Meccanica (2010) 45: 401–413
DOI 10.1007/s11012-009-9260-9

# Mechanical loads on a generalized thermoelastic medium with diffusion

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Received: 19 January 2008 / Accepted: 29 September 2009 / Published online: 16 October 2009


**Abstract** The disturbance caused by the application of continuous mechanical source on the free surface of a homogeneous, isotropic elastic half space in the context of the theory of generalized thermoelastic diffusion with one relaxation time parameter is investigated in the Laplace-Fourier transform domain for a two dimensional problem using eigenvalue approach. The integral transforms are inverted by using a numerical technique. The expressions for displacement components, stresses, temperature field, concentration and chemical potential so obtained in the physical domain are computed numerically and illustrated graphically at different times, for copper like material. As a special case the effect of diffusion on various expressions has also been obtained analytically and depicted graphically.

**Keywords** Thermoelastic diffusion · Generalized thermoelasticity · Mechanical source · Eigenvalue approach · Laplace and Fourier transforms

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**Nomenclature**

| | |
|---|---|
| $\lambda, \mu$ | Lame's constants |
| $\rho$ | density of the medium |
| $\sigma_{ij}$ | components of stress tensor |
| $e_{ij}$ | components of strain tensor |
| $u_i$ | components of displacement vector |
| $C_E$ | specific heat at constant strain |
| $t$ | time |
| $T$ | absolute temperature |
| $T_0$ | reference temperature chosen so that $\frac{\lvert T-T_0\rvert}{T_0} \ll 1$ |
| $\Theta$ | $=T-T_0$ |
| $K$ | thermal conductivity |
| $e_{kk}$ | dilatation |
| $\delta_{ij}$ | Kronecker delta |
| $P$ | chemical potential per unit mass |
| $C$ | non-equilibrium concentration |
| $C_0$ | mass concentration at natural state |
| $c$ | $=C-C_0$ |
| $D$ | thermodiffusion constant |
| $\tau_0$ | thermal relaxation time |
| $\tau$ | diffusion relaxation time |
| $a$ | measure of thermodiffusion effect |
| $b$ | measure of diffusive effects |
| $\beta_1$ | $=(3\lambda+2\mu)\alpha_t$ |
| $\beta_2$ | $=(3\lambda+2\mu)\alpha_c$ |
| $\alpha_t$ | coefficient of linear thermal expansion |
| $\alpha_c$ | coefficient of linear diffusion expansion |
| $F_0$ | intensity of the applied mechanical load |
| $\mathbf{u}$ | displacement vector |
| $\phi$ | scalar potential |

$\psi$ vector potential
$\delta(\cdot)$ Dirac delta function
$H(\cdot)$ Heaviside function

## 1 Introduction

The temperature of a deformable body can vary both with time and from point. This variation can be caused both by heat exchange with external medium and by the process of deformation itself, during which a part of the mechanical energy is transformed into heat. The thermoelastic energy degradation is one of the causes of damping of elastic body vibrations.

The coupling between thermal and strain fields give rise to the coupled theory of thermoelasticity. Duhamel [4] and Neumann [7] introduced the theory of uncoupled thermoelasticity. Boit [2] developed the coupled theory of thermoelasticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. The heat equation for both theories is of the diffusion type predicting infinite speeds of propagation for heat waves contrary to physical observations. Lord and Shulman [6] introduced the theory of generalized thermo-elasticity with one relaxation time by postulating a new law of heat conduction to replace the classical Fourier's Law. This law contains the heat flux vector as well as its time derivative. It also contains a new constant that acts as a relaxation time. The heat equation of this theory is of wave-type, ensuring finite speeds of propagation for heat and elastic wave. The remaining governing equations for this theory, namely, the equations of motion and the constitutive relation remain the same as those for the coupled and the uncoupled theories. Dhaliwal and Sherief [3] extended this theory to general an isotropic media in the presence of heat sources. Because of the complicated nature of these equations, few attempts have been made to solve them. Sherief [15] solved a spherically symmetric problem with a point source of heat, and Sherief and Anwar [16] solved a cylindrically symmetric problem with a line source of heat. Sherief and Anwar [17] have studied the state space formulation for two-dimensional problem of generalized thermoelasticity with one relaxation time. Sherief and Ezzat [18] have studied the fundamental problem of thermoelasticity for an infinite spherically symmetric space using the method of potentials. All of these problems are one-dimensional. A detailed study of thermoelastic plane waves was made by [1, 14, 22].

Diffusion can be defined as the random walk of an ensemble of particles from regions of high concentration to regions of low concentration. There is now a great deal of interest in the study of this phenomenon, due to its many applications in geophysics and industrial applications. In integrated circuit fabrication, diffusion is used to introduce "dopants" in controlled amounts into the semiconductor substrate. In particular, diffusion is used to form the base and emitter in bipolar transistors, form integrated resistors, form the source/drain regions in MOS transistors and dope poly-silicon gates in MOS transistors. In most of these applications, the concentration is calculated using what is known as Fick's law. This is a simple law that does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced or the effect of the temperature on this interaction.

Nowacki [8–11] developed the theory of thermoelastic diffusion. In this theory, the coupled thermoelastic model is used. This implies infinite speeds of propagation of thermoelastic waves. The cross effects arising from the coupling of fields of temperature, mass diffusion and that of strain in an elastic cylinder have been discussed by Olesiak and Pyryev [12]. Sherief et al. [20] developed the theory of generalized thermoelastic diffusion that predicts finite speeds of propagation for thermoelastic and diffusive waves. Recently, Sherief and Saleh [19] solved a half space problem in the theory of generalized thermoelastic diffusion. The reflection phenomena of P and SV waves from free surface of an elastic solid with thermodiffusion was considered by Singh [21]. The thermodiffusion process helps investigation in the field associated with the advent of semiconductor devices and the advancement of microelectronics. The phenomenon of diffusion is also used to improve the conditions of oil extractions (seeking ways of more efficiently recovering oil from oil deposits). These days, oil companies are interested in the process of thermoelastic diffusion for more efficient extraction of oil from oil deposits. The present study is motivated by the importance of thermoelastic diffusion process in the field of oil extraction.

## 2 Basic equations and problem formulation

Following, Sherief et al. [20], the governing equations for an isotropic, homogeneous elastic solid with generalized thermodiffusion at uniform temperature $T_0$ in the undisturbed state, in the absence of body forces and heat loads are:

(i) the equation of motion

$$\rho \ddot{u}_i = \mu u_{i,jj} + (\lambda+\mu) u_{j,ij} - \beta_1 \Theta,_i - \beta_2 c,_i \,, \tag{1}$$

(ii) the generalized energy equation

$$K\Theta,_{ii} = \rho C_E(\dot{\Theta} + \tau_0 \ddot{\Theta}) + \beta_1 T_0(\dot{e_{kk}} + \tau_0 \ddot{e_{kk}}) + aT_0(\dot{c} + \tau_0 \ddot{c}), \tag{2}$$

(iii) the generalized diffusion equation

$$D\beta_2 e_{kk},_{ii} + Da\Theta,_{ii} + \dot{c} + \tau \ddot{c} - Dbc,_{ii} = 0, \tag{3}$$

(iv) the constitutive equations

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}(\lambda e_{kk} - \beta_1 \Theta - \beta_2 c), \tag{4}$$

$$P = -\beta_2 e_{kk} + bc - a\Theta, \tag{5}$$

where $\tau_0$, the thermal relaxation time ensures that heat conduction equation satisfied by temperature $\Theta$ predicts finite speed of heat propagation and $\tau$, the diffusion relaxation time ensures that the equation satisfied by the concentration $c$ also predicts finite speed of propagation of matter from one medium to the other. The superposed dot denotes the derivative with respect to time.

