a_{30}}\cos(\theta_{50}) - \frac{(\eta_4-\eta_1)}{8\omega_3 a_{30}}a_{20}^2\cos(\theta_{50}) + \frac{\eta_2}{8\omega_3 a_{30}}a_{10}^2\cos(\theta_{50}) - \frac{\zeta_{12}}{a_{30}}\sin(\theta_{50}) + \frac{f_{21}}{4\omega_3 a_{30}}\cos(\theta_{20}-\theta_{50})\right]a_{11} \\ &+ \left[\zeta_{13}\frac{f_{11}}{4\omega_3 a_{30}}\cos(\theta_{10}-\theta_{50}) - \zeta_{14}\frac{f_{11}}{4\omega_3 a_{30}}\sin(\theta_{10}-\theta_{50}) - \frac{f_{21}\theta_{21}}{4\omega_3\theta_{11}a_{30}}a_{10}\sin(\theta_{20}-\theta_{50})\right]\theta_{11} \\ &+ \left[-\frac{\zeta_{11}}{a_{30}}a_{10}\sin(\theta_{50}) + \frac{(\eta_4-\eta_1)}{8\omega_3 a_{30}}a_{10}a_{20}^2\sin(\theta_{50}) - \frac{\eta_2}{8\omega_3 a_{30}}a_{10}^3\sin(\theta_{50}) - \frac{\zeta_{12}}{a_{30}}a_{10}\cos(\theta_{50}) \right. \\ &\left. - \zeta_{13}\frac{f_{11}}{4\omega_3 a_{30}}\cos(\theta_{10}-\theta_{50}) + \zeta_{14}\frac{f_{11}}{4\omega_3 a_{30}}\sin(\theta_{10}-\theta_{50}) + \frac{f_{21}}{4\omega_3 a_{30}}a_{10}\sin(\theta_{20}-\theta_{50})\right]\theta_{51}\end{aligned} \tag{23b}$$

The eigenvalues of the above system of equations are given by the equation

$$\lambda^6 + r_1\lambda^5 + r_2\lambda^4 + r_3\lambda^3 + r_4\lambda^2 + r_5\lambda + r_6 = 0 \tag{24}$$

where $(r_1, r_2, r_3, r_4, r_5, r_6)$ are functions of the parameters $(a_1, a_2, a_3, \omega_1, \omega_2, \omega_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, c_1, c_2, c_3, c_4, \beta, r, f_{11}, f_{21}, f_{31}, f_{41}, \alpha_1, \alpha_2, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$. If and only if the real part of the eigenvalue is negative, then the periodic solution is stable; otherwise, it is unstable. According to the Routh-Huriwitz criterion, the necessary and sufficient conditions for all the roots of (24) to have negative real parts if and only if the determinant D and all its principle minors are positive.

$$D = \begin{vmatrix} r_1 & 1 & 0 & 0 & 0 & 0 \\ r_3 & r_2 & r_1 & 1 & 0 & 0 \\ r_5 & r_4 & r_3 & r_2 & r_1 & 1 \\ 0 & r_6 & r_5 & r_4 & r_3 & r_2 \\ 0 & 0 & 0 & r_6 & r_5 & r_4 \\ 0 & 0 & 0 & 0 & 0 & r_6 \end{vmatrix} \tag{25}$$

4 Results and discussions

In this section, (1)–(3) are numerically integrated using fourth order Rung-Kutta algorithm. The nu-

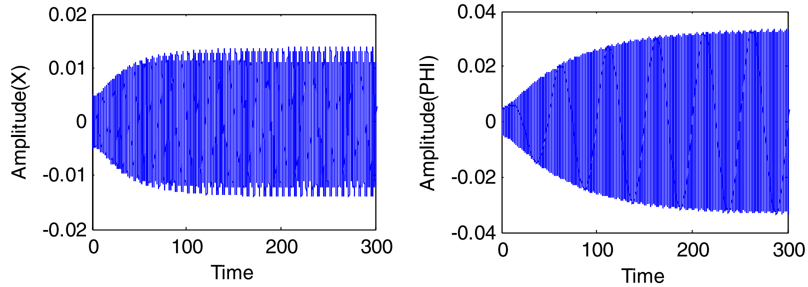

Fig. 2 Response of the system without absorber at simultaneous primary and sub-harmonic resonance case ($\Omega_{11} \cong \omega_1$, $\Omega_{21} \cong 2\omega_1$, $\Omega_{31} \cong \omega_2$, $\Omega_{41} \cong 2\omega_2$)

