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The Rayleigh Stokes problem for rectangular pipe in Maxwell and second grade fluid

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Abstract The Stokes and Rayleigh Stokes problems for a flat plate in a viscoelastic fluid has recently been generalized to an edge and an exact analytical solution is obtained. In this paper, the edge problem has further been extended to the case of a rectangular pipe and exact solutions are obtained for Maxwell and second grade fluids. Also, the flow due to an oscillating edge problem is extended to generalized Maxwell fluid.

Keywords Maxwell fluid · Second grade fluid · Rayleigh Stokes problem · Flow in rectangular pipe · Exact solutions

1 Introduction

Much attention has been attached to flat plate and edge problems because of their practical importance and attempts have been made to find the exact analytical solutions in various model. Stokes [1] solved the problem for a plate in a viscous fluid in 1851. Rajagopal [2] considered flows in second grade fluid and Erdogan [3] studied the flat plate set in motion in third grade fluid. Further, investigations are made by Rajagopal [4] and Hayat et al. [5] for an Oldroyd-B model of viscoelastic fluid. In continuation Zierep [6] presented the solution for a plate and an edge in Newtonian fluid in 1979. Since the non-Newtonian fluids are generally recognized more appropriate in industrial applications, the extension of plate and edge problem made an immediate headway in non-Newtonian fluids and a number of papers appeared in this direction [7–9]. Fetecau and Zierep [7] presented exact solution of a plate and an edge in second grade fluid which was then extended to a heated plate by Fetecau and Fetecau [10]. The solution for Maxwellian type of fluid was also given by [8].

The fluids appearing in many fields such as in food industry, bio-engineering, mixtures of different materials, amorphous polymers and organic glasses show strong non-Newtonian characteristics and are treated as viscoelastic fluids. In these fluids the viscosity function varies non-linearly with the shear rate. Many mathematical models have been proposed to fit well with the experimental observations. The simplest model for the rheological effects of viscoelastic fluids is the Maxwell model. But this model has its inadequacies in that it predicts a linear relationship between shear rate and shear stress which is likely in low strain rates. It is now known that a unidirectional Maxwell model modified with the fractional calculus has found to explain experimental data for viscoelastic fluids.

The stability behaviors of the constitutive equations generalizing the fractional derivative models

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have been given [11] with a caution that care should be taken to adopt fractional derivative models in rheology with a suggestions that these models are successful to study unidirectional behavior only under initial conditions.

In this paper the edge problem is further generalized to a rectangular pipe that is confined by the four walls and Stokes-I problem for Maxwell and second grade fluid is discussed. Thus extending the class of problems amiable to exact analytic solution for standard viscoelastic models. This problem has more practical applications in industry and to the best of our knowledge first for a rectangular pipe. Further, keeping in mind the importance of generalized models and for better understanding and better rapport with the empirical results, Stokes-II problem for an edge making an arbitrary periodic oscillations has been presented for generalized Maxwell fluid. Thus rectangular pipe and an edge problems have been discussed for standard viscoelastic fluid and generalized Maxwell fluid respectively. One such problem of harmonic oscillations for generalized Maxwell fluid has been addressed by Wenchang et al. [12]. The flow induced by an oscillating plate has been addressed by Khalid and Vafai [13] by taking slip boundary condition. These problems will further explain viscoelastic fluids and will be a step forward in obtaining exact solutions for more realistic generalized fluid models.

2 Flow of a Maxwell fluid

Suppose that Maxwell fluid occupies the space between a rectangular pipe ($-\infty < x < \infty$, $0 < y < L$, $0 < z < H$). The pipe is initially at rest and suddenly starts moving with a constant velocity U_0 in the *x-direction*. The unsteady governing equation for Maxwell fluid in two dimensions is given by

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \lambda \frac{\partial^2 u}{\partial t^2} \quad (y, z > 0), \tag{1}$$

where u is the x-component of velocity, ν is the kinematic viscosity and λ is the Maxwellian parameter.

The initial and boundary conditions of the problem are

$$u(y, z, 0) = 0, \quad y, z > 0, \tag{2}$$

$$u(0, z, t) = u(y, 0, t) = U_0, \tag{3}$$

$$u(L, z, t) = u(y, H, t) = U_0. \tag{4}$$

2.1 Method of solution

Taking Laplace transform of (1) to (4), gives

$$\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} + \left(\frac{\lambda s^2 - s}{\nu}\right) F = 0, \tag{5}$$

$$F(0, z, s) = F(y, 0, s) = \frac{U_0}{s}, \tag{6}$$

$$F(L, z, s) = F(y, H, s) = \frac{U_0}{s}, \tag{7}$$

where F is transform of u and s is the transform variable. We note that F satisfies homogeneous equation with nonhomogeneous boundary conditions at the four faces of the pipe. For that we construct four independent problems satisfying these homogeneous conditions and one nonhomogeneous boundary condition. We take

$$F(y, z, s) = F_1(y, z, s) + F_2(y, z, s) + F_3(y, z, s) + F_4(y, z, s), \tag{8}$$

where F_i $(i = 1, 2, 3, 4)$ satisfies (5) and the boundary conditions for F_1 will be

$$F_1(0, z, s) = \frac{U_0}{s}, \qquad F_1(y, 0, s) = F_1(L, z, s) = F_1(y, H, s) = 0. \tag{9}$$