We use a fixed Cartesian coordinate system $(x, y, z)$ with origin on the surface $z = 0$, which is stress free and with $z$-axis directed vertically into the medium. The region $z > 0$ is occupied by the elastic solid with generalized thermodiffusion. A mechanical (normal or tangential) load of magnitude $F_0$ is assumed to be acting at a point on the surface $z = 0$ of the medium.

We restrict our analysis parallel to $xz$-plane. The boundary of the medium is assumed to be thermally insulated. The chemical potential is also assumed to be a known function of time.

We shall use the following non-dimensional variables

$$\begin{aligned} &x^* = \frac{\omega}{c_1}x, \quad z^* = \frac{\omega}{c_1}z, \quad t^* = \omega t, \\ &\tau^* = \omega\tau, \quad \tau_0^* = \omega\tau_0, \quad u_x^* = \frac{\rho\omega c_1}{\beta_1 T_0}u_x, \\ &u_z^* = \frac{\rho\omega c_1}{\beta_1 T_0}u_z, \quad \sigma_{ij}^* = \frac{\sigma_{ij}}{\beta_1 T_0}, \\ &c^* = \frac{c}{C_0}, \quad P^* = \frac{P}{\beta_2}, \quad \Theta^* = \frac{\Theta}{T_0}, \end{aligned} \tag{6}$$

where

$$c_1^2 = \frac{\lambda+2\mu}{\rho}, \qquad \omega = \frac{\rho C_E c_1^2}{K}. \tag{7}$$

Using the quantities given by (6) in (1)–(3), we obtain the equations in dimensionless form (dropping the asterisks for convenience) as

$$\frac{\partial^2 u_x}{\partial x^2} + a_1 \frac{\partial^2 u_z}{\partial x \partial z} + a_2 \frac{\partial^2 u_x}{\partial z^2} - \frac{\partial \Theta}{\partial x} - a_3 \frac{\partial c}{\partial x} - \frac{\partial^2 u_x}{\partial t^2} = 0, \tag{8}$$

$$a_2 \frac{\partial^2 u_z}{\partial x^2} + a_1 \frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_z}{\partial z^2} - \frac{\partial \Theta}{\partial z} - a_3 \frac{\partial c}{\partial z} - \frac{\partial^2 u_z}{\partial t^2} = 0, \tag{9}$$

$$\tau_m \frac{\partial \theta}{\partial t} + b_1 \tau_m \frac{\partial}{\partial t}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right) + b_2 \tau_m \frac{\partial c}{\partial t} - b_3 \nabla^2 \Theta = 0, \tag{10}$$

$$\frac{\partial}{\partial x}(\nabla^2 u_x) + \frac{\partial}{\partial z}(\nabla^2 u_z) + b_4 \nabla^2 \Theta + b_5 \tau_n \frac{\partial c}{\partial t} - b_6 \nabla^2 c = 0, \tag{11}$$

where

$$\begin{aligned} &a_1 = \frac{\lambda+\mu}{\lambda+2\mu}, \quad a_2 = \frac{\mu}{\lambda+2\mu}, \quad a_3 = \frac{\beta_2 C_0}{\beta_1 T_0}, \\ &b_1 = \frac{\beta_1^2 T_0}{\rho^2 c_1^2 C_E}, \quad b_2 = \frac{aC_0}{C_E \rho}, \quad b_3 = \frac{K\omega}{\rho c_1^2 C_E}, \\ &b_4 = \frac{a\rho c_1^2}{\beta_1 \beta_2}, \quad b_5 = \frac{\rho C_0 c_1^4}{D\beta_1 \beta_2 T_0 \omega}, \end{aligned} \tag{12}$$

$$b_6 = \frac{b\rho C_0 c_1^2}{\beta_1\beta_2 T_0}, \qquad \tau_m = \left(1 + \tau_0 \frac{\partial}{\partial t}\right),$$

$$\tau_n = \left(1 + \tau \frac{\partial}{\partial t}\right), \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

With the aid of the expressions relating displacement components $u_x, u_z$ to the scalar potential $\phi$ and vector potential $\psi$ in dimensionless form given by

$$u_x = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}, \qquad u_z = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x} \tag{13}$$

in (8)–(11), we obtain

$$\left[\nabla^2 - \frac{\partial^2}{\partial t^2}\right]\phi - \Theta - a_3 c = 0, \tag{14}$$

$$\nabla^2\psi - \frac{1}{a_2}\frac{\partial^2\psi}{\partial t^2} = 0, \tag{15}$$

$$\left[\nabla^2 - \frac{1}{b_3}\tau_m \frac{\partial}{\partial t}\right]\Theta - \frac{b_1}{b_3}\tau_m \frac{\partial}{\partial t}\nabla^2\phi - \frac{b_2}{b_3}\tau_m \frac{\partial c}{\partial t} = 0, \tag{16}$$

$$\nabla^2\phi + b_4\nabla^2\Theta + \left[b_5\tau_n \frac{\partial}{\partial t} - b_6\nabla^2\right]c = 0. \tag{17}$$

## 3 Solution of the problem

### 3.1 Formulation of a vector-matrix differential equation in transform domain

We now apply the Laplace and Fourier transforms defined by

$$\hat{f}(x, z, p) = \int_0^\infty f(x, z, t), e^{-pt} dt, \tag{18}$$

$$\tilde{f}(\xi, z, p) = \int_{-\infty}^{\infty} \hat{f}(x, z, p) e^{\iota\xi x} dx, \tag{19}$$

where $p$ and $\xi$ are the Laplace and Fourier transform variables respectively, so that (14)–(17) reduce to the form

$$\frac{d^2\tilde{\phi}}{dz^2} = R_{11}\tilde{\phi} + R_{12}\tilde{\Theta} + R_{13}\tilde{c}, \tag{20}$$

$$\frac{d^2\tilde{\Theta}}{dz^2} = R_{21}\tilde{\phi} + R_{22}\tilde{\Theta} + R_{23}\tilde{c}, \tag{21}$$

$$\frac{d^2\tilde{C}}{dz^2} = R_{31}\tilde{\phi} + R_{32}\tilde{\Theta} + R_{33}\tilde{c}, \tag{22}$$

