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Deformations and stresses in annular disks made of functionally graded materials subjected to internal and/or external pressure

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Abstract In this paper, an analytical solution is developed to determine deformations and stresses in circular disks made of functionally graded materials subjected to internal and/or external pressure. Taking mechanical properties of the materials of circular disks to be linear variations, the governing equation is derived from basic equations of axisymmetric, plane stress problems in elasticity. By transforming the governing equation into a hypergeometric equation, an accurate analytical solution of deformations and stresses in circular disks is obtained. The comparison with the numerical solution indicates that both approaches give very agreeable results, indicating correctness of the proposed analytical solution. The obtained analytical solution is employed to determine the radial displacement and stresses in circular disks subjected to external pressure, internal pressure, and internal and external pressure, respectively. How the radius ratio of circular disks affects deformations and stresses is also investigated.

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1 Introduction

Analysis of circular disks subjected to external and/or internal pressure is an interesting topic in solid mechanics and engineering applications. It has attracted a lot of research attentions.

Elastic analysis of circular disks under external and/or internal pressure can be found in some textbooks of elasticity. Elastic-plastic analysis of the disks with strain-hardening was firstly investigated by Gamer. Introducing a linear strain-hardening material model, Gamer determined deformations and stresses in a circular ring due to external pressure [1] and in the rotating annulus and in the annulus under external pressure [2]. In his work, only circular disks with constant thickness were considered. Using the same linear strain-hardening stress-plastic strain relation, Güven gave an accurate analytical solution of annular disks with the thickness function $h = h_0(\frac{r}{b})^{-n}$ [3]. Later on, he extended this work to annular disks with the thickness variations $h = h_0 e^{1-(r/r_a)^k}$ [4] and $h = h_0(1 - tr)^k$ and $h = h_0(1 - tr^k)$ [5]. This material model was also applied in elastic-plastic analysis of rotating disks [6–9]. By changing the material model from linear strain-hardening to nonlinear strain-hardening, a numerical method was presented for elastic-plastic analysis of annular disks with arbitrary thickness functions under external pressure [10]. This nonlinear material model was also applied in elastic-plastic analysis of rotating disks [11, 12].

The above analysis only discussed the disks with constant material properties. Recently, functionally graded materials have been widely investigated in many engineering applications

For example, Dao et al. performed a micromechanical study of residual stresses in functionally graded materials [13]. Qian et al. calculated static and dynamic deformations of thick functionally graded elastic plates [14]. Güven determined stress distributions in isotropic spheres made of functionally graded materials subjected to internal pressure [15]. Durodola and Attia presented a numerical integration technique to determine deformations and stresses in functionally graded rotating disks [16]. Güven and Çelik took Poisson's ratio to be a constant, obtained variations of Young's modulus which lead to favorable stress distributions, and examined transverse vibrations of elastic rotating solid disks made of functionally graded isotropic materials [17]. You et al. investigated creep deformations and stresses in thick-walled cylindrical vessels made of functionally graded materials subject to internal pressure [18].

In this paper, we will introduce linear variations of Young's modulus and Poisson's ratio, and develop an accurate analytical solution to determine deformations and stresses in annular disks made of functionally graded materials subjected to external and/or internal pressure.

2 Formulation

As shown in Fig. 1, an annular disk, with an inner radius a and an outer radius b, is subjected to an internal pressure p_a and/or an external pressure p_b. The disk is made of a functionally graded material whose Young's modulus E and Poisson's ratio ν vary linearly in the radial direction r. The deformations and stresses of the disk can be determined as follows.

According to the geometry and loads of the disk, the deformations and stresses in the annular disk are axisymmetric. The geometric equations describing the relations between radial and circumferential strains and the radial displacement are

$$\varepsilon_r = \frac{du}{dr}, \qquad \varepsilon_\theta = \frac{u}{r} \tag{1}$$

where u is the radial displacement, ε indicates the strain, and the subscripts r and θ stand for the radial and circumferential directions.

