Meccanica (2010) 45: 671–680
DOI 10.1007/s11012-009-9262-7

SIMULATION, OPTIMIZATION & IDENTIFICATION

Optimal material orientation of linear and non-linear elastic 3D anisotropic materials

J. Majak · M. Pohlak

Received: 2 February 2009 / Accepted: 26 October 2009 / Published online: 6 November 2009

Abstract Optimal material orientation problems of linear and non-linear elastic three-dimensional anisotropic materials are studied. Most commonly, the energy based formulation is applied for solving orientational design problems of anisotropic materials, considering elastic energy density as a measure of the stress strain state. The same approach is used in the current study, but the strength criteria based approaches are also discussed. A simple relation between the stationary conditions in terms of Euler angles and the optimality conditions in terms of strains is pointed out. The complexity analysis of the different existing optimality conditions has been performed. The solution of the posed optimization problem is decomposed into the strain level solution, search for global extremes and evaluation of Euler angles (parameters). The results obtained are extended to some nonlinear elastic material models.

Keywords Optimal material orientation · Decomposition method · GA

J. Majak (✉) · M. Pohlak
Department of Machinery, Tallinn University of Technology, Ehitajate tee 5, 19086 Tallinn, Estonia
e-mail: jmajak@staff.ttu.ee

1 Introduction

Design of the structures (or materials) with maximal and minimal stiffness are both problems of practical interest. Latter problem is actual in design of energy absorbing structures (crash modeling). In the case of advanced composite materials the stiffness/flexibility of structure is affected among size, shape, topology, thermal effects [1, 2] etc. by the material orientation.

The problem of optimal orientation of anisotropic materials is studied most commonly in energy based formulation, considering elastic energy density as a measure of the stress strain state. Such an approach has been introduced by Banichuk [3] and utilized for optimal design of 2D linear elastic orthotropic materials by Pedersen [4, 5], Sacci and Rovati [6] etc. In [4] the closed form analytical solution for optimal material orientation problem of linear elastic 2D orthotropic materials is given including analysis of global and local extremes. The results for non-orthotropic 2D linear elastic materials are given in [7].

In [8–10] different non-linear elastic material models are proposed for solving oriental design problems. In all these papers the effective strain (stress) has been used as a scalar measure of the strain (stress) state. In [8] and [10] the nonlinear material behavior is simulated on the basis of a power law with one term and a series expansion, respectively. In [9] the stress strain relation is described by more general function of effective strain, given in the implicit form.

Orientational design problems of 3D orthotropic materials are studied by Seregin and Troitski [11], Rovati and Taliercio [12, 13], Cowin [14], etc. In [11] and [12–14] the potential energy of deformation and the specific elastic energy density are subjected to minimization, respectively. In all these papers it is pointed out, that the stress and strain tensors are coaxial at the optimum. In [11] the optimality conditions in terms of stresses for general orthotropic material are derived and solution regimes are discussed. In [12] the optimization problem is formulated in terms of Euler angles. Collinearity of principal directions of stress and strain at the optimum is derived from the stationary condition of the strain energy density. Complete analytical solution for body with cubic symmetry in terms of strains is given. In both, Rovati and Taliercio [13] and Cowin [14] thoroughgoing analysis of the 3D optimal material orientation problem is performed. In these papers the general non-orthotropic material with 21 independent coefficients is considered and the coaxiality condition of the stress and strain tensors is converted into simplified form using decomposition of the elasticity tensor. The final form of the optimality conditions, given in [13] and [14] in terms of compliance and constitutive tensors, respectively, is a quite similar (see details in Sect. 4), but the interpretation is principally different. In [14] the search for optimal solution is performed for all stress states, which leads to condition that all compliance coefficients (mutually rotated) must vanish. In latter case the closed formed solution of optimality conditions is not too complicated, but the unique solution does not exist for all material symmetries. In [13] a fixed strain (stress) state is considered. In the latter case the solution of optimality conditions is complicated task in the case of general orthotropic or non-orthotropic materials.

In the current paper the optimality conditions in terms of strains are derived for general 3D orthotropic material proceeding from stationary conditions of the strain energy density represented in terms of Euler angles. It is shown, that the optimality conditions derived are simplified form of the direct stationary conditions in terms of Euler angles. The complexity analysis of the existing optimality conditions is performed. It is shown that the optimality conditions derived in [11, 13, 14] and current paper have the same complexity with respect to Euler angles (orthotropic symmetry considered). The solution of the optimality conditions is decomposed into the strain level solution, search for global extremes and evaluating of Euler angles (parameters). According to latter approach the optimality conditions derived are combined with strain invariants in order to obtain fully determined system of algebraic equations in terms of strains (Majak [15]). The order of optimality conditions given in terms of strains is reduced to solving one sixth order algebraic equation and the closed form analytical solutions are determined for 15 different extremes of the strain energy density including solutions corresponding to singular solutions in terms of Euler angles. Numerical algorithm based on global optimization technique (hybrid GA) is treated as an alternative solution and the results are compared. The problem studied can be considered as sub-problem in structural optimization of complex composite structures [16].