Making use of separation of variables, the solution for each F_i can be expressed as

$$F_1(y, z, s) = \sum_{m=1}^{\infty} \frac{2(1-(-1)^m)\sin\frac{m\pi}{H}z \sinh\sqrt{\frac{\lambda s^2 - s}{\nu} + (\frac{m\pi}{H})^2}(y - L)}{m\pi s \sinh\sqrt{\frac{\lambda s^2 - s}{\nu} + (\frac{m\pi}{H})^2}L}, \tag{10}$$

$$F_2(y, z, s) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)\sin\frac{n\pi}{L}y \sinh\sqrt{\frac{\lambda s^2 - s}{\nu} + (\frac{n\pi}{L})^2}(z - H)}{n\pi s \sinh\sqrt{\frac{\lambda s^2 - s}{\nu} + (\frac{n\pi}{L})^2}H}, \tag{11}$$

$$F_3(y,z,s)=\sum_{p=1}^{\infty}\frac{2(1-(-1)^p)\sin\frac{p\pi}{H}z\sinh\sqrt{\frac{\lambda s^2-s}{\nu}+(\frac{p\pi}{H})^2}y}{p\pi s\sinh\sqrt{\frac{\lambda s^2-s}{\nu}+(\frac{p\pi}{H})^2}L}, \tag{12}$$

$$F_4(y,z,s)=\sum_{q=1}^{\infty}\frac{2(1-(-1)^q)\sin\frac{q\pi}{L}y\sinh\sqrt{\frac{\lambda s^2-s}{\nu}+(\frac{q\pi}{L})^2}z}{q\pi s\sinh\sqrt{\frac{\lambda s^2-s}{\nu}+(\frac{q\pi}{L})^2}H}. \tag{13}$$

The inverse Laplace transform now yields

$$u(y,z,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F_i(y,z,s)e^{st}ds = \sum_{i=1}^{4}u_i(y,z,t), \tag{14}$$

where, say $u_1(y,z,t)$ is the inverse Laplace transform of $F_1(y,z,s)$ given by (10). The integral in (14) can be evaluated using residue theorem of complex analysis.

The residue at simple pole $s=0$ in the expression of F_1 (see (10)) is given by

$$\operatorname{Re}s(s=0) = \sum_{m=1}^{\infty}\frac{2(1-(-1)^m)\sin\frac{m\pi}{H}z\sinh(\frac{m\pi}{H})(y-L)}{m\pi\sinh(\frac{m\pi}{H})L}. \tag{15}$$

The other singular points are the roots of the transcendental equation

$$\sinh\sqrt{\frac{\lambda s^2-s}{\nu}+\left(\frac{m\pi}{H}\right)^2}=0. \tag{16}$$

If $s_{1M}, s_{2M}, M=1,2,3,4,\dots$ are the zeros of (16), then

$$s_{1M}=\frac{1+\sqrt{1-4\nu\lambda(\delta^2+(\frac{m\pi}{H})^2)}}{2\lambda}, \tag{17}$$

$$s_{2M}=\frac{1-\sqrt{1-4\nu\lambda(\delta^2+(\frac{m\pi}{H})^2)}}{2\lambda}. \tag{18}$$

The residues related to poles s_{1M}, s_{2M} are

$$\operatorname{Re}s(s_{1M})=\sum_{M=1}^{\infty}\sum_{m=1}^{\infty}\frac{4(1-(-1)^m)\sqrt{\frac{\lambda s_{1M}^2-s_{1M}}{\nu}+(\frac{m\pi}{H})^2}\sin\frac{m\pi}{H}z\sin\delta(y-L)}{m\pi Ls_{1M}(\frac{2\lambda s_{1M}-1}{\nu})\sinh\delta L}e^{s_{1M}t}, \tag{19}$$

$$\operatorname{Re}s(s_{2M})=\sum_{M=1}^{\infty}\sum_{m=1}^{\infty}\frac{4(1-(-1)^m)\sqrt{\frac{\lambda s_{2M}^2-s_{2M}}{\nu}+(\frac{m\pi}{H})^2}\sin\frac{m\pi}{H}z\sin\delta(y-L)}{m\pi Ls_{1M}(\frac{2\lambda s_{2M}-1}{\nu})\sinh\delta L}e^{s_{2M}t}. \tag{20}$$