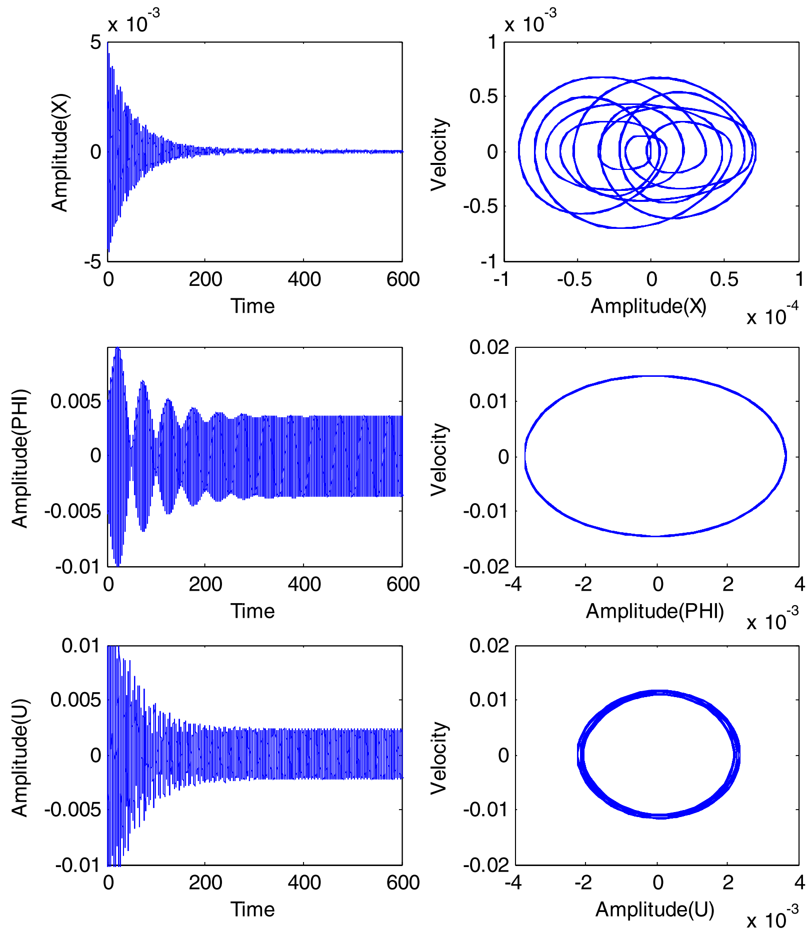

Fig. 3 Response of the system with absorber at simultaneous primary, sub-harmonic and internal resonance case ($\Omega_{11} \cong \omega_1$, $\Omega_{21} \cong 2\omega_1$, $\Omega_{31} \cong \omega_2$, $\Omega_{41} \cong 2\omega_2$, $\omega_3 \cong \omega_1$)

merical solutions of the spring pendulum with and without absorber at the simultaneous primary, sub-harmonic and internal resonance case are obtained as shown in Figs. 2 and 3 at the selected values ($\omega_1 = 5, \omega_2 = 4$, $\Omega_{11} \cong \omega_1, \Omega_{21} \cong 2\omega_1, \Omega_{31} \cong \omega_2$, $\Omega_{41} \cong 2\omega_2, \omega_3 \cong \omega_1, c_1 = 0.08, c_2 = 0.03$, $c_3 = 0.000008, c_4 = 0.00008, \alpha_1 = 0.004, \alpha_2 = 0.006$, $r = 0.5, \beta = 0.09, f_{11} = 0.005, f_{21} = 0.0025$, $f_{31} = 0.004$, $f_{41} = 0.002$).

Figure 2, illustrates the behavior of the dynamic system without absorber, where the steady state amplitude is about 200% of the excitation force amplitude

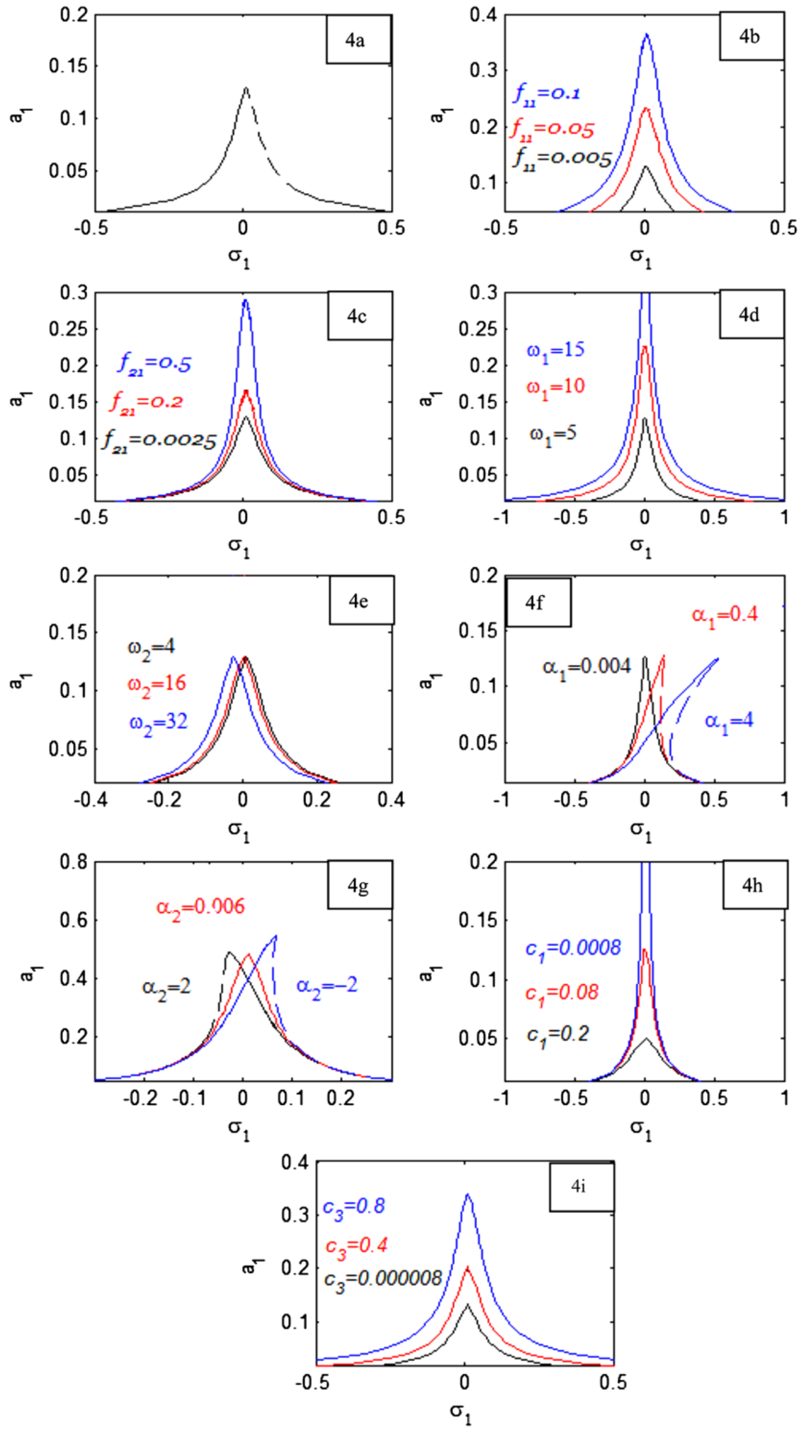

Fig. 4 Frequency response curves at selected values (a_1 against σ_1)

