Meccanica (2010) 45: 199–212
DOI 10.1007/s11012-009-9237-8

A historical perspective of Menabrea's theorem in elasticity

Danilo Capecchi · Giuseppe Ruta

Received: 25 September 2008 / Accepted: 1 July 2009 / Published online: 21 July 2009

Abstract This paper presents the theorem proposed by Luigi Federico Menabrea to study linear elastic redundant systems. Some of Menabrea's papers on the subject are examined, as well as the criticism and the corrections brought to his first proof. We consider Menabrea's work in the frame of the studies of his contemporaries; we try to provide a historical and epistemological background for Menabrea's theorem and for its consequences in modern mechanics.

Keywords Menabrea's theorem · Linear elasticity · Least work

1 Introduction

Menabrea's theorem [1–4] reads: 'In a linear elastic system subjected to a compatible deformation the elastic complementary energy attains an absolute minimum, when the stress varies remaining balanced with external actions, in correspondence with the actual stress'. This is a version of the theorem of minimum complementary energy in absence of dislocations and was the basis for the solution methods of redundant frames proposed by Castigliano [5, 6] and Müller-Breslau [7]. The theorem was stated by Luigi Federico Menabrea in 1858 [1] with the aim of providing a general tool of solution for the linear elastic problems met in engineering practice. The tool was to be more efficient than the *ad hoc* procedures known at the time, based on the method of forces introduced by Navier [8]. Menabrea's theorem was perfected by Castigliano [5, 6] and only recently, thanks to the rational assessment of the theory of structures, was recognized to be but a particular statement of a theorem attributable to no single author.

Menabrea's theorem emerged in the Italian school of engineering, yet its origins lie in the studies of rational mechanicians, at that time referred to as 'geometers'. The route leading to the statement of the theorem appears a linear continuous accumulation of knowledge. It seems that the first origin of the theorem is in the paper by Euler on the distribution of 'pressures' (i.e. contact forces) due to a massive body over more than three coplanar point-shaped supports.[1] Such a problem does not admit solution by means of the balance equations for rigid bodies, and further equations,

D. Capecchi
Dipartimento di ingegneria strutturale e geotecnica, "Sapienza" Università di Roma, via A. Gramsci 53, 00197 Roma, Italy
e-mail: danilo.capecchi@uniroma1.it

G. Ruta (✉)
Dipartimento di ingegneria strutturale e geotecnica, "Sapienza" Università di Roma, via Eudossiana 18, 00184 Roma, Italy
e-mail: giuseppe.ruta@uniroma1.it

[1] Euler L, De pressione ponderis in planum cui incumbit, Novi commentarii Academiae Scientiarum Imperialis Petropolitanae, vol. 18, 1773, 289–329.

derived from a different principle, shall be used. Euler assumed that 'pressure' is linearly distributed over the plane support; then, balance equations provide 'pressures' on the supports, irrespective of their number.

Euler's paper, as it happens with pioneer works, contains an ambiguity: it is not clear if the linear distribution of 'pressures' is actually a constitutive relation or a new principle of statics, suitable for contact problems and describing a particular repartition of forces. Even if in our opinion Euler's actual thought matches the former interpretation,[2] his paper can be—and in fact was—interpreted in both ways. A series of papers followed Euler's, proposing different views of the new principle [9], but were inconclusive.

Navier [8] showed that interpreting Euler's position as a new law of statics is inconsistent: for a simple redundant structure (for instance, a beam resting on three supports), the constraint reactions are easily obtained by ordinary balance equations when considering elastic deformations. Navier's reasoning was not accepted by many scientists coping with the problem of finding constraint reactions of supports; indeed, until mid 1800's papers appeared, searching for a new principle of statics, among which many by Italian mechanicians.[3]

Cournot [12–14] was among the first to explicitly introduce the elasticity of supports, considering proportionality between pressures and deflections and stating a principle of minimum, called "théorème générale".

In 1857 the italian scholar Dorna extended Cournot's theorem from a rigid body on supports to an elastic system, always focusing on constraint reactions.[4] In 1858 Menabrea formulated his "équation d'élasticité", without exhibiting a satisfactory proof.

Timoshenko only hints on Menabrea in his historical monograph,[5] while a slightly more extended comment is found in Todhunter and Pearson.[6] The history of Menabrea's theorem was followed in some detail by Benvenuto in [9, 10] and Nascé in [11].

In this paper attention is focused mainly on the logical structure of Menabrea's proofs, not thoroughly studied in the papers appeared so far, which sometimes underestimate Menabrea's originality. Indeed, as far as we know, all historians of mechanics have addressed their attention only to the first statement of Menabrea's theorem, stressing the weakness of the proof. Our aim is to put into evidence how the other papers by Menabrea on the subject, not satisfactorily examined in the literature, contain apparent improvements.

We present Menabrea's first statement (1858), its positives and drawbacks, and the criticism of his contemporaries. We compare Menabrea's proof with those of his immediate forerunners; the aim is the search for a motivation of some ambiguities by Menabrea, not a philological reconstruction of the genesis of the theorem. Then we present the proof of 1865, on new bases and satisfactory enough for us—at least if compared with the standards of the engineers of the age. We do not present the proof of 1875, which only refines that of 1865, since, being published after Castigliano's thesis (1873), it might have been influenced by the latter.

We leave aside the mechanical interpretations of the theorem, not universally accepted yet, and the disputes on the priority of the formulation, in particular that between Menabrea and Castigliano. More informations on these other subjects can be found in [9–11, 15, 16].

2 Menabrea's "Nouveau principe sur la distribution des tensions"

Here [1] and in Menabrea's following papers an elastic system is modelled as a set of points, connected by elastic links which can be bars, undergoing displacements small enough to use linear equations. This may be seen as an extension of Cauchy and Poisson's molecular model (body-points exchanging central forces). In his studies on elasticity later on [17], Maxwell adopted a similar ideal truss (a set of elastic bars hinged to each other). It is simple conceptually and mathematically: indeed, it is described by

[2] Confirmed also by some of Euler's later works on beams.

[3] [9], vol. 2, pp. 447–466.

[4] Dorna A, Memoria sulle pressioni supportate dai punti di appoggio di un sistema equilibrato ed in istato prossimo al moto, Memorie R. Accademia delle Scienze di Torino, 18, 4–40 (1857).

[5] Timoshenko SP, A history of the theory of elasticity, New York, McGraw-Hill, 1953, p. 289.

[6] Todhunter I, Pearson K, A history of the theory of elasticity and of the strength of materials: from Galileo to the present time, Cambridge, University Press, 1886–1893, vol. 2.1, artt. 604–606.

algebraic equations, not differential as it is for continua. The mechanicians adopting the model were certain that most of the obtained results were valid both for the trusses in the industrial structures of 19th century and for more complex linear elastic systems, under suitable modifications.

From statics a truss with n nodes and m bars is redundant if $m > 3n - 6$, i.e., inner actions cannot be found by balance equations alone and a statical indeterminacy is left: there are infinite values of forces in the bars-links ('tensions') that keep the truss balanced with external forces. Menabrea states that

> The values of actual tensions depend on the elasticity of the links [the bars], and once this is determined, the same happens with the tensions.[7]

Thus, the key idea for finding the solution of the problem is to keep strain and mechanical properties (i.e, compatibility equations and constitutive prescriptions) into account. By means of the set of equations further obtained the problem is solvable, as it had been shown, among the first, by Navier in [8].