2 Constitutive relations

In the current study, both linear and non-linear elastic material models are considered. The stress-strain relation for a three-dimensional orthotropic linearly elastic material can be expressed as (Hooke's law)

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}. \tag{1}$$

In (1) $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ stand for the stress and strain vectors, respectively and the matrix $\mathbf{C}$ is a symmetric positive definite constitutive matrix with nine independent material parameters

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{3131} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix}, \qquad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix}. \tag{2}$$

The nonlinear elastic material behavior is modeled most commonly by use of the following power law stress-strain relation

$$\boldsymbol{\sigma} = E\varepsilon_e^{p-1}\mathbf{C}\boldsymbol{\varepsilon}, \tag{3}$$

introduced in Pedersen and Taylor [8]. The values of the power p and elasticity coefficient E in (3) should be determined experimentally from tensile tests. The scalar quantity ε_e stands for the effective strain and is defined as

$$\varepsilon_e^2 = \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{C}^*\boldsymbol{\varepsilon}, \tag{4}$$

where $\mathbf{C}^*$ stand for nondimensional constitutive matrix.

In the current paper the relationship for nonlinear elastic material is assumed in the form

$$\boldsymbol{\sigma} = F(\varepsilon_e, E_i)\mathbf{C}\boldsymbol{\varepsilon}, \tag{5}$$

where $F(\varepsilon_e, E_i)$ is a given differentiable function depending on the effective strain and on elasticity coefficients. The relation (5) introduced in [9] is more general and covers the stress-strain behavior described by power-law, logarithmic, series expansion and etc. relations. The form of the function $F(\varepsilon_e, E_i)$ and the elasticity parameters can be specified in accordance with the stress-strain behavior of the particular problem considered.

The 3D transversally isotropic material is considered in (1)–(2) as particular case, where

$$\begin{aligned} &C_{2222} = C_{1111}, \qquad C_{2233} = C_{1133},\\ &C_{3131} = C_{2323}, \qquad C_{1212} = (C_{1111} - C_{1122})/2. \end{aligned} \tag{6}$$

Transversally isotropic material contains a plane of isotropy (the symmetry plane is chosen perpendicular to e_3) and different properties in the direction normal to this plane. The number of independent material parameters is reduced to 5.

3 Optimality criteria

In the following it is assumed that the optimality of the objective is local (strength) based on the assumption of fixed strains (or fixed stresses). Two principally different formulations of the problem of optimal material orientation are considered. The energy-based formulation, considering the elastic energy density as a measure of stress-strain state

$$J_{Energy} = \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon} \to \min(\max), \tag{7}$$

and strength criteria based approaches (Hill, Tsai-Wu, etc., strength criteria)

$$J_{Str\_Hill} = \frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{R}\boldsymbol{\sigma} \to \min(\max), \tag{8}$$

$$J_{Str\_Tsai\text{-}Wu} = \frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{T}\boldsymbol{\sigma} + \frac{1}{2}\mathbf{W}^{\mathrm{T}}\boldsymbol{\sigma} \to \min(\max). \tag{9}$$

The coefficient $1/2$ is introduced in strength criteria in order to obtain similar expressions with the strain energy based formulation. The matrices $\mathbf{R}$ and $\mathbf{T}$ in (8)–(9) characterize the plastic anisotropy of the material and can be written in terms of yield strengths F, G, H, L, M, N (Hill criterion) and F_{ij} (Tsai-Wu criterion) as

$$\mathbf{R} = \begin{bmatrix} G+H & -H & -G & 0 & 0 & 0\\ -H & F+H & -F & 0 & 0 & 0\\ -G & -F & F+G & 0 & 0 & 0\\ 0 & 0 & 0 & 2L & 0 & 0\\ 0 & 0 & 0 & 0 & 2M & 0\\ 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix}, \tag{10}$$

$$\mathbf{T} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0\\ F_{12} & F_{22} & F_{23} & 0 & 0 & 0\\ F_{13} & F_{23} & F_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & F_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & F_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & F_{66} \end{bmatrix}.$$

The vector $\mathbf{W}$ contains the coefficients of the linear term of the Tsai-Wu criterion given as

$$\mathbf{W}^{\mathrm{T}} = \{ F_1 \quad F_2 \quad F_3 \quad 0 \quad 0 \quad 0 \}. \tag{11}$$

The linear term of the Tsai-Wu criterion characterizes different properties of the material in tension and compression. In the following, the Hill and Tsai-Wu yield criteria are utilized as anisotropic strength criteria (limits of linear elastic behavior) and yield strengths in (10)–(11) are regarded as failure strengths.

Recently, it is shown by Groenwold and Haftka [17] that the objective (9) has considerable limitations. It is pointed out, that the objective function will not maximize the failure load, when it is carried at a load which is different from the failure load. Latter limitations hold good for non-homogeneous criteria like Tsai-Wu criterion (9). The use of a safety factor for the objective function is suggested in order to overcome such a limitation. In the following, the main attention is paid to application of the criteria (7) and (8).

4 The necessary optimality conditions

Let us assume that the solid is subjected to a fixed state of strain assigned through the principal strains. According to objective (7), the local orientations of the material axes with respect to principal directions of strains, which correspond to extreme values of the strain energy density, should be determined.

As mentioned above, in literature different approaches have been utilized for the derivation of the necessary optimality conditions. In the following an approach proposed by Majak [15] is employed. Latter approach is based on the solution method developed by Pedersen [4] for linear elastic 2D materials.

In the following the Euler angles are introduced in order to describe the finite rotation of local system x_1, x_2, x_3 as a composition of rotations from a reference frame. The first rotates the local system about x_3 axes by an angle θ_1, next the resulting system of axes is rotated about new x_1 axes by an angle θ_2, finally the resulting system of axes is rotated about new x_3 axes by an angle θ_3.