When the annular disk is very thin, the stress in the axial direction of the disk can be regarded as zero. The relationship between stresses and strains can be represented with Hooke's law in elasticity which has the

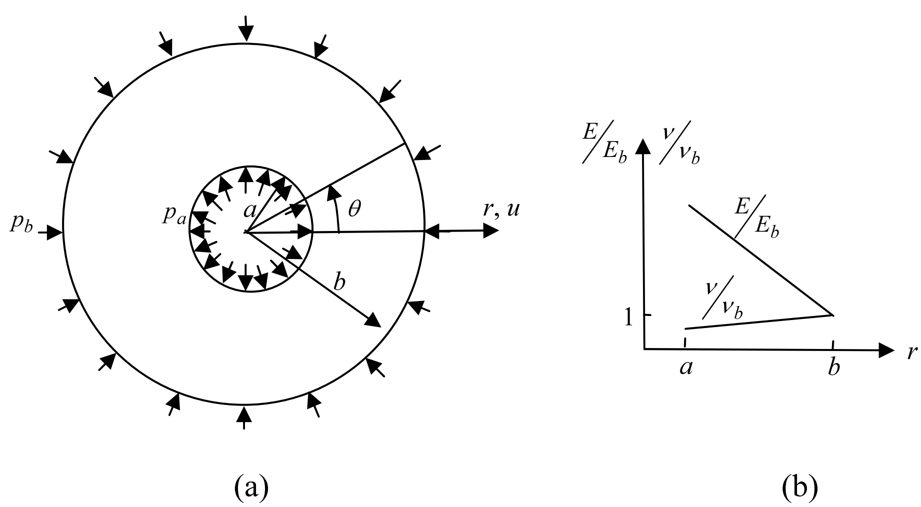

Fig. 1 Geometry, loads and material properties of annular disks

form of

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$
$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \quad (2)$$

where E and ν are Young's modulus and Poisson's ratio of the annular disk, respectively, and σ indicates the stress.

For functionally graded materials, Young's modulus and Poisson's ratio are the functions of the radial coordinate. A linear variation of material properties has been widely adopted. With such a treatment, Young's modulus and Poisson's ratio of the annular disk can be formulated as

$$t = a_t + b_t r \quad (t = E, \nu) \quad (3)$$

Taking the inner and outer radii of the circular disk to be a and b, and Young's modulus and Poisson's ratio at these two positions to be E_a and E_b and ν_a and ν_b, respectively, the unknown constants a_E, b_E, a_ν and b_ν in the above equation are found to be

$$a_t = \frac{bt_a - at_b}{b-a}$$
$$b_t = \frac{t_b - t_a}{b-a} \quad (t = E, \nu) \quad (4)$$

For axisymmetric, plane stress problems, the circumferential stress can be represented with the radial stress and its first derivative with respect to the radial coordinate through the following equilibrium equation

$$\sigma_\theta = \sigma_r + r\frac{d\sigma_r}{dr} \quad (5)$$

Substituting (3) and (5) into (2), the radial and circumferential strains are related to the radial stress and its first derivative as follows

$$\varepsilon_r = \frac{1}{a_E + b_E r}\left[(1 - a_\nu - b_\nu r)\sigma_r - (a_\nu + b_\nu r)r\frac{d\sigma_r}{dr}\right] \quad (6)$$
$$\varepsilon_\theta = \frac{1}{a_E + b_E r}\left[(1 - a_\nu - b_\nu r)\sigma_r + r\frac{d\sigma_r}{dr}\right]$$

The deformation compatibility equation of the annular disk under internal and/or external pressure can be obtained below from (1)

$$r\frac{d\varepsilon_\theta}{dr} + \varepsilon_\theta - \varepsilon_r = 0 \quad (7)$$

Differentiating the second of (6) with respect to the radial coordinate, and substituting it and (6) into (7), the deformation compatibility equation is transformed into a second order ordinary differential equation

$$\begin{aligned}&[bE_a - aE_b + (E_b - E_a)r](b-a)r\frac{d^2\sigma_r}{dr^2}\\&\quad + [3(bE_a - aE_b) + 2(E_b - E_a)r](b-a)\frac{d\sigma_r}{dr}\\&\quad - [(b-a)(E_b - E_a) + (bE_a - aE_b)(\nu_b - \nu_a)\\&\quad - (E_b - E_a)(b\nu_a - a\nu_b)]\sigma_r = 0\end{aligned} \quad (8)$$

The above equation can be solved with numerical methods such as that given in [19]. In this paper, we develop an analytical solution.

