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ORIGINAL ARTICLE

Some helical flows of a Burgers' fluid with fractional derivative

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Abstract In this article, fractional calculus approach is used in the constitutive relationship of a Burgers' fluid model. Integral transforms are used to calculate the velocity and the stress fields for some helical flows of a Burgers' fluid with fractional derivative. Moreover, the behavior of different physical parameters involve in the Burgers' fluid model is analyzed through several graphs.

Keywords Exact solutions · Burgers' fluid · Fractional calculus · Concentric cylinders · Helical flow

1 Introduction

The flow of a non-Newtonian fluid through concentric annuli represents an idealization of several industrially important processes. One important example is in oil well drilling where a heavy drilling mud is circulated through the annular space around the drill pipe in order to carry drilling debris to the surface. These drilling mud are typically non-Newtonian fluids. Others examples include the extrusion of plastic tubes and pipes in which the molten polymer is forced through an annular die, and the flow in double-pipe heat exchanger. Non-Newtonian fluids are also extremely encountered in many disciplinary fields, such as chemical engineering, foodstuff, biomedicine etc. Typical non-Newtonian characteristics include shear thinning, viscoplasticity, viscoelasticity and shear thickening behavior. In the category of non-Newtonian fluids the fluids of differential types have acquired special status as well as much controversy [1]. These fluids cannot describe the influence of relaxation and retardation times. Recently, one of the rate type fluids model proposed by Burgers [2] has become very popular amongst researchers. This is because of it success in describing asphalt in geomechanics and food products such as cheese etc. There are numerous examples of the use of Burgers' model to study asphalt and asphalt mixes [3].

In Refs. [4–7] Fetecau has discussed the flows of the non-Newtonian fluid in cylindrical geometry for ordinary viscoelastic fluid models. In Refs. [8–11] Fetecau and Fetecau constructed the exact solution for some unsteady helical flows of a second grade, Maxwell and Oldroyd-B fluid models. Moreover, in order to describe the viscoelasticity [12] the fractional calculus approach is very important. The starting point of the fractional calculus is usually a classical differential equation, which is modified by replacing the time derivative of an integer order by the so-called Riemann-Liouville fractional calculus operator. This generalization allows one to describe precisely non-

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integer order derivative [13, 14]. So far very little efforts [15–26] have been made to discuss the flows of viscoelastic fluids with a fractional calculus approach.

To the best of my knowledge, it seems to me that no attempt is available in the literature, which describes the helical flows for fractional Burgers' fluid model. The aim of this article is to discuss some helical flows of a Burgers' fluid model with fractional derivative. The article is managed in the following manner. Firstly, it is to develop modeling of flow equations for fractional Burgers' fluid model. Secondly, it is to obtain exact analytical solutions for two types of flow problems: (i) helical flow between concentric cylinders and (ii) helical flow within an infinite cylinder. The explicit expressions for the velocity and stress fields are constructed by using Hankel and Laplace transforms.

2 Governing equations

The flow of the incompressible viscoelastic fluid is governed by the set of conservation and constitutive equations. In the absences of body force, the momentum and mass equations can be written as follows

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}\right]=-\nabla p+\nabla\cdot\mathbf{S}, \tag{1}$$

$$\nabla\cdot\mathbf{V}=\mathbf{0}, \tag{2}$$

where ρ is the fluid density, $\mathbf{V}$ the velocity vector, p the pressure and $\mathbf{S}$ the extra-stress tensor. The extra stress tensor $\mathbf{S}$ for a Burgers' fluid satisfies the following constitutive equation

$$\mathbf{S}+\lambda_1^{\alpha}\widetilde{D}_t^{\alpha}\mathbf{S}+\lambda_2^{\alpha}\widetilde{D}_t^{2\alpha}\mathbf{S}=\mu(\mathbf{A}+\lambda_3^{\beta}\widetilde{D}_t^{\beta}\mathbf{A}), \tag{3}$$

where $\mathbf{A}=\mathbf{L}+\mathbf{L}^{\mathrm{T}}$ is the first Rivlin-Ericksen tensor with $\mathbf{L}$ the velocity gradient, μ the dynamic viscosity, λ_1 and λ_3 ($<\lambda_1$) are the relaxation and retardation times, respectively, λ_2 a new material constant of Burgers' fluid having the dimension of t^2, α and β are fractional calculus parameters such that $0\le\alpha\le\beta\le 1$ and $\widetilde{D}_t$ is the upper convected time derivative defined by [22]

$$\widetilde{D}_t^{\alpha}\mathbf{S}=D_t^{\alpha}\mathbf{S}+(\mathbf{V}\cdot\nabla)\mathbf{S}-\mathbf{L}\mathbf{S}-\mathbf{S}\mathbf{L}^{\mathrm{T}}\quad\text{and}\quad \widetilde{D}_t^{2\alpha}\mathbf{S}=\widetilde{D}_t^{\alpha}(\widetilde{D}_t^{\alpha}\mathbf{S}), \tag{4}$$

$$\widetilde{D}_t^{\beta}\mathbf{A}=D_t^{\beta}\mathbf{A}+(\mathbf{V}\cdot\nabla)\mathbf{A}-\mathbf{L}\mathbf{A}-\mathbf{A}\mathbf{L}^{\mathrm{T}}, \tag{5}$$

where D_t^{α} is the fractional differential operator, which is defined as [13, 14]

$$D_t^{\alpha}[f(t)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}f(\tau)d\tau \quad (0\le\alpha\le 1) \tag{6}$$

with $\Gamma(\cdot)$ as the Gamma function.