(f_{11}) for the first degree-of-freedom (x) and about 750 % of the excitation force amplitude (f_{31}) for the second degree-of-freedom (φ). Figure 3 shows the response for the dynamical system with absorber. It is clear from the figure that the steady state amplitude of the first degree-of-freedom (x) is reduced to 0.024% of

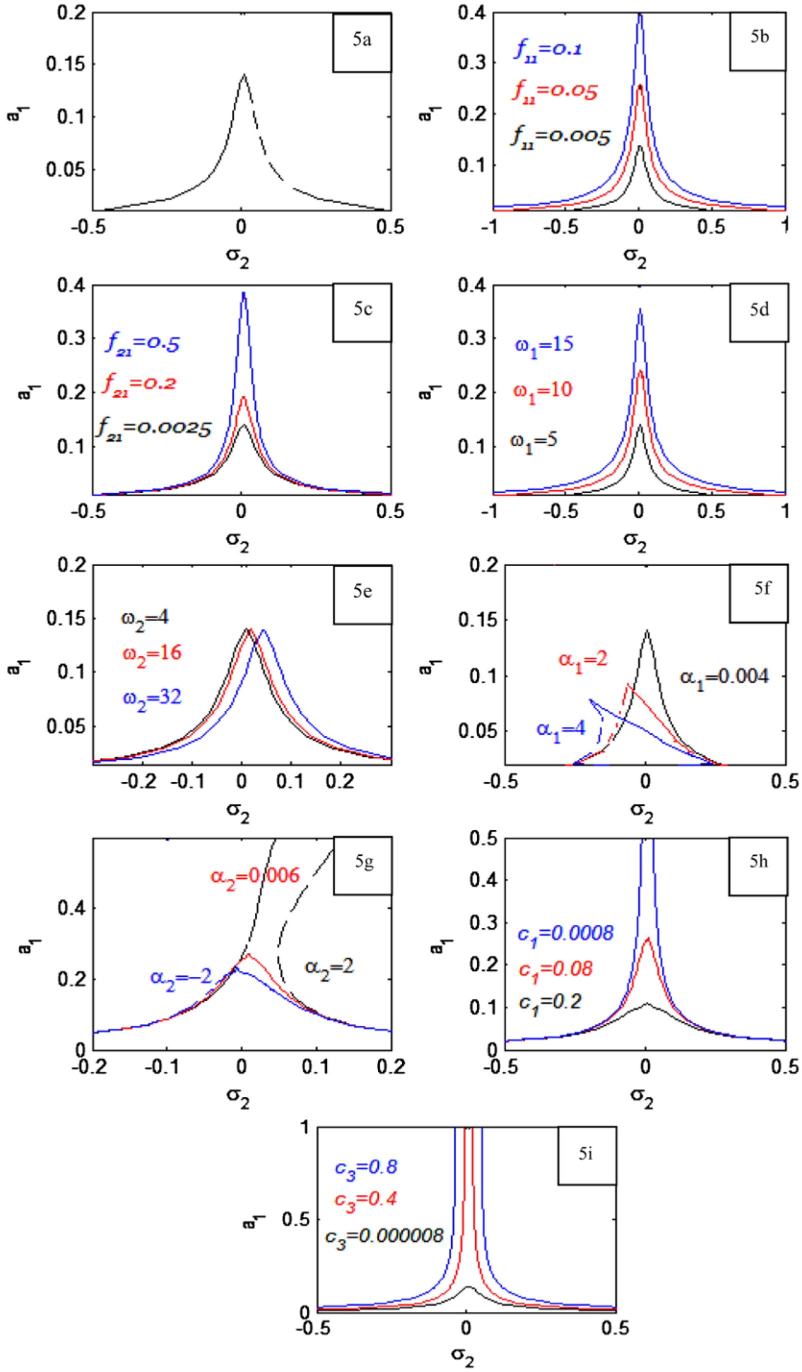

Fig. 5 Frequency response curves at selected values (a_1 against σ_2)

the excitation forces amplitude (f_{11}) where the steady state amplitude of the absorber (u) is about 40% of the excitation force amplitude (f_{11}). It is worth to notice that the effectiveness of the absorber E_a ($E_a =$ steady state amplitude of the main system without absorber/steady state amplitude of the main system with absorber) is about 8400 for the first degree-of-freedom (x) and 10 for the second degree-of-freedom (φ). Both degree-of-freedom of the system are stable with chaotic limit cycle.

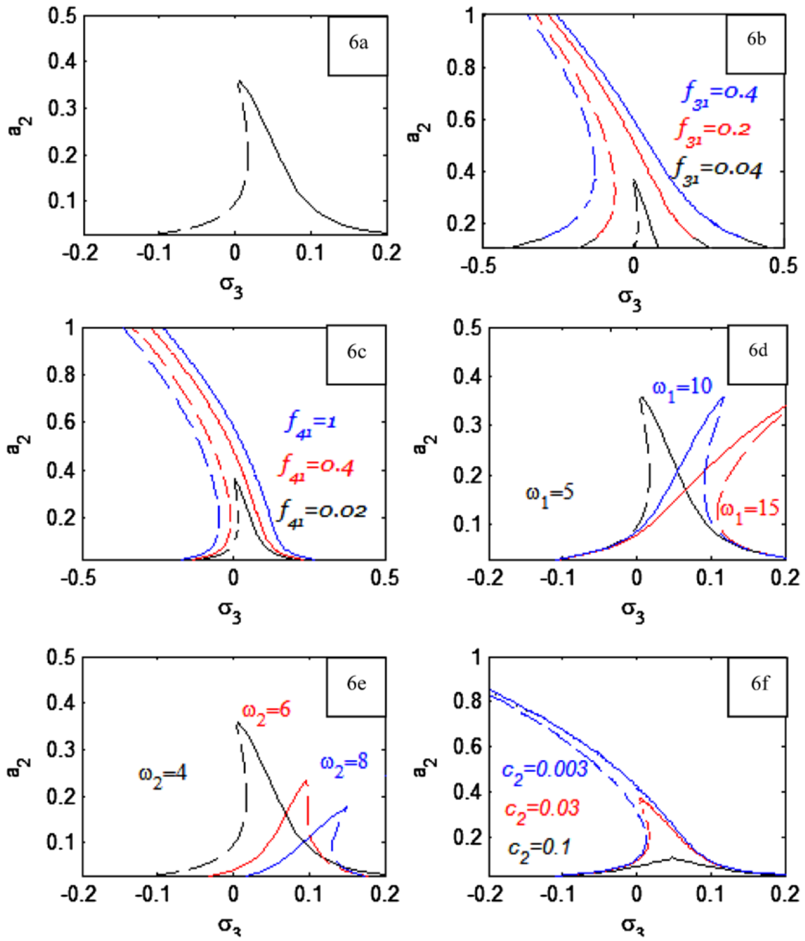

Fig. 6 Frequency response curves at selected values (a_2 against σ_3)

