Meccanica (2007) 42:365–374
DOI 10.1007/s11012-006-9050-6

A Julia set for the Kepler problem

Maria Dina Vivarelli

Received: 6 October 2005 / Accepted: 5 November 2006 / Published online: 27 April 2007

Abstract Following our previous works (M. D. Vivarelli, Celest. Mech. Dyn. Astr., 60:291–305, 1994; Meccanica, 35:55–67, 2000; The amazing **S**-code of the conic sections and the Kepler problem, Polipress, Milano, 2005) on the unified **S**-description of the family of confocal conic sections, we show how the unit circle, which reveals to play an exceptional role in the family, emerges naturally as the Julia set of a quadratic complex map which is strictly related to the regularization of the classical three-dimensional Kepler problem.

Keywords Kepler problem · Fractal geometry and dynamics · Conic sections · General mechanics

1 Introduction

It is well known that the orbits of the classical three-dimensional Kepler problem are conic sections. Precisely, in an inertial frame of the ordinary three-dimensional Euclidean space $R^3 - \{0\}$, the motion (Kepler problem) of a particle in a central, attractive, inverse-square law field is expressed by the Newtonian nonlinear differential equation

M. D. Vivarelli (✉)
Dipartimento di Matematica, F. Brioschi Politecnico di Milano, Piazza L. da Vinci 32, 20133 Milano, Italy
e-mail: marviv@mate.polimi.it

$$\ddot{\mathbf{x}} + \frac{K^2}{r^3}\mathbf{x} = \mathbf{0} \quad (\cdot \equiv \mathrm{d}/\mathrm{d}t,\ \mathbf{r} = |\ \mathbf{x}\ |),$$

where $\mathbf{x}$ represents the particle position vector with respect to the attractive centre and K is a dimensional constant.

The Keplerian orbits (belonging to the fixed plane through the attractive centre and orthogonal to the constant angular momentum vector per unit of mass $\mathbf{\Gamma}$) are conic sections with focus at the attractive centre and may be classified by their constant scalar total mechanical energy per unit of mass E

$$E < 0, \quad E = 0, \quad E > 0,$$

respectively, as

ellipse, parabola, hyperbola (one branch).

In this classification an *exceptional* role is played by the *circular orbit*, related to the particular negative value

$$E = -K^4(2\Gamma^2)^{-1},$$

which corresponds to the balance of the centrifugal force with the effective force.

On the other hand, since a comparison with the general equation of a conic section shows that the energy E of the particle is related to the eccentricity e of a conic section by

$e^2 - 1 = 2E\,\Gamma^2\,K^{-4}$,

the explicit existence of the *circular* orbit (corresponding to $e = 0$) turns out to be automatically related to the negative value $E = -K^4(2\Gamma^2)^{-1}$.

From the above, we notice that the e-classification appears as a refinement of the E-classification: it shows explicitly *all* the *four* types of the family of confocal orbits, namely

circle, ellipse, parabola, hyperbola

corresponding to

$e = 0, \quad 0 < e < 1, \qquad e = 1, \quad e > 1.$

Of course we may consider the *explicit* e-account of the circular shape due to the fact that the interpretation of mathematical results may depend very much on the form in which they are stated.

But, as we are going to show in this paper, the peculiar *exceptional* role of the circular orbit in the conic family is a foretaste of the fact that a circular orbit (or its paradigma, *the unit circle*) has a *hidden feature*.

This will be unravelled—through the new description of the conic sections via the vector **S** and its connection to the regularization of the physical Keplerian motion [11,12]—by relating the classical circular shape to a different geometry of shapes, that is to the fractal geometry of Mandelbrot. In the following two Sections, we review the main facts about the genetic vector **S** of the conic family and, although in section "The $K(\phi, r)$-Rotodilation map" we develop a map which coincides with the one we presented in Vivarelli [10], this is done from a different and autonomous standpoint.

2 The S-equation of the conic sections

Let us recall the plane polar definition of a conic section as the locus of points in a plane, all of which satisfy the well known peculiar *equation of constraint*

$$\mathbf{x} = \frac{p\,\boldsymbol{\rho}}{1 + e\cos\theta}, \qquad (1)$$

where **x** represents the position vector of each point in the standard plane polar coordinate system (r, θ), with the origin at the fixed point F (the *focus*), the two orthogonal unit vectors $\boldsymbol{\rho} = \boldsymbol{\rho}(\theta)$, $\boldsymbol{\tau} = \boldsymbol{\tau}(\theta)$ pointing in the direction of increasing r and θ and the reference line being $\theta = 0$ (see Fig. 1).