The flow under consideration has the following velocity and the stress fields

$$\mathbf{V}(r,t)=v(r,t)\mathbf{e}_{\theta}+w(r,t)\mathbf{e}_z,\qquad \mathbf{S}=\mathbf{S}(r,t), \tag{7}$$

where $\mathbf{e}_{\theta}$ and $\mathbf{e}_z$ are the unit vectors in θ- and z-directions.

The velocity field (7) automatically satisfies the incompressibility condition. The initial condition $\mathbf{S}(r,0)=\mathbf{0}$ yields $S_{rr}=0$. In the absence of body forces and pressure gradient in the axial direction, (3) and the balance of linear momentum leads to the following equations [22]

$$\begin{aligned}&(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\tau_1=\mu(1+\lambda_3^{\beta}D_t^{\beta})(\partial_r-1/r)v,\\ &\rho\partial_t v=(\partial_r+2/r)\tau_1\end{aligned} \tag{8}$$

and

$$\begin{aligned}&(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\tau_2=\mu(1+\lambda_3^{\beta}D_t^{\beta})(\partial_r w),\\ &\rho\partial_t w=(\partial_r+1/r)\tau_2\end{aligned} \tag{9}$$

where $\tau_1=S_{r\theta}$ and $\tau_2=S_{rz}$. Eliminating τ_1 and τ_2 from (8) and (9), we get

$$(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\frac{\partial v}{\partial t} =\nu(1+\lambda_3^{\beta}D_t^{\beta})\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)v(r,t), \tag{10}$$

and

$$(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\frac{\partial w}{\partial t} =\nu(1+\lambda_3^{\beta}D_t^{\beta})\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)w(r,t), \tag{11}$$

where $\nu=\mu/\rho$ is the kinematic viscosity of the fluid.

3 First problem

Consider the helical flow of a Burgers' fluid initially at rest between two infinitely long coaxial cylinders of radii a and $b(> a)$. At time $t = 0^+$ the two cylinders begin to rotate about its axis with the constant angular velocities ω_1 and ω_2 and to slides with the velocities V_1 and V_2. The flow is governed by (10) and (11), and its corresponding initial and boundary conditions are

$$v(r,0) = \partial_t v(r,0) = \partial_{tt} v(r,0) = w(r,0) = \partial_t w(r,0) = \partial_{tt} w(r,0) = 0 \quad \text{for } a < r < b, \tag{12}$$

$$v(a,t) = a\omega_1, \qquad v(b,t) = b\omega_2, \qquad w(a,t) = V_1, \qquad w(b,t) = V_2 \quad \text{for } t \geq 0. \tag{13}$$

To find the solutions for the above equations (10) and (11) subject to the initial and boundary conditions (12) and (13), we first make the change of unknown function

$$v(r,t) = U_1(r) + u_1(r,t) \quad \text{and} \quad w(r,t) = U_0(r) + u_0(r,t), \tag{14}$$

where

$$U_1(r) = r\omega_1 - \frac{(b^2 - r^2)}{(b^2 - a^2)}(\omega_2 - \omega_1) \quad \text{and} \quad U_0(r) = V_1 - \frac{\ln(b/r)}{\ln(b/a)}(V_2 - V_1). \tag{15}$$

So we attain the next two problems with initial and boundary conditions

$$(1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha})\frac{\partial u_n}{\partial t} = \nu(1 + \lambda_3^\beta D_t^\beta)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n}{r^2}\right)u_n(r,t), \tag{16}$$

$$u_n(r,t) = -U_n(r), \qquad \partial_t u_n(r,t) = \partial_{tt} u_n(r,t) = 0, \quad \text{at } t = 0, \tag{17}$$

$$u_n(a,t) = u_n(b,t) = 0 \quad \text{for } t \geq 0, n = 0, 1. \tag{18}$$

To obtain the exact analytical solution of the above problem (16)–(18), we firstly apply Weber transform (A.1) from appendix to (16) and (17) keeping in mind (18), we find

$$(1 + \lambda_1^\alpha D_t^\alpha + \lambda_2^\alpha D_t^{2\alpha})\partial_t \widetilde{u}_n(r_{nm},t) + \nu r_{nm}(1 + \lambda_3^\beta D_t^\beta)\widetilde{u}_n(r_{nm},t) = 0. \tag{19}$$

$$\widetilde{u}_n(r_{nm},0) = -\widetilde{U}_{nm}, \qquad \partial_t \widetilde{u}_n(r_{nm},0) = \partial_{tt}\widetilde{u}_n(r_{nm},0) = 0 \tag{20}$$

where

$$\widetilde{U}_{0m} = \frac{2}{\pi r_{0m}^2}\left[\frac{J_0(r_{0m}a)}{J_0(r_{0m}b)}V_2 - V_1\right] \quad \text{and} \quad \widetilde{U}_{1m} = \frac{2}{\pi r_{1m}^2}\left[\frac{J_1(r_{1m}a)}{J_1(r_{1m}b)}b\omega_2 - a\omega_1\right]. \tag{21}$$

Now applying Laplace transform formula for sequential fractional derivative to (19) and using (A.3) from appendix and the initial conditions (20), we get

$$\bar{\widetilde{u}}_n(r_{nm},s) = -\frac{1 + \lambda_1^\alpha s^\alpha + \lambda_2^\alpha s^{2\alpha}}{s + \lambda_1^\alpha s^{\alpha+1} + \lambda_2^\alpha s^{2\alpha+1} + \nu r_{nm}^2(1 + \lambda_3^\beta s^\beta)}\widetilde{U}_{nm}, \tag{22}$$

where $\bar{\widetilde{u}}_n(r_{nm},s)$ is the Laplace transform of $\widetilde{u}_n(r_{nm},t)$.