The frequency response equations (19) are nonlinear algebraic equations, which are solved numerically as shown in Figs. 4, 5, 6, 7, 8. From those figures, the results are presented as the steady state amplitudes a_1, a_2 and a_3 against detuning parameters $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and σ_5 for the selected practical case (where, $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0$). From Figs. 4(a), 5(a), 6(a), 7(a) and 8(a), the solid lines stand for the stable solution and the dashed lines for the unstable solution. Also, the curves are bent to left (soft spring) as shown in Fig. 6(a) or bent to right (hard spring) as shown in Fig. 7(a) leading to multi-valued solutions and jump phenomenon occurrence.

Figures 4(b), 5(b), 4(d), 5(d) and 4(i) show that the steady-state amplitudes are monotonic increasing functions to the parameters f_{11}, ω_1 and c_3 respectively, and the separation distance of non zero solution at $\sigma_1 = 0$ is increased also. The steady-state amplitude is a monotonic increasing function to f_{21} and c_3, monotonic decreasing function to c_1 and c_3 as shown in Figs. 4(c), 5(c), 5(i), 4(h) and 5(h) respectively. For increasing values of ω_2, the frequency response curve is shifted to the left as shown in Fig. 4(e) and shifted to right as shown in Fig. 5(e). From Figs. 4(f) and 4(g), we found that the frequency response curve is bent to the right when $\alpha_1 > 0.004$ and $\alpha_2 < 0.006$, indicating hardening-type of spring nonlinearity and is bent to the left when $\alpha_2 > 0.006$ and $\alpha_1 > 0.004$, indicating softening-type spring nonlinearity as shown in Figs. 4(g) and 5(f) respectively. Also, we have the steady-state amplitude is a monotonic increasing function to α_2 as shown in Fig. 5(g).

Furthermore, Figs. 6(b), 7(b), 6(c) and 7(c) show that the steady-state amplitudes are monotonic in-

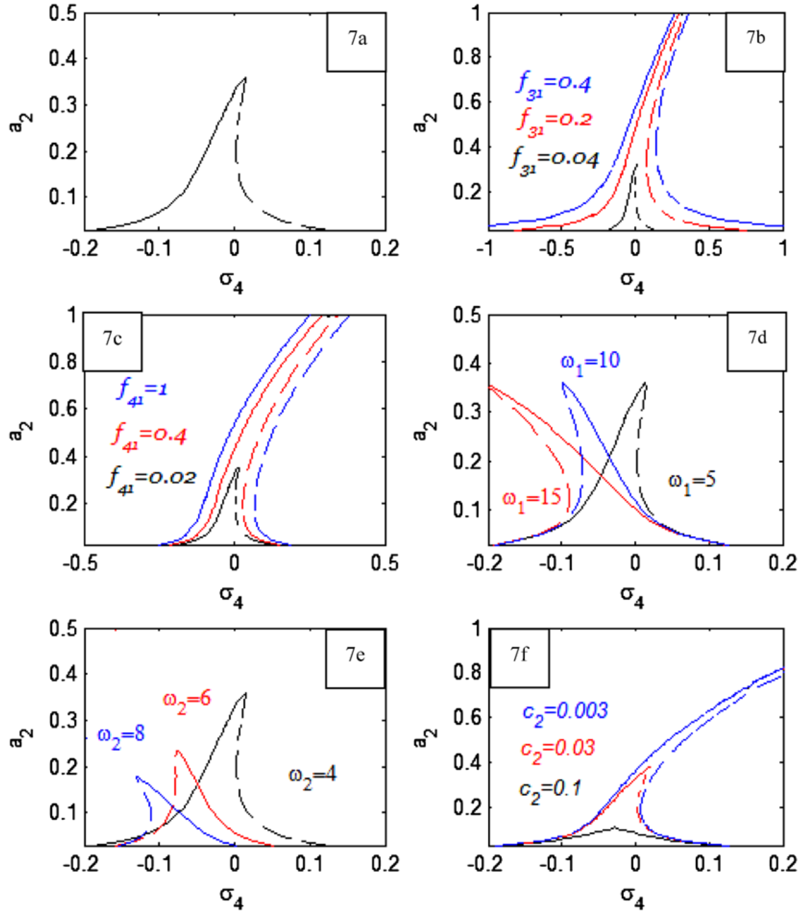

Fig. 7 Frequency response curves at selected values (a_2 against σ_4)

creasing functions to excitation amplitudes f_{31}, f_{41} respectively and the unstable region is increasing also. The steady-state amplitudes are monotonic decreasing function to ω_2 and c_2 as shown in Figs. 6(e), 7(e), 6(f) and 7(f) respectively. When ω_1 is increasing the frequency response curves are bent to right as shown in Fig. 6(d) and bent to left as shown in Fig. 7(d).

Also, Fig. 8(b) shows that the steady-state amplitude is a monotonic increasing function to excitation amplitudes f_{11} and the unstable region is increasing. The steady-state amplitudes are monotonic decreasing function to ω_3 and c_3 as shown in Figs. 8(c) and 8(d) respectively.