Menabrea claims to prove in a few lines that the additional equations are obtained applying the following 'elasticity equation' (équation d'élasticité)

> When an elastic system is balanced under [given] external forces, the work spent by the tensions or compressions of the links joining the various points of the system is a minimum.[8]

Let us consider in full Menabrea's proof:

> Since [...] tensions may vary without altering equilibrium, one shall admit that these variations occur independently of the work spent by external forces; they always cause lengthening or shortening of the different corresponding links, which implies, in each one of them, a work spent. The variations of length of the links shall be supposed very small so that the relative placement of the different points of the system is not sensibly altered. However, since in this internal movement equilibrium still holds and the work spent by external forces is nil, it follows that the total work of the tensions so developed is equally nil.
>
> To express this, let T be the tension of a generic link, δl the elementary variation of length of this link; the work spent because of the corresponding variation of tension will be $T\delta l$, and, consequently, for the whole system one will have
>
> $$\sum T\,\delta l = 0. \tag{1}$$
>
> Let l be the lengthening or shortening originally underwent by the link under the action of the tension T, it is, modulo a sign,
>
> $$T = \varepsilon l \tag{2}$$
>
> where ε is a coefficient that I will call coeffiecient of elasticity, function of the elasticity modulus and of the section and length of the link.
>
> The work spent to produce this variation of length l will equal $1/2\varepsilon l^2$, thus the total work of the system will equal $1/2\sum \varepsilon l^2$.
>
> Because of (1) and (2) one has
>
> $$\sum T\,\delta l = \sum \varepsilon l\,\delta l = \delta\frac{1}{2}\sum \varepsilon l^2 = 0. \tag{3}$$
>
> This is the proof of the stated principle [...]. It is equally possible to express it in another way, since one has[9]
>
> $$\sum T\,\delta l = \sum \frac{1}{\varepsilon} T\,\delta T = \delta\frac{1}{2}\sum \frac{1}{\varepsilon} T^2 = 0. \tag{4}$$

[7] Les valeurs des tensions effectives dépendent de l'élasticité respective des liens, et lorsque celle-ci est déterminée, il doit en être de même des tensions [1], p. 1057.

[8] Lorsqu'un système élastique se met en équilibre sous l'action de forces extérieures, le travail développé par l'effet des tensions ou des compressions des liens qui unissent les divers points du système est un minimum [1], p. 1056.

[9] Puisque [...] les tensions peuvent varier sans que l'équilibre cesse d'exister, on devra admettre que ces variations s'effectuent indépendamment de tout travail des forces extérieures; elles sont toujours accompagnées d'allongements ou d'accourcissements dans les divers liens correspondants, ce qui donne lieu, dans chacun d'eux, à un développement de travail. Les variations de longueur des liens doivent être supposées très petites pour que les positions respectives des divers points du système ne soient pas sensiblement altérées. Mais, puisque pendant ce petit mouvement intérieur l'équilibre continue à exister et que le travail des forces extérieures est nul, il s'ensuit que le travail total élémentaire des tensions ainsi développé est également nul. Pour exprimer cette conséquence, soient T la tension d'un lien quelconque, δl la variation élémentaire de la longueur de ce lien; le travail développé par suite de la variation de tension correspondante sera $T\delta l$, et par conséquent, pour l'ensemble du système, on aura (eq. (1)).

The proof rests on the consideration that if the variation δT of the 'tensions' is infinitesimal so is also the variation δl of the length of the bars, hence the variation δu of the position of the nodes is negligible. Menabrea assumes it to be zero; this opinion, singular to say the least, is stated again almost twenty years after:

> Given an equilibrium configuration, if we suppose that the system moves to another one very close, the set of external forces (X, Y, Z) will maintain equilibrium, independently on inner forces; since this equilibrium state does not depend only on the direction and magnitude of forces, but also on the positions of their points of application, it follows that each node shall remain in the same position, no matter what the variations are in the tensions of the corresponding links.[10]

Menabrea implicitely uses the equation of virtual work, i.e., the work of inner forces ($L^i = \sum T\delta l$) equals that of external ones ($L^e = \sum f\delta u$) for any admissible change of shape. Since δu is admitted nil, the external virtual work L^e vanishes and, by the equation of virtual work ($L^i = L^e$), the internal virtual work L^i also vanishes. The proof is thus complete.

A modern reader questions the statement that the displacements of the nodes are negligible with respect to the variation in length of the bars, which is false. It is easy to prove, also on the basis of other relations in Menabrea's following papers, that δu and δl are of the same order of magnitude. Even if the statement were true Menabrea's proof is not satisfactory: for the equation of virtual work (1) to hold, the strains δl must be compatible with the constraints; on the other hand, the null of the variation of the work of inner forces in (3) and (4) is obtained when inner forces vary in the set of those balancing external ones. The resulting strains δl are not, in general, compatible with the constraints.

A minor criticism is about Menabrea's statement of a property of minimum with the only comment, without explanations, that the work of inner forces in the actual solution is not a maximum.[11] For a modern reader this result clearly derives from the potential, or complementary, energy in a linear elastic system being quadratic and positive definite. However, Menabrea never studies the properties of the epression of work and the possibility of the stationarity conditions in (3) and (4) providing a maximum or saddle point. Apparently he was not much interested in a fully rigourous proof, maybe following the italian engineering schools.

The modern reader, used to high standards of rigour, finds another weakness in this proof: does the minimum of inner work always provide the solution of the linear elastic static problem? In other words, Menabrea claims to prove that for a balanced, compatible with the constraints linear elastic system the work of inner forces is a minimum, but never cares about the converse (which is of course of great importance).

The drawbacks in Menabrea's proof are similar to those of others of his time studying redundant systems. They are due to a non correct use either of infinitesimals or of the equation of virtual work. For instance, Dorna[12] underwent fallacies of the first kind, Cournot [12–14] and Villarceau [18] underwent fallacies of the second kind. This is surprising and one asks himself how is it possible that such well cultured scientists could undergo so gross errors: the doubt may arise, if we understood their writings correctly. Our opinion is, however, that these are genuine errors in the procedure and do not represent the outcome of different points of view with respect to those commonly accepted in our time. That errors derive from not well defined mathematical tools is supported by

Soit l l'extension ou l'accourcissement qu'a primitivement éprouvé le lien sous l'action de la tension T, on a, indépendamment du signe, (eq. (2)), ou ε est un coefficient que j'appellerai coefficient d'élasticité, e qui est fonction du module d'élasticité, de la section et de la longueur du lien.

Le travail développé pour produire cette variation de longueur l sera égal à $1/2\varepsilon l^2$, et par suite le travail totale du système sera égal a $1/2\sum \varepsilon l^2$.

Mais en vertu des équations (1) et (2) on a: (eq. (3)).

Ce qui est la démonstration du principe énoncé [...]. Il est également possible de l'exprimer d'une autre manier, car on a (eq. (4)) [1], pp. 1057–1058.