Introducing the following notations,

$$\begin{aligned}&\bar{r} = b - a\\&E_0 = bE_a - aE_b\\&E_1 = E_b - E_a\\&\nu_0 = b\nu_a - a\nu_b\\&\nu_1 = \nu_b - \nu_a\end{aligned} \quad (9)$$

Equation (8) is changed into

$$(E_0 + E_1 r)\bar{r}r\frac{d^2\sigma_r}{dr^2} + (3E_0 + 2E_1 r)\bar{r}\frac{d\sigma_r}{dr} - (\bar{r}E_1 + E_0\nu_1 - E_1\nu_0)\sigma_r = 0 \quad (10)$$

Dividing the above equation by $\bar{r}E_0$, and using the notations

$$n_1 = \frac{E_1}{E_0}$$
$$n_2 = \frac{n_1\bar{r} + \nu_1 - n_1\nu_0}{\bar{r}} \quad (11)$$

(10) becomes

$$(1 + n_1 r)r\frac{d^2\sigma_r}{dr^2} + (3 + 2n_1 r)\frac{d\sigma_r}{dr} - n_2\sigma_r = 0 \quad (12)$$

Introducing a new variable $z = -n_1 r$, (12) can be further transformed into the following hypergeometric equation

$$(1-z)z\frac{d^2\sigma_r}{dz^2} + (3-2z)\frac{d\sigma_r}{dz} + \frac{n_2}{n_1}\sigma_r = 0 \quad (13)$$

With the method given by Abramowitz and Stegun [20], the solution of the above hypergeometric equation can be written as

$$\sigma_r = c_1 F(\beta,\gamma,\delta,z) + c_2(1-z)^{\delta-\beta-\gamma} \times F(\delta-\beta,\delta-\gamma,1-\beta-\gamma+\delta,1-z) \quad (14)$$

where c_1 and c_2 are unknown constants,

$$\beta = 0.5\left(1-\sqrt{1+\frac{4n_2}{n_1}}\right)$$
$$\gamma = 0.5\left(1+\sqrt{1+\frac{4n_2}{n_1}}\right) \quad (15)$$
$$\delta = 3$$

and $F(\beta,\gamma,\delta,z)$ is called the hypergeometric series of variable z with parameters β, γ and δ whose mathematical representation is

$$F(\beta,\gamma,\delta,z) = 1 + \frac{\beta\gamma}{1!\delta}z + \frac{\beta(\beta+1)\gamma(\gamma+1)}{2!\delta(\delta+1)}z^2 + \frac{\beta(\beta+1)(\beta+2)\gamma(\gamma+1)(\gamma+2)}{3!\delta(\delta+1)(\delta+2)} \times z^3 + \cdots \quad (16)$$

Substituting the relationship $z = -n_1 r$ back into (14), the radial stress is described with the equation of

$$\sigma_r = c_1 F(\beta,\gamma,\delta,-n_1 r) + c_2(1+n_1 r)^{\delta-\beta-\gamma} \times F(\delta-\beta,\delta-\gamma,1-\beta-\gamma+\delta,1+n_1 r) \quad (17)$$

Inserting the above radial stress and its first derivative into the equilibrium equation (5), the circumferential stress is obtained below

$$\sigma_\theta = c_1\left[F(\beta,\gamma,\delta,-n_1 r) - \frac{\beta\gamma n_1 r}{\delta} \times F(\beta+1,\gamma+1,\delta+1,-n_1 r)\right] + c_2(1+n_1 r)^{\delta-\beta-\gamma} \times \left\{F(\delta-\beta,\delta-\gamma,1-\beta-\gamma+\delta,1+n_1 r) \times\left[1+(\delta-\beta-\gamma)\frac{n_1 r}{1+n_1 r}\right] + \frac{(\delta-\beta)(\delta-\gamma)n_1 r}{1-\beta-\gamma+\delta} \times F(\delta-\beta+1,\delta-\gamma+1, 2-\beta-\gamma+\delta,1+n_1 r)\right\} \quad (18)$$