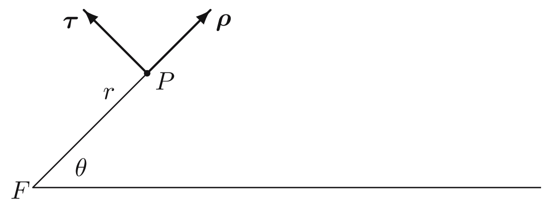

Fig. 1 The plane polar coordinate frame

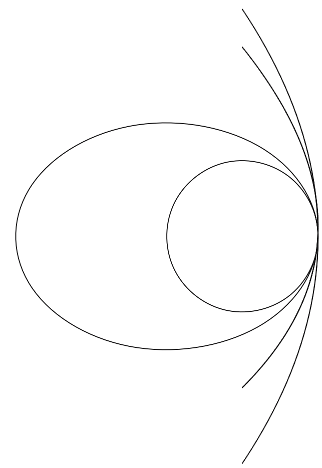

Fig. 2 The family of confocal conics

The constant scalar parameter $p > 0$ is called the *semi-latus rectum*. The constant scalar parameter e represents the *eccentricity* of the conic section: the *four types* of the family of the conics with common focus are shown in Fig. 2 (for $e = 0$ the conic is a circle, for $e < 1$ an ellipse, for $e = 1$ a parabola, for $e > 1$ an hyperbola, the *right branch* being excluded by the condition $r \geq 0$).

The reference line $\theta = 0$ may be characterized by the *eccentricity vector* $\mathbf{e} = e\,\boldsymbol{\rho}(0)$ which has the tail at the focus F and points towards the pericentre (the nearest point to F).

The new sum vector **S** and the vector **S**-equation.

In Vivarelli [11,12], we have presented a new approach to the conic sections which leads to a *new vector* **S** and to a *new vector equation* for the family of the conic sections.

We sketch here that the leading idea and the main results were obtained by the following:

(1) Reinvestigating the *general equation of constraint for a point in a plane*, commonly written in polar form as

$$\mathbf{x} = r\boldsymbol{\rho}, \tag{2}$$

which *explicitly* involves the radial unit vector $\boldsymbol{\rho} = \boldsymbol{\rho}(\theta)$, which makes a counterclockwise angle θ with a suitably chosen but *implicit* fixed direction through F.

(2) Characterizing this *implicit* fixed direction by a unit vector, say $\boldsymbol{\iota}$, so that $\boldsymbol{\iota} = \boldsymbol{\rho}(0)$, localized at F.

The two unit vectors $\boldsymbol{\iota}$ and $\boldsymbol{\rho}$ are *fundamental* in classifying all the conic sections with a focus at F (through the joint action of their sum vector $\boldsymbol{\iota} + \boldsymbol{\rho}$ and through two suitable scalar parameters p and e).

In fact we have presented in Vivarelli [11,12]:

THE VECTOR **S**-EQUATION OF THE CONIC SECTIONS.

Let

(a) p be a real positive parameter;

(b) **e** denote the vector

$$\mathbf{e} \equiv e\boldsymbol{\iota} \tag{3}$$

with, respectively, $e = 0, 0 < e < 1, e = 1, e > 1$;

(c) **S** be the new sum vector defined by

$$\mathbf{S} \equiv p^{-1}(\mathbf{e} + \boldsymbol{\rho}) \tag{4}$$

or $\mathbf{S} = \mathbf{S}(\theta) = B - F$ with tip point B and $p^{-1}\mathbf{e} = p^{-1}e\,\boldsymbol{\iota} = H - F$ (see Fig. 3).

The equation for the four types of the conic sections may be written in the simple vector form

$$\mathbf{S} \cdot \mathbf{x} = 1, \tag{5}$$

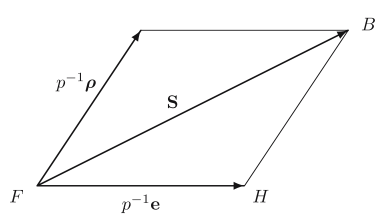

Fig. 3 The sum vector **S**

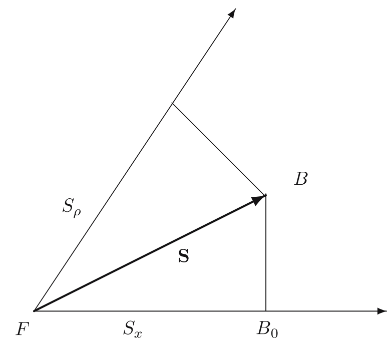

Fig. 4 The covariant components S_x and S_ρ

which yields the standard polar equation

$$r = \frac{p}{1 + e\cos\theta}, \tag{6}$$

where e is the eccentricity of the conic (shape), the parameter p is the semi-latus rectum (dimension) and $\theta = 0$ is the reference line.

Notice that the vector **S** encompasses in a natural way the triple $\{p, e, \theta\}$ which characterizes a conic section.

Several geometrical properties of the conics are **S**-encoded. For instance, in the case of an elliptic section, the ratio S_x/S_ρ of the two covariant components of the vector **S** (Fig. 4)

$$S_x = FB_0 = p^{-1}(e + \cos\theta), \tag{7}$$

$$S_\rho = p^{-1}(1 + e\cos\theta) = r^{-1} \tag{8}$$

encodes the well-known Kepler's *eccentric anomaly* (see [11]).