[10] Data una di quelle disposizioni d'equilibrio, se si suppone che il sistema passi gradatamente ad un'altra vicinissima, il complesso delle forze esterne (X, Y, Z) non dovrà cessare di essere in equilibrio per ognuna di queste disposizioni, indipendentemente dalle forze interne; e siccome questo stato di equilibrio non dipende soltanto dalle intensità e direzione rispettiva delle forze, ma anche dalle posizioni de' punti di applicazione, ne segue che ogni nodo deve mantenersi costantemente nella stessa posizione, malgrado le variazioni che possono succedere nelle tensioni de' legami che vi corrispondono [4], p. 213.

[11] [1], p. 1058.

[12] Dorna A, Memoria sulle pressioni..., cit.

the fact that all the authors examined by us made similar approaches, and the critical point of their reasonings is when they try to argue that the work of external forces equals zero. These errors of procedure were not repeated by more careful scientists of the time, like Bertrand and Castigliano.

These are examples of how in the history of mathematics and mechanics the proof of a statement believed as true becomes of secondary importance.[13] Indeed, the scientists were certain of the statement and their search for a rigorous proof was sometimes weak; they were satisfied enough by ambiguous rhetorical tricks. In the case of Menabrea, this stems quite probably also from his engineering education, in which relative importance was given to the formal aspects of logical deduction.

2.1 The comments to Menabrea's first work

The weaknesses in Menabrea's proof were immediately recorded by his contemporaries and he himself, in a letter to the president of the Accademia delle Scienze of Turin in 1870, quoted some of the criticism he received:

> It seems that my paper has in general been well accepted by scientists [...], but for mr. Emilio SABBIA who, in the note *Errore del principio di elasticità formolato dal signor L. Federigo* MENABREA, *Cenno critico di Emilio* SABBIA, *Torino* 1869, strongly questions the correctness of the statement [...].
>
> [...] I would not have delayed my response to his criticism [...] but I was informed of a note by the savant Adolfo PARODI [...]. He so clearly replies to mr. SABBIA's points that I would not have known better [...].
>
> [...my proof] was judged, as it may be seen in the attached notes, rigourous enough, and at least simple and clear.[14]

Menabrea never openly admitted to doubt of his first proof and in his following works [2–4] perfected the proof and used polite, firm words to defend the paper of 1858. For instance, in 1875 he wrote:

> Even though the coincidence of the results obtained by applying the principle of elasticity with those derived from special, uncontested method, was confirmed by numerous examples in my second memoir [that read in 1865, [2]], and should have induced to conclude that the principle and the method are correct, yet both were subject to harsh criticism by some [scientists], while many of the most notable mathematicians of our time better accepted the principle.[15]

Menabrea, behaving in line with his political career, did not actually examine the points of view of the critics to his proof and presented some of them as a support to his statements. He later kept the same attitude, assured by his renewed proof of 1865, of which we shall discuss in the next section. This may be explained, at least in part, by the desire to affirm the paternity of the proof, in particular with respect to Alberto Castigliano, with whom there was a dispute.

The strongest criticism to Menabrea's first proof was by lieutenant Emilio Sabbia. In spite of our attempts, we could not find the paper quoted by Menabrea, but two other short notes by Sabbia,[16] from

[13] A similar example in the history of mathematics of 19th century is the discussion on the so-called Dirichlet's principle for harmonic problems. See Klein F, Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert, Berlin, 1926.

[14] Sembra che il mio scritto venisse generalmente accolto con favore dagli scienziati [...], fuorché dal sig. Emilio SABBIA il quale, in un opuscolo intitolato: *Errore del principio di elasticità formolato dal signor L. Federigo* MENABREA, *Cenno critico di Emilio* SABBIA, *Torino* 1869, impugna, con particolare vivacità, la verità di quel principio [...].

[...] non avrei tardato a rispondere alle sue critiche, [...] quando mi fu comunicato uno scritto del valente cultore delle scienze matematiche il sig. Comm. Adolfo PARODI [...]. Egli così nitidamente ribatte gli appunti del sig. SABBIA che non saprei come meglio [...].

[...la mia dimostrazione] venne giudicata, come si rileverà da uno degli scritti qui uniti, rigorosa abbastanza, e che ha almeno il pregio della semplicità e della chiarezza [3], pp. 687–688.

[15] Sebbene la coincidenza de' risultati ottenuti dalla applicazione del principio di elasticità, con quelli ricavati da altri metodi speciali e non contestati, fosse nella mia seconda memoria, confermata da moltiplici esempi, e dovesse indurre ad ammettere che il principio ed il metodo che ne derivava erano esatti, tuttavia l'uno e l'altro furono per parte di alcuni, oggetto di aspre [...] denegazioni, mentre parecchi fra i più eminenti matematici di nostra epoca accolsero il principio con maggiore benevolenza [4], p. 203.

[16] Sabbia E, Nuove delucidazioni sul principio d'elasticità. Risposta ad un opuscolo contenente delucidazioni di L. Federico Menabrea sullo stesso principio, Torino, Baglione, 1870; Lettera al professore sig. S. L., place and date of publication unknown.

which we can reconstruct his points. Sabbia says that Menabrea makes confusion in the choice of the quantities to vary in order to attain the minimum of elastic work: in particular, Menabrea does not make it clear if they must be node displacements or bar forces. This is important, since in modern terms the difference consists in searching the minimum of potential or complementary elastic energy, respectively. Only in this last case we may affirm, and Sabbia does as well, that Menabrea's principle is correct. Another criticism that Sabbia puts forth is that Menabrea did not explicitly state that his principle holds true only if the system has a natural state (free of tensions in the bars). These criticisms, according to us correct and important, were considered seriously neither by his contemporaries nor by modern historians of mechanics. Indeed, a (partial) reconstruction of Sabbia's arguments and Menabrea's answers are in the letter by the Italian savant Adolfo Parodi who, for what can be suspected 'political' reasons, supports Menabrea.[17] Coming to modern times, Timoshenko does not even mention the matter, and so do Todhunter and Pearson; even in Benvenuto and Nascé [10, 11], Sabbia is considered as a disturbing fellow, possibly motivated by non-scientific reasons.