The Euler angles are considered as design variables (in the case of singular solution, the Euler parameters are used instead of angles). Thus, the gradient of the strain energy density with respect to Euler angles θ_i (parameters) is equalized to zero

$$\frac{\partial J_{Energy}}{\partial \theta_i} = \boldsymbol{\varepsilon}^{\mathrm{T}} \mathbf{C} \frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{\theta}_i} = 0. \tag{12}$$

The Euler angles (parameters) are introduced in strain energy density through the strain components. In the case of 3D linear elasticity the transformation formulas for strains are given as

$$\varepsilon_{ik} = \varepsilon_j Q_{ji} Q_{jk} \tag{13}$$

where ε_j $(j = I, II, III)$ denotes the principal strains and Q_{ji} are the components to a second-order orthogonal tensor, representing the coordinate transformation

$$\mathbf{Q} = \begin{pmatrix} \cos(\theta_3)\cos(\theta_1) - \sin(\theta_3)\cos(\theta_2)\sin(\theta_1) & -\cos(\theta_3)\sin(\theta_1) - \sin(\theta_3)\cos(\theta_2)\cos(\theta_1) & \sin(\theta_3)\sin(\theta_2) \\ \sin(\theta_3)\cos(\theta_1) + \cos(\theta_3)\cos(\theta_2)\sin(\theta_1) & -\sin(\theta_3)\sin(\theta_1) + \cos(\theta_3)\cos(\theta_2)\cos(\theta_1) & -\cos(\theta_3)\sin(\theta_2) \\ \sin(\theta_2)\sin(\theta_1) & \sin(\theta_2)\cos(\theta_1) & \cos(\theta_2) \end{pmatrix}. \tag{14}$$

Obviously, the orthogonality constraint $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$ is trivially satisfied for matrix $\mathbf{Q}$ and there is no reason for including it in extended functional (Lagrange multipliers method). In order to represent the optimality conditions in compact form, the partial derivatives of the strain components with respect to Euler angles $\frac{\partial \boldsymbol{\varepsilon}}{\partial \theta_i}$ are expressed in terms of strain components (similarly to 2D case in [4]). The detailed formulas for all strain components and Euler angles are omitted here for conciseness sake. However, it is correct to note that the partial derivatives with respect to Euler angle θ_1 contain the strain components only, but with respect to θ_2 and θ_3 also Euler angles ($\frac{\partial \varepsilon_{11}}{\partial \theta_1} = 2\varepsilon_{12}$, $\frac{\partial \varepsilon_{11}}{\partial \theta_2} = 2\sin(\theta_1)\varepsilon_{13}$, $\frac{\partial \varepsilon_{11}}{\partial \theta_3} = 2\cos(\theta_2)\varepsilon_{12} - 2\sin(\theta_2)\cos(\theta_1)\varepsilon_{13}$). Inserting the expressions of the partial derivatives in (12), one obtains the stationary condition of the strain energy density as

$$\frac{\partial J_{Energy}}{\partial \theta_1} = 2EQ1 = 0 \tag{15}$$

$$\frac{\partial J_{Energy}}{\partial \theta_2} = 2(\cos(\theta_1)EQ2 + \sin(\theta_1)EQ3) = 0, \tag{16}$$

$$\begin{aligned} \frac{\partial J_{Energy}}{\partial \theta_3} &= 2\cos(\theta_2)EQ1 + 2\sin(\theta_2)(\sin(\theta_1)EQ2 \\ &\quad - \cos(\theta_1)EQ3) = 0, \end{aligned} \tag{17}$$

where

$$\begin{aligned} EQ1 = {} & [(C_{1111} - C_{1122} - 2C_{1212})\varepsilon_{11} \\ & - (C_{2222} - C_{1122} - 2C_{1212})\varepsilon_{22} \\ & + (C_{1133} - C_{2233})\varepsilon_{33}]\varepsilon_{12} \\ & + 2(C_{3131} - C_{2323})\varepsilon_{23}\varepsilon_{31}, \end{aligned} \tag{18}$$

$$EQ2 = [(C_{1122} - C_{1133})\varepsilon_{11} + (C_{2222} - C_{2233} - 2C_{2323})\varepsilon_{22} - (C_{3333} - C_{2233} - 2C_{2323})\varepsilon_{33}]\varepsilon_{23} + 2(C_{1212} - C_{3131})\varepsilon_{12}\varepsilon_{31}, \quad (19)$$

$$EQ3 = [(C_{1133} - C_{1111} + 2C_{3131})\varepsilon_{11} + (C_{2233} - C_{1122})\varepsilon_{22} + (C_{3333} - C_{1133} - 2C_{3131})\varepsilon_{33}]\varepsilon_{31} + 2(C_{2323} - C_{1212})\varepsilon_{12}\varepsilon_{23}. \quad (20)$$

Proposition 1 *The optimality conditions* (15)–(20) *and the following optimality conditions in terms of strains*

$$EQ1 = 0; \qquad EQ2 = 0; \qquad EQ3 = 0; \quad (21)$$

are equivalent.

Proof A. Assume that the conditions (18)–(21) are valid. The stationary conditions (15)–(20) are filled obviously.

B. Assume that (15)–(20) are valid. Obviously $EQ1 = 0$ and it implies from (16)–(17) that $\sin(\theta_2) = 0$ or

$$\begin{cases} \cos(\theta_1)EQ2 + \sin(\theta_1)EQ3 = 0, \\ \sin(\theta_1)EQ2 - \cos(\theta_1)EQ3 = 0. \end{cases} \quad (22)$$

B.1. Case $\sin(\theta_2) = 0$. This solution is included in (18)–(21), since the second and third equations of (21) can be expressed as (see relation (13))

$$\begin{cases} EQ2 = \sin(\theta_2)G_1(C_{klmn}, \theta_1, \theta_2, \theta_3), \\ EQ3 = \sin(\theta_2)G_2(C_{klmn}, \theta_1, \theta_2, \theta_3), \end{cases} \quad (23)$$

where G_1 and G_2 are given functions (determined by (13)) of Euler angles and components of the elasticity tensor C_{klmn}.