3 Unit circle inversion map and genetic S-code of the 'conic sections family'

The vector **S** stems from the standard *polar* definition (1) of a conic section. But it is intimately related to another well-known definition (see Fig. 5) by which a conic section is the locus of points in a plane such that the ratio of their distances from a fixed point F in the plane and from a fixed line (*directrix*) in the plane is a constant value e, that is

$$\frac{r}{d - r\cos\theta} = e, \tag{9}$$

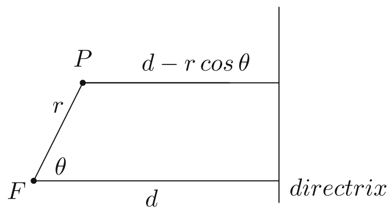

Fig. 5 The directrix

the *circle* $e = 0$ *being included*, for after setting

$$p = de, \tag{10}$$

the circle may be considered as an ellipse with directrix 'tending to the infinity' and the focus 'tending to the centre' of the ellipse.

Now, noticing that the vector **S** has (recall Fig. 3) two scalar contravariant components:

p^{-1} along the $\boldsymbol{\rho}$-direction,

$p^{-1}e$ along the **e**-direction,

which define *two points* (I and H) along each direction, such that

$$p^{-1} = \mid I - F \mid, \qquad p^{-1}e = \mid H - F \mid,$$

we may consider:

(1) the point I as the image (under the operation of *inversion* with respect to the *unit* circle centered at F) of the point I^* which lies on the same ray at distance p from F (such that $p^{-1}p = 1$). We shall call the point I^* a *typical point* of the conic orbit, for $I^* - F = p\boldsymbol{\rho}$ appears in the polar equation (1).
(2) the point H as the image, under the unit circle inversion map, of the point H^* which lies on the direction of **e** at a distance pe^{-1} from F, that is at exactly the distance d of the *directrix* of the conic orbit from F (recall (10), and Fig. 5). The point H^* is the intersection of the directrix with the line $\boldsymbol{\iota} = \boldsymbol{\rho}(0)$ and thus is a *typical point* of the conic orbit.

Accordingly (see Fig. 6) we may state that:
The vector **S** is the sum of two vectors, localized at the same focus F, with tip points which are the images, under the inversion map in the unit circle with centre at F, of the two typical points H^* and I^* of the conic orbit.

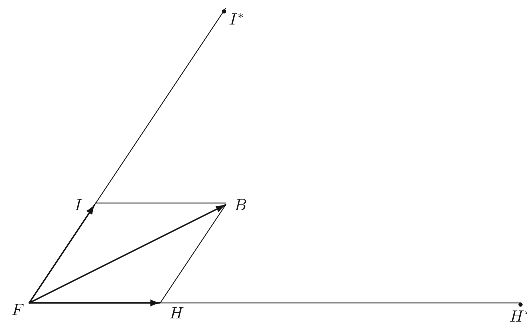

Fig. 6 The typical points I^* and H^* of a conic orbit

As shown in Vivarelli [12], the amazing fact about the diagonal sum vector **S** is that it appears as a *sort of geometric DNA of the family of the conic sections*, in the sense that it encodes all the information regarding each of the four members of the family.

This 'genetic' code is written by the *ordered* couple of vectors $p^{-1}\mathbf{e}$, $p^{-1}\boldsymbol{\rho}$.
The couple:

(1) characterizes the shape, the dimension and the counterclockwise orientation of each conic member;
(2) generates (besides the sum vector **S**) the complementary difference vector $\mathbf{S}_{-}=p^{-1}\mathbf{e}-p^{-1}\boldsymbol{\rho}$ and therefore creates a *double stranded*-**S** *structure*, in perfect analogy with the well-known complementary *two stranded structure* of the *DNA* double helix in molecular biology (see [11,12]).

Mitochondrial and physical E-encoding

Let us highlight the interesting fact that, for a given p, all the members of the family share the radial vector $p^{-1}\boldsymbol{\rho}$, inherited by the family as a *sort of mitochondrial (Eve) element* and shown [12] to be related to the *energy E* of the orbits of the classical Kepler problem.

Several properties stored in a natural way by the vector **S** have an immediate relationship to the *physical* properties of the *classical Kepler problem*, governed by the Newton's law of gravitation [12]. We mention here the stocking of the mechanical energy E (through the scalar product $\mathbf{S}\cdot\mathbf{S}_{-}$) and of the famous Laplace–Runge–Lenz vector (through the vector product $\mathbf{S}\wedge\mathbf{S}_{-}$).

4 The exceptional unit circle and the Julia set

So far, while revisiting the **S**-structure of the family of the four conic sections, we have found that one of its members—namely the circle (or its paradigma, the unit circle, hereafter briefly denoted *U-circle*)—shows up more than once as a *peculiar* member of the family. In fact:

(1) the U-circle is a pivotal member of the family, being the only one which appears explicitly, via its *unit* radius, in the conics **S**-equation (5);
(2) the U-circle characterizes the **S**-vector via the inversion map;
(3) the U-circle is an exceptional element in the physical energy E-classification of the Kepler orbits.
On the other hand we notice here the following peculiarity:
(4) all the four conic sections are invariant with respect to scaling (identified as *self-similarity*, that is invariance of shape against changes in size, the scaling factor being the multiplicative factor p, as explicitly shown by the **S**-definition (4)).