The comments by Villarceau and Bertrand, included in Menabrea's self-defence of 1870,[18] have a different nature. The letter by Villarceau at first sight seems to support Menabrea and suggests to consider dynamics and apply the equation of conservation of the *vis viva*:

$$L^a - L^i = \Delta K \tag{5}$$

with L^a, L^i the work of external and internal forces, respectively, and ΔK the first-order variation of the kinetic energy during the process.[19] Villarceau states that since ΔK is negligible, being a higher-order infinitesimal with respect to L^a, L^i, one has from (5)

$$L^a - L^i = 0. \tag{6}$$

Villarceau then follows Menabrea's argument:

> Now, *if* one imagines that the work [of external forces] remains constant [...], while there is a possible [virtual] variation of the work of [inner] forces, one also has:
>
> $$L^a - L^i - \delta L^i = 0 \tag{7}$$
>
> hence[20]
>
> $$\delta L^i = \delta \sum f\,\rho\,\Delta\rho = 0 \tag{8}$$

with $f\rho$ the inner force between two molecules the distance of which is ρ. Many symbols are not explained, nor is it clear which inner forces are considered (mutual forces? balanced? derived from a linearized elastic law or similar?). Villarceau italicizes the word "si" (if): it seems to the modern reader that he puts into question one of the weakest point in Menabrea's proof, that the work of external forces is not altered by the displacement of the nodes of the ideal truss. Hence, a careful reader would ask himself how this letter can support Menabrea's proof: it contains errors in itself, and Villarceau's result actually does not coincide with Menabrea's. Indeed, in (8) the first-order variation operator is applied to the whole sum expressing inner work, without specifying which are the fields subject to variation (forces or strains). Menabrea's statement is different, in that the first-order variation of strain is considered, see (3) and (4). It is difficult to accept Villarceau's thesis, but, he writes to Menabrea, he is more or less as satisfied of the proof as Menabrea: rigour is only a matter of style if we are certain of the result.[21]

Bertrand's polite letter to Menabrea is much clearer:

> [...] one is led to the following statement, free of ambiguity.
>
> *The sum of the squares of tensions, respectively divided by the elasticity coefficient of the corresponding link is a minimum*; that is, that sum is less than for all other systems of tensions assuring equilibrium, once one neglects the conditions of extensibility of the links.

[17] [3], pp. 690–696.

[18] [3], pp. 702–705.

[19] The original equation (5) is written using a different notation, the meaning of which appears to be that quoted above.

[20] Maintenant, *si* l'on imagine que le travail reste constant [...], malgré la variation possible du travail des forces, on aura aussi: (eq. (7)) d'où (eq. (8)) [3], p. 705.

[21] Malgré ce que l'une ou l'autre démonstration peut laisser à désirer pour les esprits très-rigoureux, il me semble que la concordance des résultats obtenus par la nouvelle méthode, avec ceux que fournissent les méthodes connues, ne doit laisser de doute dans l'esprit d'aucun géométre, ou mécanicien sur la généralité du théorème [3], p. 705.

Let me, Sir, propose you [...] a very simple proof of your [principle of elasticity].

Let l be the length of one link, λ its strain in the equilibrium configuration, $T = \varepsilon\lambda$ its tension, $T + \Delta T$ the tension in the same link corresponding to another solution of balance equations, the links being supposed non-extensible; if the forces ΔT acted alone, they would be self-balanced, since the forces T and $T + \Delta T$, by hypothesis, balance external forces (the system is that where the extensible links disappeared). The sum of virtual works of the forces ΔT is thus zero for all displacements compatible with the constraints other than the non-extensibility of the links. Yet one of these displacements is the one actually occurring for which the link l has strain $\lambda = \frac{T}{\varepsilon}$, so that

$$\sum \frac{T\,\Delta T}{\varepsilon} = 0. \tag{9}$$

This is [...] the principle of elasticity [...].[22]

Bertrand's use of the equation of virtual work (the net amount of the virtual work of all actions vanishes) is apparent. Unlike Menabrea and Villarceau, Bertrand makes it clear that one shall consider the variation of forces from T to $T + \Delta T$, fixing the virtual displacements. Since the virtual work of the T equals that of the $T + \Delta T$ and that of external forces, it follows that the work of the ΔT vanishes. Bertrand does not answer the question if the minimum of the inner work always provides the solution to the elastic static problem.

It is also worth mentioning the criticism to Menabrea by the Italian mechanician Valentino Cerruti [19]:

> I do not question the simplicity of mr. MENABREA's proofs, but their rigour. Indeed, [the proofs of 1858 and of 1865, respectively] are based on the fact that in an elastic truss, when the number of elements linking n points in space is greater than $3n - 6$, one may conceive infinitely many repartitions of tensions, while it is well known that elastic problems are always uniquely determined (see Clebsch [...]).[23]

This statement and the reference to Clebsch are not clear: Cerruti seems to make confusion between the unique solution to a linear elastic problem (proved by Kirchhoff and discussed by Clebsch) and the infinite balanced solutions in a redundant system. Moreover, by reading in full [19], it seems that at that time Cerruti did not distinguish between potential and complementary energy.[24]

2.2 The origins of Menabrea's statement

Menabrea himself in more than one occasion tried to frame his statement among the scientific literature of the time; the more extended version of this view is in [4]:

> [...] the genesis originated, as far as I know, in a memoir of Mr. *Vène* [...], who since 1818 and in 1836 (*Mémoire sur les lois que suivent les pressions*) stated the following theorem for the special case of the pressures exerted by

[22][...] on est conduit à l'énoncé suivant qui n'offre plus aucune ambiguité.

La somme des quarrés des tensions, divisés respectivement par le coefficient d'élasticité du lien correspondant est un minimum; c'est-à-dire que cette somme est moindre que pour tout autre système de tensions capable d'assurer l'équilibre, lorsqu'on néglige les conditions relatives a l'extensibilité des liens.

Permettez-moi, Monsieur, de vous soumettre [...] une démonstration fort simple de votre [...].

Soit l la longueur de l'un des liens, λ son allongement dans la position d'équilibre, T sa tension égale à $\varepsilon\lambda$, $T + \Delta T$ la tension du même lien à une autre solution des équations d'équilibre, lorsque les liens sont supposes inextensibles; les forces ΔT, si elles étaient seules, se feraient équilibre sur le système, puisque les forces T et les forces $T + \Delta T$, font, par hypothèse, équilibre aux mêmes forces extérieures (le système est celui dont le liens extensibles ont disparu). La somme des moments virtuels des forces ΔT est donc nulle pour tous les déplacements compatibles avec les liaisons autres que l'inextensibilité des liens. Mais, un de ces déplacements est celui qui se produit réellement et dans lequel le lien l s'allonge de λ égal à $\frac{T}{\varepsilon}$, on a par conséquent (eq. (9)). C'est [...] le principe d'élasticité [...] [3], pp. 702–703.

[23]Io non dubito punto che le dimostrazioni del sig. MENABREA siano molto elementari, ma non parmi che si debbano anche riguardare come assolutamente rigorose. Imperocchè la prima e la terza poggiano sul supposto che in un sistema elastico articolato, quando il numero dei pezzi congiungenti n punti dello spazio sia superiore a $3n - 6$, si possano concepire infinite maniere diverse di ripartizione delle tensioni, mentre è cosa notoria che i problemi dell'elasticità sono sempre determinati ed in un modo solo (CLEBSCH, *Theorie der Elasticität fester Körper*—pag. 67–70) [19], pp. 570–571.

[24]In addition, it is not clear if the boundary conditions are on displacements or on stress, see [19], pp. 571–572.

a weight on homogeneous supports: *The sum of the squares of the weights shall be a minimum*. This new principle was mentioned in the *Bulletin [. . .] de FERUSSAC tome neuviéme pag.* 7 in a paper signed S. In a following paper in the same volume page 10 and signed A. C. the aforementioned principle is extended to non-homogeneous supports and to pressures due to rigid bars over the supports. It had been supposed that the author A. C. was Augustin Cauchy; but then it was more probably attributed to Mr. *A. Cournot*.—*Pagani* dealt with the special case of elastic threads fixed at one end and joined at the other, to which a force is applied. Mossotti in his *Meccanica* dealt with all these subjects.[25]

The sources suggested by Menabrea have been considered in detail by Benvenuto,[26] hence we examine in some detail only the work of Cournot, not fully commented and fundamental to understand Menabrea's papers. We will provide a hint also on the work by Dorna.