B.2. Case $\sin(\theta_2) \neq 0$.

B.2.1. Case $\sin(\theta_1) \neq 0$ and $\cos(\theta_1) \neq 0$.

Summarizing the first and second equations of system (22) multiplied by $\cos(\theta_1)$ and $\sin(\theta_1)$, respectively one obtains that $EQ2 = 0$ and $EQ3 = 0$.

B.2.2. Case $\sin(\theta_1) = 0$ or $\cos(\theta_1) = 0$.

It implies from (22) that $EQ2 = 0$ and $EQ3 = 0$.

Proposition 1 is proved. □

The conditions (18)–(21) in terms of strains coincide formally with those derived by Seregin and Troitski [11] proceeding from minimum potential energy criterion, but latter conditions are given in terms of stresses. The obtained results confirm the commutativity of the product of stress and strain tensors at the stationary points of the strain energy density from which implies that the stress and strain tensors are coaxial [12, 13]. An approach proposed above allows to form some conclusions.

Proposition 2 *The optimality conditions in terms of strains* (18)–(21) *are simplified form of stationary conditions* (15)–(20).

Proof Obviously, the system (15)–(20) is higher order algebraic system than (18)–(21) (both conditions can be expressed in terms of Euler angles by use of the transformation formulas for strain components (13)–(14)).

The Proposition 2 is proved. □

Cowin [14] and Rovati [13] expressed the coaxiality condition of the stress and strain tensors in the following form

$$\begin{cases} S_{1144}\sigma_I + S_{2244}\sigma_{II} + S_{3344}\sigma_{III} = 0 \\ S_{1155}\sigma_I + S_{2255}\sigma_{II} + S_{3355}\sigma_{III} = 0 \quad \text{(Cowin)}, \\ S_{1166}\sigma_I + S_{2266}\sigma_{II} + S_{3366}\sigma_{III} = 0 \end{cases}$$
$$\begin{cases} C_{1144}\varepsilon_I + C_{2244}\varepsilon_{II} + C_{3344}\varepsilon_{III} = 0 \\ C_{1155}\varepsilon_I + C_{2255}\varepsilon_{II} + C_{3355}\varepsilon_{III} = 0 \quad \text{(Rovati)}, \\ C_{1166}\varepsilon_I + C_{2266}\varepsilon_{II} + C_{3366}\varepsilon_{III} = 0 \end{cases} \quad (24)$$

where S_{ijkl} and C_{ijkl} stand for components of the compliance and constitutive tensors, respectively. The Euler angles are included in constitutive (compliance) tensor components through orthogonal rotation tensor Q ($C_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{iq}\bar{C}_{mnpq}$, see (14)). In [14] it is assumed that all compliance coefficients must vanish in (24), since the optimal solution is searched for all stress states. In [13] a fixed state of strain is considered and the solution of system (24) is more complicated. In latter case the system (24) can be solved with respect to the components of the rotation tensor Q or Euler angles. In [13] the solution in terms of Euler angles is preferred (unconstraint optimization problem) and additionally to (24) the condition

Table 1 Complexity analysis of the optimality conditions (18)–(21) and (24)

Equation	$\cos(\theta_2)$	$\cos(2\theta_3)$	$\cos(2\theta_1)$	$\sin(2\theta_3)$	$\sin(2\theta_1)$
Eq. 1 in (24)	3	2	2	1	1
Eq. 2 in (24)	3	2	2	1	1
Eq. 3 in (24)	4	2	2	1	1
$\|C^*\|=0$	4	2	3	1	1
Eq. 1 in (21)	4	2	2	1	1
Eq. 2 in (21)	3	2	2	1	1
Eq. 3 in (21)	3	2	2	1	1

$|C^*|=0$ is pointed out, where $|C^*|$ stands for determinant of the system in (24) (Rovati). Latter condition is obviously necessary for existence of non-trivial solutions for system (24) and it can be used for determining Euler angle(s) (instead of any equation of the system (24)).

It is interesting to compare the complexity of the optimality conditions (18)–(21) and (24). Let us consider orthotropic symmetry and assume that some preliminary simplification of the equations is performed (both optimality conditions are previously factorized using MAPLE 10 software package, also the rank of the equations is reduced by introducing double angles for θ_1 and θ_3). In Table 1 the ranks of the algebraic equations with respect Euler angles are pointed out.

It can be seen from Table 1, that in general the complexity of the optimality conditions (18)–(21) and (24) is the same. The system (24) is linear with respect to principal strains, but the system (18)–(21) is quadratic. The equation $|C^*|=0$ has higher rank with respect to Euler angles, but it does not depend on principal strains. The equation $|C^*|=0$.

It is shown by authors that complexity of the optimality conditions (21) can be reduced significantly (details in next section).