Using the residue theorem, $u_1(y,z,t)$ is finally expressed as

$$\begin{aligned}u_1(y,z,t)&=\sum_{m=1}^{\infty}\frac{2(1-(-1)^m)\sin\frac{m\pi}{H}z\sinh(\frac{m\pi}{H})(y-L)}{m\pi\sinh(\frac{m\pi}{H})L}\\&\quad+\sum_{M=1}^{\infty}\sum_{m=1}^{\infty}\frac{4(1-(-1)^m)\sqrt{\frac{\lambda s_{1M}^2-s_{1M}}{\nu}+(\frac{m\pi}{H})^2}\sin\frac{m\pi}{H}z\sin\delta(y-L)}{m\pi Ls_{1M}(\frac{2\lambda s_{1M}-1}{\nu})\sin\delta L}e^{s_{1M}t}\\&\quad+\sum_{M=1}^{\infty}\sum_{m=1}^{\infty}\frac{4(1-(-1)^m)\sqrt{\frac{\lambda s_{2M}^2-s_{2M}}{\nu}+(\frac{m\pi}{H})^2}\sin\frac{m\pi}{H}z\sin\delta(y-L)}{m\pi Ls_{1M}(\frac{2\lambda s_{2M}-1}{\nu})\sin\delta L}e^{s_{2M}t}.\end{aligned} \tag{21}$$

Employing the same procedure as for u_1 the solutions for $u_i(y,z,t)$ $(i=2,3,4)$ can be written as:

$$u_2(y,z,t)=\sum_{n=1}^{\infty}\frac{2((-1)^n-1)\sin\frac{n\pi}{L}y\sinh(\frac{n\pi}{L})(z-H)}{n\pi\sinh(\frac{n\pi}{L})H}$$
$$+\sum_{N=1}^{\infty}\sum_{n=1}^{\infty}\frac{4((-1)^n-1)\sqrt{\frac{\lambda s_{1N}^2-s_{1N}}{\nu}+(\frac{n\pi}{L})^2}\sin\frac{n\pi}{L}y\sin\delta_1(z-H)}{n\pi H s_{1N}(\frac{2\lambda s_{1N}-1}{\nu})\sin\delta_1 H}e^{s_{1N}t}$$
$$+\sum_{N=1}^{\infty}\sum_{n=1}^{\infty}\frac{4((-1)^n-1)\sqrt{\frac{\lambda s_{2N}^2-s_{2N}}{\nu}+(\frac{n\pi}{L})^2}\sin\frac{n\pi}{L}y\sin\delta_1(z-H)}{n\pi H s_{2N}(\frac{2\lambda s_{2N}-1}{\nu})\sin\delta_1 H}e^{s_{2N}t}, \qquad (22)$$

$$u_3(y,z,t)=\sum_{p=1}^{\infty}\frac{2((-1)^p-1)\sin\frac{p\pi}{H}z\sinh(\frac{p\pi}{H})y}{p\pi\sinh(\frac{p\pi}{H})L}$$
$$+\sum_{P=1}^{\infty}\sum_{p=1}^{\infty}\frac{4((-1)^p-1)\sqrt{\frac{\lambda s_{1P}^2-s_{1P}}{\nu}+(\frac{p\pi}{H})^2}\sin\frac{p\pi}{H}z\sin\delta_2 y}{p\pi L s_{1P}(\frac{2\lambda s_{1P}-1}{\nu})\sin\delta_2 L}e^{s_{1P}t}$$
$$+\sum_{P=1}^{\infty}\sum_{p=1}^{\infty}\frac{4((-1)^p-1)\sqrt{\frac{\lambda s_{2P}^2-s_{2P}}{\nu}+(\frac{p\pi}{H})^2}\sin\frac{p\pi}{H}z\sin\delta_2 y}{p\pi L s_{2P}(\frac{2\lambda s_{2P}-1}{\nu})\sin\delta_2 L}e^{s_{2P}t}, \qquad (23)$$

$$u_4(y,z,t)=\sum_{q=1}^{\infty}\frac{2((-1)^q-1)\sin\frac{q\pi}{L}y\sinh(\frac{q\pi}{L})z}{q\pi\sinh(\frac{q\pi}{L})H}$$
$$+\sum_{Q=1}^{\infty}\sum_{q=1}^{\infty}\frac{4((-1)^q-1)\sqrt{\frac{\lambda s_{1Q}^2-s_{1Q}}{\nu}+(\frac{q\pi}{L})^2}\sin\frac{q\pi}{L}y\sin\delta_3 z}{q\pi H s_{1Q}(\frac{2\lambda s_{1Q}-1}{\nu})\sin\delta_3 H}e^{s_{1Q}t}$$
$$+\sum_{Q=1}^{\infty}\sum_{q=1}^{\infty}\frac{4((-1)^q-1)\sqrt{\frac{\lambda s_{2Q}^2-s_{2Q}}{\nu}+(\frac{q\pi}{L})^2}\sin\frac{q\pi}{L}y\sin\delta_3 z}{q\pi H s_{2Q}(\frac{2\lambda s_{2Q}-1}{\nu})\sin\delta_3 H}e^{s_{2Q}t} \qquad (24)$$

in which

$$s_{1N}=\frac{1+\sqrt{1-4\nu\lambda(\delta_1^2+(\frac{n\pi}{L})^2)}}{2\lambda}, \qquad (25)$$

$$s_{2N}=\frac{1-\sqrt{1-4\nu\lambda(\delta_1^2+(\frac{n\pi}{L})^2)}}{2\lambda}, \qquad (26)$$