Cournot faces the problem of finding the interactions at the contact points between two rigid bodies, of which one is the 'ground', in three papers, two in 1827[27] and one in 1828.[28] In 1827 he states, for a problem of impact among bodies, the principle according to which "the sum of the squares of the percussions [the contact forces] is a *minimum*".[29] In 1828 Cournot presented a proof, which he presented independently of the dynamical problem of impact.

Cournot considers a rigid body supported by a deformable layer in some points. If active forces are exerted upon the body, the contact points will exchange "pressions". Cournot spends some time in this definition, even if at his time more or less all scientists gave to the noun "pression" the meaning of a contact force:

> These pressures [. . .] are quantities other than the forces genrating them [. . .].
>
> The determination of pressures must be considered another branch of dynamics [. . .] which may be called latent [. . .].
>
> In a system with several points [subjected] to fixed obstacles [constraints], each obstacle will undergo a pressure proportional to the infinitely small straight line which the corresponding point would describe in the unity of time.[30]

Cournot's statement of 'pressures' proportional to infinitesimal displacements lets some doubts arise, since it could appear a law of statics not depending on the linear elasticity of the support. Indeed, Cournot says that 'pressures' are other than the forces generated by the external ones. A careful reading of the paper makes it clear that Cournot actually does not postulate a different law of statics but what we now call a constitutive relation, admitting that "le coefficient de l'élasticité" may vary "de l'un [point] à l'autre".[31] Eventually, Cournot admits that "pressions", even if are not forces in strict sense, may be treated in the same way:

[25] [. . .] la genesi [. . .] ebbe origine, per quanto mi consta, in una memoria del Sig. *Vène* [. . .], il quale fin dal 1818 e quindi nel 1836 (*Mémoire sur les lois que suivent les pressions*) enunziava il seguente teorema per il caso speciale di pressioni esercitate da pesi sopra punti d'appoggio omogenei: *La somme des Quarrés des poids doit être un minimum*. Di questo nuovo principio si faceva cenno nel *Bulletin [. . .] de FERUSSAC tome neuviéme pag.* 7 in un articolo firmato S. In un altro articolo che fa seguito al precedente, nello stesso tomo pag. 10 e firmato A. C. il principio anzidetto venne esteso al caso di punti d'appoggio non omogenei ed a quello di pressioni prodotte sopra i punti d'appoggio per mezzo di spranghe rigide. L'Autore A. C. di quell'articolo si supponeva essere Augustin Cauchy; ma ulteriormente desso venne con maggiore probabilità attribuito al S. *A. Cournot*.—*Pagani* trattava il caso speciale di cordoni elastici fissi rispettivamente in una delle loro estremità e riuniti nell'altra in un nodo al quale era applicata una forza. Il Mossotti trattò nella sua *Meccanica* gli argomenti precedenti [4], p. 202.

[26] [9], Chaps. 14, 16, [10].

[27] [12, 13].

[28] [14].

[29] [. . .] la somme des quarrées des percussions soit un *minimum* [13], p. 87.

[30] Ces pressions [. . .] sont des grandeurs hétérogènes aux forces par lesquelles sont engendrées [. . .].

La détermination des pressions doit être considérée comme une autre branche de la dynamique [. . .] qui pourrait prendre le nom de dynamique latente [. . .].

S'il s'agit d'un système ayant plusieurs points par des obstacles fixes, chaque obstacle subira une pression proportionnelle à la droite infiniment petite que le point correspondant décrirait pendant l'élément du temps [14], pp. 11–12.

[31] [14], p. 18.

> The opposite of these pressures may be considered as forces applied to the system to keep it balanced, once the constraints are neglected.[32]

Thus, Cournot's attempt to define 'pressures' as heterogeneous to external forces and find them by the physical principle of "dynamique latente" fails. His attempt, however, reveals a strong resistance to accept the 'metaphysical' idea of contact force and constraint reaction.

Cournot applies the equation of virtual work considering the rigid body subjected to active forces $F, F', \ldots$ acting along the directions $f, f', \ldots$ and to constraint reactions $P, P', \ldots,$ opposite to the 'pressures', acting along the directions $p, p', \ldots$. Since the body is supposed balanced, the total amount of the virtual work of active forces and constraint reactions shall vanish:

$$F\delta f + F'\delta f' + \cdots - (P\delta p + P'\delta p' + \cdots) = 0 \quad (10)$$

> formula providing the balance equations after having reduced to the less possible the number of independent variations, keeping into account the inner constraints of the system but not those resulting form the presence of obstacles, now replaced by the forces P, P', etc.[33]

Cournot then makes an important statement:

> When considering the presence of these obstacles to reduce the number of variations, it simply is
>
> $$F\ \delta f + F'\ \delta f' + \cdots = 0; \quad (11)$$
>
> and then, in the same case:
>
> $$P\ \delta p + P'\ \delta p' + \cdots = 0, \quad (12)$$
>
> which immediately descends from the systems (F) and (P) being equivalent.[34]

Such statements are quite ambiguous.[35] It is interesting, however, to compare Cournot's obscure statement with the corresponding one by Menabrea, quoted above with the proof of 1858. Contrarily to Cournot, Menabrea provides an explicit explanation (find it satisfactory or not) for the vanishing of the virtual work of external forces, stating that it derives from the displacements of the nodes of his truss model are negligible.

Once admitted that the virtual work of external forces vanishes, the passages in Cournot are easy to follow. From (12), since 'pressures' are proportional to displacements, one has

$$p\ \delta p + p'\ \delta p' + \cdots = 0, \quad (13)$$

> relation according to which the sum of the quantities p^2, p'^2, etc., or, by hypothesis, that of the squares of the pressures P^2, P'^2, etc. is a *minimum*, since it is easy to see that the case of *maximum* cannot take place here.[36]

It is not evident why a maximum is not attainable. Cournot can now state his "Théorème générale":

> As a consequence, the equations completing in all cases the number of those necessary for the full determination of pressures result from the condition that the sum of the squares of these pressures be a *minimum*.[37]

Menabrea is apparently deeply indebted with Cournot: both their statements are more or less identical. Menabrea, however, considers an elastic system suitably modelled, not a rigid body on supports, like Cournot.

[32] Ces pressions, prises en sens contraires, pourront être considérées comme des forces appliquées au système, et qui le maintiennent en équilibre, abstraction faite des obstacles [14], p. 13.

[33] [...] formule qui donnera les relations de l'équilibre, après qu'on aura réduit, au plus petit nombre possible, les variations indépendants, en tenant compte des liaisons propres du système, mais non pas de celle qui résultent de la présence des obstacles, maintenant remplacées par les forces P, P', etc. [14], p. 18.

[34] Quand on a regard à la présence de ces obstacles pour réduire le nombre des variations, il vient simplement: (eq. (11)); donc aussi, dans le même cas: (eq. (12)), ce qui résulte immédiatement de ce que le deux systèmes (F) et (P) sont équivalents [14], p. 18.