Now, *self-similarity* is a typical phenomenon in the geometry of shape (the *fractal* geometry of Mandelbrot [6,3]: since there is no widely accepted definition of the term *fractal* (see for instance [4]), we focus on self-similarity as the unifying concept underlying objects presenting scaling invariance of shape, against changes in size.

Moreover, in the fractal theory of Mandelbrot a relevant role is played by the *iteration maps in the complex plane*, maps which generate crucial and beautiful sets of boundary points, commonly called *Julia sets*.

Incredibly, the *unit circle* is found to be the Julia set for the most famous iteration map, the *simplest, paradigmatic complex quadratic map*

$$f(z) = z^2, \qquad z \in \mathcal{C},$$

which iterates a point z in the complex plane by squaring its magnitude $\mid z \mid$ and by doubling its angle: in the common complex parameterization, if $z = r\mathrm{e}^{\mathrm{i}\theta}$, then $z^2 = r^2\mathrm{e}^{2\mathrm{i}\theta}$.

The squaring and doubling properties of the map $f(z) = z^2$ *defines the U-circle structure*, since the points z which satisfy:

$\mid z \mid < 1$, that is the points lying within the U-circle, converge to 0,
$\mid z \mid > 1$, that is lying outside the U-circle, converge to infinity,
$\mid z \mid = 1$, that is lying on the U-circle, remain on the circle forever.

Thus, in the fractal language, the U-circle is the *Julia set* of the map, for it maps onto itself and it is the separating boundary between two basins of attractions (the inside of the circle with fixed attracting point 0 and the outside of the circle with attracting point ∞).

The powerful application of the Julia sets in the study of dynamical systems (that is physical systems, such as the planetary orbits, analyzed by means of iterative processes) is enhanced by the beautiful and colorful visualization techniques realized by modern computer graphics. A fine color-coded information may be found, for instance, in Alpigini and Russel [1].

At this point, being convinced that there is a hidden aspect lurking behind all the above characteristic topics (i.e. the geometrical and physical Keplerian peculiarities presented by the U-circle as a conic orbit, together with its self-similarity exhibited as an object of the complex map $f(z) = z^2$ of the fractal theory) *we expect to find this paradigmatic complex map also in the context of the classical Euclidean conic orbits arena*.

By following the chain of features

classical $- - \to$ *fractal* $- - \to$ *dynamical*,

we find the ultimate felicitous link, provided by the following considerations:

(1) The family of the conic sections lives in a *plane*, say the plane orthogonal to the unit vector $\mathbf{K}$ of the fixed orthogonal right-handed unit frame $\{F, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$ with the origin at F in the 3-dimensional Euclidean space.
(2) The unit position vector $\boldsymbol{\rho}$ of an arbitrary point on the U-circle may be obtained by an active *rotation* about the origin F applied to the unit vector $\mathbf{K}$.
(3) The position vector $r\boldsymbol{\rho}$ of a point on a conic section in the plane $\mathbf{I}F\mathbf{J}$ may be obtained by the previous active *rotation* (applied to the unit vector $\mathbf{K}$ and about the origin F) followed by a *dilation* by the scale factor r.

The genesis of this *compound roto-dilation*, which rotates the unit vector $\mathbf{K}$ about the origin F and stretches it by the radial distance factor r, is the subject of the following sections. The standard, powerful and hypercomplex quaternion description for roto-dilations is reformulated in a *particular and simple* form.

5 The $\mathbf{K}(\phi, r)$-roto-dilation map

As it is well known, a *general* active rigid *rotation* in the ordinary Euclidean space R^3 about a direction (axis of rotation) through a fixed point O is characterized by the unit vector $\mathbf{n}$ along the axis of rotation and by a right-handed angle Φ. Thus a rotation is characterized by the four real Euler–Rodrigues β-parameters $(\beta_0, \beta_1, \beta_2, \beta_3) = (\beta_0, \boldsymbol{\beta})$ defined by

$$\beta_0 = \cos \Phi/2, \qquad \boldsymbol{\beta} = \sin(\Phi/2)\mathbf{n},$$

which satisfy the normalizing condition $\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1$, that is define the surface of the unit sphere S^3 in a four-dimensional Euclidean space R^4.

If $b = (\beta_0 + \boldsymbol{\beta}) = \beta_0 + (\beta_1\mathbf{I} + \beta_2\mathbf{J} + \beta_3\mathbf{K})$ denotes a *unit quaternion* with *vector* part $\boldsymbol{\beta}$ and with *conjugate* $b^{-1} = (\beta_0 - \boldsymbol{\beta})$, a rotation acting on $\mathbf{K}$ and bringing it to an arbitrary unit vector $\mathbf{x}$ *in the space* is given by the product of the three unit quaternions $b, \mathbf{K}, b^{-1}$, or explicitly by:

$$\mathbf{x} = b\,\mathbf{K}\,b^{-1}. \tag{11}$$

But we may rewrite this standard rotation-product (and therefore a standard compound roto-dilation acting on $\mathbf{K}$) in terms of only *two* quaternions.