Substituting the radial stress and its first derivative into the second of (6), then substituting the circumferential strain into the second of (1), the radial displacement is found to be

$$u = c_1\frac{\bar{r}r}{E_0+E_1 r}\left[\left(1-\frac{\nu_0+\nu_1 r}{\bar{r}}\right)F(\beta,\gamma,\delta,-n_1 r) - \frac{\beta\gamma n_1 r}{\delta}F(\beta+1,\gamma+1,\delta+1,-n_1 r)\right] + c_2\frac{\bar{r}r}{E_0+E_1 r}(1+n_1 r)^{\delta-\beta-\gamma} \times\left\{\left[1+(\delta-\beta-\gamma)\frac{n_1 r}{1+n_1 r} - \frac{\nu_0+\nu_1 r}{\bar{r}}\right] \times F(\delta-\beta,\delta-\gamma,1-\beta-\gamma+\delta,1+n_1 r) + \frac{(\delta-\beta)(\delta-\gamma)n_1 r}{1-\beta-\gamma+\delta} \times F(\delta-\beta+1,\delta-\gamma+1, 2-\beta-\gamma+\delta,1+n_1 r)\right\} \quad (19)$$

Subjected to an internal pressure p_a and external pressure p_b, boundary conditions of the annular disk are

$$\begin{aligned} r = a \quad \sigma_r = -p_a \\ r = b \quad \sigma_r = -p_b \end{aligned} \quad (20)$$

Substituting (17) into the above equation, solving for the unknown constants c_1 and c_2, and using the notation

$$D = (1+n_1 b)^{\delta-\beta-\gamma}F(\beta,\gamma,\delta,-n_1 a) \times F(\delta-\beta,\delta-\gamma,1-\beta-\gamma+\delta,1+n_1 b) - (1+n_1 a)^{\delta-\beta-\gamma}F(\beta,\gamma,\delta,-n_1 b) \times F(\delta-\beta,\delta-\gamma,1-\beta-\gamma+\delta,1+n_1 a) \quad (21)$$

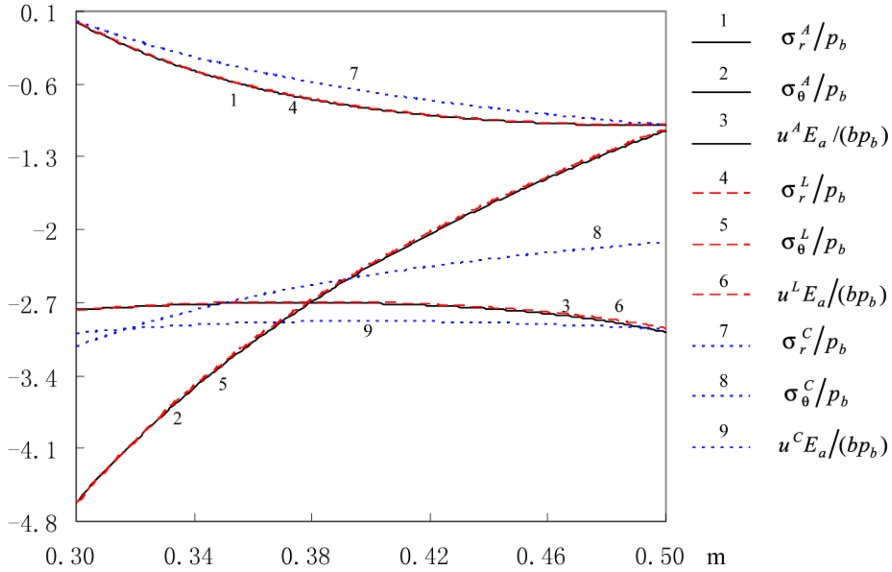

Fig. 2 Comparison of stresses and displacement in annular disks under external pressure

we obtain

$$
\begin{aligned}
c_1 &= \frac{1}{D}[(1+n_1a)^{\delta-\beta-\gamma} \\
&\quad \times F(\delta-\beta,\delta-\gamma,1-\beta-\gamma+\delta,1+n_1a)p_b \\
&\quad - (1+n_1b)^{\delta-\beta-\gamma} \\
&\quad \times F(\delta-\beta,\delta-\gamma,1-\beta-\gamma+\delta,1+n_1b)p_a] \\
c_2 &= \frac{1}{D}[F(\beta,\gamma,\delta,-n_1b)p_a - F(\beta,\gamma,\delta,-n_1a)p_b]
\end{aligned} \tag{22}
$$

Substituting (22) into (17)–(19), the radial displacement and all the stresses in the circular disk are determined.