[35] The italian mechanician Mossotti (Mossotti OF, Lezioni di meccanica razionale, Pisa, 1858, pp. 97–98) provided a possible motivation of Cournot's assumption (11): since (10) holds for any virtual displacement, it is possible to choose particular ones yielding (11) and (12), respectively. Still, this reasoning does not seem fully clear to us.

[36] Relation en vertu de laquelle la somme des quantités p^2, p'^2, etc., ou, par l'hypothèse, celle des carrés des pressions P^2, P'^2, etc. est un *minimum*; car il est facile de s'assurer que le case du *maximum* ne peut avoir lieu ici [14], p. 18.

[37] Par conséquence, les équations qui complétent, dans tous les cas, le nombre de celles qui sont nécessaires pour l'entière détermination des pressions, résultent de la condition que la somme des carrés de ces pressions soit un *minimum* [14], p. 18.

Moreover, Menabrea avoids speaking of rigid bodies supported by rigid point-wise constraints, affirming that this is not an actual case.

It is worth examining also the work by Dorna,[38] a colleague of Menabrea, which is another example of how an improper use of infinitesimals leads to errors. Dorna, before Menabrea, considers a more general problem than Cournot: not a rigid body but a deformable frame with elastic constraints.

Dorna writes the equation of virtual work for the equilibrium of such a frame: $L_e - L_i + L_v = 0$, where L_e is the work of external active forces, L_i that of inner forces and L_v that of constraint reactions. Dorna states that the virtual displacements of the constraints, consisting of very stiff springs, are higher-order infinitesimals with respect to those in L_i and L_e, then L_v is negligible. Hence it is $L_e + L_i = 0$; and the equation of virtual work, $L_e - L_i + L_v = 0$, provides $L_v = 0$.

This procedure contains two statements subject to strong criticism; (a) the virtual work of constraint reactions is a higher-order infinitesimal and negligible, (b) as a consequence of (a) and of the equation of virtual work the constraint reactions spend exactly no work. Indeed, if the support is an elastic spring, however stiff, there is in general no reason to admit that the displacements of the constrained points are negligible with respect to those of free points. The statement (b) is a paralogism, since if L_v is a higher-order infinitesimal in the sum $L_e - L_i + L_v = 0$ then it correctly follows that $L_e - L_i \simeq 0$, but in no way can we affirm that $L_v = 0$.

3 Menabrea's "Étude de statique physique"

Menabrea read a second memoir on the "principle of elasticity" at the Academy of Sciences of Turin in 1865, published only in 1871 in the proceedings of the Academy. Menabrea presents a new proof, beginning with a very 'politically correct' defence of his previous paper:

> Since 1857 I have communicated to the Academy of Sciences of Turin the statement of this new principle; then in 1858 [...] I made it the subject of a communication to the Institut de France (Académie des Sciences). In the proof I gave I relied on considerations on the transmission of work in bodies. Even though, according to me, that proof was rigourous enough, it seemed to some geometers too simple to be accepted without criticism. On the other hand, the meaning of the equations deduced from this theorem had not been precised enough. This is why I believed it necessary to resume this study [...]. I now present new researches that led me to a new proof definitely precise and rigourous.[39]

Menabrea also gives a naïve hint to thermodynamics, which clearly descends from his incompetence in this branch of physics[40]:

> To provide the question of the distribution of tensions the physical generality [...], account should be taken of *thermodynamics* phenomena occurring in the change of shape of the body or elastic system; but I consider the body when equilibrium among external and internal forces is attained, supposing that temperature has not varied.[41]

3.1 The new proof

Menabrea's model is the same of 1858: a linear elastic truss undergoing small displacements and strains.

[38] A. Dorna, Memoria sulle pressioni supportate ..., cit.

[39] Dès l'année 1857 j'avais fait connaître à l'Académie des Sciences de Turin l'énoncé de ce nouveau principe; puis en 1858 [...] j'en avais fait l'objet d'une communication a l'Institut de France (Académie des Sciences). Dans la démonstration que j'en donnai je m'appuyais sur la considérations de la transmission du travail dans les corps. Quoique, selon moi, celle démonstrations fût suffisamment rigoureuse, elle parut à quelques géomètres trop subtile pour être acceptée sans contestation. D'un autre côté la signification des équations déduites de ce théorème n'etait pas suffisamment indiqué. C'est pourquoi j'ai cru devoir reprendre cette étude [...]. Je présent aujourd'hui ces nouvelles recherches qui ont eu pour résultat de me conduire à une démonstration tout-à-fait simple et rigoureuse [...] [2], p. 144.

[40] Though never explicitely stated, we infer from the text that Menabrea does not consider heat supplies or dissipative processes.

[41] Pour donner à la question de la distribution de tension toute l'étendue [...] physique, il faudrait tenir compte des phénomènes de *thermodynamique* qui se manifestent dans l'acte de changement de forme du corps ou système élastique; mais je considère le corps au moment où l'équilibre est établi entre les forces *intérieures* et extérieures, en supposant que la température n'a pas varié [2], p. 145.

Menabrea studies a system without external constraints first, writing the balance equations for each node p[42]:

$$
\begin{aligned}
X_p &= \sum_m T_{pm} \frac{x_m - x_p}{l_{pm}}, \\
Y_p &= \sum_m T_{pm} \frac{y_m - y_p}{l_{pm}}, \\
Z_p &= \sum_m T_{pm} \frac{z_m - z_p}{l_{pm}};
\end{aligned}
\tag{14}
$$

X_p, Y_p, Z_p are the components of the external force applied to p; T_{pm} is the tension (i.e., the force) in the bar joining the nodes p and m, of length l_{pm}; x, y, z are the coordinates of the nodes m and p; the sum is over all the $m = 1, 2, \ldots, n$ nodes. Since the displacements are supposed small, one may refer to the present configuration, undistinguishable from the reference one. Remark that such considerations are not explicitely stated by Menabrea but inferred from the text.

The nodes being n, the number of balance equations (14) is $3n$. Since the whole system is balanced, there are 6 global balance equations, hence the number of independent balance equations (14) is reduced to $3n - 6$. If the number N of bars is such that $N > 3n - 6$, Menabrea's model of continuum is redundant and "one may conceive infinite distributions of these tensions which can all balance external forces[43]".

Let δT_{pq} be a first-order variation of the tensions in the bars, such that $T_{pq} + \delta T_{pq}$ are still balanced with the external forces X_p, Y_p, Z_p. Since the system is redundant, there are ∞^{N-3n+6} sets of tensions balancing external actions, and the same amount of self-balanced sets of forces δT_{pq}. If the magnitude of the δT_{pq} is small one may admit that they induce a negligible change of shape and balance equations can be written in the reference configuration (this also is inferred from the text).