Now rewriting the above equation (22) into the series form, we find

$$\bar{\widetilde{u}}_n(r_{nm},s) = -\sum_{l=0}^{\infty}\frac{(-1)^l}{\lambda_2^{\alpha(l+1)}}\sum_{i+j+k=l}^{i,j,k\geq 0}\frac{[\nu r_{nm}^2]^{j+k}\lambda_3^{\beta j}}{i!j!k!} \times \frac{(1 + \lambda_1^\alpha s^\alpha + \lambda_3^\beta s^{2\alpha})s^\delta l!}{(s^\alpha + \lambda_1^\alpha/\lambda_2^\alpha)^{l+1}}\widetilde{U}_{nm}, \tag{23}$$

in which $\delta = i + \beta j - \alpha l - l - \alpha - 1$ and using next the property of the inverse Laplace transform (A.4) from appendix to (23) gives

$$\widetilde{u}_n(r_{nm},t) = -\sum_{l=0}^{\infty}\frac{(-1)^l}{\lambda_2^{\alpha(l+1)}}\sum_{i+j+k=l}^{i,j,k\geq 0}\frac{[\nu r_{nm}^2]^{j+k}\lambda_3^{\beta j}t^{\alpha(l+1)-\delta-1}}{i!j!k!} \times \left[E_{\alpha,\alpha-\delta}^{(l)}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right) + \frac{\lambda_1^\alpha}{t^\alpha}E_{\alpha,-\delta}^{(l)}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right) + \frac{\lambda_2^\alpha}{t^{2\alpha}}E_{\alpha,-\alpha-\delta}^{(m)}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right]\widetilde{U}_{nm}. \tag{24}$$

Now using (24) into (A.2), we obtain the velocity field in the following form

$$u_n(r,t) = -\frac{\pi^2}{2}\sum_{m=1}^{\infty}\frac{r_{nm}^2 J_n^2(r_{nm}b)H_n(r_{nm},r)\widetilde{U}_{nm}}{J_{nm}^2(r_{nm}a) - J_{nm}^2(r_{nm}b)} \times \sum_{l=0}^{\infty}\frac{(-1)^l}{\lambda_2^{\alpha(l+1)}}\sum_{i+j+k=l}^{i,j,k\geq 0}\frac{[\nu r_{nm}^2]^{j+k}\lambda_3^{\beta j}}{i!j!k!}$$

$$\times t^{\alpha(l+1)-\delta-1}\left[E^{(l)}_{\alpha,\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right.$$
$$+\frac{\lambda_1^\alpha}{t^\alpha}E^{(l)}_{\alpha,-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)$$
$$\left.+\frac{\lambda_2^\alpha}{t^{2\alpha}}E^{(l)}_{\alpha,-\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right]. \quad (25)$$

Substituting (25) into (14), the exact analytical solutions for the flow between concentric cylinders will take the form

$$v(r,t)=U_1(r)$$
$$-\frac{\pi^2}{2}\sum_{m=1}^{\infty}\frac{r_{1m}^2J_1^2(r_{1m}b)H_1(r_{1m},r)\widetilde{U}_{1m}}{J_{1m}^2(r_{1m}a)-J_{1m}^2(r_{1m}b)}$$
$$\times\sum_{l=0}^{\infty}\frac{(-1)^l}{\lambda_2^{\alpha(l+1)}}\sum_{i+j+k=l}^{i,j,k\geq 0}\frac{[\nu r_{1m}^2]^{j+k}\lambda_3^{\beta j}}{i!j!k!}$$
$$\times t^{\alpha(l+1)-\delta-1}\left[E^{(l)}_{\alpha,\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right.$$
$$+\frac{\lambda_1}{t^\alpha}E^{(l)}_{\alpha,-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)$$
$$\left.+\frac{\lambda_2}{t^{2\alpha}}E^{(l)}_{\alpha,-\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right], \quad (26)$$

and

$$w(r,t)=U_0(r)$$
$$-\frac{\pi^2}{2}\sum_{m=1}^{\infty}\frac{r_{0m}^2J_0^2(r_{0m}b)H_0(r_{0m},r)\widetilde{U}_{0m}}{J_{0m}^2(r_{0m}a)-J_{0m}^2(r_{0m}b)}$$
$$\times\sum_{l=0}^{\infty}\frac{(-1)^l}{\lambda_2^{\alpha(l+1)}}\sum_{i+j+k=l}^{i,j,k\geq 0}\frac{[\nu r_{0m}^2]^{j+k}\lambda_3^{\beta j}}{i!j!k!}$$
$$\times t^{\alpha(l+1)-\delta-1}\left[E^{(l)}_{\alpha,\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right.$$
$$+\frac{\lambda_1^\alpha}{t^\alpha}E^{(l)}_{\alpha,-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)$$
$$\left.+\frac{\lambda_2^\alpha}{t^{2\alpha}}E^{(l)}_{\alpha,-\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right]. \quad (27)$$