$$\left[\frac{d^2}{dz^2} - \left(\xi^2 + \frac{p^2}{a_2}\right)\right]\tilde{\psi} = 0, \tag{23}$$

where

$$\begin{aligned}
&R_{11} = (p^2 + \xi^2), \qquad R_{12} = 1, \qquad R_{13} = a_3,\\
&R_{21} = f_1, \qquad R_{22} = f_2, \qquad R_{23} = f_3,\\
&R_{31} = \frac{g_1}{b_6 - a_3}, \qquad R_{32} = \frac{g_2}{b_6 - a_3},\\
&R_{33} = \frac{g_3}{b_6 - a_3},\\
&g_1 = p^4 + f_1(1 + b_4),\\
&g_2 = p^2 + (f_2 - \xi^2)(1 + b_4),\\
&g_3 = a_3(p^2 - \xi^2) + f_3(1 + b_4) + b_5\tau_n^* p + b_6\xi^2,\\
&f_1 = \frac{b_1}{b_3}\tau_m^* p^3, \qquad f_2 = \frac{(1 + b_1)}{b_3}\tau_m^* p + \xi^2,\\
&f_3 = \frac{(b_1 a_3 + b_2)}{b_3}\tau_m^* p,\\
&\tau_m^* = 1 + \tau_0 p, \qquad \tau_n^* = 1 + \tau p.
\end{aligned} \tag{24}$$

The system of (20)–(22) can be written in the form of a vector-matrix differential equation as follows:

$$\frac{d}{dz}V(\xi, z, p) = A(\xi, p)V(\xi, z, p), \tag{25}$$

where

$$V = \begin{bmatrix} U \\ D^*U \end{bmatrix}, \qquad A = \begin{bmatrix} O & I \\ A_1 & O \end{bmatrix},$$

$$U = \begin{bmatrix} \tilde{\phi} \\ \tilde{\Theta} \\ \tilde{c} \end{bmatrix}, \qquad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{26}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix},$$

where $D^*$ denotes the differentiation with respect to $z$ i.e. $d/dz$.

### 3.2 Solution of the vector-matrix differential equation

We now proceed to solve (25) by eigenvalue approach. To solve (25), we take

$$V(\xi, z, p) = X(\xi, p)e^{qz}, \tag{27}$$

so that

$$A(\xi, p)V(\xi, z, p) = qV(\xi, z, p), \tag{28}$$

which leads to an eigenvalue problem. The characteristic equation corresponding to the matrix $A$ is given by

$$\det[A - qI] = 0, \tag{29}$$

which on expansion provides us

$$q^6 - \lambda_1 q^4 + \lambda_2 q^2 - \lambda_3 = 0, \tag{30}$$

where

$$\begin{aligned}
\lambda_1 &= R_{11} + R_{22} + R_{33},\\
\lambda_2 &= R_{11}R_{22} + R_{22}R_{33} + R_{33}R_{11} - R_{12}R_{21}\\
&\quad - R_{23}R_{32} - R_{31}R_{13},\\
\lambda_3 &= R_{11}(R_{22}R_{33} - R_{23}R_{32}) + R_{12}(R_{23}R_{31}\\
&\quad - R_{21}R_{33}) + R_{13}(R_{21}R_{32} - R_{22}R_{31}).
\end{aligned} \tag{31}$$

The roots of (30), which are the eigenvalues of the matrix $A$ are $\pm q_i$, $i = 1, 2, 3$. We assume that real parts of $q_i$ are positive. The eigen vector $X(\xi, p)$ corresponding to the eigen values $q_i$ can be determined by solving the homogeneous equation

$$[A - qI]X(\xi, p) = 0. \tag{32}$$

The set of eigen vectors $X_i(\xi, p)$ $(i = 1, 2, 3, 5, 6, 7)$ may be obtained as

$$X_i(\xi, p) = \begin{bmatrix} X_{i1}(\xi, p) \\ X_{i2}(\xi, p) \end{bmatrix}, \tag{33}$$

where

$$\begin{aligned}
&X_{i1}(\xi, p) = \begin{bmatrix} s_i \\ r_i \\ 1 \end{bmatrix}, \qquad X_{i2}(\xi, p) = \begin{bmatrix} s_i q_i \\ r_i q_i \\ q_i \end{bmatrix},\\
&q = q_i; \quad i = 1, 2, 3\\
&X_{j1}(\xi, p) = \begin{bmatrix} s_i \\ r_i \\ 1 \end{bmatrix}, \qquad X_{j2}(\xi, p) = \begin{bmatrix} -s_i q_i \\ -r_i q_i \\ -q_i \end{bmatrix},\\
&j = i + 4, q = -q_i; \quad i = 1, 2, 3,\\
&s_i = \frac{s_{i1} - s_{i2} - R_{23}s_{i3}}{R_{21}s_{i3}},\\
&r_i = \frac{R_{31}R_{13} - (R_{33} - q_i^2)(R_{11} - q_i^2)}{R_{32}(R_{11} - q_i^2) - R_{12}R_{31}},\\
&s_{i1} = (R_{11} - q_i^2)(R_{22} - q_i^2)(R_{33} - q_i^2),\\
&s_{i2} = R_{31}R_{13}(R_{22} - q_i^2),\\
&s_{i3} = (R_{32}(R_{11} - q_i^2) - R_{12}R_{31}), \quad i = 1, 2, 3.
\end{aligned} \tag{34}$$

The solution of (25) is given by

$$V(\xi, z, p) = \sum_{i=1}^{3} [B_i X_i(\xi, p) e^{q_i z} + B_{i+4} X_{i+4}(\xi, p) e^{-q_i z}] \tag{35}$$

and solution of (23) is

$$\tilde{\psi} = B_4 e^{q_4 z} + B_8 e^{-q_4 z}, \tag{36}$$

where $B_i$ $(i = 1, 2, 3, 4, 5, 6, 7, 8)$ are arbitrary constants and

$$q_4 = \sqrt{\xi^2 + \frac{p^2}{a_2}}. \tag{37}$$

The equations (35) and (36) represent the solution of the general problem in the case of generalized thermodiffusion elasticity by employing the eigenvalue approach and therefore can be applied to a broad class of problems in the domain of Laplace and Fourier transforms.

## 4 Application: interactions due to mechanical load

In this section, the general solutions for displacement, stresses, temperature field, concentration and chemical potential presented in (35) and (36) will be used to yield the response of a half space subjected to an continuous mechanical load. The constants $B_i$ will be determined by imposing the proper boundary conditions. These constants, when substituted in (35) and (36), enable us to obtain the required physical quantities in the Fourier and Laplace transformed $(\xi, z, p)$ domain. The final solution in the original domain $(x, z, t)$ is obtained by a numerical inversion of both transforms.