The first-order variation of the balance equations (14) provides the balance equations for the self-balanced δT_{pq} at each node p[44]:

$$
\begin{aligned}
0 &= \sum_q \delta T_{pq} \frac{x_q - x_p}{l_{pq}} = \sum_q \varepsilon_{pq} \delta\lambda_{pq} \frac{x_q - x_p}{l_{pq}}, \\
0 &= \sum_q \delta T_{pq} \frac{y_q - y_p}{l_{pq}} = \sum_q \varepsilon_{pq} \delta\lambda_{pq} \frac{y_q - y_p}{l_{pq}}, \\
0 &= \sum_q \delta T_{pq} \frac{z_q - z_p}{l_{pm}} = \sum_q \varepsilon_{pq} \delta\lambda_{pq} \frac{z_q - z_p}{l_{pq}},
\end{aligned}
\tag{15}
$$

where the sum is over all the $q = 1, 2, \ldots, n$ nodes and Menabrea uses the linear elastic constitutive relations $\delta T_{pq} = \varepsilon_{pq} \delta\lambda_{pq}$, with $\delta\lambda_{pq}$ and $\varepsilon_{pq} = (\frac{E\omega}{l})_{pq}$ the first-order variation of the length and the axial stiffness of the bar pq, respectively (E is Young's modulus and ω the area of the cross-section of the bar).

Menabrea expresses the bar strain (more precisely, the variation of the bar length) in terms of the displacement components α, β, γ of the nodes, by projecting the relative displacement of the nodes p and q onto the direction of the bar[45]:

$$
\lambda_{pq} = \frac{(x_q - x_p)(\alpha_q - \alpha_p)}{l_{pq}} + \frac{(y_q - y_p)(\beta_q - \beta_p)}{l_{pq}} + \frac{(z_q - z_p)(\gamma_q - \gamma_p)}{l_{pq}}. \tag{16}
$$

Such expression is basically the same of the contemporary theory of infinitesimal strain,[46] and we may assume it was well accepted at the time.

Multiplying the λ_{pq} in (16) by $\varepsilon_{pq}\delta\lambda_{pq}$ and summing over both indices p and q yields[47]:

$$
\begin{aligned}
\sum_{p,q} \lambda_{pq} \varepsilon_{pq} \delta\lambda_{pq}
&= \sum_{p,q} \varepsilon_{pq} \delta\lambda_{pq} \left\{ \frac{(x_q - x_p)(\alpha_q - \alpha_p)}{l_{pq}} \right. \\
&\quad \left. + \frac{(y_q - y_p)(\beta_q - \beta_p)}{l_{pq}} + \frac{(z_q - z_p)(\gamma_q - \gamma_p)}{l_{pq}} \right\}.
\end{aligned}
\tag{17}
$$

[42] [2], (1), p. 165.

[43] l'on peut concevoir une infinité de manières de répartition de ces tensions, qui toutes peuvent satisfaire aux conditions d'équilibre avec les forces extérieures [2], p. 167.

[44] [2], (4), p. 168.

[45] [2], (7), p. 168.

[46] Indeed, one may re-write (15) in the form $\Delta l = [(\text{grad}\,\mathbf{u})\mathbf{n}] \cdot \mathbf{n}$, where $\mathbf{u}$ is the displacement vector and $\mathbf{n}$ is the unit vector in the direction of the bar pq. A modern definition of strain is due to Saint-Venant; Menabrea did not quote him but for sure knew Saint-Venant's papers.

[47] [2], (8), p. 168.

Menabrea multiplies the balance equations (15) by α_p, β_p, γ_p and sums over all nodes p. Since in the obtained equation one finds the term $(x_q - x_p)$ and the like for the node p, while for the node q there is the term $(x_q - x_p) = -(x_p - x_q)$, the expression simplifies into[48]:

$$\sum_{p,q} \varepsilon_{pq} \delta\lambda_{pq} \left\{ \frac{(x_q - x_p)(\alpha_q - \alpha_p)}{l_{pq}} + \frac{(y_q - y_p)(\beta_q - \beta_p)}{l_{pq}} + \frac{(z_q - z_p)(\gamma_q - \gamma_p)}{l_{pq}} \right\} = 0. \tag{18}$$

Equations (17) and (18) provide[49]:

$$\sum_{p,q} \varepsilon_{pq}\, \delta\lambda_{pq}\, \lambda_{pq} = 0 = \sum_{p,q} \frac{1}{\varepsilon_{pq}} \delta T_{pq}\, T_{pq}, \tag{19}$$

> which is the equation of elasticity, from which one obtains the theorem which we have stated at the beginning of this Memoir, that is: *When an elastic system is balanced under external forces, the internal work spent on the resulting change of shape, is a* **minimum**.[50]

This proof is surely more satisfactory than the previous one; yet some criticism remains. First of all, Menabrea pretends to have shown that the work of internal elastic forces attains a minimum in a balanced configuration, while we actually may conclude from (19) that the work is stationary. Second, the strains λ_{pq} in (16) are, by construction, compatible with the displacement of the nodes; the tensions δT_{pq} in (15) are self-balanced, by hypothesis. In principle, the δT_{pq} do not provide, by elastic relations, compatible strains, nor do the λ_{pq} provide balanced tensions. Menabrea does not make this explicit in (19), so it is not clear if he considers a minimum in the set of compatible strains or of balanced tensions. Other drawbacks that can be seen are: the hypothesis of infinitesimal displacements and strain is not fully explained; the external forces are implicitely assumed independent of the change of configuration. Again, Menabrea does not prove the converse of his statement, that is, if the elastic energy attains a minimum over all possible balanced configuration, then the strains of the bars are compatible with the displacements of the nodes.

The above described proof is for a free elastic system; Menabrea then considers a constrained system, and, when the constraints are perfect and immovable he obtains the same result.

3.2 The inductive proof

Menabrea presents the proof after a series of five particular cases where he shows that his "principle of elasticity", provides the same results as other well-accepted techniques. The success of the "principle of elasticity" in these cases can be seen as an inductive proof of Menabrea's statement.

The considered cases are a plane system of three elastic threads loaded by a force in the same plane acting on their common point; another plane system consisting of six elastic bars lying on the edges and the diagonals of a parallelogram loaded by opposite forces along the diagonals; a spatial system of 16 bars forming a regular octahedron loaded by opposite forces along the diagonals; an elastic bar axially loaded by different concentrated weights along its length; a rigid plate supported by elastic bars. In each problem, the question is the distribution of inner forces.

As commented above, such an approach shows how Menabrea was certain of the truth of his statement, and did not worry much about how to be prove it. Moreover, one can see in this a defence of his paper of 1858: Menabrea suggests that what counts is to show that the statement holds in as many applications as possible, rigour is of secondary importance and sooner or later some expert in formal aspects will find the correct proof.

In the following, for sake of brevity, we will give a hint of the first example provided by Menabrea, interesting in any case. As already mentioned, it is about three elastic threads lying in the same plane, each fixed to the 'ground' at one end and joined together at the other. The system is loaded at the common node by a given ('dead') force lying in the same plane, as shown in Fig. 1.

Menabrea finds the solution with an *ad hoc* procedure first, keeping into account the linear elasticity

[48] [2], (9), p. 169.

[49] [2], (10), p. 169.

[50] qui est l'*équation d'élasticité*, de la quelle on conclut le théorème que nous avons énoncé au commencement de ce Mémoire, savoir que: *Lorsqu'un système élastique se met en équilibre sous l'action de forces extérieures, le travail intérieur, développé dans le changement de forme qui en dérive, est un* **minimum** [2], p. 169.