5 Comparison with published work

a. The present paper extends the analysis done in [15], which is limited to a single external force only for each degree-of-freedom at primary and internal resonance case ($E_a = 200$) and [20] studied the system with a transverse absorber under multi parametric forces at sub-harmonic and internal resonance case ($E_a(x) = 1300$, $E_a(\varphi) = 600$).
b. The study in this paper is expansion to illustrates the response and the stability of the two degree-of-freedom (x, φ) of the nonlinear spring-pendulum system subjected to multi external and parametric excitations. We succeeded to reduce the steady state amplitude of the first degree-of-freedom (x) to 0.012% of its maximum value via passive control method. This means that E_a of the first degree-of-freedom is approximates 8400 at the simultaneous primary, sub-harmonic and internal resonance case. Moreover, numerically the stable and unstable region are defined when studying the effects of the selected parameters.

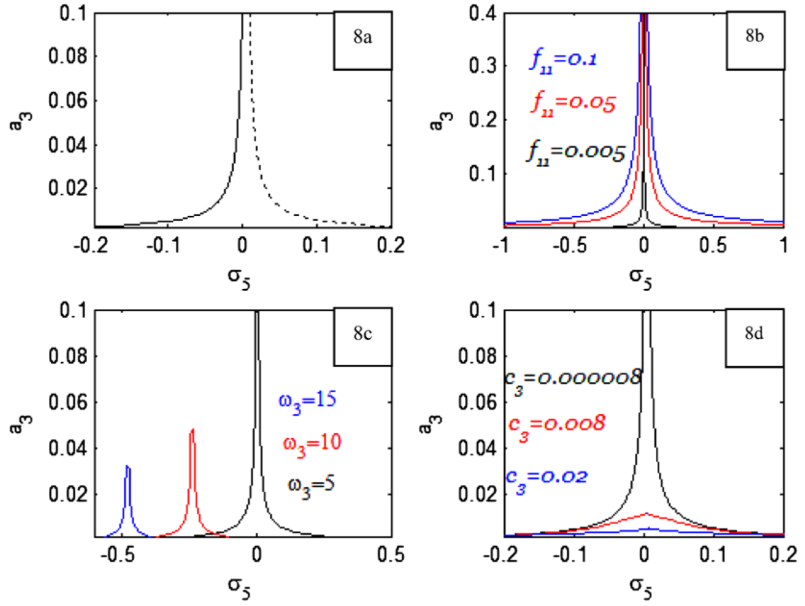

Fig. 8 Frequency response curves at selected values (a_3 against σ_5)

6 Conclusion

The vibrations and stability of the two degree-of-freedom (x, φ) of the nonlinear spring-pendulum system, simulating the ship roll motion, are considered and solved under multi external and multi parametric excitation forces. The vibration reduction examined using nonlinear longitudinal absorber (passive control). MSPT is applied to determine the frequency response equations near the primary, sub-harmonic resonances in the presence of internal resonance for the system with absorber. It is observed from the numerical study of the stability that the steady state amplitudes of the first degree-of-freedom (x) and the second degree-of-freedom (φ) are reduced to 0.012% and 10% of its maximum values respectively. This means that $E_a = 8400$ for the first degree-of-freedom and $E_a = 10$ for the second degree-of-freedom. The steady state amplitude of both degrees-of-freedom are monotonic increasing functions to the excitation force amplitudes $f_{11}, f_{21}, f_{31}, f_{41}$ and monotonic decreasing functions to the natural frequencies $\omega_1, \omega_2, \omega_3$ and the damping coefficients c_1, c_2, c_3, c_4. Increasing or decreasing the values of α_1, α_2 produced either hard or soft spring respectively leading to the occurrence of jump phenomena.

Appendix

$$\eta_1 = \left(1 + \frac{7\omega_2^2 - 4\omega_1^2}{\omega_1^2 - 4\omega_2^2} - \frac{\alpha_1}{\omega_1^2}\right)2\omega_2^2,$$

$$\eta_2 = \left(\frac{10\alpha_1^2}{3\omega_1^2} - 3\alpha_2\right),$$

$$\eta_3 = \left(1 + \frac{\omega_2^2(11\omega_1^2 - 8\omega_2^2)}{\omega_1^2(\omega_1^2 - 4\omega_2^2)}\right)\frac{\omega_2^2}{2},$$

$$\eta_4 = \left(1 + \frac{7\omega_2^2 - 4\omega_1^2}{\omega_1^2 - 4\omega_2^2} + \frac{r(10\omega_2^2 - 8\omega_1^2)}{\omega_1^2 - 4\omega_2^2}\right)2\omega_2^2,$$

$$\zeta_1 = \left[-\frac{c_1}{2} + \frac{c_1\beta\omega_3}{8\omega_1} - \frac{\beta c_1}{2}\right],$$

$$\zeta_2 = \left[\frac{c_1}{4\omega_1} - \frac{c_3}{4\omega_1} + \frac{c_3}{2\omega_3}\right],$$

$$\zeta_3 = \left[\frac{\beta\omega_3}{4\omega_1} + \frac{\sigma_1}{2\omega_1} - (\beta + 1)\right],$$

$$\zeta_4 = \left[\frac{c_1\beta\omega_3^2}{8\omega_1^2} - \frac{c_4\beta\omega_3^2}{8\omega_1^2} - \frac{\beta c_3\omega_3^2}{4\omega_1^2} - \frac{\sigma_5 c_3\omega_3}{4\omega_1^2} + \frac{c_4\beta\omega_3}{2\omega_1} + \frac{c_3\beta\omega_3}{4\omega_1} + \frac{\beta c_3\omega_3}{2\omega_1} + \frac{c_3\omega_3}{2\omega_1}\right],$$

$$\zeta_5=\left[\frac{c_1c_3\omega_3}{8\omega_1^2}-\frac{\omega_3c_3c_4}{8\omega_1^2}+\frac{\beta^2\omega_3^3}{8\omega_1^2}-\frac{c_3^2\omega_3}{8\omega_1^2}+\frac{\sigma_5\beta\omega_3^2}{4\omega_1^2}+\frac{c_3c_4}{4\omega_1}+\frac{c_3^2}{4\omega_1}-\frac{\beta^2\omega_3^2}{2\omega_1}-\frac{\beta\omega_3^2}{2\omega_1}\right],$$