Using the above equations (26) in $(8)_1$ and (27) in $(9)_1$, we find the shear stresses in the following form

$$\tau_1(r,t)=T_1(r,t)$$
$$-\frac{\mu\pi^2}{2}\sum_{m=1}^{\infty}\frac{r_{1m}^3J_1^2(r_{1m}b)\phi_1(r_{1m},r)\widetilde{U}_{1m}}{J_{1m}^2(r_{1m}a)-J_{1m}^2(r_{1m}b)}$$
$$\times\sum_{l=0}^{\infty}\frac{(-1)^l}{\lambda_2^{\alpha(l+1)}}\sum_{i+j+k=l}^{i,j,k\geq 0}\frac{[\nu r_{1m}^2]^{j+k}\lambda_3^{\beta j}}{i!j!k!}$$
$$\times t^{\alpha(l+1)-\delta-1}\left[E^{(l)}_{\alpha,\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right.$$
$$\left.+\frac{\lambda_3^\beta}{t^\beta}E^{(l)}_{\alpha,\alpha-\beta-\delta}\left(\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right], \quad (28)$$

where

$$T_1(r,t)=\frac{2\mu a^2(b^2+r^2)(\omega_2-\omega_1)}{r^2(b^2-a^2)}$$
$$\times\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{\lambda_3^{\beta l}t^{\alpha(1-l)}}{\lambda_2^{\alpha(l+1)}}$$
$$\times\left[E^{(l)}_{\alpha,\alpha-2\alpha l+1}\left(-\frac{t^\alpha}{\lambda_2^\alpha}\right)\right.$$
$$\left.+\frac{\lambda_3^\beta}{t^\beta}E^{(l)}_{\alpha,\alpha-2\alpha l-\beta+1}\left(-\frac{t^\alpha}{\lambda_2^\alpha}\right)\right]$$

and

$$\phi_1(r_{1m},r)=J_1(r_{1m}a)Y_2(r_{1m}r)$$
$$-Y_1(r_{1m}a)J_2(r_{1m}r),$$

and

$$\tau_2(r,t)=T_0(r,t)$$
$$-\frac{\mu\pi^2}{2}\sum_{m=1}^{\infty}\frac{r_{0m}^3J_0^2(r_{0m}b)\phi_0(r_{0m},r)\widetilde{U}_{0m}}{J_{0m}^2(r_{0m}a)-J_{0m}^2(r_{0m}b)}$$
$$\times\sum_{l=0}^{\infty}\frac{(-1)^l}{\lambda_2^{\alpha(l+1)}}$$
$$\times\sum_{i+j+k=l}^{i,j,k\geq 0}\frac{[\nu r_{0m}^2]^{j+k}\lambda_3^{\beta j}}{i!j!k!}t^{\alpha(l+1)-\delta-1}$$
$$\times\left[E^{(l)}_{\alpha,\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right.$$
$$\left.+\frac{\lambda_3^\beta}{t^\beta}E^{(l)}_{\alpha,\alpha-\beta-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right], \quad (29)$$