Next, the strength criterion (8) is focused on. Using the Hooke's law (1), the strength criterion (8) can be rewritten in terms of strains as

$$J_H=\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}(\mathbf{CRC})\boldsymbol{\varepsilon}\rightarrow \min(\max). \tag{25}$$

The objective function (25) can be transformed to form similar with energy based formulation by introducing the matrix **A** as

$$\mathbf{A}=\mathbf{CRC}. \tag{26}$$

The stiffness-strength matrix **A** is orthotropic since both its components, the Hill's strength matrix **R** and stiffness matrix **C** are orthotropic matrices. Thus, the minimization of the strength functional (25) can be performed similarly with the energy based formulation discussed in detail above.

5 Solution of the optimization problem

Let us proceed from optimality conditions (18)–(21) given in terms of strains. Corresponding to approach proposed in Majak [15], the solution of the optimization problem posed is divided into strain level solution, search for global extremes and determination of Euler angles (or parameters) corresponding to global extremes. In order to obtain completely determined system in terms of strains the optimality conditions (18)–(21) are combined with the strain invariants (6 strain component and 6 equations)

$$\begin{aligned}
&\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}-\varepsilon_I+\varepsilon_{II}-\varepsilon_{III}=0,\\
&\varepsilon_{11}\varepsilon_{22}+\varepsilon_{11}\varepsilon_{33}+\varepsilon_{22}\varepsilon_{33}-\varepsilon_I\varepsilon_{II}\\
&\quad-\varepsilon_I\varepsilon_{III}-\varepsilon_{II}\varepsilon_{III}=0,\\
&\varepsilon_{11}\varepsilon_{22}\varepsilon_{33}+2\varepsilon_{12}\varepsilon_{23}\varepsilon_{31}\\
&\quad-\varepsilon_{11}\varepsilon_{23}^2-\varepsilon_{22}\varepsilon_{31}^2-\varepsilon_{33}\varepsilon_{12}^2-\varepsilon_I\varepsilon_{II}\varepsilon_{III}=0.
\end{aligned} \tag{27}$$

Obviously, the solution of the system (18)–(21), (27) is covered with the following cases (similar regimes for stresses are given in [11])

$$\text{(a)}\quad \varepsilon_{12}=0,\qquad \varepsilon_{23}=0,\qquad \varepsilon_{31}=0, \tag{28}$$

$$\text{(b)}\quad \varepsilon_{12}=0,\qquad \varepsilon_{31}=0,\qquad f_2=0, \tag{29}$$

(c) $\varepsilon_{23}=0, \quad \varepsilon_{31}=0, \quad f_1=0,$ (30)

(d) $\varepsilon_{12}=0, \quad \varepsilon_{23}=0, \quad f_3=0,$ (31)

(e) $\varepsilon_{12}\neq 0, \quad \varepsilon_{23}\neq 0, \quad \varepsilon_{31}\neq 0,$ (32)

where

$$f_1=(C_{1111}-C_{1122}-2C_{1212})\varepsilon_{11}-(C_{2222}-C_{1122}-2C_{1212})\varepsilon_{22}+(C_{1133}-C_{2233})\varepsilon_{33}, \tag{33}$$

$$f_2=(C_{1122}-C_{1133}\varepsilon_{11}+(C_{2222}-C_{2233}-2C_{2323})\varepsilon_{22}-(C_{3333}-C_{2233}-2C_{2323})\varepsilon_{33}, \tag{34}$$

$$f_3=(C_{1133}-C_{1111}+2C_{3131})\varepsilon_{11}+(C_{2233}-C_{1122})\varepsilon_{22}+(C_{3333}-C_{1133}-2C_{3131})\varepsilon_{33}. \tag{35}$$

In the cases (a)–(d) the equations (18)–(21) are satisfied trivially and the nonzero strain components can be determined from the strain invariants (27) and equation $F_i=0$ (cases (b)–(d)) analytically. All these cases lead to solutions where at least one (cases (b)–(d)) or all (case (a)) strain component(s) are equal to principal strain. Here the details of the solution are omitted due to conciseness sake (the strain invariants have reduced form and the solution is simplified). The results of the solution for cases (a) and (b) are given in Tables 2 and 3, respectively, where

$$r_1=C_{1122}-C_{1133},$$
$$r_2=C_{2222}-C_{2233}-2C_{2323}, \tag{36}$$
$$r_3=C_{2222}+C_{3333}-2C_{2333}-4C_{2323}.$$

The cases (c) and (d) can be treated similarly to the case (b). Obviously, the regime (a) generates 6 and regimes (b), (c), (d) each 3 different extremes of the strain energy density, respectively (the values of the shear strains with sign $\pm$ in Table 3 correspond to same value of the strain energy density). Thus, the regimes (a)–(d) together generate 15 extreme values of the strain energy density with closed form analytical solution in terms of strains.

In the case of regime (e) the rank of the algebraic system (18)–(21), (27) can be reduced by solving the system (18)–(21) with respect to the shear strains

$$\varepsilon_{23}=\pm\sqrt{\frac{f_1}{D_1}\frac{f_3}{D_3}}, \qquad \varepsilon_{12}=\pm\sqrt{\frac{f_3}{D_3}\frac{f_2}{D_2}},$$
$$\varepsilon_{31}=\pm\sqrt{\frac{f_1}{D_1}\frac{f_2}{D_2}} \tag{37}$$
$$\left(\varepsilon_{12}\varepsilon_{23}\varepsilon_{31}=\frac{f_1}{D_1}\frac{f_2}{D_2}\frac{f_3}{D_3}\right)$$

and substituting the results in strain invariants (27)