$$\zeta_6=\left[-\frac{c_1^2}{8\omega_1}+\frac{c_1c_3}{8\omega_1}-\frac{c_1c_3}{4\omega_3}\right],$$

$$\zeta_7=\left[-\frac{c_2}{2}+\frac{c_2\beta}{2}(r^2+2r)\right],$$

$$\zeta_8=\left[\frac{3}{2}\beta r^2+\frac{11}{2}\beta r+\frac{\sigma_3}{\omega_2}-2\right],$$

$$\zeta_9=\left[-\frac{c_2^2}{8\omega_2}+\frac{\omega_2}{2}\left(\frac{3}{4}\beta r^2+\frac{7}{4}\beta r-1\right)(\beta r^2+\beta r)\right],$$

$$\zeta_{10}=\left[-\frac{c_4}{2}-\frac{c_3}{2}+\frac{c_1\beta}{8}-\frac{c_1\beta\omega_3}{8\omega_1}+\frac{c_4\beta\omega_3}{8\omega_1}+\frac{\beta c_3\omega_3}{4\omega_1}+\frac{\sigma_5c_3}{4\omega_1}-\frac{c_4\beta}{2}-\frac{\beta c_3}{2}\right],$$

$$\zeta_{11}=\left[\frac{c_1c_3\omega_1}{8\omega_3^2}-\frac{c_1^2\omega_1}{8\omega_3^2}+\frac{c_1c_4\omega_1}{8\omega_3^2}+\frac{c_1^2}{8\omega_3}-\frac{c_1c_3}{8\omega_3}\right],$$

$$\zeta_{12}=\left[\frac{3\beta c_1\omega_1}{8\omega_3}+\frac{\sigma_5c_1\omega_1}{4\omega_3^2}-\frac{c_1\beta}{8}+\frac{c_1\omega_1}{2\omega_3}\right],$$

$$\zeta_{13}=\left[-\frac{c_4}{2\omega_3}-\frac{c_3}{2\omega_3}+\frac{c_1}{2\omega_3}-\frac{c_1}{2\omega_1}+\frac{c_3}{2\omega_1}\right],$$

$$\zeta_{14}=\left[\frac{3\beta}{2}-\frac{(\sigma_1-\sigma_5)}{\omega_3}-\frac{\beta\omega_3}{2\omega_1}-\frac{\sigma_1}{\omega_1}+2\right],$$

$$\zeta_{15}=\left[\frac{\beta\omega_3}{2}-\frac{c_4^2}{8\omega_3}+\frac{3\beta^2\omega_3}{8}-\frac{c_3^2}{8\omega_3}+\frac{c_1c_3}{8\omega_3}-\frac{c_1c_3}{8\omega_1}+\frac{c_3c_4}{8\omega_1}-\frac{\beta^2\omega_3^2}{8\omega_1}+\frac{c_3^2}{8\omega_1}-\frac{\sigma_5\beta\omega_3}{4\omega_1}-\frac{c_3c_4}{4\omega_3}\right],$$

$$\zeta_{16}=\left(\zeta_{10}-\frac{\sigma_5c_3}{4\omega_1}\right),$$

$$\zeta_{17}=\left(\zeta_{12}-\frac{\sigma_5c_1\omega_1}{4\omega_3^2}\right),$$

$$\zeta_{18}=\left(\zeta_{14}-\frac{\sigma_5}{\omega_3}\right),$$

$$\zeta_{19}=\left(\zeta_{15}+\frac{\sigma_5\beta\omega_3}{4\omega_1}\right),$$

$$\zeta_{20}=1-\frac{\beta\omega_3}{2\omega_1}+\frac{\beta^2\omega_3^2}{16\omega_1^2}+\frac{c_3^2}{16\omega_1^2}-\frac{c_1^2\omega_1^2}{16\omega_3^4a_3^2}a_1^2-\frac{f_{11}^2}{16\omega_3^4a_3^2}+\left(\frac{c_1\omega_1f_{11}}{8\omega_3^4a_3^2}a_1\right)\sin(\theta_1),$$

$$H_1=\left[2\zeta_6-\frac{\eta_1a_2^2}{4\omega_1}-\frac{\eta_2a_1^2}{4\omega_1}-2\sigma_2-\frac{f_{11}^2\beta\omega_3}{16\omega_1^4a_1^2}+\frac{f_{11}^2}{4\omega_1^3a_1^2}+\frac{f_{11}^2\beta}{4\omega_1^3a_1^2}-\frac{\zeta_5f_{11}}{2\omega_1^2a_1^2}a_3\cos(\theta_1-\theta_5)+\frac{\zeta_4f_{11}}{2\omega_1^2a_1^2}a_3\sin(\theta_1-\theta_5)+\frac{f_{11}f_{21}}{8\omega_1^3a_1}\cos(\theta_1-\theta_2)\right]\Big/\left(1-\frac{f_{11}^2}{16\omega_1^4a_1^2}\right),$$