$$s_{1P}=\frac{1+\sqrt{1-4\nu\lambda(\delta_2^2+(\frac{p\pi}{H})^2)}}{2\lambda}, \qquad (27)$$

$$s_{2P}=\frac{1-\sqrt{1-4\nu\lambda(\delta_2^2+(\frac{p\pi}{H})^2)}}{2\lambda}, \qquad (28)$$

$$s_{1Q}=\frac{1+\sqrt{1-4\nu\lambda(\delta_3^2+(\frac{q\pi}{H})^2)}}{2\lambda}, \qquad (29)$$

$$s_{2Q}=\frac{1-\sqrt{1-4\nu\lambda(\delta_3^2+(\frac{q\pi}{H})^2)}}{2\lambda}, \qquad (30)$$

$\delta_1, \delta_2, \delta_3$ are roots of the transcendental equations appearing in the equation for u_i $(i=2,3,4)$.

The final velocity field can now be expressed as

$$u(y,z,t)=\sum_{i=1}^{4}u_i(y,z,t), \qquad (31)$$

where u_i $(i=2,3,4)$ are given by (21)–(24).

3 Flow of a second grade fluid

In this, we discuss the rectangular pipe problem for second grade fluid. The governing equation as given by [7]

$$\frac{\partial u}{\partial t} = \left(\nu + \frac{\alpha_1}{\rho}\frac{\partial}{\partial t}\right)\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{32}$$

where α_1 is the material parameter of second grade fluid and ρ is the density of the fluid. To solve (32) subject to initial and boundary conditions (2)–(4), we use the same procedure as for the Maxwell fluid and the solution is finally written as

$$\begin{aligned} u(y,z,t) &= \sum_{m_1=1}^{\infty} \frac{2(1-(-1)^{m_1})\sin\frac{m_1\pi}{H}z\sinh(\frac{m_1\pi}{H})(y-L)}{m_1\pi\sinh(\frac{m_1\pi}{H})L} + \sum_{n_1=1}^{\infty} \frac{2((-1)^{n_1}-1)\sin\frac{n_1\pi}{L}y\sinh(\frac{n_1\pi}{L})(z-H)}{n_1\pi\sinh(\frac{n_1\pi}{L})H} \\ &+ \sum_{p_1=1}^{\infty} \frac{2((-1)^{p_1}-1)\sin\frac{p_1\pi}{H}z\sinh(\frac{p_1\pi}{H})y}{p_1\pi\sinh(\frac{p_1\pi}{H})L} + \sum_{q_1=1}^{\infty} \frac{2((-1)^{q_1}-1)\sin\frac{q_1\pi}{L}y\sinh(\frac{q_1\pi}{L})z}{q_1\pi\sinh(\frac{q_1\pi}{L})H} \\ &+ \sum_{M_1=1}^{\infty}\sum_{m_1=1}^{\infty} \frac{4(1-(-1)^{m_1})(\nu+\frac{\alpha_1}{\rho}\tilde{s}_{1M})^2\sqrt{\frac{\tilde{s}_{1M}}{\nu+\frac{\alpha_1}{\rho}\tilde{s}_{1M}}+(\frac{m_1\pi}{H})^2}\sin\frac{m_1\pi}{H}z\sin\delta_4(y-L)}{m_1\pi L\tilde{s}_{1M}\sin\delta_4 L} e^{\tilde{s}_{1M}t} \\ &+ \sum_{N_1=1}^{\infty}\sum_{n_1=1}^{\infty} \frac{4((-1)^{n_1}-1)(\nu+\frac{\alpha_1}{\rho}\tilde{s}_{1N})^2\sqrt{\frac{\tilde{s}_{1N}}{\nu+\frac{\alpha_1}{\rho}\tilde{s}_{1N}}+(\frac{n_1\pi}{L})^2}\sin\frac{n_1\pi}{L}y\sin\delta_5(z-H)}{n_1\pi H\tilde{s}_{1N}\sin\delta_5 H} e^{\tilde{s}_{1N}t} \\ &+ \sum_{P_1=1}^{\infty}\sum_{p_1=1}^{\infty} \frac{4((-1)^{p_1}-1)(\nu+\frac{\alpha_1}{\rho}\tilde{s}_{1P})^2\sqrt{\frac{\tilde{s}_{1P}}{\nu+\frac{\alpha_1}{\rho}\tilde{s}_{1P}}+(\frac{p_1\pi}{H})^2}\sin\frac{p_1\pi}{H}z\sin\delta_6 y}{p_1\pi L\tilde{s}_{1P}\sin\delta_6 L} e^{\tilde{s}_{1P}t} \\ &+ \sum_{Q_1=1}^{\infty}\sum_{q_1=1}^{\infty} \frac{4((-1)^{q_1}-1)(\nu+\frac{\alpha_1}{\rho}\tilde{s}_{1Q})^2\sqrt{\frac{\tilde{s}_{1Q}}{\nu+\frac{\alpha_1}{\rho}\tilde{s}_{1Q}}+(\frac{q_1\pi}{L})^2}\sin\frac{q_1\pi}{L}y\sin\delta_7 z}{q_1\pi H\tilde{s}_{1Q}\sin\delta_7 H} e^{\tilde{s}_{1Q}t}, \end{aligned} \tag{33}$$