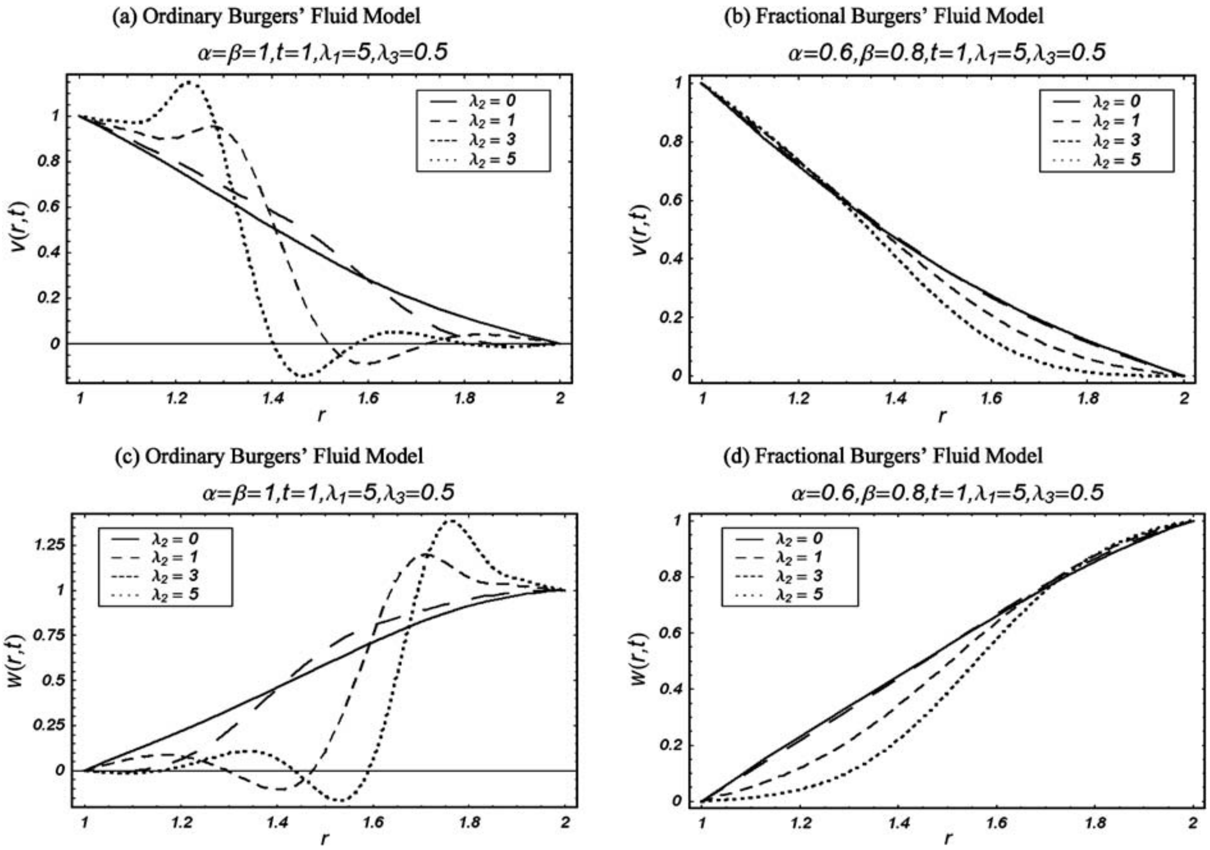

Fig. 1 Variation of velocities $v(r,t)$ and $w(r,t)$ for different values of λ_2

where

$$T_0(r,t) = \frac{\mu(V_2 - V_1)}{r\ln(b/a)} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \frac{\lambda_3^{\beta l} t^{\alpha(1-l)}}{\lambda_2^{\alpha(l+1)}} \times \left[E^{(l)}_{\alpha,\alpha-2\alpha l+1}\left(-\frac{t^\alpha}{\lambda_2^\alpha}\right) + \frac{\lambda_3^\beta}{t^\beta} E^{(l)}_{\alpha,\alpha-2\alpha l-\beta+1}\left(-\frac{t^\alpha}{\lambda_2^\alpha}\right)\right]$$

and

$$\phi_0(r_{0m}, r) = J_0(r_{0m}a)Y_1(r_{0m}r) - Y_0(r_{0m}a)J_1(r_{0m}r),$$

respectively.

4 Second problem

Here we suppose that the Burgers' fluid is in an infinitely long cylinder which is initially at rest and sets in motion suddenly due to rotation of the cylinders about its axis with constant angular velocity ω and slides with constant velocity V. So for this situation the governing equations will be same except the initial and boundary conditions (12) and (13) we use

$$v(r,0) = \partial_t v(r,0) = \partial_{tt} v(r,0) = w(r,0) = \partial_t w(r,0) = \partial_{tt} w(r,0) = 0 \quad \text{for } r < b, \tag{30}$$

$$|v(0,t)| < \infty, \qquad v(b,t) = b\omega, \qquad |w(0,t)| < \infty, \qquad w(b,t) = V \quad \text{for } t \geq 0. \tag{31}$$

Applying the finite Hankel and Laplace transform to solve the above problem, we attain the solutions in the following forms

$$v(r,t) = r\omega - 2\omega \sum_{m=1}^{\infty} \frac{J_1(r_{1m}b)}{r_{1m}J_2(r_{1m}b)} \sum_{l=0}^{\infty} \frac{(-1)^l}{\lambda_2^{\alpha(l+1)}} \times \sum_{i+j+k=l}^{i,j,k\geq 0} \frac{[\nu r_{1m}^2]^{j+k} \lambda_3^{\beta j} t^{\alpha(l+1)-\delta-1}}{i!j!k!}$$

$$\times\left[E^{(l)}_{\alpha,\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)+\frac{\lambda_1^\alpha}{t^\alpha}E^{(l)}_{\alpha,-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)+\frac{\lambda_2^\alpha}{t^{2\alpha}}E^{(l)}_{\alpha,-\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right], \tag{32}$$

and

$$v(r,t)=V-\frac{2V}{b}\sum_{m=1}^{\infty}\frac{J_0(r_{0m}r)}{r_{0m}J_1(r_{0m}b)}\sum_{l=0}^{\infty}\frac{(-1)^l}{\lambda_2^{\alpha(l+1)}}\times\sum_{\substack{i,j,k\geq 0\\ i+j+k=l}}\frac{[\nu r_{0m}^2]^{j+k}\lambda_3^{\beta j}t^{\alpha(l+1)-\delta-1}}{i!j!k!}$$
$$\times\left[E^{(l)}_{\alpha,\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)+\frac{\lambda_1^\alpha}{t^\alpha}E^{(l)}_{\alpha,-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)+\frac{\lambda_2^\alpha}{t^{2\alpha}}E^{(l)}_{\alpha,-\alpha-\delta}\left(-\frac{\lambda_1^\alpha}{\lambda_2^\alpha}t^\alpha\right)\right]. \tag{33}$$

5 Numerical results and discussion

In this section numerical results for the velocity and the stress fields are illustrated graphically for the first problem. For the sake of simplicity all the graphs are plotted by taken $a=1$, $b=2$, $\omega_1=1$, $\omega_2=0$, $V_1=0$, $V_2=1$. Through many diagrams the variations of the velocity field and the shear stress are shown. The effects of various parameters especially the fractional parameters α,β and the rheological parameter λ_2 of the Burgers' fluid are interpreted. In diagrams, comparison is drawn between ordinary and fractional models. Special attention has been focused to analyze the difference between the velocity and stress profiles for an Oldroyd-B ($\lambda_2=0$) and Burgers' fluid ($\lambda_2\neq 0$) models.