$$H_2=\Bigg\{\sigma_2^2+\zeta_1^2+\zeta_6^2+\frac{\eta_2^2a_1^4}{64\omega_1^2}+\frac{\eta_1^2a_2^4}{64\omega_1^2}+\left(-2\zeta_6+\frac{\eta_2a_1^2}{4\omega_1}+\frac{\eta_1a_2^2}{4\omega_1}\right)\sigma_2-\frac{\zeta_6\eta_2a_1^2}{4\omega_1}+\frac{\eta_1\eta_2a_1^2a_2^2}{32\omega_1^2}-\frac{\zeta_6\eta_1a_2^2}{4\omega_1}-\frac{\zeta_2^2f_{11}^2}{4\omega_1^2a_1^2}-\frac{f_{11}^2(\beta+1)^2}{4\omega_1^2a_1^2}-\frac{f_{11}^2\beta^2\omega_3^2}{64\omega_1^4a_1^2}+\frac{f_{11}^2\beta\omega_3(\beta+1)}{8\omega_1^3a_1^2}-\frac{f_{21}^2}{16\omega_1^2}-\left(\frac{\zeta_4^2+\zeta_5^2}{a_1^2}\right)a_3^2+\left[\frac{\beta\omega_3}{16\omega_1^3a_1}-\frac{(\beta+1)}{4\omega_1^2a_1}\right]f_{11}f_{21}\cos(\theta_1-\theta_2)+\frac{\zeta_2f_{11}f_{21}}{4\omega_1^2a_1}\sin(\theta_1-\theta_2)-\left[\frac{\beta\omega_3\zeta_5}{4\omega_1^2a_1^2}-\frac{(\beta+1)\zeta_5}{\omega_1a_1^2}+\frac{\zeta_2\zeta_4}{\omega_1a_1^2}\right]\times f_{11}a_3\cos(\theta_1-\theta_5)+\left[\frac{\beta\omega_3\zeta_4}{4\omega_1^2a_1^2}-\frac{(\beta+1)\zeta_4}{\omega_1a_1^2}-\frac{\zeta_2\zeta_5}{\omega_1a_1^2}\right]\times f_{11}a_3\sin(\theta_1-\theta_5)+\frac{\zeta_5f_{21}}{2\omega_1a_1}a_3\cos(\theta_2-\theta_5)+\frac{\zeta_4f_{21}}{2\omega_1a_1}a_3\sin(\theta_2-\theta_5)\Bigg\}\Big/\left(1-\frac{f_{11}^2}{16\omega_1^4a_1^2}\right)$$

$$H_3=\left(\frac{\eta_2a_1^2}{4\omega_1}+\frac{\eta_1a_2^2}{4\omega_1}-2\zeta_6-2\sigma_1\right)$$

$$H_4 = \sigma_1^2 + \zeta_1^2 + \left(2\zeta_6 - \frac{\eta_2 a_1^2}{4\omega_1} - \frac{\eta_1 a_2^2}{4\omega_1}\right)\sigma_1 + \zeta_6^2 + \frac{\eta_2^2 a_1^4}{64\omega_1^2} + \frac{\eta_1^2 a_2^4}{64\omega_1^2} - \frac{\zeta_6\eta_2 a_1^2}{4\omega_1} - \frac{\zeta_6\eta_1 a_2^2}{4\omega_1} + \frac{\eta_1\eta_2 a_1^2 a_2^2}{32\omega_1^2} - \frac{\zeta_2^2 f_{11}^2}{4\omega_1^2 a_1^2} - \frac{\zeta_3^2 f_{11}^2}{4\omega_1^2 a_1^2} - \frac{f_{21}^2}{16\omega_1^2} - \left(\frac{\zeta_4^2+\zeta_5^2}{a_1^2}\right)a_3^2 + \frac{\zeta_3 f_{11} f_{21}}{4\omega_1^2 a_1}\cos(\theta_1-\theta_2) + \frac{\zeta_2 f_{11} f_{21}}{4\omega_1^2 a_1}\sin(\theta_1-\theta_2) - \left(\frac{\zeta_3\zeta_5}{\omega_1 a_1^2} + \frac{\zeta_2\zeta_4}{\omega_1 a_1^2}\right) \times f_{11} a_3 \cos(\theta_1-\theta_5) + \left(\frac{\zeta_3\zeta_4}{\omega_1 a_1^2} - \frac{\zeta_2\zeta_5}{\omega_1 a_1^2}\right) \times f_{11} a_3 \sin(\theta_1-\theta_5) + \frac{\zeta_5 f_{21}}{2\omega_1 a_1} a_3\cos(\theta_2-\theta_5) + \frac{\zeta_4 f_{21}}{2\omega_1 a_1} a_3 \sin(\theta_2-\theta_5)$$

$$H_5 = \left[-2\sigma_4 - \frac{\eta_1}{4\omega_2}a_1^2 + 2\zeta_9 - \frac{\eta_3}{4\omega_2}a_2^2 - \frac{(3\beta r^2+11\beta r-4) f_{31}^2}{16\omega_2^3 a_2^2} + \frac{f_{31} f_{41}}{8\omega_2^3 a_2}\cos(\theta_3-\theta_4)\right]\Big/\left(1-\frac{f_{31}^2}{16 a_2^2\omega_2^4}\right)$$

$$H_6 = \left\{\zeta_7^2 + \sigma_4^2 + \left(\frac{\eta_1 a_1^2}{4\omega_2} - 2\zeta_9 + \frac{\eta_3 a_2^2}{4\omega_2}\right)\sigma_4 + \zeta_9^2 + \frac{\eta_3^2 a_2^4}{64\omega_2^2} + \frac{\eta_1^2 a_1^4}{64\omega_2^2} - \frac{\zeta_9\eta_3 a_2^2}{4\omega_2} + \frac{\eta_1\eta_3 a_1^2 a_2^2}{32\omega_2^2} - \frac{\zeta_9\eta_1 a_1^2}{4\omega_2} - \frac{c_2^2 f_{31}^2}{64\omega_2^4 a_2^2} - \frac{(3\beta r^2+11\beta r-4)^2 f_{31}^2}{32\omega_2^2 a_2^2} - \frac{f_{41}^2}{16\omega_2^2} + \frac{c_2 f_{31} f_{41}}{16\omega_2^3 a_2}\sin(\theta_3-\theta_4) + \frac{(3\beta r^2+11\beta r-4) f_{31} f_{41}}{16\omega_2^2 a_2}\cos(\theta_3-\theta_4)\right\} \Big/\left(1-\frac{f_{31}^2}{16 a_2^2\omega_2^4}\right)$$

$$H_7 = \left(\frac{\eta_1 a_1^2}{4\omega_2} - 2\zeta_9 + \frac{\eta_3 a_2^2}{4\omega_2} - 2\sigma_3\right)$$