Precisely, we identify the space R^4 with the *real algebra of quaternions*, $\mathcal{H}$, in such a way that a quaternion $q \in \mathcal{H}$ is given by

$$q = (u_1 + u_2\mathbf{I} + u_3\mathbf{J}) + u_4\mathbf{K},$$

where the term between parentheses denotes the 3-vector part (being $R^3 = im\{1, \mathbf{I}, \mathbf{J}\}$).

It follows that:

(1) a vector $\mathbf{x} = x_1 + x_2\mathbf{I} + x_3\mathbf{J}$ is a quaternion with the *fourth* null component;

(2) the norm-preserving correspondence between a *standard* (non-unit) quaternion

$$m = \sqrt{r}\, b = \sqrt{r}\,(\beta_0, \boldsymbol{\beta}) = \sqrt{r}\,(\beta_0 + \beta_1\mathbf{I} + \beta_2\mathbf{J} + \beta_3\mathbf{K})$$

and our (non-unit) quaternion q, is given by

$$m \to q: \quad q = \sqrt{r}\,(\beta_3 + \beta_1\mathbf{I} + \beta_2\mathbf{J} - \beta_0\mathbf{K}) \tag{12}$$

(for *unit* quaternions the correspondence requires $r = 1$);

(3) by (11), a roto-dilation reads $\mathbf{x} = r(b\mathbf{K}b^{-1}) = m\mathbf{K}m^{-1}$.

Consequently, after having defined for any quaternion q the anti-involute q_* as $q_* = \mathbf{K}\,\bar{q}\,\mathbf{K}^{-1}$, that is, explicitly,

$$q_* \equiv u_1 + u_2\mathbf{I} + u_3\mathbf{J} - u_4\mathbf{K}$$

we find that a general compound roto-dilation $\mathbf{K} \to \mathbf{x}$ in R^3 given by

$$\mathbf{x} = m\,\mathbf{K}\,m^{-1}$$

may be expressed, via the correspondence $m \to q$, by the simple product qq_* of two non-unit quaternions

$$\mathbf{x} = qq_* \tag{13}$$

that is, explicitly by

$$\mathbf{x} = (u_1 + u_2\mathbf{I} + u_3\mathbf{J} + u_4\mathbf{K})\,(u_1 + u_2\mathbf{I} + u_3\mathbf{J} - u_4\mathbf{K})$$

or equivalently in real form

$$x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2, \quad x_2 = 2(u_1u_2 - u_3u_4)$$
$$x_3 = 2(u_1u_3 + u_2u_4), \quad x_4 = 0,$$

which corresponds to the compact matrix form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 & -u_2 & -u_3 & -u_4 \\ u_2 & u_1 & -u_4 & u_3 \\ u_3 & u_4 & u_1 & -u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

(recall that $\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = -1$; $\mathbf{IJ} = \mathbf{K} = -\mathbf{JI}$ and so on, in a cyclic order). Notice that

$$|\,\mathbf{x}\,| = r = |\,q\,|^2 .$$

In Table 1, we summarize the hypercomplex forms of the $\mathbf{K}(\phi, r)$-map which describes a $\mathbf{K} \to \mathbf{x}$

Table 1 The hypercomplex $\mathbf{K}(\phi, r)$-roto-dilation map

Hypercomplex notation	Real notation	Matrix notation
$x_1 + x_2\mathbf{I} + x_3\mathbf{J} = qq_*$	$x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2$ $x_2 = 2(u_1u_2 - u_3u_4)$ $x_3 = 2(u_1u_3 + u_2u_4)$ $x_4 = 0$	$\begin{pmatrix} u_1 & -u_2 & -u_3 & -u_4 \\ u_2 & u_1 & -u_4 & u_3 \\ u_3 & u_4 & u_1 & -u_2 \\ u_4 & -u_3 & u_2 & u_1 \end{pmatrix}$

roto-dilation which carries the unit vector $\mathbf{K}$ onto the position vector $\mathbf{x}$ of an *arbitrary* point of R^3.

Geometrical comments. The hypercomplex quaternion map qq_* is a *Hopf fibering* $S^3 \to S^2$ of the sphere S^3 since it is unaffected by the gauge transformation $q \to qe^{k\theta} = q(\cos\theta + k\sin\theta)$ (with $k = \mathbf{K}$), whence the whole one-dimensional set (fiber) of *unit* quaternions $we^{k\theta}$ with $w\bar{w} = 1$—corresponding to a unit circle $S^1 = \{we^{k\theta}\}$ of the unit 3-sphere $S^3 = \{\mathbf{u} \mid u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1\}$ in the space R^4—is mapped by ww_* onto the single point $\mathbf{x}$ of S^2 in the ordinary space R^3.