where

$$\tilde{s}_{1M} = -\nu\left[\frac{\delta_4^2 + (\frac{m_1\pi}{H})^2}{1+\frac{\alpha_1}{\rho}(\delta_4^2+(\frac{m_1\pi}{H})^2)}\right], \tag{34}$$

$$\tilde{s}_{1N} = -\nu\left[\frac{\delta_5^2 + (\frac{n_1\pi}{L})^2}{1+\frac{\alpha_1}{\rho}(\delta_5^2+(\frac{n_1\pi}{L})^2)}\right], \tag{35}$$

$$\tilde{s}_{1P} = -\nu\left[\frac{\delta_6^2 + (\frac{p_1\pi}{H})^2}{1+\frac{\alpha_1}{\rho}(\delta_6^2+(\frac{p_1\pi}{H})^2)}\right], \tag{36}$$

$$\tilde{s}_{1Q} = -\nu\left[\frac{\delta_7^2 + (\frac{q_1\pi}{L})^2}{1+\frac{\alpha_1}{\rho}(\delta_7^2+(\frac{q_1\pi}{L})^2)}\right], \tag{37}$$

$\delta_4, \delta_5, \delta_6, \delta_7$ satisfies the roots of transcendental equation.

4 Flow of a generalized Maxwell fluid

In this section we consider a generalized Maxwell fluid which occupies the space on the rectangular edge defined by $-\infty < x, \infty,\ y \geq 0,\ z \geq 0$. The flow is assumed to be generated by the periodic oscillations of the edge. The unsteady governing equation in two dimensions with appropriate boundary conditions is expressed as

$$\rho\frac{\partial u}{\partial t}+\rho\lambda^{\gamma}\frac{\partial^{\gamma+1}u}{\partial t^{\gamma+1}}=G\lambda^{\beta}\frac{\partial^{\beta-1}}{\partial t^{\beta-1}}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \quad (38)$$

$$u(0,z,t)=u(y,0,t)=U_0 f(t), \quad (39)$$

$$u(y,z,t)\to 0, \quad \text{as } y^2+z^2\to\infty, \quad (40)$$

where $f(t)$ are general periodic oscillations. The function $f(t)$ being periodic can be expressed as complex Fourier Series $f(t)=\sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 t}$, where the Fourier coefficient a_k are given by

$$a_k=\frac{1}{T_0}\int_{T_0} f(t)e^{-ik\omega_0 t}dt, \quad (41)$$

with non zero fundamental frequency $\omega_0=\frac{2\pi}{T_0}$. Here $\lambda=\frac{\mu}{G}$ is relaxation time, G is the shear modulus, μ is the constant coefficient of viscosity and γ and β are fractional calculus parameters such that $0\le\gamma\le\beta\le 1$. For $\gamma>\beta$ the relaxation fraction is increasing, which is general not responsible [12], and has require that $\gamma\le\beta$. It should be noted that model includes the ordinary Maxwell model as a special case for $\alpha=\beta=1$ and to the Navier-Stokes model when $\alpha=0$, $\beta=1$.

4.1 Solution of the problem

The temporal Fourier transform pair is defined by

$$\Psi(y,z,\omega)=\int_{-\infty}^{\infty} u(y,z,t)e^{-i\omega t}dt, \quad (42)$$

$$u(y,z,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(y,z,\omega)e^{i\omega t}d\omega, \quad (43)$$

ω being the temporal frequency. We can obtain the Fourier transform of the fractional derivative [14]

$$\int_{-\infty}^{\infty} D_t^{\beta}[u(y,z,\omega)]e^{-i\omega t}dt=(i\omega)^{\beta}\Psi(y,z,\omega), \quad (44)$$

where

$$(i\omega)^{\beta}=|\omega|^{\beta}e^{i\beta\pi/2\,\mathrm{sign}\,\omega}=|\omega|^{\beta}\left(\cos\frac{\beta\pi}{2}+i\,\mathrm{sign}\,\omega\sin\frac{\beta\pi}{2}\right). \quad (45)$$

The problem gives by (38) to (40), after taking the Fourier transform with respect to t takes the form

$$\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}-(a+ib)^2\Psi(y,z,\omega)=0, \quad (46)$$

$$\Psi(0,z,\omega)=\Psi(y,0,\omega)=U_0\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0), \quad (47)$$

$$F\to 0, \quad \text{as } y^2+z^2\to\infty \quad (48)$$

in which

$$(a+ib)^2=\frac{\rho+\rho\lambda^{\gamma}|\omega|^{\gamma}(\cos\frac{\gamma\pi}{2}+i\,\mathrm{sign}\,\omega\sin\frac{\gamma\pi}{2})}{G\lambda^{\beta-1}|\omega|^{\beta-2}(\cos\frac{(\beta-2)\pi}{2}+i\,\mathrm{sign}\,\omega\sin\frac{(\beta-2)\pi}{2})}. \quad (49)$$