$$H_8 = \zeta_7^2 + \sigma_3^2 + \left(-\frac{\eta_1}{4\omega_2}a_1^2 + 2\zeta_9 - \frac{\eta_3}{4\omega_2}a_2^2\right)\sigma_3 + \zeta_9^2 + \frac{\eta_3^2 a_2^4}{64\omega_2^2} + \frac{\eta_1^2 a_1^4}{64\omega_2^2} - \frac{\zeta_9\eta_3 a_2^2}{4\omega_2} + \frac{\eta_1\eta_3 a_1^2 a_2^2}{32\omega_2^2} - \frac{\zeta_9\eta_1 a_1^2}{4\omega_2} - \frac{c_2^2 f_{31}^2}{64\omega_2^4 a_2^2} - \frac{\zeta_8^2 f_{31}^2}{16\omega_2^2 a_2^2} - \frac{f_{41}^2}{16\omega_2^2} + \frac{c_2 f_{31} f_{41}}{16\omega_2^3 a_2}\sin(\theta_3-\theta_4) + \frac{\zeta_8 f_{31} f_{41}}{8\omega_2^2 a_2}\cos(\theta_3-\theta_4)$$

$$H_9 = \left[2\sigma_1 - \frac{\omega_2^2}{2\omega_3}a_2^2 + \frac{\omega_2^2\beta}{8\omega_1}a_2^2 + 2\zeta_{19} - \frac{\sigma_1\beta\omega_3}{2\omega_1} + \frac{\sigma_2\beta\omega_3}{2\omega_1} - 2\sigma_2 - \frac{\zeta_{19}\beta\omega_3}{2\omega_1} + \frac{\zeta_{16}c_3}{2\omega_1} - \frac{\zeta_{17}c_1\omega_1}{2\omega_3^2 a_3^2}a_1^2 - \frac{\zeta_{18} f_{11}^2}{8\omega_3^3 a_3^2} - \left(\frac{\zeta_{11} f_{11}}{2\omega_3^2 a_3^2}a_1 - \frac{(\eta_4-\eta_1) f_{11}}{16\omega_3^3 a_3^2}a_1 a_2^2 + \frac{\eta_2 f_{11}}{16\omega_3^3 a_3^2}a_1^3 + \frac{c_1\omega_1\zeta_{13} f_{11}}{8\omega_3^3 a_3^2}a_1\right)\cos(\theta_1) - \left(\frac{\zeta_{17} f_{11}}{2\omega_3^2 a_3^2}a_1 + \frac{c_1\omega_1\zeta_{18} f_{11}}{8\omega_3^3 a_3^2}a_1\right)\sin(\theta_1) + \left(\frac{c_1\omega_1 f_{21}}{8\omega_3^3 a_3^2}a_1^2\right)\sin(\theta_2) - \left(\frac{f_{11} f_{21}}{8\omega_3^3 a_3^2}a_1\right)\cos(\theta_1-\theta_2)\right]\Big/\zeta_{20}$$

$$H_{10} = \left\{\zeta_{16}^2 + \sigma_1^2 + \left(2\zeta_{19} - \frac{\omega_2^2}{2\omega_3}a_2^2\right)\sigma_1 - 2\sigma_1\sigma_2 + \sigma_2^2 + \left(-2\zeta_{19} + \frac{\omega_2^2}{2\omega_3}a_2^2\right)\sigma_2 - \zeta_{19}\frac{\omega_2^2}{2\omega_3}a_2^2 + \zeta_{19}^2 + \frac{\omega_2^4}{16\omega_3^2}a_2^4 - \left[\frac{\zeta_{11}}{a_3}a_1 - \frac{(\eta_4-\eta_1)}{8\omega_3 a_3}a_1 a_2^2 + \frac{\eta_2}{8\omega_3 a_3}a_1^3\right]^2 - \frac{\zeta_{17}^2}{a_3^2}a_1^2 - \frac{\zeta_{13}^2 f_{11}^2}{16\omega_3^2 a_3^2} - \frac{\zeta_{18}^2 f_{11}^2}{16\omega_3^2 a_3^2}\right.$$

$$
\begin{aligned}
&-\frac{f_{21}^2}{16\omega_3^2 a_3^2}a_1^2-\left[\frac{\zeta_{11}\zeta_{18}f_{11}}{2\omega_3 a_3^2}a_1\right.\\
&-\frac{(\eta_4-\eta_1)\zeta_{18}f_{11}}{16\omega_3^2 a_3^2}a_1a_2^2+\frac{\eta_2\zeta_{18}f_{11}}{16\omega_3^2 a_3^2}a_1^3\\
&\left.+\frac{\zeta_{17}\zeta_{13}f_{11}}{2\omega_3 a_3^2}a_1\right]\cos(\theta_1)-\left[-\frac{\zeta_{17}\zeta_{18}f_{11}}{2\omega_3 a_3^2}a_1\right.\\
&+\frac{\zeta_{11}\zeta_{13}f_{11}}{2\omega_3 a_3^2}a_1-\frac{(\eta_4-\eta_1)\zeta_{13}f_{11}}{16\omega_3^2 a_3^2}a_1a_2^2\\
&\left.+\frac{\eta_2\zeta_{13}f_{11}}{16\omega_3^2 a_3^2}a_1^3\right]\sin(\theta_1)-\left[\frac{\zeta_{11}f_{21}}{2\omega_3 a_3^2}a_1^2\right.\\
&\left.-\frac{(\eta_4-\eta_1)f_{21}}{16\omega_3^2 a_3^2}a_1^2a_2^2+\frac{\eta_2 f_{21}}{16\omega_3^2 a_3^2}a_1^4\right]\cos(\theta_2)\\
&+\left[\frac{\zeta_{17}f_{21}}{2\omega_3 a_3^2}a_1^2\right]\sin(\theta_2)\\
&-\left[\frac{\zeta_{13}f_{11}f_{21}}{8\omega_3^2 a_3^2}a_1\right]\sin(\theta_1-\theta_2)\\
&-\left[\frac{\zeta_{18}f_{11}f_{21}}{8\omega_3^2 a_3^2}a_1\right]\cos(\theta_1-\theta_2)\Bigg\}\Big/\zeta_{20}
\end{aligned}
$$

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