$$\varepsilon_{11}\varepsilon_{22}+\varepsilon_{22}\varepsilon_{33}+\varepsilon_{33}\varepsilon_{11}-\frac{f_2}{D_2}\frac{f_3}{D_3}-\frac{f_1}{D_1}\frac{f_2}{D_2}-\frac{f_1}{D_1}\frac{f_3}{D_3}=\varepsilon_I\varepsilon_{II}+\varepsilon_{II}\varepsilon_{III}+\varepsilon_{III}\varepsilon_I. \tag{38}$$

$$\varepsilon_{11}\varepsilon_{22}\varepsilon_{33}+2\frac{f_1}{D_1}\frac{f_2}{D_2}\frac{f_3}{D_3}-\varepsilon_{11}\frac{f_1}{D_1}\frac{f_3}{D_3}-\varepsilon_{22}\frac{f_1}{D_1}\frac{f_2}{D_2}-\varepsilon_{33}\frac{f_2}{D_2}\frac{f_3}{D_3}=\varepsilon_I\varepsilon_{II}\varepsilon_{III}. \tag{39}$$

Table 2 Solutions in terms of strains (case (a))

ε_{11}	ε_{22}	ε_{33}
ε_I	ε_{II}	ε_{III}
ε_I	ε_{III}	ε_{II}
ε_{II}	ε_I	ε_{III}
ε_{II}	ε_{III}	ε_I
ε_{III}	ε_I	ε_{II}
ε_{III}	ε_{II}	ε_I

Table 3 Solutions in terms of strains (case (b))

ε_{11}	ε_{22}	ε_{33}	ε_{23}
ε_I	$(r_1\varepsilon_I+r_2(\varepsilon_{II}+\varepsilon_{III}))/r_3$	$\varepsilon_{II}+\varepsilon_{III}-\varepsilon_{22}$	$\pm\sqrt{\varepsilon_{22}\varepsilon_{33}-\varepsilon_{II}\varepsilon_{III}}$
ε_{II}	$(r_1\varepsilon_{II}+r_2(\varepsilon_I+\varepsilon_{III}))/r_3$	$\varepsilon_I+\varepsilon_{III}-\varepsilon_{22}$	$\pm\sqrt{\varepsilon_{22}\varepsilon_{33}-\varepsilon_I\varepsilon_{III}}$
ε_{III}	$(r_1\varepsilon_{III}+r_2(\varepsilon_{II}+\varepsilon_I))/r_3$	$\varepsilon_I+\varepsilon_{II}-\varepsilon_{22}$	$\pm\sqrt{\varepsilon_{22}\varepsilon_{33}-\varepsilon_I\varepsilon_{II}}$

In (37)–(39) F_i $(i = 1, 2, 3)$ are determined with the relations (33)–(35) and

$$D_1 = 2(C_{2323} - C_{3131}),$$
$$D_2 = 2(C_{3131} - C_{1212}), \tag{40}$$
$$D_3 = 2(C_{1212} - C_{2323}).$$

The equations (38)–(39) can be solved together with first strain invariant with respect to the strain components ε_{11}, ε_{22} and ε_{33}. The total rank of this system is 6 and its solution can be transformed into solution of the one 6-th order equation (for example with respect to ε_{11}) and evaluation of other strain components (ε_{22} and ε_{33}) from linear expressions (using CAS-es method). Thus, the complexity of the optimality conditions (24) or (18)–(21) is reduced significantly (see Table 1). When ε_{11}, ε_{22} and ε_{33} are determined, the shear strains can be computed using relation (37).

Next, the global extremes of the strain energy density can be determined by evaluating the functional (7) for all solutions corresponding to regimes (a)–(e). The Euler angles corresponding to global extremes of the strain energy density can be obtained by solving system (13) with respect to Euler angles. The latter system has closed form analytical solution. Here the details are omitted (for conciseness sake) and the results are pointed out for most general case (e) where $\varepsilon_{12} \neq 0$, and $\varepsilon_{23} \neq 0$ and $\varepsilon_{31} \neq 0$.

$$\sin^2(\theta_2) = \frac{(\varepsilon_{33} - \varepsilon_{III})(\varepsilon_{33} + \varepsilon_{III} - \varepsilon_I - \varepsilon_{II}) + \varepsilon_{31}^2 + \varepsilon_{23}^2}{(\varepsilon_I - \varepsilon_{III})(\varepsilon_{III} - \varepsilon_{II})}, \tag{41}$$

$$\sin^2(\theta_1) = \frac{\varepsilon_{23}^2 - \varepsilon_{22}\varepsilon_{33} - \varepsilon_{11}\varepsilon_{III} + \varepsilon_{III}(\varepsilon_I - \varepsilon_{II})}{\sin^2(\theta_2)(\varepsilon_I - \varepsilon_{III})(\varepsilon_{III} - \varepsilon_{II})}, \tag{42}$$

$$\cos(2\theta_3) = 2\frac{\varepsilon_{III} - \frac{1}{2}(\varepsilon_I + \varepsilon_{II}) - (\varepsilon_{III} - \varepsilon_{33})/\sin^2(\theta_2)}{(\varepsilon_{II} - \varepsilon_I)}. \tag{43}$$

The singular solutions, corresponding to the case $\sin(\theta_2) = 0$, can be determined from (13) by replacing the matrix $\mathbf{A}$ with the following transformation matrix given in terms of Euler parameters

$$\mathbf{A}^{\mathrm{P}} = 2\begin{pmatrix} e_0^2 + e_1^2 - 1/2 & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - 1/2 & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - 1/2 \end{pmatrix}. \tag{44}$$

The closed form analytical expressions for 32 singular solutions are obtained, which determine 16 optimal material directions, since the solutions with sign $\pm$ ahead of e_0 correspond the same direction but opposite straight. In the case $e_0 = 0$ the opposite straight is determined by sign e_i $(i = 1, 2, 3; i \neq 0)$.