Taking Fourier Sine transform of (46) and conditions (47) and (48) with respect to y, we obtain

$$\frac{d^2\bar{\Psi}}{dz^2}-(a+ib)^2\bar{\Psi}(y,z,\omega)=-U_0\sqrt{\frac{2}{\pi}}\xi\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0), \quad (50)$$

$$\bar{\Psi}(\xi,z,\omega)=U_0\sum_{k=-\infty}^{\infty}\frac{2\pi a_k\delta(\omega-k\omega_0)}{\xi}, \quad \text{at } z=0, \quad (51)$$

$$\bar{\Psi}\to 0, \quad \text{as } z\to\infty.$$

The solution of (50) satisfying the boundary conditions (51) is

$$\bar{\Psi}=U_0\sum_{k=-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{2\pi a_k\delta(\omega-k\omega_0)\xi}{(a+ib)^2+\xi^2}+U_0\sum_{k=-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{2\pi a_k\delta(\omega-k\omega_0)e^{-z\sqrt{(a+ib)^2+\xi^2}}}{\{(a+ib)^2+\xi^2\}\xi}. \quad (52)$$

Fourier Sine inversion of above equation yields

$$\Psi = U_0 \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) e^{-(a+ib)y}$$
$$+ U_0 \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} 2(-1)^{n+1} \pi a_k \delta(\omega - k\omega_0) \frac{(a+ib)^2}{n!} (a+ib)^{2n} y z^n$$
$$\times \left[\frac{1}{\text{Gamma}(1-n)} \left(\frac{\sqrt{\pi}\,\text{Gamma}(\frac{1}{2}-n)}{(a+ib)} \text{HypergeometricPFQ}\left[\frac{1}{2}, \left(\frac{3}{2}, \frac{1}{2}+n\right), \frac{(a+ib)^2 y^2}{4}\right]\right)\right.$$
$$\left. - \frac{2y^{1-2n}\,\text{Gamma}(2n-2)}{(a+ib)^n} \text{HypergeometricPFQ}\left[(1-n), \left(\frac{3}{2}-n, 2-n\right), \frac{(a+ib)^2 y^2}{4}\right] \sin n\pi \right]. \quad (53)$$

Substituting (54) in (44) and then using the property of Delta function, we arrive at

$$u(y,z,t) = U_0 \sum_{k=-\infty}^{\infty} a_k e^{-m_k y + i(k\omega_0 t - n_k y)}$$
$$+ U_0 \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} a_k (-1)^n \frac{(m_k + i n_k)^2}{\pi n!} (m_k + i n_k)^{2n} y z^n e^{i\omega_0 k t}$$
$$\times \left[\frac{1}{\text{Gamma}(1-n)} \frac{\sqrt{\pi}\,\text{Gamma}(\frac{1}{2}-n)}{(m_k + i n_k)} \text{HypergeometricPFQ}\left[\frac{1}{2}, \left(\frac{3}{2}, \frac{1}{2}+n\right), \frac{(m_k + i n_k)^2 y^2}{4}\right]\right.$$
$$- \frac{2y^{1-2n}\,\text{Gamma}(2n-2)}{(m_k + i n_k)^n}$$
$$\left. \times \text{HypergeometricPFQ}\left[(1-n), \left(\frac{3}{2}-n, 2-n\right), \frac{(m_k + i n_k)^2 y^2}{4}\right] \sin n\pi \right], \quad (54)$$

where

$$m_k = \sqrt{\frac{\sqrt{L_r^2 + 4L_i^2} + L_r}{2}}, \quad (55)$$

$$n_k = \sqrt{\frac{\sqrt{L_r^2 + 4L_i^2} - L_r}{2}}, \quad (56)$$

$$L_r = \frac{\rho + \rho\lambda^\alpha |\omega k|^\alpha [\cos\frac{\alpha\pi}{2} \cos\frac{(\beta-2)\pi}{2} + (\operatorname{sign}\omega k)^2 \sin\frac{\alpha\pi}{2} \sin\frac{(\beta-2)\pi}{2}]}{G\lambda^{\beta-1} |\omega k|^{\beta-2} [(\cos\frac{(\beta-2)\pi}{2})^2 + (\operatorname{sign}\omega k \sin\frac{(\beta-2)\pi}{2})^2]},$$

$$L_i = \frac{\rho + \rho\lambda^\alpha |\omega|^\alpha [\operatorname{sign}\omega k \sin\frac{\alpha\pi}{2} \cos\frac{(\beta-2)\pi}{2} - \operatorname{sign}\omega k \cos\frac{\alpha\pi}{2} \sin\frac{(\beta-2)\pi}{2}]}{G\lambda^{\beta-1} |\omega k|^{\beta-2} [(\cos\frac{(\beta-2)\pi}{2})^2 + (\operatorname{sign}\omega k \sin\frac{(\beta-2)\pi}{2})^2]}.$$

Equation (55) gives the complete analytic solution for the velocity field due to general periodic oscillations. As a special case of this oscillation, the flow field for the different edge oscillations is obtained by an appropriate choice of the Fourier coefficients which give rise to the different edge oscillations. The periodic oscilla-