6 The $\mathbf{K}(\pi/2, r)$-roto-dilation map in the conic sections arena

Getting back to the *plane* conic sections, we first restrict our attention to the particular $\mathbf{K}$-rotation $\mathbf{K} \to \boldsymbol{\rho}$, which carries the unit vector $\mathbf{K}$ onto the unit vector $\mathbf{x} = \boldsymbol{\rho}$ which *lies in the* $\{\mathbf{I}, \mathbf{J}\}$*-plane* (that is on a point on the U-circle). This rotation is characterized by

$$\Phi = \frac{\pi}{2}, \qquad \mathbf{n} = n_x\mathbf{I} + n_y\mathbf{J}$$

with ${n_x}^2 + {n_y}^2 = 1$.

If one works via the standard β-quaternion description and thus adopts the rotation relation (11), one has:

$$\boldsymbol{\rho} = b\mathbf{K}b^{-1} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}n_x\mathbf{I} + \frac{\sqrt{2}}{2}n_y\mathbf{J}\right)$$

$$\mathbf{K}\ \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}n_x\mathbf{I} - \frac{\sqrt{2}}{2}n_y\mathbf{J}\right),$$

which upon calculation gives the unit vector

$$\boldsymbol{\rho} = n_y\,\mathbf{I} - n_x\mathbf{J}\,.$$

Of course the two resulting components of the vector $\boldsymbol{\rho}$ keep track of the fact that, by construction, the unit vectors $\boldsymbol{\rho}$ and $\mathbf{n}$ are orthogonal vectors in the $(\pi/2)$-rotation of $\mathbf{K}$ about $\mathbf{n}$.

But what is important here is that the same result may be obtained as the product of only two unit quaternions.

Owing to our quaternion map (13) and to the correspondence (12) with $r = 1$, we recover the unit vector $\boldsymbol{\rho}$ by the product of the unit quaternion $w = \frac{\sqrt{2}}{2}n_x\mathbf{I} + \frac{\sqrt{2}}{2}n_y\mathbf{J} - \frac{\sqrt{2}}{2}\mathbf{K}$ and its anti-involute w_*, that is explicitly by

$$\boldsymbol{\rho} = ww_*$$

or

$$\boldsymbol{\rho} = \left(\frac{\sqrt{2}}{2}n_x\mathbf{I} + \frac{\sqrt{2}}{2}n_y\mathbf{J} - \frac{\sqrt{2}}{2}\mathbf{K}\right) \times \left(\frac{\sqrt{2}}{2}n_x\mathbf{I} + \frac{\sqrt{2}}{2}n_y\mathbf{J} + \frac{\sqrt{2}}{2}\mathbf{K}\right) = n_y\,\mathbf{I} - n_x\mathbf{J}.$$

Let us now effect a dilation by r, in order to obtain a *compound roto-dilation*

$$\mathbf{K} \to \boldsymbol{\rho} \to r\boldsymbol{\rho}$$

for each $\mathbf{n}$.

The collection of all these roto-dilations (for each $\mathbf{n}$) may be represented by a ‘crazy fountain’ picture (see Fig. 7) where circular sprays, spreading in a circular fashion from the arrowed point of $\mathbf{K}$, reach the point $\boldsymbol{\rho}$ on the plane and then spring away *horizontally* to reach the point $\mathbf{x} = r\boldsymbol{\rho}$ on the same plane.

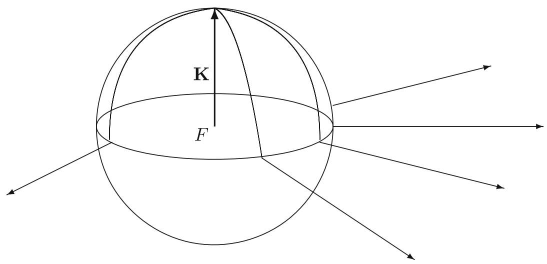

Fig. 7 The crazy fountain picture

The compound roto-dilation $\mathbf{K} \to \boldsymbol{\rho} \to r\boldsymbol{\rho}$, characterized by the quaternion

$$q = \sqrt{r}w$$

and which unfolds as

$$r\boldsymbol{\rho} = qq_* = r\,(n_y\mathbf{I} - n_x\mathbf{J})\,, \tag{14}$$

is the key map which plays a crucial role in finding the hidden character of the U-circle.

7 The K($\pi/2\,,r$)-map: a complex quadratic map. Its Julia set: the *U*-circle

If, as usual, we identify the plane $\{\mathbf{I},\mathbf{J}\}$ with the complex plane $\{1,i\}$ we find that the vector $r\boldsymbol{\rho}$ given by (14) corresponds to the complex number $z \in \mathcal{C}$

$$z = r\,n_y - r\,n_x i\ . \tag{15}$$

On the other hand, being $x_3 = 0$, we have that (see Table 1):

$$u_3 = 0, \qquad u_4 = 0\,. \tag{16}$$

From (15) and (16) and Table 1, we find that

$$r\,n_y \ = u_1^2 - u_2^2,\ -r\,n_x = 2u_1u_2, \tag{17}$$

which, upon resolution with respect to the unknowns u_1, u_2, gives:

$$u_1 = -\frac{\sqrt{2r}}{2}\,\frac{n_x}{\sqrt{1-n_y}}, \qquad u_2 = \frac{\sqrt{2r}}{2}\,\sqrt{1-n_y}\ .$$

We may now write (17) as

$$x_1 = u_1^2 - u_2^2, \qquad x_2 = 2u_1u_2 \tag{18}$$

which may be epitomized in the following complex form

$$z = x_1 + ix_2 = (u_1 + \mathrm{i}u_2)^2, \tag{19}$$

which reveals as a complex quadratic squaring map.