Figure 1 displays the behavior of the rheological parameter λ_2 of the Burgers' fluid; it is observed that the Burgers' fluid model shows the sharp change in velocity when compared with Oldroyd-B fluid model. It

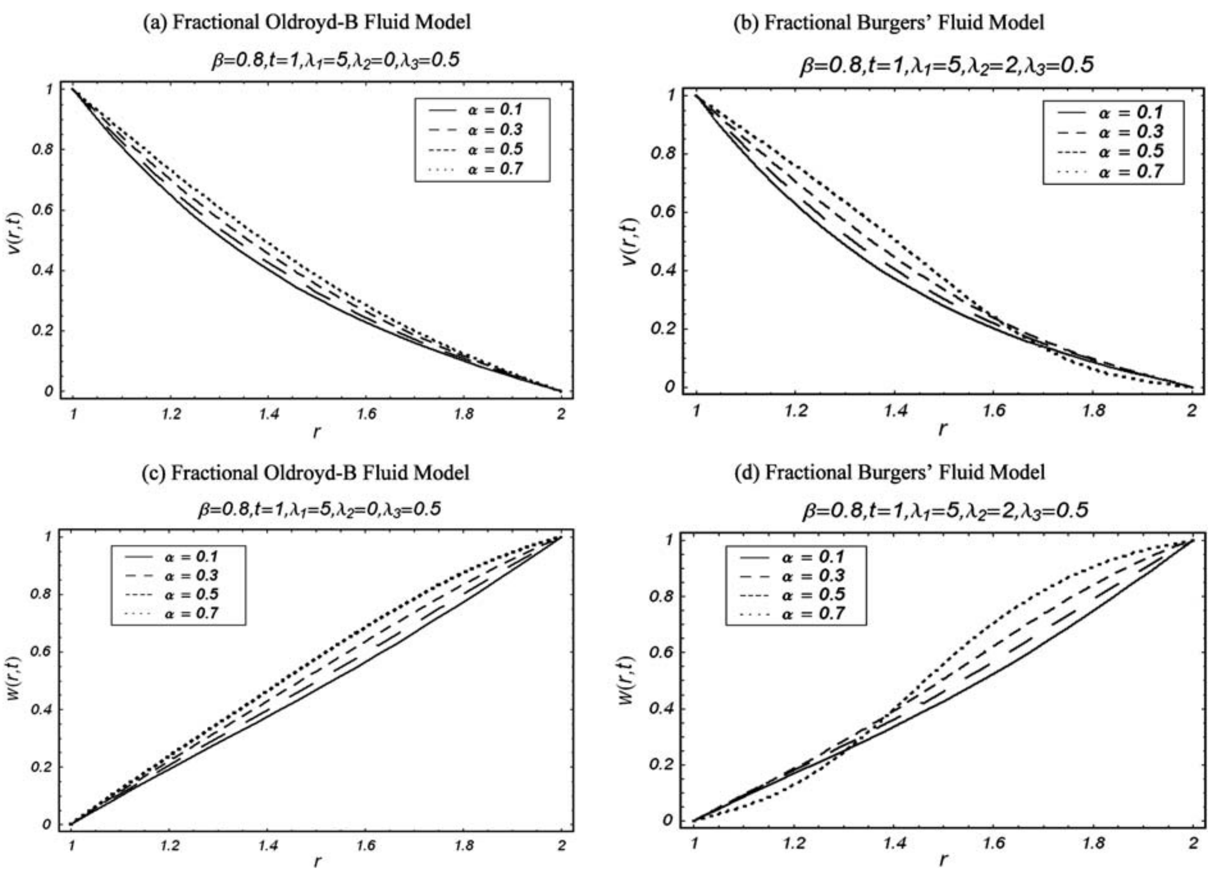

Fig. 2 Variation of velocities $v(r,t)$ and $w(r,t)$ for different values of α

is also seen that the behavior of rheological parameter λ_2 of the Burgers' fluid is non-monotonous. This shows that the flow is the strong function of the rheological parameter λ_2 of the Burgers' fluid model. It is also observed that the effect of the reology of the fluid is much stronger in ordinary models than those of fractional models. Figures 2 and 3 are plotted to show the effects of non-integers fractional parameters α and β on the velocity fields. It is observed in Fig. 1 that for time $t = 1$ the velocity profiles will

