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Mixed convection boundary layers in the stagnation-point flow toward a stretching vertical sheet

A. Ishak · R. Nazar · I. Pop

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Abstract An analysis is made for the steady mixed convection boundary layer flow near the two-dimensional stagnation-point flow of an incompressible viscous fluid over a stretching vertical sheet in its own plane. The stretching velocity and the surface temperature are assumed to vary linearly with the distance from the stagnation-point. Two equal and opposite forces are impulsively applied along the x-axis so that the wall is stretched, keeping the origin fixed in a viscous fluid of constant ambient temperature. The transformed ordinary differential equations are solved numerically for some values of the parameters involved using a very efficient numerical scheme known as the Keller-box method. The features of the flow and heat transfer characteristics are analyzed and discussed in detail. Both cases of assisting and opposing flows are considered. It is observed that, for assisting flow, both the skin friction coefficient and the local Nusselt number increase as the buoyancy parameter increases, while only the local Nusselt number increases but the skin friction coefficient decreases as the Prandtl number increases. For opposing flow, both the skin friction coefficient and the local Nusselt number decrease as the buoyancy parameter increases, but both increase as Pr increases. Comparison with known results is excellent.

A. Ishak · R. Nazar (✉)
School of Mathematical Sciences, National University of Malaysia, 43600 UKM Bangi, Selangor, Malaysia
e-mail: rmn72my@yahoo.com

I. Pop
Faculty of Mathematics, University of Cluj, R-3400 Cluj, CP 253, Romania

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Nomenclature

a, b, c	constants
C_f	skin friction coefficient
f	dimensionless stream function
g	acceleration due to gravity (ms^{-2})
Gr_x	local Grashof number
k	thermal conductivity ($\mathrm{Wm}^{-1}\mathrm{K}$)
Nu_x	local Nusselt number
Pr	Prandtl number
q_w	heat transfer from the stretching surface (Wm^{-2})
Re_x	local Reynolds number
T	fluid temperature (K)
$T_w(x)$	temperature of the stretching surface (K)
T_∞	ambient temperature (K)
u, v	velocity components along x and y directions, respectively ($\mathrm{m\,s}^{-1}$)
$u_\mathrm{e}\,(x)$	velocity of external flow (ms^{-1})

$u_w(x)$ velocity of the stretching surface (ms^{-1})
x, y Cartesian coordinates along the surface and normal to it, respectively (m)

Greek letters
α thermal diffusivity (m^2s^{-1})
β thermal expansion coefficient (K^{-1})
η pseudo-similarity variable
θ dimensionless temperature
λ buoyancy parameter
μ dynamic viscosity ($kgm^{-1}s^{-1}$)
ν kinematic viscosity (m^2s^{-1})
ρ fluid density (kgm^{-3})
τ_w skin friction from the surface of the sheet (Pa)
ψ stream function (m^2s^{-1})

Superscript
$'$ differentiation with respect to η

Subscripts
w condition at the stretching sheet
∞ condition at infinity

1. Introduction

Flow and heat transfer of a viscous and incompressible fluid over a continuously moving surface through a quiescent fluid has attracted many investigations during the past several decades. This is because of its wide applications in many practical applications in manufacturing processes, such as extrusion of polymers, continuous casting, cooling of metallic plates, glass fiber production, hot rolling, paper production, wire drawing, aerodynamic extrusion of plastic sheets, crystal growing, etc. The study of heat transfer and flow field is necessary for determining the quality of the final products of such processes as explained by Karwe and Jaluria [1]. The flow induced by a semi-infinite horizontally moving wall in an ambient fluid was first investigated by Sakiadis [2]. Later, Crane [3] studied the flow over a linearly stretching sheet in an ambient fluid and gave a similarity solution in closed analytical form for the steady two-dimensional problem. Many authors such as Carragher and Crane [4], Elbashbeshy and Bazid [5], Gupta and Gupta [6], Magyari and Keller [7, 8], Magyari et al. [9], Liao and Pop [10], and Nazar et al. [11] investigated this problem by taking into account different aspects, such as uniform heat flux, permeability of the surface, flow and heat transfer unsteadiness, etc. Other physical features such as magnetic field, viscoelasticity of the fluid, suction and three-dimensional flow have been considered by Pop [12], Andersson [13], Takhar and Nath [14], and Nazar et al. [15]. However, for a stretching vertical plate there are much less work published. We mention to this class of problems those by Chen [16, 17], Lin and Chen [18], Ali and Al-Yousef [19, 20], Ali [21, 22], and Abo-Eldahab [23]. It is worth mentioning that the unsteady boundary layer flow and heat transfer over a stretching vertical sheet has been studied only by Ishak et al. [24] in a very recent paper.