Table 1 Functions and their Fourier coefficients

Oscillations	Fourier coefficients
$f(t)$	a_k
(i) $e^{i\omega_0 t}$	$a_1 = 1$ and $a_k = 0 (k \neq 1)$,
(ii) $\cos\omega_0 t$	$a_1 = a_{-1} = \frac{1}{2}$ and $a_k = 0$, otherwise,
(iii) $\sin\omega_0 t$	$a_1 = a_{-1} = \frac{1}{2i}$ and $a_k = 0$, otherwise,
(iv) $\begin{bmatrix} 1, \lvert t\rvert < T_1 \\ 0, T_1 < \lvert t\rvert < \frac{T_0}{2} \end{bmatrix}$	$a_0 = 2T_1/T_0$, $a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}$, for all $k \neq 0$,
(v) $\sum_{k=-\infty}^{\infty} \delta(t - kT_0)$	$a_k = \frac{1}{T\,0}$ for all k.

tions and their corresponding Fourier coefficients are given in Table 1.

The results in the above five cases can be obtained by using successively the appropriate Fourier coefficients in (54) and are given below

$$
\begin{aligned}
u_1(y,z,t) = {} & U_0 e^{-m_1 y + i(\omega_0 t - n_1 y)} \\
& + U_0 \sum_{n=0}^{\infty} (-1)^n \frac{(m_1 + i n_1)^2}{\pi n!} (m_1 + i n_1)^{2n} y z^n e^{i\omega_0 t} \\
& \times \left[\frac{1}{\text{Gamma}(1-n)} \frac{\sqrt{\pi}\,\text{Gamma}(\frac{1}{2} - n)}{(m_k + i n_k)} \text{HypergeometricPFQ}\left[\frac{1}{2}, \left(\frac{3}{2}, \frac{1}{2} + n\right), \frac{(m_1 + i n_1)^2 y^2}{4}\right] \right. \\
& - \frac{2 y^{1-2n}\,\text{Gamma}(2n-2)}{(m_1 + i n_1)^n} \\
& \left. \times \text{HypergeometricPFQ}\left[(1-n), \left(\frac{3}{2} - n, 2 - n\right), \frac{(m_1 + i n_1)^2 y^2}{4}\right] \sin n\pi \right],
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
u_2(y,z,t) = {} & \frac{U_0}{2} [e^{-m_1 y + i(\omega_0 t - n_1 y)} + e^{-m_{-1} y - i(\omega_0 t - n_{-1} y)}] \\
& + \frac{U_0}{2} \sum_{n=0}^{\infty} (-1)^n \left[\frac{(m_1 + i n_1)^2}{\pi n!} (m_1 + i n_1)^{2n} y z^n e^{i\omega_0 t} \right. \\
& \times \left[\frac{1}{\text{Gamma}(1-n)} \frac{\sqrt{\pi}\,\text{Gamma}(\frac{1}{2} - n)}{(m_1 + i n_1)} \text{HypergeometricPFQ}\left[\frac{1}{2}, \left(\frac{3}{2}, \frac{1}{2} + n\right), \frac{(m_1 + i n_1)^2 y^2}{4}\right] \right. \\
& \left. - \frac{2 y^{1-2n}\,\text{Gamma}(2n-2)}{(m_1 + i n_1)^n} \text{HypergeometricPFQ}\left[(1-n), \left(\frac{3}{2} - n, 2 - n\right), \frac{(m_1 + i n_1)^2 y^2}{4}\right] \sin n\pi \right] \\
& + \frac{(m_{-1} + i n_{-1})^2}{\pi n!} (m_{-1} + i n_{-1})^{2n} y z^n e^{-i\omega_0 t} \\
& \times \left[\frac{1}{\text{Gamma}(1-n)} \frac{\sqrt{\pi}\,\text{Gamma}(\frac{1}{2} - n)}{(m_{-1} + i n_{-1})} \text{HypergeometricPFQ}\left[\frac{1}{2}, \left(\frac{3}{2}, \frac{1}{2} + n\right), \frac{(m_{-1} + i n_{-1})^2 y^2}{4}\right] \right.
\end{aligned}
$$

$$- \frac{2y^{1-2n}\,\text{Gamma}(2n-2)}{(m_{-1}+in_{-1})^n}$$

$$\times \text{HypergeometricPFQ}\left[(1-n), \left(\frac{3}{2}-n, 2-n\right), \frac{(m_{-1}+in_{-1})^2 y^2}{4}\right] \sin n\pi \Bigg]\Bigg], \tag{58}$$

$$u_3(y,z,t) = U_0 e^{-m_1 y + i(\omega_0 t - n_1 y)}$$

$$+ \frac{U_0}{2i} \sum_{n=0}^{\infty} (-1)^n \Bigg[\frac{(m_1+in_1)^2}{\pi n!} (m_1+in_1)^{2n} y z^n e^{i\omega_0 t}$$