*Case 1. Load in normal direction* In the half-space, the load $F(x)$ is applied in the normal direction at the origin of the co-ordinate system. The surface $z = 0$ is assumed to be thermally insulated so that there is no variation of temperature and concentration on it. Therefore, for this loading case the boundary conditions are

$$\begin{aligned}
&\sigma_{zz} = -F(x)H(t), \qquad \sigma_{zx} = 0,\\
&\frac{\partial \Theta}{\partial z} = 0, \qquad \frac{\partial c}{\partial z} = 0, \text{ at } \quad z = 0,
\end{aligned} \tag{38}$$

where $F(x) = F_0\delta(x)$.

*Case 2. Load in tangential direction* In the half-space, the load $F(x)$ is applied in the tangential direction at the origin of the co-ordinate system. The boundary conditions in this case are

$$\sigma_{zz}=0, \qquad \sigma_{zx}=-F(x)H(t), \\ \frac{\partial\Theta}{\partial z}=0, \qquad \frac{\partial c}{\partial z}=0, \text{ at } \quad z=0. \tag{39}$$

It can be seen that eight unknowns are to be determined in (35) and (36) and only four boundary conditions appear in each case. For the half-space the radiation conditions imply outgoing waves with decreasing amplitudes in the positive $z$-direction. Therefore, the radiation conditions require that $B_1=B_2=B_3=B_4=0$.

We obtain the expressions for the displacement components, stresses, temperature field, concentration and potential as

$$\tilde{\sigma_{zz}}=e_1B_5e^{-q_1z}+e_2B_6e^{-q_2z} \\ +e_3B_7e^{-q_3z}+e_4B_8e^{-q_4z}, \tag{40}$$

$$\tilde{\sigma_{zx}}=k_1B_5e^{-q_1z}+k_2B_6e^{-q_2z} \\ +k_3B_7e^{-q_3z}-k_4B_8e^{-q_4z}, \tag{41}$$

$$\tilde{u_x}=-\iota\xi[s_1B_5e^{-q_1z}+s_2B_6e^{-q_2z}+s_3B_7e^{-q_3z}] \\ +q_4B_8e^{-q_4z}, \tag{42}$$

$$\tilde{u_z}=-[s_1q_1B_5e^{-q_1z}+s_2q_2B_6e^{-q_2z}+s_3q_3B_7e^{-q_3z}] \\ -\iota\xi B_8e^{-q_4z}, \tag{43}$$

$$\tilde{\Theta}=r_1B_5e^{-q_1z}+r_2B_6e^{-q_2z}+r_3B_7e^{-q_3z}, \tag{44}$$

$$\tilde{c}=B_5e^{-q_1z}+B_6e^{-q_2z}+B_7e^{-q_3z}, \tag{45}$$

$$\tilde{P}=M_1B_5e^{-q_1z}+M_2B_6e^{-q_2z}+M_3B_7e^{-q_3z}, \tag{46}$$

where

$$B_{i+4}=\frac{\Delta_i}{p\Delta}, \quad i=1,2,3,4,$$

$$M_i=-e^*(q_i^2-\xi^2)s_i+\frac{bC_0}{\beta_2}-\frac{aT_0r_i}{\beta_2}, \quad i=1,2,3$$

$$\Delta=(r_3-r_2)q_2q_3(e_1k_4+e_4k_1) \\ +(r_1-r_3)q_1q_3(e_2k_4+e_4k_2) \\ +(r_2-r_1)q_1q_2(e_3k_4+e_4k_3),$$

$$e_i=q_i^2s_i-a^*s_i-r_i-b^*; \quad i=1,2,3,$$

$$e_4=\iota\xi q_4\left(1-\frac{\lambda}{\rho c_1^2}\right), \tag{47}$$

$$k_i=\frac{2\iota\xi s_iq_i\mu}{\rho c_1^2}; \quad i=1,2,3,$$

$$k_4=\frac{\mu}{\rho c_1^2}(\xi^2+q_4^2),$$

$$e^*=\frac{\beta_1T_0}{\rho c_1^2}, \qquad a^*=\frac{\lambda\xi^2}{\rho c_1^2}, \qquad b^*=\frac{\beta_2C_0}{\beta_1T_0}.$$

*Case 1. In normal direction* The values of $\Delta_i;\ i=1,2,3,4,$ when the load is acting in normal direction are

$$\Delta_1=F_0k_4(r_2-r_3)q_2q_3, \\ \Delta_2=F_0k_4(r_3-r_1)q_3q_1, \\ \Delta_3=F_0k_4(r_1-r_2)q_1q_2, \\ \Delta_4=F_0[k_1(r_2-r_3)q_2q_3+k_2(r_3-r_1)q_1q_3 \\ +k_3(r_1-r_2)q_1q_2]. \tag{48}$$

*Case 2. In tangential direction* The solution for this case as in (40)–(46), only with the replacement of $\Delta_i;\ i=1,2,3,4$ as:

$$\Delta_1=F_0e_4(r_2-r_3)q_2q_3, \\ \Delta_2=F_0e_4(r_3-r_1)q_3q_1, \\ \Delta_3=F_0e_4(r_1-r_2)q_1q_2, \\ \Delta_4=-F_0[e_1(r_2-r_3)q_2q_3+e_2(r_3-r_1)q_1q_3 \\ +e_3(r_1-r_2)q_1q_2]. \tag{49}$$

## 4.1 Particular case I

By taking $c=D=a=b=\beta_2=0$, we obtain the expressions for displacement components, stresses and temperature field in the generalized thermoelastic medium as:

$$\tilde{\sigma_{zz}}=e_1^*B_4^*e^{-q_1^*z}+e_2^*B_5^*e^{-q_2^*z}+e_3^*B_6^*e^{-q_3^*z} \tag{50}$$

$$\tilde{\sigma_{zx}}=k_1^*B_4^*e^{-q_1^*z}+k_2^*B_5^*e^{-q_2^*z}-k_3^*B_6^*e^{-q_3^*z} \tag{51}$$

$$\tilde{u_x}=-\iota\xi[s_1^*B_4^*e^{-q_1^*z}+s_2^*B_5^*e^{-q_2^*z}]+q_3^*B_6^*e^{-q_3^*z}, \tag{52}$$

$$\tilde{u_z}=-[s_1^*q_1^*B_4^*e^{-q_1^*z}+s_2^*q_2^*B_5^*e^{-q_2^*z}]-\iota\xi B_6^*e^{-q_3^*z}, \tag{53}$$

$$\tilde{\Theta}=B_4^*e^{-q_1^*z}+B_5^*e^{-q_2^*z}, \tag{54}$$

where

$$q_i^{*2} = \frac{\lambda_1^* + (-1)^{i+1}\sqrt{\lambda_1^{*2} - 4\lambda_2^*}}{2}; \quad i = 1, 2, \tag{55}$$

are the roots of the equation

$$q^4 - \lambda_1^* q^2 + \lambda_2^* = 0, \tag{56}$$

where

$$\begin{aligned}
&\lambda_1^* = R_{11} + R_{22}, \qquad \lambda_2^* = R_{11}R_{22} - R_{21}R_{12}\\
&q_3^* = q_4^2, \qquad B_{i+3}^* = \Delta_i^*/p\Delta^*; \quad i = 1, 2, 3\\
&\Delta^* = q_1^*(e_2^* k_3^* + e_3^* k_2^*) - q_2^*(e_3^* k_1^* + e_1^* k_3^*),\\
&e_i^* = q_i^{*2} s_i^* - a^* s_i^* - 1; \quad i = 1, 2,\\
&e_3^* = \iota\xi q_3^*\left(1 - \frac{\lambda}{\rho c_1^2}\right),\\
&k_i^* = \frac{2\mu}{\rho c_1^2}(\iota\xi q_i^* s_i^*); \quad i = 1, 2,\\
&k_3^* = \frac{\mu}{\rho c_1^2}(q_3^{*2} + \xi^2),\\
&s_i^* = -\frac{R_{22} - q_i^{*2}}{R_{21}}; \quad i = 1, 2.
\end{aligned} \tag{57}$$