3 Numerical applications

With the above-developed method, in this section, we determine the radial displacement and stress distributions in circular disks subject to external pressure, internal pressure, and external and internal pressure, respectively, and investigate the effects of the radius ratio of the disks on deformations and stresses.

Young's modulus and Poisson's ratio of circular disks at the inner radius are set to $E_a = 440$ GPa and $\nu_a = 0.25$, and those at outer radius to $E_b = 110$ GPa and $\nu_b = 0.3$, respectively. The inner and outer radii of circular disks are taken to be $a = 0.3$ m and $b = 0.5$ m.

The normalized stresses and radial displacement of circular disks subjected to an external pressure of $p_b = 100$ MPa were given in Fig. 2, those from an internal pressure of $p_a = 100$ MPa were depicted in Fig. 3, and those caused by an internal pressure $p_a = 100$ MPa and an external pressure $p_b = 150$ MPa were shown in Fig. 4.

In order to demonstrate correctness of the obtained analytical solution, a numerical method has also been introduced to calculate the disks. For this, the disks were divided into a number of thin circular rings. For each ring, Young's modulus and Poisson's ratio are taken to be the average values of the ring. The equations determining the radial displacement and stresses in the ith elastic annular ring are

$$
\begin{aligned}
u_i &= \frac{1}{E_i}\left[(1-\nu_i)d_i r - (1+\nu_i)e_i r^{-1}\right] \\
\sigma_{ri} &= d_i + e_i r^{-2} \\
\sigma_{\theta i} &= d_i - e_i r^{-2} \\
&(i = 1, 2, 3, \ldots, N)
\end{aligned} \tag{23}
$$

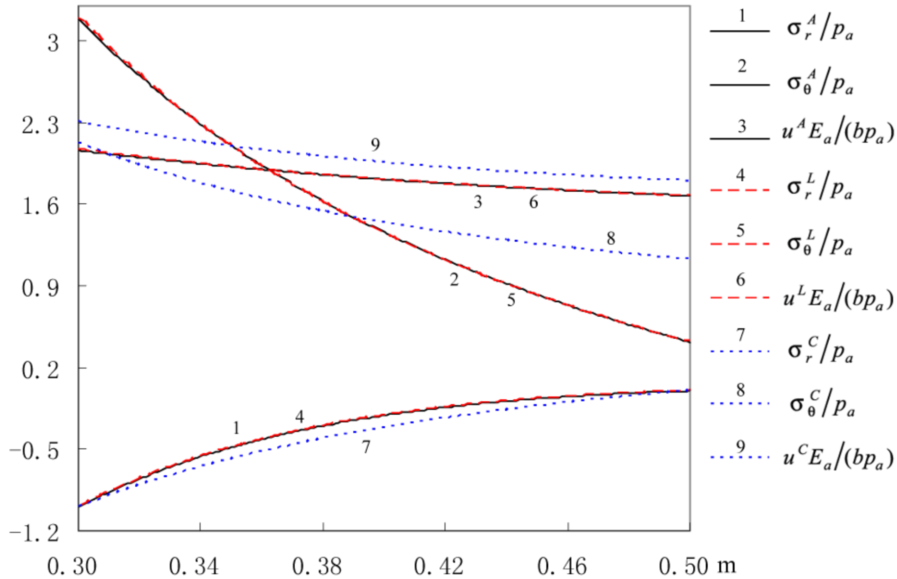

Fig. 3 Comparison of stresses and displacement in annular disks under internal pressure