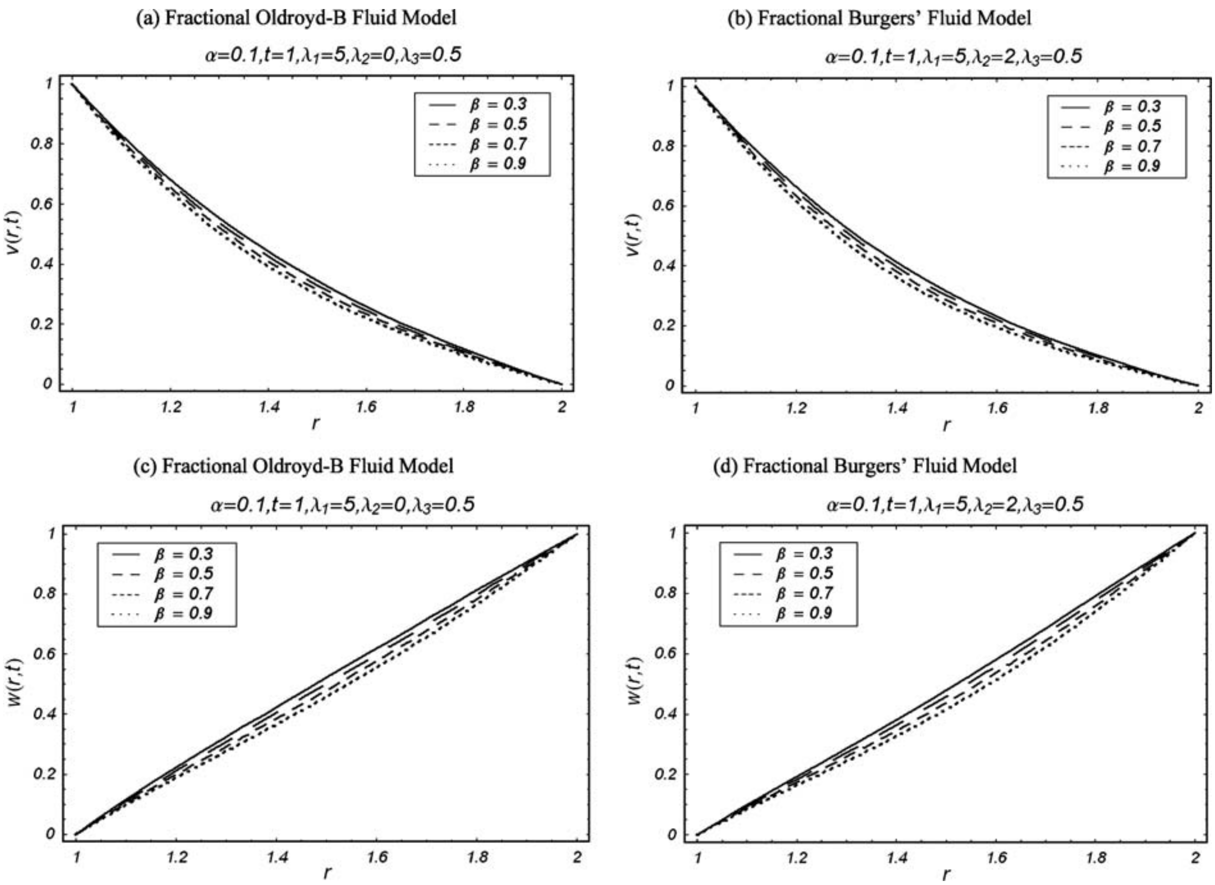

Fig. 3 Variation of velocities $v(r,t)$ and $w(r,t)$ for different values of β

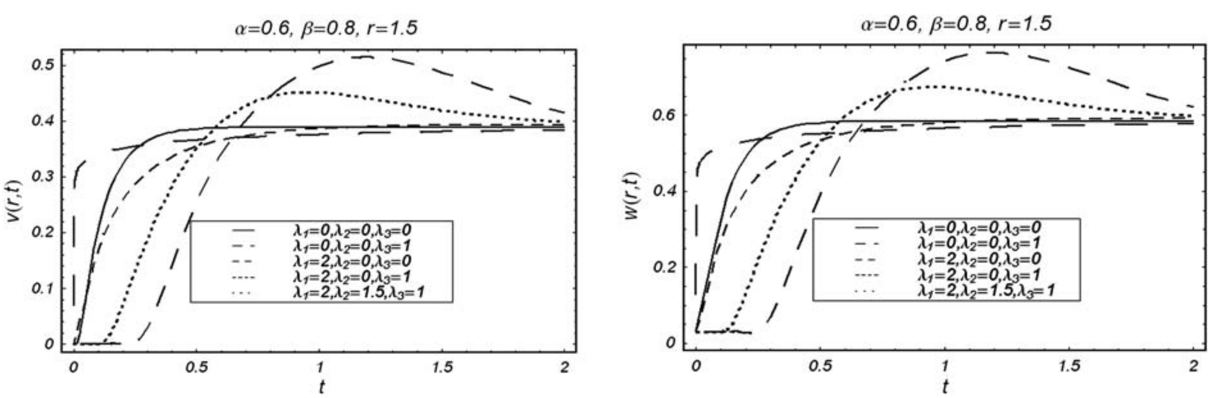

Fig. 4 Variation of velocities $v(r,t)$ and $w(r,t)$ for different values of t

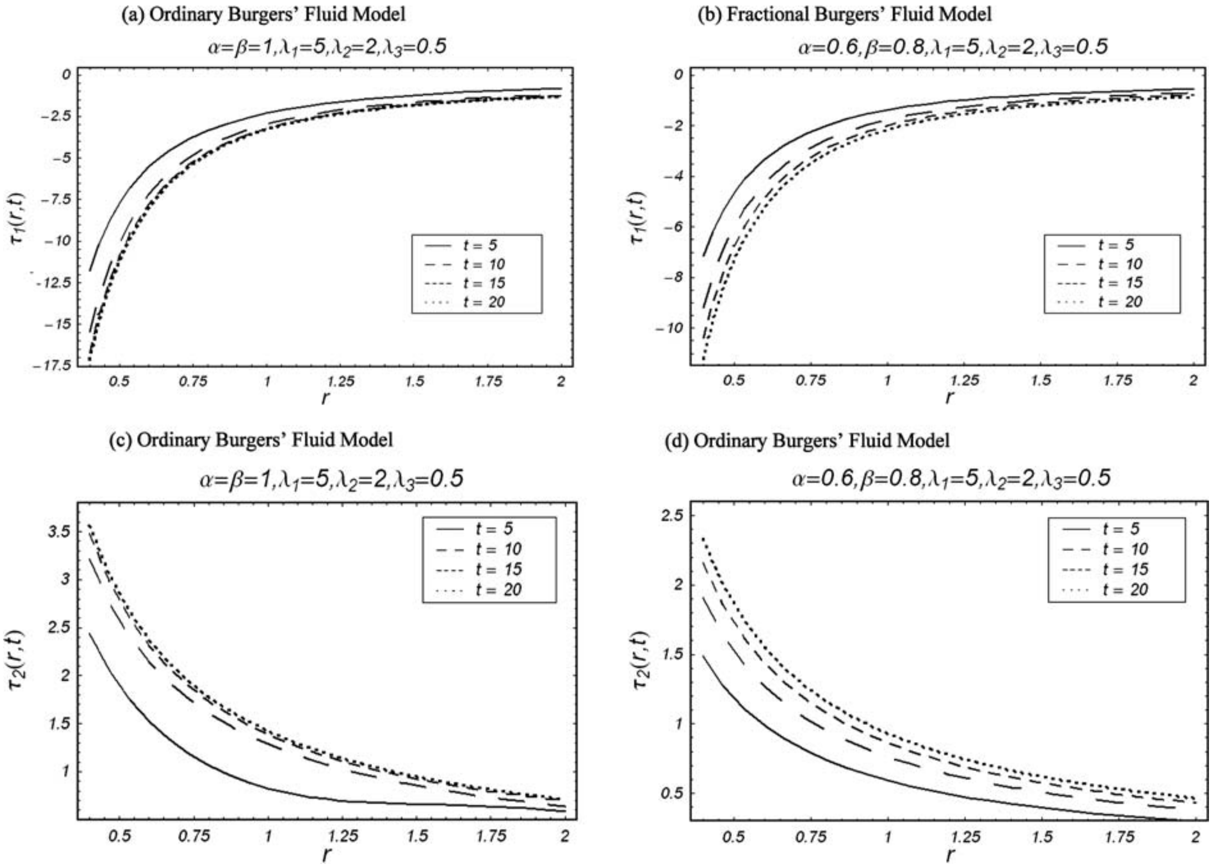

Fig. 5 Variation of shear stresses $\tau_1(r,t)$ and $\tau_2(r,t)$ for different values of t

increase by increasing the fractional parameter α for both the Oldroyd-B and Burgers' fluid models. While in Fig. 3 the opposite effects are observed for β. Thus from these figures it is obvious that the order of the fractional parameters have strong effects on the velocity profiles. In Fig. 4 the velocity fields in between concentric cylinders are plotted as a function of time t for Newtonian, generalized second grade, generalized Maxwell, generalized Oldroyd-B and generalized Burgers' fluid models by keeping $r = 1.5$. It is shown through this figure that for increasing time t the non-Newtonian effects ultimately become weaker. Figure 5 is plotted to show the behavior of shear stresses $\tau_1(r,t)$ and $\tau_2(r,t)$ for ordinary and fractional Burgers' fluid model, the effects of time t on the stress fields can be observed from this figure.

Conclusively, in this work we have firstly modeled the equation for some helical flows for a viscoelastic fluid between coaxial cylinders. The expressions for the velocity and the stress fields are constructed by using Hankel and Laplace transforms for helical flow between concentric cylinders and for flow within an infinite cylinder. Moreover, the graphs are plotted for helical flow between concentric cylinders through these graphs the behavior of many physical parameters of interest involved in the velocity and stress fields are discussed.