*Case 1. In normal direction* The values of $\Delta_i^*$; $i = 1, 2, 3$, when load is in normal direction, are

$$\begin{aligned}
&\Delta_1^* = F_0 q_2^* k_3^*,\\
&\Delta_2^* = -F_0 q_1^* k_3^*,\\
&\Delta_3^* = F_0[q_2^* k_1^* - q_1^* k_2^*].
\end{aligned} \tag{58}$$

*Case 2. In tangential direction* The solution for this case as in (50)–(54), only with the replacement of $\Delta_i^*; i = 1, 2, 3$ as:

$$\begin{aligned}
&\Delta_1^* = F_0 q_2^* e_3^*,\\
&\Delta_2^* = -F_0 q_1^* e_3^*,\\
&\Delta_3^* = F_0[q_1^* e_2^* - q_2^* e_1^*].
\end{aligned} \tag{59}$$

### 4.2 Particular case II

If we neglect the thermodiffusion effect from the medium considered, the corresponding expressions for displacement components and stresses are given by:

$$\tilde{\sigma_{zz}} = e_1' B_3' e^{-q_1' z} + e_2' B_4' e^{-q_2' z}, \tag{60}$$

$$\tilde{\sigma_{zx}} = k_1' B_3' e^{-q_1' z} - k_2' B_4' e^{-q_2' z}, \tag{61}$$

$$\tilde{u_x} = -(\iota\xi) B_3' e^{-q_1' z} + q_2' B_4' e^{-q_2' z}, \tag{62}$$

$$\tilde{u_z} = -q_1' B_3' e^{-q_1' z} + \iota\xi B_4' e^{-q_2' z}, \tag{63}$$

where

$$\begin{aligned}
&q_1' = \sqrt{p^2 + \xi^2}, \qquad q_2' = \sqrt{\frac{p^2}{a_2} + \xi^2},\\
&B_{i+2}' = \Delta_i'/p\Delta'; \quad i = 1, 2,\\
&\Delta' = -(e_1' k_2' + e_2' k_1'),\\
&e_1' = q_1'^2 - a^*, \qquad e_2' = \iota\xi q_2'\left(1 - \frac{\lambda}{\rho c_1^2}\right),\\
&k_1' = \frac{2\mu}{\rho c_1^2}(\iota\xi q_1'), \qquad k_2' = \frac{\mu}{\rho c_1^2}(\xi^2 + q_2'^2).
\end{aligned} \tag{64}$$

*Case 1. In normal direction* The values of $\Delta_i'$; $i = 1, 2$, when load is in normal direction are

$$\Delta_1' = F_0 k_2', \qquad \Delta_2' = F_0 k_1'. \tag{65}$$

*Case 2. In tangential direction* The solution for this case are as in (60)–(63), only with the replacement of $\Delta_i'; i = 1, 2$ as:

$$\Delta_1' = F_0 e_2', \qquad \Delta_2' = -F_0 e_1'. \tag{66}$$

## 5 Inversion of transforms

The transformed displacements, stresses, temperature field, concentration and chemical potential (40)–(46), (50)–(54) and (60)–(63) are functions of $z$, the parameters of Laplace and Fourier transforms $p$ and $\xi$, respectively and hence are of the form $\tilde{f}(\xi, z, p)$. To get the function $f(x, z, t)$ in the physical domain, first we invert the Fourier transform using

$$\begin{aligned}
\hat{f}(x, z, p) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\iota\xi x}\tilde{f}(\xi, z, p)d\xi,\\
&= \frac{1}{\pi}\int_0^{\infty}\{\cos(\xi x)\tilde{f}_e - \iota\sin(\xi x)\tilde{f}_o\}d\xi,
\end{aligned} \tag{67}$$

where $\tilde{f}_e$ and $\tilde{f}_o$ are even and odd parts of the function $\tilde{f}(\xi, z, p)$ respectively. Thus, expressions (67) gives us the Laplace transform $\hat{f}(x, z, p)$ of function $f(x, z, t)$. Following Honig and Hirdes [5], the

Laplace transform function $\hat{f}(x,z,p)$ can be converted to $f(x,z,t)$.

The last step is to evaluate the integral in (67). The method for evaluating this integral by Press et al. [13], which involves the use of Romberg's integration with adaptive step size. This also uses the results from successive refinement of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero.

## 6 Numerical results and discussion

The copper material was chosen for the purpose of numerical evaluations. The material constants of the problem are thus given by Thomas [23] in SI units

$$T_0 = 293 \text{ K}, \qquad \rho = 8954 \text{ kg/m}^3,$$
$$\tau_0 = 0.02 \text{ s}, \qquad \tau = 0.2 \text{ s},$$
$$C_E = 383.1 \text{ J/(kg K)},$$
$$\alpha_t = 1.78 \times 10^{-5} \text{ K}^{-1},$$
$$K = 386 \text{ W/(m K)},$$
$$\lambda = 7.76 \times 10^{10} \text{ kg/(m s}^2),$$
$$\mu = 3.86 \times 10^{10} \text{ kg/(m s}^2),$$
$$\alpha_c = 1.98 \times 10^{-4} \text{ m}^3/\text{kg},$$
$$D = 0.85 \times 10^{-8} \text{ kg s/m}^3,$$
$$a = 1.2 \times 10^4 \text{ m}^2/(\text{s}^2 \text{ K}),$$
$$b = 0.9 \times 10^6 \text{ m}^5/(\text{kg s}^2).$$

The comparison of dimensionless normal displacement $u_z(=u_z/F_0)$, normal stress $\sigma_{zz}(=\sigma_{zz}/F_0)$, temperature $\Theta(=\Theta/F_0)$, and deviation of concentration $c(=c/F_0)$ for two different cases; a solid with thermoelastic diffusion (THED) and a thermoelastic solid (THE) due to normal and tangential continuous loads have been studied and shown in Figs. 1–8. The computations are carried out for two values of non-dimensional time, namely for $t = 0.075$ and $t = 0.10$ and initial concentration $C_0 = 1$ at $z = 1.0$ in the range $0 \le x \le 10$.