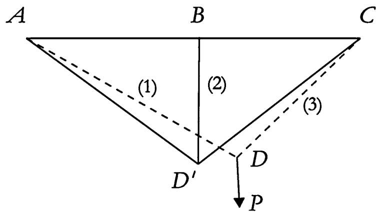

Fig. 1 Menabrea's first example

of the threads. He writes the balance equations at the point D among the forces T_1, T_2, T_3 exerted by the threads and the external force P. The problem is redundant since we have two balance equations in three unknowns (the tensions in the threads). Menabrea then introduces the linear elastic constitutive relation between tension and strain in the threads, replacing the tensions T_i by the strains λ_i. The compatibility condition that the strains must be such that the threads still have a point in common provides the additional equation required to solve the problem. Nowadays such a procedure would be called a method of forces. Menabrea then applies his "principle of elasticity" and easily obtains the same values of strain and tension in the threads.

4 Final remarks

We have seen how Menabrea's "principle of elasticity" represents but a step in the progressive precise statement of what we now call theorem of minimum of complementary energy. Menabrea collected previous statements of similar kind by the French and Italian schools of mechanics, and re-arranged them, with a touch of originality, in what he thought could be a definitive tool for the solution of problems in mechanics of structures. Still, he was son of his time and education, and his statement was not free from drawbacks, put into evidence by his contemporaries and in part reviewed here. Yet he tried to improve himself and we must admit that some of the gross ambiguities in his proof were overpassed. In any case we think, on the basis of the whole of his production which we have examined, that Menabrea's contribution is more important than what is usally admitted. Indeed, even the most known historians of structural mechanics limit their judgement to Menabrea's first paper, which is clearly unsatisfactory. They neglect Menabrea's other papers, which, as we have seen, are much more suggesting, even if not perfectly rigourous.

As it often happens in the history of science, it is possible to follow also for the "principle of elasticity" a cumulative but irregular path. Some of the ways followed to solve the problem of the support reactions were unfruitful, especially those limiting to apply the laws of statics of rigid bodies with some trick. We may see some forerunners, that is, those who seem to attain the result before, but whose results have not been acknowledged by the contemporaries. Among them we shall quote James Henry Cotterill, who, in three papers of 1865,[51] found some results later presented by Menabrea and Castigliano and used them for applications to redundant elements in compression and bending, almost ten years before Castigliano. Cotterill was not quoted in Menabrea's historical introduction to his papers, and gained not so much credit among the engineers in continental Europe. From a certain point of view, Cotterill's work could be ignored in a history of Menabrea's statement, and we actually did.

There was a dispute on the priority of the proof of the "principle of elasticity" between Menabrea and Castigliano, well studied in the literature [9–11]. The conclusions of these studies, as already remarked, are however based on a limited amount of the whole of Menabrea's papers on the subject, and usually present comments more positive towards Castigliano, quite negative towards Menabrea. Without pretending to provide a definite word on the matter, we think it necessary to review the subject on wider basis.

We found it interesting, both from the historical and epistemological points of view, to examine Menabrea's papers preceding the dispute with Castigliano, in order to better understand in which environment the statement took origin and which were the roots of the criticisms that can be brought. Menabrea's figure emerges as one of the people who participate to a great enterprise, of which none in particular is the main character, but without the help of whom the result would not have been the same.

The conclusion of the history of Menabrea's statement is indeed not so sharply marked. Indeed, even if Castigliano's work marked a well defined point, there

[51] Cotterill JH, On an extension of the dynamical principle of least action, Philosophical Magazine, 29, 299–305 (1865); On the equilibrium of arched ribs of uniform section, *ibidem*, 380–389; Further applications of least action, *ibidem*, 430–436.

are still some aspects to clarify from the theorical point of view. A contribution to keep into account is for sure that of a friend of Castigliano, Francesco Crotti [11].

Acknowledgements The authors wish to acknowledge the support of the grant "Progetto di ricerca d'Università" for the year 2008 of the "Sapienza" University, Rome.

References

1. Menabrea LF (1858) Nouveau principe sur la distribution des tensions dans les sistèmes élastiques. C R Séances Acad Sci Paris 46:1056–1061
2. Menabrea LF (1871) Étude de statique physique: principe général pour déterminer les pressions et les tensions dans un système élastique (1865). Mem Reale Accad Sci Torino, Ser 2, 25:141–178
3. Menabrea LF (1870) Sul principio di elasticità. Atti Reale Accad Sci Torino 5:686–710
4. Menabrea LF (1875) Sulla determinazione delle tensioni e delle pressioni ne' sistemi elastici. Atti Reale Accad Lincei, Roma, Ser 2 2:201–225
5. Castigliano CA (1873) Intorno ai sistemi elastici. Thesis, University of Turin
6. Castigliano CA (1873) Théorie de l'équilibre des systèmes élastiques et ses applications. Negro, Turin
7. Müller-Breslau H (1927) La scienza delle costruzioni. Hoepli, Milano (Italian translation of the fifth German edition of "Die neuren Methoden der Festigkeitslehre und der Baukonstruktionen, ausgehend von dem Gesetze der virtuellen Verschiebungen und des Lehrsätzen über die Formänderungsarbeit", Leipzig, 1925)
8. Navier CLMH (1825) Sur des questions de statique dans lesquelles on considère un corps supporté par un nombre de points d'appui surpassant trois. Bull Sci Soc Philomat 11:35–37
9. Benvenuto E (1981) An introduction to the history of structural mechanics. Springer, New York
10. Benvenuto E (1984) A brief outline of the scientific debate that preceded the works of Castigliano. Meccanica 19(supplement):19–32. Also in: Castigliano A, Selecta 1984, Levrotto e Bella, Torino (1984)
11. Nascé V (1981) Alberto Castigliano, railway engineer: his life and times. Meccanica 19(supplement):5–14. Also in: Castigliano A, Selecta 1984, Levrotto e Bella, Torino (1984)
12. Cournot A (1827) Sur les percussions entre deux corps durs qui se choquent en pleusiers points. Bull Sci [...] (Férussac) Paris 7:4–11
13. Cournot A (1827) Sur les percussions entre deux corps durs qui se choquent en pleusiers points (part 2). Bull Sci [...] (Férussac) Paris 7:85–90
14. Cournot A (1828) Sur la théorie des pressions. Bull Sci [...] (Férussac), Paris 9:10–22
15. Charlton TM (1982) A history of theory of structures in the nineteenth century. The University Press, Cambridge
16. Charlton TM (1984) Least work theory according to Menabrea, Castigliano, and Fränkel. Struct Eng 62A:345–347
17. Maxwell JC (1864) On the calculation of the equilibrium and stiffness of frames. Philos Mag 27:294–299
18. Villarceau Y (1870) Lettre à M. Menabrea. Atti Reale Accad Sci Torino 5:704–705
19. Cerruti V (1875) Sopra un teorema del Sig Menabrea. Atti Reale Accad Lincei, Ser 2, 2:570–581; reprinted with the same title, Salviucci, Roma (1875)