Finally, the solutions determined in terms of Euler angles can be transformed to Euler parameters using the relation between Euler angles and parameters. An alternative solution is treated by use of hybrid GA algorithm and the results are compared (details in next section).

Note, that closed form theoretical analysis of the global and local extremes remains open, since the global extremes may correspond to numerical solutions (3D orthotropic material). However, using (18)–(20), it can be shown, that the solutions depend not more than seven non-dimensional material parameters.

6 Numerical results

Let us consider the 3D optimal material orientation problem in the case of a E-Glass/vinylester as an example. The properties of the material are described with the following values of engineering parameters $E_1 = 25$ GPa, $E_2 = 24.8$ GPa, $E_3 = 8.5$ GPa, $G_{12} = 6.5$ GPa, $G_{13} = 4.2$ GPa, $G_{23} = 4.5$ GPa, $\nu_{12} = 0.1$, $\nu_{13} = 0.28$, $\nu_{23} = 0.3$ (stiffness characteristics).

The plot of the strain energy density is given in Fig. 1 in order to get some perception on its behavior ($\varepsilon_I = 8$, $\varepsilon_{II} = -7$, $\varepsilon_{III} = -6$). The value of the Euler angle θ_1 is fixed ($\theta_1 = \pi/4$) in order to obtain 3D plot and the values of the principal strains are chosen so that the most general case of the strain level solution (32) realizes.

Obviously, the objective function has a number of local extremes that refers to the complexity of the posed problem. The optimal material orientations corresponding to the global minimum of the strain energy density are given in Table 4. In order to validate the results, an alternative solution has been performed by the use of hybrid GA (global search by the use of GA and

Table 4 Minimum value of the strain energy density and corresponding values of the Euler angles

Method	Min strain energy den.	θ_1^*	θ_2^*	θ_3^*
Decomposition method	1590.8347	0.7456	1.5261	0.5326
GA + Gradient	1590.8347	0.7456	1.5261	0.5326
GA	1590.9651	0.7508	1.5344	0.5423

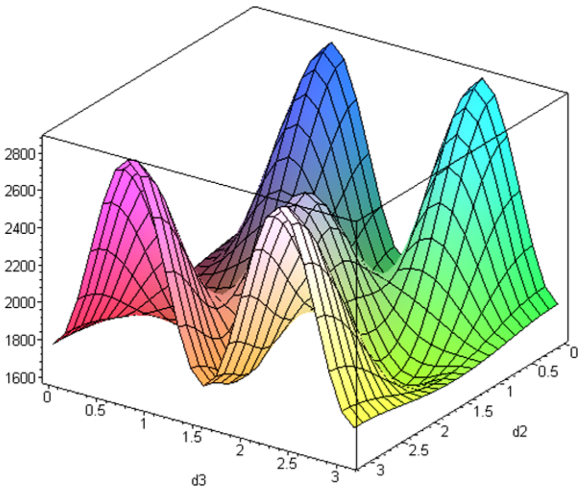

Fig. 1 The strain energy density as function of Euler angles θ_2 and θ_3 ($\theta_1 = \pi/4$) and principal strains

local search by the use of gradient method) [18, 19]. The hybrid GA algorithm is realized in terms of Euler angles (by substituting the strain components (13) in optimality conditions (21)) in order to keep optimization problem unconstrained.

It can be seen from Table 4 that solutions obtained by the use of decomposition method proposed and hybrid GA algorithm, respectively coincide and are close to the solution obtained by the use of GA algorithm (last row of the table in Table 4). It is correct to note, that the content of the last row of the table in Table 4 (GA) depends on the random variables (mutation rate, random variables used for parent selection) and may vary in different runs of the program.

In the case of values of the material parameters and principal strains used above, the system (38)–(39) produce 4 real and 2 complex solutions in terms of strains. The global extremes of the energy density are determined by comparing the values of the objective function corresponding to these 4 real-valued solutions and all solutions corresponding to regimes (a)–(d). The Euler angles are determined for global extremes of the energy density only using closed form analytical relations ((41)–(43) for regime (e)). The system (41)–(43) determines four solutions in terms of Euler angles in interval $[0, \pi]$ satisfying the conditions (37). These solutions in terms of Euler angles correspond to the same extreme value of the strains energy density. The corresponding solutions in terms of strains differ by signs of the shear strains. Table 4 only one of the solutions in terms of strains is given. The remaining three solutions in terms of Euler angles are determined as

$$\theta_i = \theta_i^*, \quad \theta_j = \pi - \theta_j^*, \\ i = 1, 2, 3, \ j \in \{1, 2, 3\}, \ j \neq i, \tag{45}$$

where θ_i^* stands for solution given in Table 4.

The solution corresponding to maximal value of the strain energy density is determined by regime (a) and is given in terms of strains as $\varepsilon_{11} = \varepsilon_I$, $\varepsilon_{22} = \varepsilon_{II}$, $\varepsilon_{33} = \varepsilon_{III}$, $\varepsilon_{12} = \varepsilon_{31} = \varepsilon_{23} = 0$. This solution corresponds to singular case where $\sin(\theta_2) = 0$ (see (41)) and cannot be determined uniquely in terms of Euler angles. Corresponding solutions can be determined in terms of Euler parameters using formulas (13) and (44) as $e_1 = e_2 = e_3 = 0$, $e_0 = \pm 1$ and $e_1 = e_2 = e_0 = 0, e_3 = \pm 1$. The hybrid GA algorithm converges in latter case to the solution where $\theta_1 = \theta_2 = \theta_3 = 0$. Note, that the maximal value of the strain energy density corresponds to the most flexible structure and may be useful in design of energy absorbing structures.