### 6.1 Case I: Normal load applied

The comparison of dimensionless normal displacement $u_z$, normal stress $\sigma_{zz}$, temperature $\Theta$ and deviation in mass concentration $c$ for the two different cases are studied in Figs. 1–4. Figure 1 represents the variation of normal displacement $u_z$ with $x$. The values of displacement $u_z$ for THED and THE theories decrease with distance $x$. At time $t = 0.075$ the variations of normal displacement lie in a higher range in comparison to time $t = 0.1$ for both the theories. The behaviour of variations is similar in nature for both the theories and for both values of time. Very near to the point of application of source the values of normal displacement are smaller for THE theory than that of THED theory. It can be depicted from figure that the magnitude of displacement is greater due to presence of diffusion in thermoelastic medium.

Figure 2 shows the variation of normal stress $\sigma_{zz}$ with $x$ due to normal load. For both the times, the values of $\sigma_{zz}$ for THE theory are greater than the corresponding values for THED solids in the initial range. The flow of variations for THED theory is similar to that for THE theory in almost whole of the range. The diffusion effect causing difference between two curves at any fix point as well as fixed time for the two theories is clearly visible from this figure. The distribution of temperature for both the theories i.e. THED and THE, is observed from Fig. 3, for both the times. It is clear that diffusion in thermoelastic medium plays an important role in temperature distribution in the medium as $\Theta$ varies very smoothly in thermoelastic medium without diffusion whereas its values lie comparatively in large range in thermoelastic medium with diffusion. Variation of concentration about initial concentration is represented by Fig. 4 for THED theory. The difference in the values of $c$ at a particular point for two different times can easily be observed from the curves. It is also clearly depicted from figure that the values of concentration $c$ are maximum at the origin for both the times and after a small range of $x$, seem to be vanishing in the range far from origin.

### 6.2 Case II: Tangential load applied

The comparison of dimensionless normal displacement $u_z$, normal stress $\sigma_{zz}$, temperature $\Theta$ and concentration deviation $c$ for the two different cases are studied in Figs. 5–8. Figure 5 shows the variation of normal displacement $u_z$ with $x$ due to tangential load. The behaviour of variations of displacement is different to that due to normal load as in Fig. 1. For both the theories, this function remains close to zero in the considered domain of the distance, except near the vicinity of the load where slight variations are noticed. The