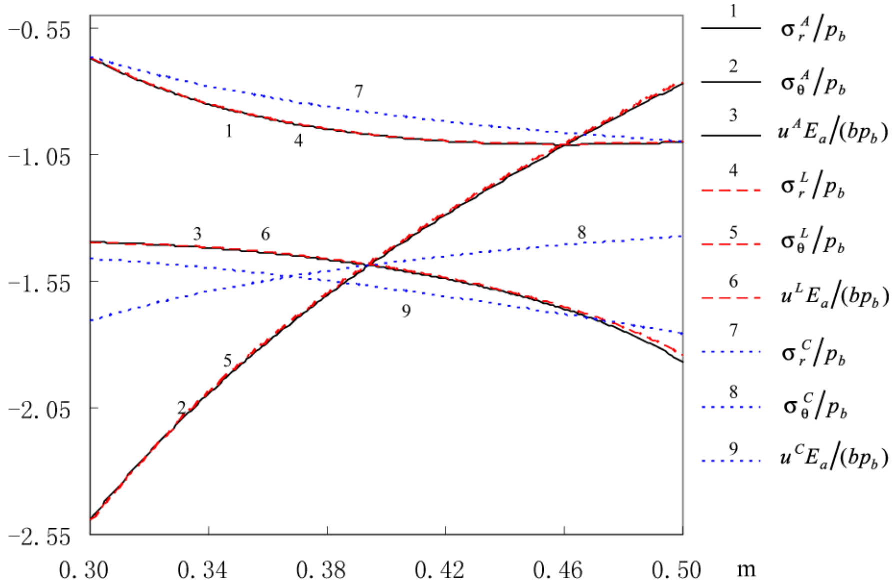

Fig. 4 Comparison of stresses and displacement in annular disks under internal and external pressure

where d_i and e_i are unknown constants, E_i and ν_i are average Young's modulus and Poisson's ratio of the ith ring, and N is the total number of the divided rings.

The unknown constants d_i and e_i in the above equation can be determined with the continuity conditions of radial stress and displacement between the adjacent rings and the boundary conditions of annular disks which can be written as

$$\begin{aligned} &r = a \quad \sigma_{r1} = -p_a \\ &r = r_i \quad \sigma_{ri-1} = \sigma_{ri} \\ &u_{i-1} = u_i \\ &r = b \quad \sigma_{rN} = -p_b \\ &\quad (i = 2, 3, 4, \ldots, N) \end{aligned} \tag{24}$$

where

$$r_i = a + (i-1)\Delta r$$
$$\Delta r = \frac{b-a}{N} \tag{25}$$

Taking $N = 100$, the radial displacement and stresses obtained with the above numerical method were also depicted in Fig. 2 to Fig. 4 where the superscript A indicates the accurate analytical solution and L stands for the numerical solution.

In order to indicate the differences of stresses and radial displacement from varying and constant material properties, the disks subjected to the same loads but with constant material properties were also considered here. Young's modulus and Poisson's ratio of the disks with constant material properties were taken to be the average values of those at the inner and outer surfaces of the disks with varying Young's modulus and Poisson's ratio. The calculated stresses and radial displacement were depicted in Figs. 2, 3 and 4 as well, and indicated by the superscript C.

From these figures, it can be observed that both approaches give very agreeable results of the radial displacement and radial and circumferential stresses in the disks. For annular disks under the external pressure $p_b = 100$ MPa, the errors of the radial displacement, and radial and circumferential stresses are 0.43%, 0% and 0.05% at the inner radius, and 1.08%, 0% and 0.27% at the outer radius. For annular disks subjected to the internal pressure $p_a = 100$ MPa, these errors become 0.42%, 0% and 0.04% at the inner radius, and 0.33%, 0% and 1.17% at the outer radius. For annular disks under both internal pressure $p_a = 100$ MPa and external pressure $p_b = 150$ MPa, these figures are changed to 0.43%, 0% and 0.06% at the inner radius, and 1.52%, 0%, and 0.05% at the outer radius.

When circular disks are subjected to the external pressure $p_b = 100$ MPa, the radial displacement, and radial and circumferential stresses in the disks are compressive. The compressive radial stress is the smallest at the inner radius, and the largest at the outer radius. Oppositely, the compressive circumferential stress is the biggest at the inner radius, and the smallest at the outer radius. Along the radius direction, the variation of the radial displacement is not obvious. When the internal pressure $p_a = 100$ MPa is applied, the radial displacement and circumferential stress become tensile while the radial stress still keeps compressive. Same as before, the radial displacement does not change much along the radial direction. Quite differently, both the compressive radial stress and tensile circumferential stress decrease from the inner radius to outer radius. When the disks are subjected to an internal pressure $p_a = 100$ MPa and an external pressure $p_b = 150$ MPa, the variation of the radial displacement becomes obvious. Especially in the region near the outer radius, the compressive radial displacement increases more and more quickly. Similar to those of the disks under the external pressure, the compressive radial stress rises from the inner radius to outer radius. However, the compressive circumferential stress drops along this direction. For all three cases, the maximum absolute values and variations of the circumferential stress along the radial direction are the largest among all stresses and radial displacement.