Appendix

The Weber transform is define as

$$\widetilde{u}_n(r_{nm},t) = \int_a^b r u_n(r,t) H_n(r_{nm},r) dr, \tag{A.1}$$

and its inverse is

$$u_n(r,t) = \sum_{m=1}^{\infty} \widetilde{u}_n(r_{nm},t) \frac{H_n(r_{nm},r)}{N(r_{nm})}, \tag{A.2}$$

in which $H_n(r_{nm},r)=J_n(r_{nm}r)Y_n(r_{nm}a)-J_n(r_{nm}a)Y_n(r_{nm}r)$, r_{nm} are the positive root of $H_n(r_{nm},b)$and

$$\frac{1}{N(r_{nm})}=\frac{\pi^2}{2}\frac{r_{nm}^2 J_n^2(r_{nm}b)}{J_n^2(r_{nm}a)-J_n^2(r_{nm}b)},$$

where $J_n(\cdot)$ and $Y_n(\cdot)$ are the Bessel functions of the first and second kinds of order n.

The Laplace transforms of $L\{D_t^\alpha \partial_t \widetilde{u}_n(r_{nm},t)\}$ and $D_t^\beta \widetilde{u}_n(r_{nm},t)$ is

$$\begin{aligned} &L\{D_t^\alpha \partial_t \widetilde{u}_n(r_{nm},t)\}=s^{\alpha+1}\overline{\widetilde{u}}_n(r_{nm},s)-s^\alpha \widetilde{u}_n(r_{nm},0),\\ &L\{D_t^\beta \widetilde{u}_n(r_{nm},t)\}=s^\beta \overline{\widetilde{u}}_n(r_{nm},s). \end{aligned} \tag{A.3}$$

$$L^{-1}\left\{\frac{k!s^{\lambda-\mu}}{(s^\lambda \mp c)^{k+1}}\right\}=t^{\lambda k+\mu-1}E_{\lambda,\mu}^{(k)}(\pm ct^\lambda)\quad (\mathrm{Re}(s)>|c|^{1/\lambda}), \tag{A.4}$$

where

$$E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\lambda n+\mu)},\qquad \lambda,\mu>0,$$

$$E_{\lambda,\mu}^{(k)}(z)=\frac{d^k}{dz^k}E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{(n+k)!z^n}{\Gamma(\lambda(n+k)+\mu)}.$$

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