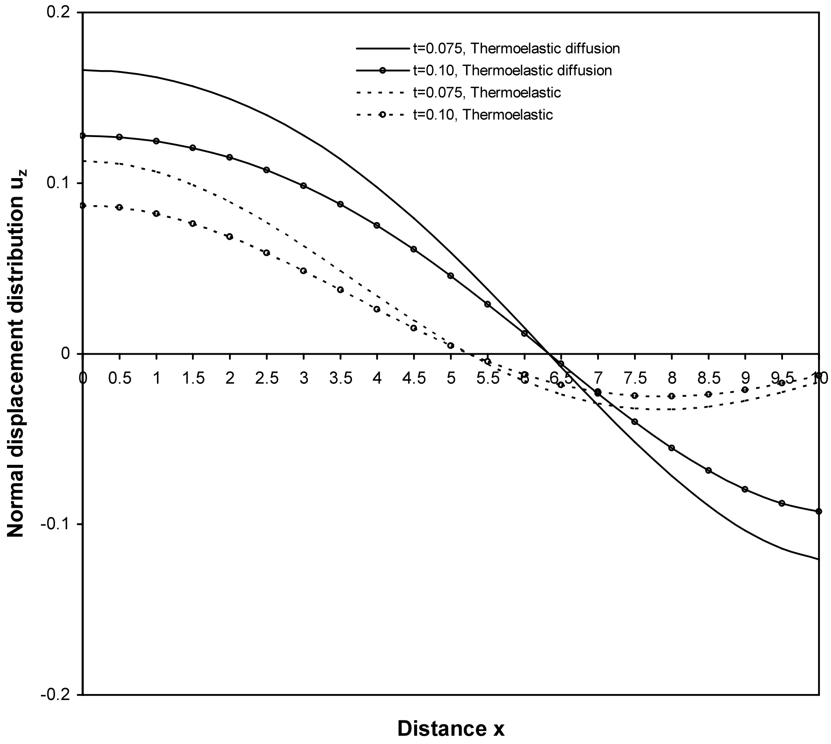


**Fig. 1** Distribution of normal displacement $u_z$ (due to normal load) versus distance

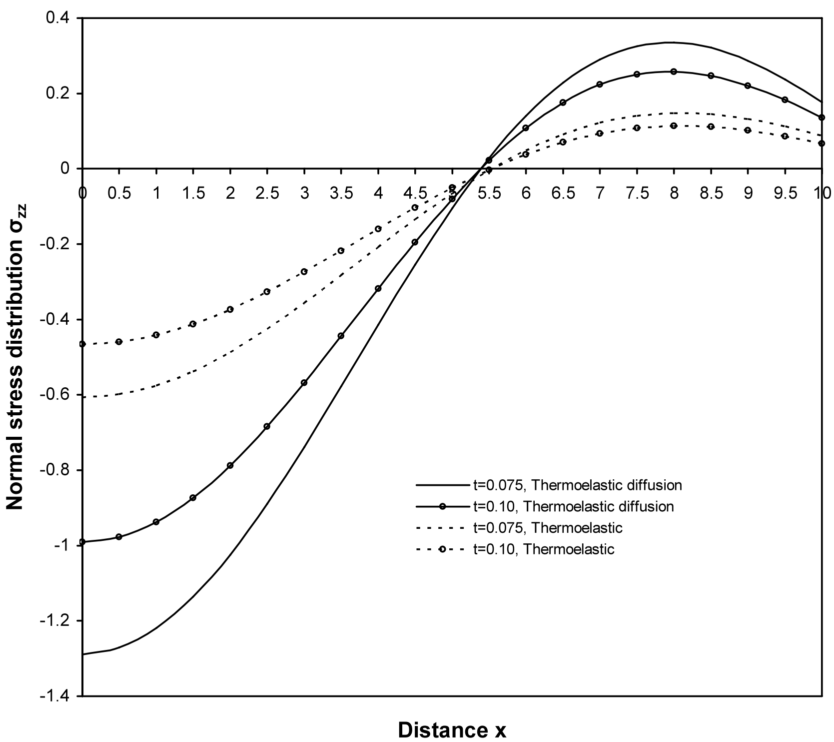


**Fig. 2** Distribution of normal stress $a_{zz}$ (due to normal load) versus distance

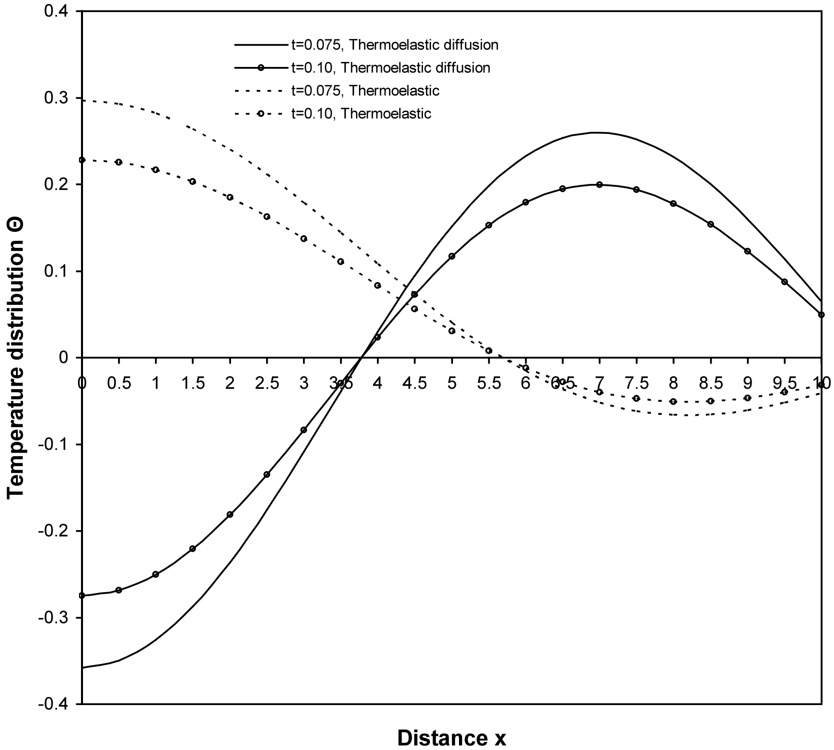


**Fig. 3** Distribution of temperature Θ (due to normal load) versus distance

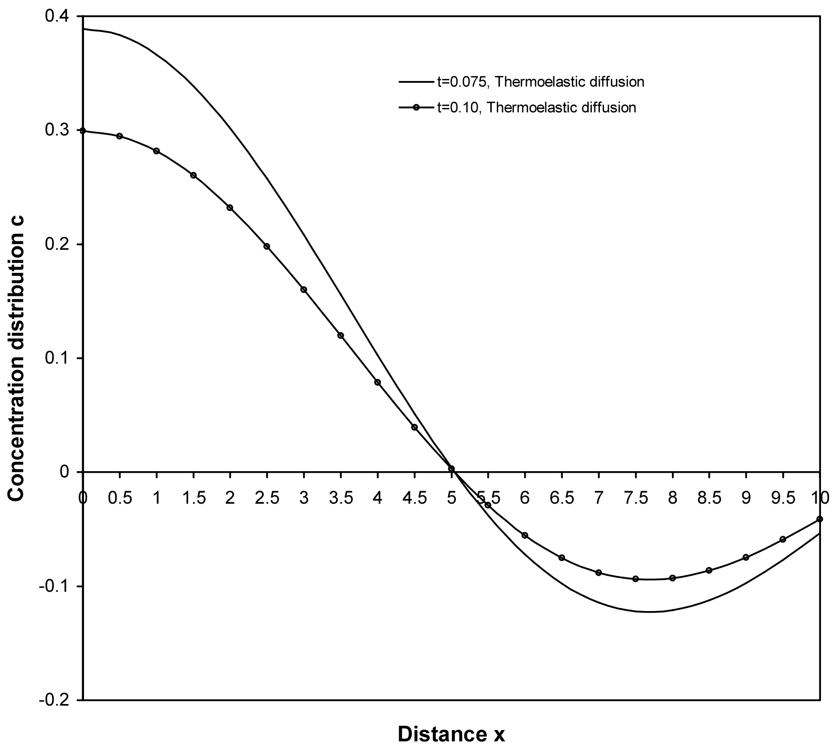


**Fig. 4** Distribution of concentration $c$ (due to normal load) versus distance

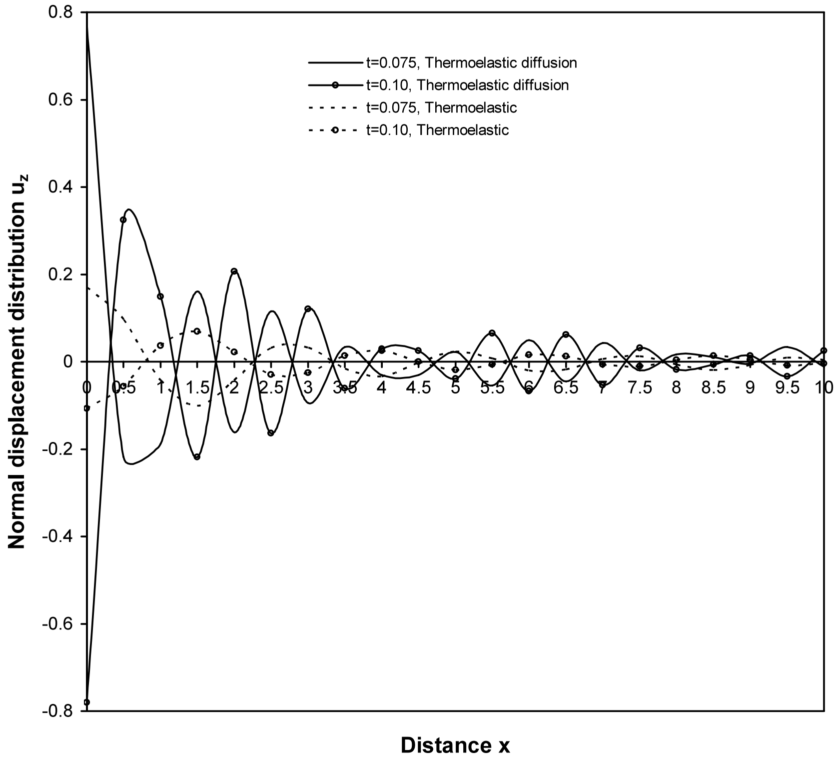


**Fig. 5** Distribution of normal displacement $u_z$ (due to tangential load) versus distance

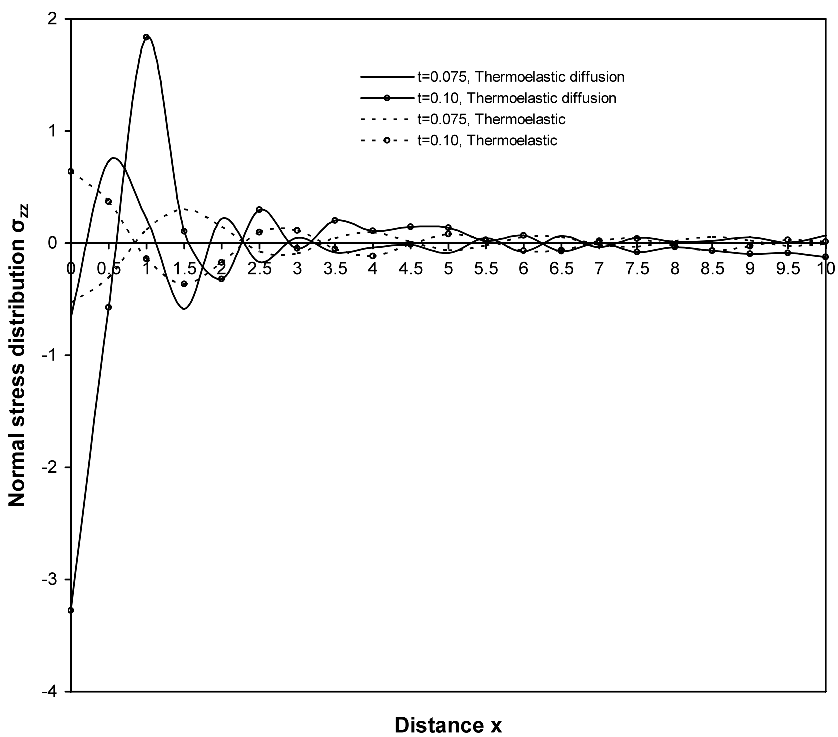


**Fig. 6** Distribution of normal stress $\sigma_{zz}$ (due to tangential load) versus distance

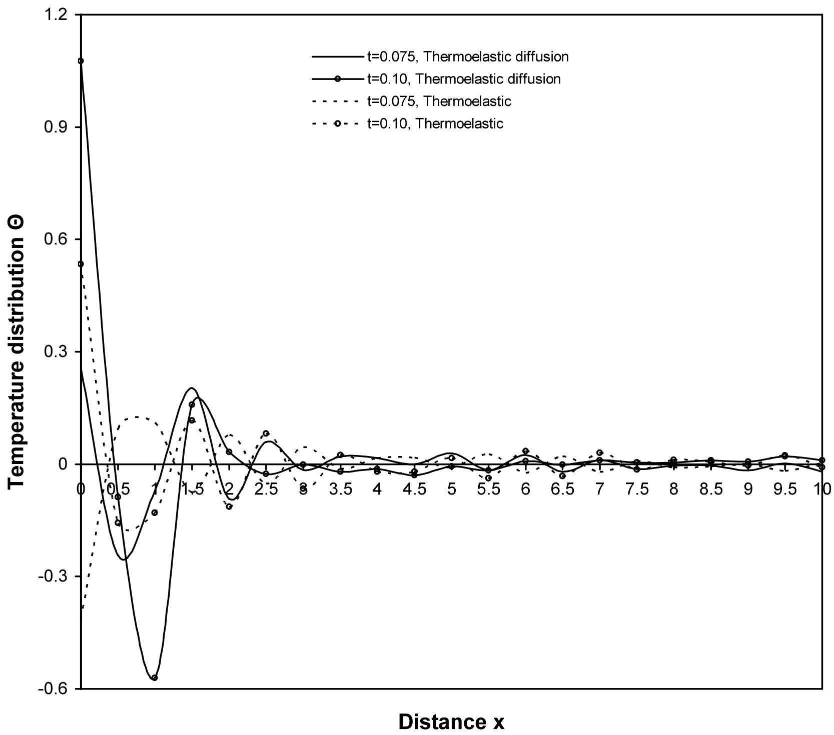


**Fig. 7** Distribution of temperature Θ (due to tangential load) versus distance

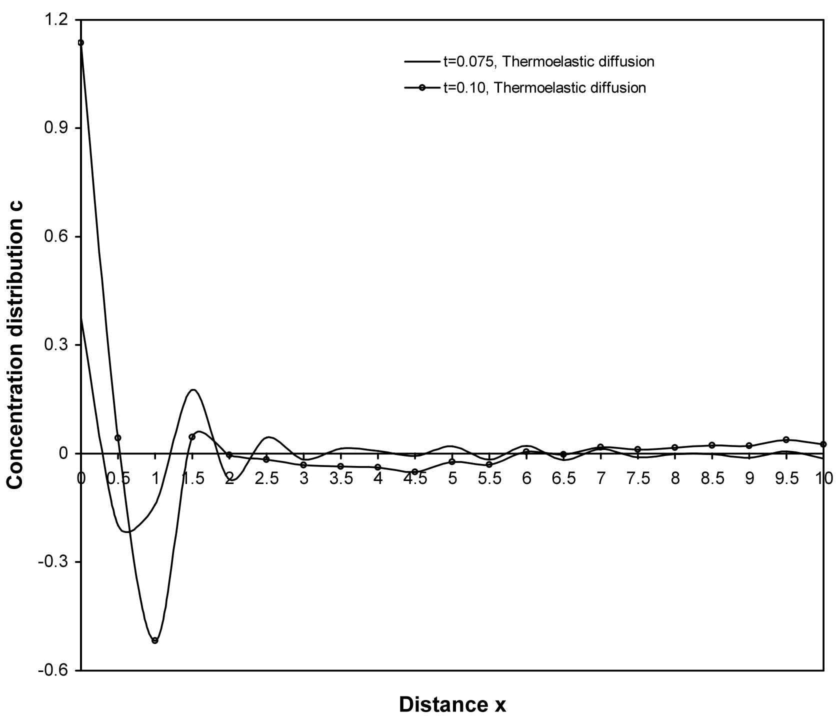


**Fig. 8** Distribution of concentration *c* (due to tangential load) versus distance

range of magnitude of normal displacement is greater in the thermoelastic medium with diffusion effect than that in thermoelastic medium without this effect.

Figure 6 depicts the variations of normal stress $\sigma_{zz}$ with $x$ due to tangential load. As the value of $x$ increases, the values of stress function for both the cases approach zero. Very near to the point of the application of the source there is a great difference in magnitude of normal stress for both the considered media. The effect of diffusion can be observed clearly from this figure by comparing the curve for the THED and THE theories. From a perusal of Fig. 7 it is clear that the trend of variation of temperature change in the THED theory is similar to that of THE theory at time 0.075 whereas at time 0.10 it is reverse in nature. The deviation of concentration from the mean value for THED theory has been depicted in Fig. 8. The values of concentration for a particular range show considerable difference for the two times.

## 7 Conclusion

Analysis of normal displacement, normal stress component, temperature and mass concentration developed in a body due to a continuous mechanical load (normal and tangential) is an interesting problem of mechanics having its applications in determining the stability of a medium. The results of all the functions for THED theory are distinctly different from those obtained for THE theory. This is due to the presence of diffusion in thermoelastic solid. The method used in the present article is applicable to a wide range of problems in thermodynamics. Near the point of application of load remarkable effect of time has been obtained.

In the frame work of this article, one also finds that the response to the mechanical load is not felt instantly at very large distance from the bounding plane of the medium. This is not the case for uncoupled and coupled theories of thermoelasticity where the response to the shock is felt instantly at infinitely far points from the source of disturbance. From the distribution of all the field variables, it can be found that wave type heat propagates in the medium. This indicates that the generalized heat conduction mechanism is completely different from the classical one. From all the figures, it can be noticed that all the field variables experience less and less disturbance after some distance from the source of disturbance.

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