This simple squaring map (19) is a particular case of the *famous complex quadratic map* often rendered in the form

$$z_{n+1} = z_n^2 + c, \qquad c \in \mathcal{C},$$

which is paradigmatic in *the fractal theory* of Mandelbrot [6,3], representing a nonlinear iteration procedure in $\mathcal{C}$, which constructs the sequence $\{z_n\}$ of complex numbers via a repeated application (squaring and addition by c).

Our quadratic map (19), being of the type $f(z) = z^2$, corresponds to the value $c = 0$ for the complex growth parameter (unperturbed map) and shows up as the first iteration. Moreover, the U-circle is its Julia set (see section "The exceptional unit circle and the Julia set").

Thus, by (19), *we have reached the crucial result* searched for in section "The exceptional unit circle and the Julia set": by identifying the Euclidean $\{\mathbf{I},\mathbf{J}\}$-plane of a conic section with the standard complex $\{1,i\}$-plane, we have found that the peculiar roto-dilation in R^3 (which rotates the unit vector $\mathbf{K}$ onto the plane position vector $\mathbf{x} = r\boldsymbol{\rho}$ of a point on the conic) is exactly the complex quadratic squaring map paradigmatic in the fractal theory, the U-circle showing up as its Julia set.

8 The complex map and the regularization of the Kepler problem

Now, chained to the above strictly geometrical peculiarities of the complex map (19), there is a *physical* one, related to the Kepler problem.

If we identify the algebra of complex numbers $\mathcal{C}$ with $im\{1,i\}$ in the quaternion algebra $\mathcal{H}$, we have that, by definition, the anti-involute of a complex number $c \in \mathcal{C}$ satisfies $c = c_*$ whence, for each $c = u_1 + \mathrm{i}u_2 \in \mathcal{C}$ the square $(u_1 + \mathrm{i}u_2)^2$ may be expressed by the *product* cc_*, that is

$$(u_1 + \mathrm{i}u_2)^2 = cc_*\,.$$

Table 2 The hypercomplex roto-dilation hierarchy

Hypercomplex notation	Real notation	Matrix
$x_1 + x_2 i + x_3 j = qq_*$	$\begin{cases} x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ x_2 = 2(u_1u_2 - u_3u_4) \\ x_3 = 2(u_1u_3 + u_2u_4) \\ x_4 = 0 \end{cases}$	$\begin{pmatrix} u_1 & -u_2 & -u_3 & -u_4 \\ u_2 & u_1 & -u_4 & u_3 \\ u_3 & u_4 & u_1 & -u_2 \\ u_4 & -u_3 & u_2 & u_1 \end{pmatrix}$
$x_1 + x_2 i = (u_1 + u_2 i)^2 = cc_*$	$\begin{cases} x_1 = u_1^2 - u_2^2 \\ x_2 = 2u_1u_2 \end{cases}$	$\begin{pmatrix} u_1 & -u_2 \\ u_2 & -u_1 \end{pmatrix}$
$x_1 = u_1^2 = u_1 u_{1*}$	$x_1 = u_1^2$	(u_1)

Thus the complex map (19) is a particular case of the hypercomplex quaternion map qq_*. In Table 2, we summarize the natural hierarchical structure of the roto-dilation map in its three mathematical forms (for simplicity the boldfaced symbols, say **I**, are replaced by the standard quaternion symbols, say *i*).

Now, the complex map (19) coincides exactly with the peculiar map constructed by Levi-Civita [5] in order to *regularize at the origin* the differential equations of the physical *planar* Kepler problem (which are *singular* at the collision point $r = 0$). As clearly remarked by Levi–Civita, the plane Levi–Civita map is not extensible to the three dimensions, since a matrix of the Levi–Civita's type is only either a 2×2, or 4×4 or 8×8 matrix. A 3-dimensional extension was successfully constructed by Kustaanheimo (see [9]) who found a regularizing *matrix* map (the KS-map). Remarkably, our map qq_* is the simple *quaternion* representation of the *matrix* regularizing KS-map (see [10]).

Summing up: the exceptional U-circle is the Julia set of the map (19) which (from a geometrical point of view) is a roto-dilation map in R^3 and (from a physical point of view) regularizes at the origin the planar Kepler problem.

Let us finally report that each regularizing map (in the 'bottom to top order': $\mathcal{R} \to \mathcal{C} \to \mathcal{H}$ of Table 2) may be derived from the previous one by a technique which Deprit, Elipe and Ferrer [2] call a *doubling technique* and which, (implemented in software [2] by fashioning hypercomplex numbers as saturated binary trees) provides interesting extensions of the KS-map, the algebraic difficulties being solved by a current good symbolic processor.