The differences of stresses and radial displacement from the constant and varying material properties are obvious for all three cases. Among them, the differences of the circumferential stress are the largest and those of the radial stress are the smallest. Compared to the circumferential stress from the varying material properties, the error of the circumferential stress from the constant material properties is 32.28%, 33.28% and 31.43% at the inner radius of the disks subjected to external pressure, internal pressure, and both external and internal pressure, respectively, and changes to 103.06%, 168.82% and 79.15% at the outer radius of the disks. These data indicate that using the formulae of constant material properties to calculate stresses and deformations in the disks with varying material properties will introduce very large errors.

At last, we examine how the radius ratio of circular disks affects deformations and stresses. The outer radius kept $b = 0.5$ m unchanged. The inner radius was taken to be $a = 0.2$ m and $a = 0.4$ m, respectively. The above three cases of the load were considered and the obtained radial displacement, and radial and circumferential stresses were depicted in Fig. 5 to Fig. 7 where the numbers 1, 2, and 3 indicate the disks with an inner radius $a = 0.2$ m and 4, 5, and 6 stand for the disks with an inner radius $a = 0.4$ m.

It can be observed from these figures that the radius ratio does not influence the radial stress obviously. However, it changes the radial displacement and circumferential stress greatly for all three cases. When the inner radius varies from $a = 0.4$ m to $a = 0.2$ m, i.e., the radius ratio b/a is raised from 1.25 to 2.5, the radial displacement is increased from -1.2451,

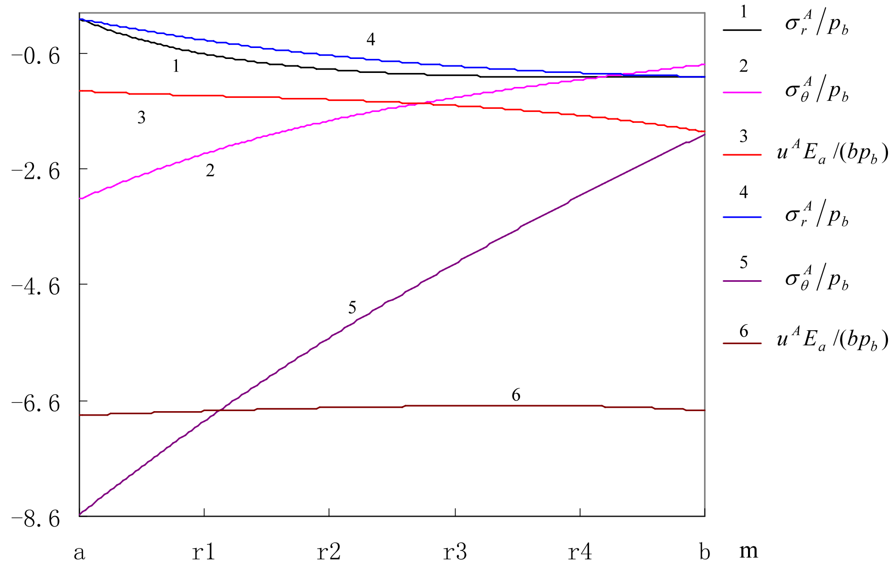

Fig. 5 Stresses and displacement in annular disks under external pressure from different radius ratios