7 Reference discussions with 2D solutions

An approach proposed in the current paper has been tested by authors on 2D orthotropic materials in [20]. The decomposition method based solution procedure allows to reduce the optimization problem to solving set of linear equations. Firstly, the closed form analytical solutions have been determined terms of strains. Next the transformation formula for strain component has been utilized in order to determine the rotation angle. The obtained results coincide with well-known solution derived by Pedersen for linear elastic 2D orthotropic material in [4].

In the case of current approach, here are several possibilities how to manage the constraints. In certain cases it may be simpler to solve some constraints with respect to design variables and substitute these variables in objective function. For example, expressing the shear strain ε_{12}^2 and ε_{22} (or ε_{11}) from the strain invariants and substituting in objective function results a quadratic function of one variable (2D orthotropic material). It can be concluded that decomposition method proposed allows to simplify the solution of the optimal material optimization problem also in the case of 2D materials.

8 Conclusions

Current paper is focused on solving optimal material orientation problems of linear elastic three-dimensional orthotropic materials. The existing optimality conditions given by Seregin and Troitski [11], Rovati and Taliercio [13], Cowin [14] are analyzed and compared. The optimality conditions in terms of strains are derived. The decomposition method, including the strain level solution, search for global extremes and evaluating of Euler angles, is developed. The solutions, corresponding to regimes (a)–(d) and (e) are determined analytically and numerically, respectively. The numerical solution of the complex nonlinear system is reduced to the solution of the one sixth order algebraic equation. The obtained results are validated against an alternative solution performed by the use of hybrid GA algorithm.

Although there are at least four different approaches used in order to derive optimality conditions for optimization problem considered, the most general approach is still not used in opinion of authors of the current paper. Namely, the optimization problem can be formulated in strain space, considering the strain components as design variables. In latter case, the strain invariants should be included as constrains. In the case of 3D orthotropic linear elastic material such an approach results the optimality conditions derived in the current paper (here other considerations have been used).

Following [9] it can be shown, that the results obtained for linear elastic 3D material hold good also for some nonlinear elastic material models.

Acknowledgement The work has been supported by Estonian Science Foundation grant G6835.

References

1. Gupta AK, Kumar L (2008) Thermal effect on vibration of non-homogenous visco-elastic rectangular plate of linearly varying thickness. Meccanica 43:47–54
2. Gupta AK, Kaur H (2008) Study of the effect of thermal gradient on free vibration of clamped visco-elastic rectangular plates with linearly thickness variation in both directions. Meccanica 43:449–458
3. Banichuk NV (1981) Optimization problems for elastic anisotropic bodies. Arch Mech 33:347–363
4. Pedersen P (1989) On optimal orientation of orthotropic materials. Struct Optim 1:101–106
5. Pedersen P (1990) Bounds on elastic energy in solids of orthotropic materials. Struct Optim 2:55–62
6. Sacci Landriani G, Rovati M (1991) Optimal design for 2-D structures made of composite materials. ASME J Eng Mater Technol 113:88–92
7. Majak J (2001) Orientational design of nonorthotropic two-dimensional materials. Mech Compos Mater 37:27–38
8. Pedersen P, Taylor JE (1993) Optimal design based on power law nonlinear elasticity. In: Optimal design with advanced materials. Elsevier, Amsterdam, pp 51–66
9. Lellep J, Majak J (2000) Nonlinear constitutive behavior of orthotropic materials. Mech Compos Mater 36:261–266
10. Lellep J, Majak J (1997) On optimal orientation of nonlinear elastic orthotropic materials. Struct Optim 14:116–120
11. Seregin GA, Troitski VA (1982) On the best position of elastic symmetry planes in an orthotropic body. PMM USSR 45:139–142
12. Rovati M, Taliercio A (1989) Optimal orientation of the symmetry axes of orthotropic 3-D materials. Lect Notes Eng 63:127–134
13. Rovati M, Taliercio A (2003) Stationarity of the strain energy density for some classes of anisotropic solids. Int J Solids Struct 40:6043–6075
14. Cowin SC (1994) Optimization of the strain energy density in linear anisotropic elasticity. J Elast 34:45–68
15. Majak J (1995) On optimal material orientation of transversally isotropic and orthotropic 3D materials. In: Proc of the 8-th Nordic seminar on computational mechanics. Chalmers University of Technology, pp 39–42
16. Marannano G, Mariotti V (2008) Structural optimization and experimental analysis of composite material panels for naval use. Meccanica 43:251–262
17. Groenwold AA, Haftka RT (2006) Optimization with non-homogeneous failure criteria line Tsai-Wu for composite laminates. Struct Optim 32:183–190
18. Bassir DH, Zapico JL, Gonzales MP, Alonso R (2007) Identification of a spatial linear model based on earthquake-induced data and genetic algorithm with parallel selection. Int J Simul Multidiscipl Des Optim 1:39–48
19. Pohlak M, Majak J, Eerme M (2007) Optimization of car frontal protection system. Int J Simul Multidiscipl Des Optim 1:31–38
20. Majak J, Pohlak M (2009) Stationary of the strain energy density in anisotropic solids. In: Proc of the 8th World Congress on structural and multidisciplinary optimization (CD-ROM)