9 Comments and outlook

Needless to say, the fact that a 'unit circle' is the Julia set of the complex quadratic map $z \to z^2$ is a well-known result. What is then the novelty of this paper? It is the discovery that a particular *conic section* emerges from its own family and reveals a hidden, new facet: the 'Julia set' facet. And this through the natural introduction in the arena of the roto-dilation map (19) and through a simple, natural recourse to the **S**-approach (which has gently opened the door to the remarkable 'Julia set feature' exhibited by one of the members of the conic family).

It is remarkable that the U-circle (although a classical object and not a fractal one) shows the coexistence of both the traditional aspects and the recent fractal ones. The vector **S** itself exhibits by definition a *two-fold character* through its 'real' component $p^{-1}\mathbf{e}$ and its 'complex' $p^{-1}\boldsymbol{\rho}$ one.

The U-circle defines the vector **S** via the inversion map, which interchanges the two attracting regions $\mid z \mid <1$ and $\mid z \mid >1$ of the complex plane (the 'prisoner' and the 'escape' set of the fractal theory). Adding to this that concentric circles are the equipotential curves in the attractive *gravitation* field, it may be interesting to study the physical implications, on the same line followed for the electrostatic field by Douady and Hubbard (see [8]). At this juncture, we remark that the real and imaginary parts (18) of the complex map satisfy

$$\frac{\partial x_1}{\partial u_1} = \frac{\partial x_2}{\partial u_2}, \quad \frac{\partial x_1}{\partial u_2} = -\frac{\partial x_2}{\partial u_1},$$

that is the Laplace equation $\nabla^2 x_1 = \nabla^2 x_2 = 0$, which is paradigmatic in potential theory [7].

Last, but not least, it is interesting to note that the key relations

$$\mathbf{S} \cdot \mathbf{x} = 1, \quad x = cc_*$$

have an aesthetic appeal, being expressed in a very simple form (the recourse to the complex domain revealing itself fundamental, according to the motto 'complexify to simplify').

Let us close by observing that the **S**-approach, in that it brings together traditional and recent aspects, may open a door to applications centered around the 'elusive origin of complexity' in geometrical and dynamical phenomena. For instance, recall that A.A.Turing proposes a *symmetry breaking* characterization as the origin of morphogenesis. Thus, we do not think that it is casual that the 'first point' of our vector **S** is exactly the *focus* of the conic sections: the geometric centre of the conics being abandoned in favour of this centre of attraction. The perfect symmetry is abandoned.

Remark In the literature (see [6,7]) one encounters an array of several phenomena related to the famous complex quadratic map $f(z) = z^2$.

In this paper this famous map is stricly related to the peculiar roto-dilation $\mathbf{K}(\pi/2, r)$ in R^3 and to the regularization of the unperturbed Kepler problem.

In this setting it will be interesting to study the 'perturbed Kepler problem'

$$\ddot{\mathbf{x}} + \frac{K^2}{r^3}\mathbf{x} = \mathbf{f}$$

(where **f** is the perturbing force per unit of mass) by connecting it to the 'perturbed complex map'

$$f(z) = z^2 + c,$$

where, starting from $c \neq 0$ small enough, the Julia set boundary is a U-circle which slightly distorts into a 'quasi'-circle and so on, becoming a fractal curve. In this context, modern computer graphics (see section The exceptional unit circle and the Julia set) will be crucial in suggesting, finding and understanding hidden features.

References

1. Alpigini JJ, Russel DW (2003) Dynamical systems visualization and analysis via performance maps. Control Engineering Practice 11:493–504
2. Deprit A, Elipe A, Ferrer S (1994) Linearization: Laplace vs. Stiefel. Celest Mech and Dynam Astr 58:151-201
3. Devaney RL (1997) An introduction to chaotic dynamical systems, 2nd edn. Addison–Wesley, Reading, MA
4. Freiberg UR (2005) Analysis on fractal objects. Meccanica 40:419–436
5. Levi-Civita T (1973) Opere matematiche. vol. 6, Zanichelli, Bologna, pp 111–133
6. Mandelbrot BB (1982) The fractal geometry of nature. W.H.Freeman, New York
7. Moon FC (1992) Chaotic and fractal dynamics. Wiley, New York
8. Peitgen H-O, Jurgens H, Saupe D (1991) Fractals for the classroom. Springer-Verlag, Berlin Hieldleberg, New York
9. Stiefel EL, Scheifele G (1971) Linear and regular celestial mechanics. Springer, Berlin Hieldleberg, New York
10. Vivarelli MD (1994) The Kepler problem: a unifying view. Celest Mech Dyn Astr 60:291–305
11. Vivarelli MD (2000) The sum vector **S** and the fictitious time s in the Kepler problem. Meccanica 35: 55–67
12. Vivarelli MD (2005) The amazing **S**-code of the conic sections and the Kepler problem. Polipress, Milano