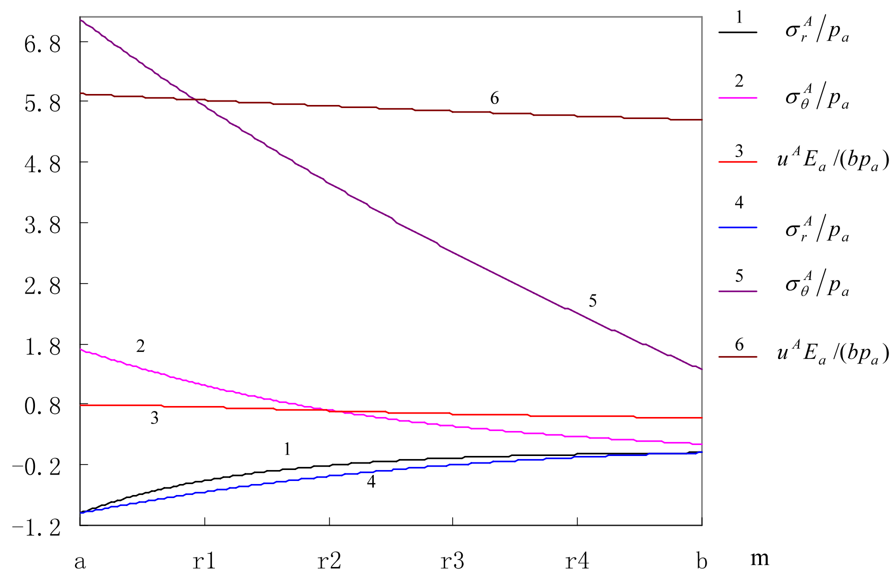

Fig. 6 Stresses and displacement in annular disks under internal pressure from different radius ratios

0.7821 and −0.7243 to −6.856, 5.9282 and −2.9039 at the inner radius, and from −1.9597, 0.5781 and −1.5743 to −6.7788, 5.4954 and −3.1147 at the outer radius for the three cases, respectively. And the maximum circumferential stress occurs at the inner radius and is changed from −3.1129 to −8.5701 for the external pressure $p_b = 100$ MPa, from 1.703 to 7.1603 for the internal pressure $p_a = 100$ MPa, and from −1.9775 to −3.7966 for the external pressure $p_b = 150$ MPa and internal pressure $p_a = 100$ MPa. These numbers indicate that a large radius ratio causes a large radial displacement and a large circumferential stress. In addition, the variation of the circumferential stress becomes more significant along the radial direction for all three cases when the radius ratio is increased.

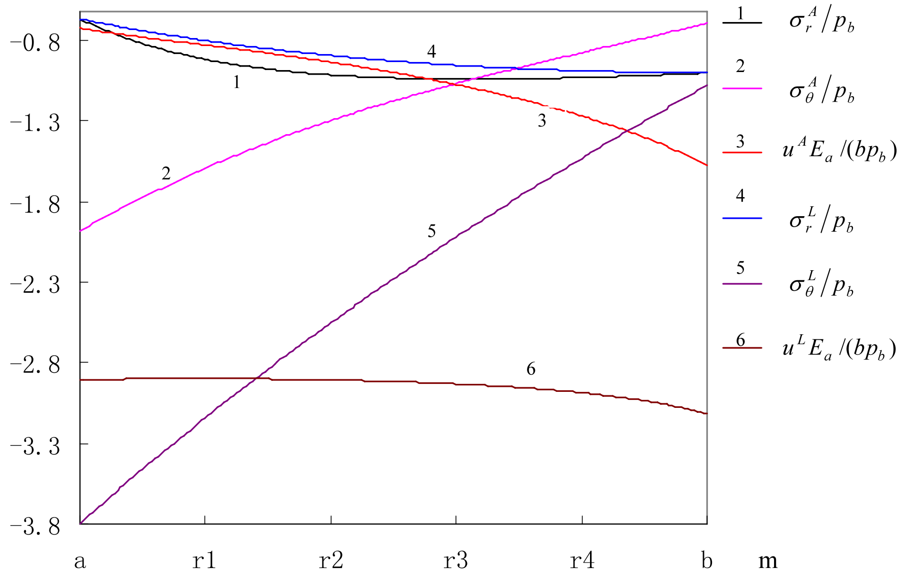

Fig. 7 Stresses and displacement in annular disks under internal and external pressure from different radius ratios

4 Conclusions

An accurate analytical solution has been presented to investigate deformations and stresses in circular disks subjected to internal and/or external pressure made of functionally graded materials. The comparison with the corresponding numerical solution indicates that the results calculated by both approaches are in excellent agreement, indicating the proposed analytical solution is correct. The proposed method is used to determine distributions of the radial displacement and stresses in circular disks subjected to external pressure, internal pressure, and both internal and external pressure, respectively. The effects of the radius ratio on the deformation and stresses were also investigated. The bigger the radius ratio, the larger the radial displacement and circumferential stress.

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