$$\times \left[\frac{1}{\text{Gamma}(1-n)} \frac{\sqrt{\pi}\,\text{Gamma}(\frac{1}{2}-n)}{(m_1+in_1)} \text{HypergeometricPFQ}\left[\frac{1}{2}, \left(\frac{3}{2}, \frac{1}{2}+n\right), \frac{(m_1+in_1)^2 y^2}{4}\right]\right.$$

$$\left. - \frac{2y^{1-2n}\,\text{Gamma}(2n-2)}{(m_1+in_1)^n} \text{HypergeometricPFQ}\left[(1-n), \left(\frac{3}{2}-n, 2-n\right), \frac{(m_1+in_1)^2 y^2}{4}\right] \sin n\pi \right]$$

$$- \frac{(m_{-1}+in_{-1})^2}{\pi n!} (m_{-1}+in_{-1})^{2n} y z^n e^{-i\omega_0 t}$$

$$\times \left[\frac{1}{\text{Gamma}(1-n)} \frac{\sqrt{\pi}\,\text{Gamma}(\frac{1}{2}-n)}{(m_{-1}+in_{-1})} \text{HypergeometricPFQ}\left[\frac{1}{2}, \left(\frac{3}{2}, \frac{1}{2}+n\right), \frac{(m_{-1}+in_{-1})^2 y^2}{4}\right]\right.$$

$$- \frac{2y^{1-2n}\,\text{Gamma}(2n-2)}{(m_{-1}+in_{-1})^n}$$

$$\times \text{HypergeometricPFQ}\left[(1-n), \left(\frac{3}{2}-n, 2-n\right), \frac{(m_{-1}+in_{-1})^2 y^2}{4}\right] \sin n\pi \Bigg]\Bigg], \tag{59}$$

$$u_4(y,z,t) = \frac{U_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\sin(k\omega_0 T_1)}{k} e^{-m_k y + i(k\omega_0 t - n_k y)},$$

$$+ \frac{U_0}{\pi} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{\sin(k\omega_0 T_1)}{k} (-1)^n \frac{(m_k+in_k)^2}{\pi n!} (m_k+in_k)^{2n} y z^n e^{i\omega_0 k t}$$

$$\times \left[\frac{1}{\text{Gamma}(1-n)} \frac{\sqrt{\pi}\,\text{Gamma}(\frac{1}{2}-n)}{(m_k+in_k)} \text{HypergeometricPFQ}\left[\frac{1}{2}, \left(\frac{3}{2}, \frac{1}{2}+n\right), \frac{(m_k+in_k)^2 y^2}{4}\right]\right.$$

$$- \frac{2y^{1-2n}\,\text{Gamma}(2n-2)}{(m_k+in_k)^n}$$

$$\times \text{HypergeometricPFQ}\left[(1-n), \left(\frac{3}{2}-n, 2-n\right), \frac{(m_k+in_k)^2 y^2}{4}\right] \sin n\pi \Bigg], \quad k \neq 0, \tag{60}$$

$$u_5(y,z,t) = \frac{U_0}{T_0} \sum_{k=-\infty}^{\infty} e^{-m_k^* y + i(2\pi k t/T_0 - n_k^* y)},$$

$$+ \frac{U_0}{T_0} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{(m_k^*+in_k^*)^2}{\pi n!} (m_k^*+in_k^*)^{2n} y z^n e^{2i\pi k t/T_0}$$

$$\times\left[\frac{1}{\text{Gamma}(1-n)}\frac{\sqrt{\pi}\,\text{Gamma}(\frac{1}{2}-n)}{(m_k^*+in_k^*)}\text{HypergeometricPFQ}\left[\frac{1}{2},\left(\frac{3}{2},\frac{1}{2}+n\right),\frac{(m_k^*+in_k^*)^2y^2}{4}\right]\right.$$
$$-\frac{2y^{1-2n}\,\text{Gamma}(2n-2)}{(m_k^*+in_k^*)^n}$$
$$\left.\times\,\text{HypergeometricPFQ}\left[(1-n),\left(\frac{3}{2}-n,2-n\right),\frac{(m_k^*+in_k^*)^2y^2}{4}\right]\sin n\pi\right],\quad k\neq 0 \tag{61}$$

where

$$m_k^* = m_k|\omega=\omega_0, \qquad n_k^* = n_k|\omega=\omega_0.$$

5 Conclusion

The analytic solutions of unsteady problems are very rare because of the complexity of the governing equations and nonlinearity of nature, even in the case of simple geometries such as single plate or disk in which the governing equations are one dimensional. In this work, we have studied the flow due to a rectangular pipe for non-Newtonian fluids such as Maxwell and second grade. The geometry requires the governing equations to be two dimensional (which are often difficult to handle) for which an exact analytical solution is presented. Further keeping in view the importance of generalized Maxwell fluid, which have better report with empirical results and the industrial applications of edges and finite plates, an exact solution is obtained for the flow due to an edge oscillating in a generalized Maxwell fluid. This may be remembered that this problem also contributes a two dimensional problem. The problem considered are not only important in finding in finding analytical solution of non-Newtonian fluids as such but have a great practical applications in industry.

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