Recently, Mahapatra and Gupta [25, 26] studied the heat transfer in the steady two-dimensional stagnation-point flow of a viscous and incompressible Newtonian and viscoelastic fluids over a horizontal stretching sheet considering the case of constant surface temperature.

In this paper, the steady two-dimensional stagnation-point flow of a viscous and incompressible fluid over a stretching vertical sheet in its own plane is investigated theoretically. It is assumed that the velocity and temperature of the stretching sheet is proportional to the distance from the stagnation-point. To our best knowledge this problem has not been studied before.

2. Mathematical model

Consider the steady, two-dimensional flow of a viscous and incompressible fluid near the stagnation point on a stretching vertical surface placed in the plane $y = 0$ of a Cartesian system of coordinates Oxy ($y = 0$) with the x-axis along the sheet as shown in Fig. 1. The fluid occupies the half plane ($y > 0$). It is assumed that the velocity $u_w(x)$ and the temperature $T_w(x)$ of the stretching sheet is proportional to the distance x from the stagnation-point, where $T_w(x) > T_\infty$ with T_∞ being the uniform temperature of the ambient fluid. The velocity of the flow external to the boundary layer is $u_e(x)$.

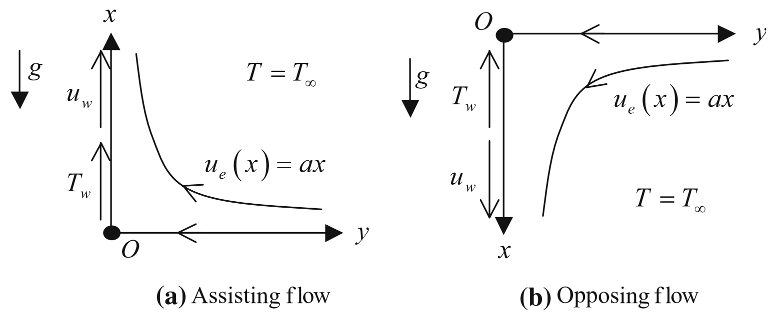

Fig. 1 Physical model and coordinate system

Under these assumptions along with the Boussinesq and boundary layer approximations, the system of equations, which model the boundary layer flow are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \pm g\beta(T - T_\infty), \quad (2)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \quad (3)$$

where u and v are the velocity components along x and y axes, respectively, T is the fluid temperature, g is the gravity acceleration, α, ν and β are the thermal diffusivity, kinematic viscosity and thermal expansion coefficient, respectively, and the "+" and "−" signs in Eq. 2 correspond to assisting buoyant flow and to opposing buoyant flow, respectively. We shall assume that the boundary conditions of Eqs. 1–3 are

$$\begin{aligned} &v = 0, \quad u = u_w(x) = cx, \\ &T = T_w(x) = T_\infty + bx \quad \text{at } y = 0, \\ &u = u_e(x) = ax, \quad T = T_\infty \quad \text{as } y \to \infty, \end{aligned} \quad (4)$$

where a, b, and c are positive constants.

We look for a similarity solution of Eqs. 1–3 of the form

$$\begin{aligned} &\psi = (c\nu)^{1/2}\, x f(\eta), \\ &\theta(\eta) = (T - T_\infty)/(T_w - T_\infty), \\ &\eta = (c/\nu)^{1/2}\, y, \end{aligned} \quad (5)$$

where ψ is the stream function which is defined in the usual form as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Substituting variables (5) into Eqs. 2 and 3, we get the following ordinary differential equations

$$f''' + ff'' - f'^2 + \frac{a^2}{c^2} \pm \lambda\theta = 0, \quad (6)$$

$$\frac{1}{\Pr}\theta'' + f\theta' - f'\theta = 0, \quad (7)$$

subject to the boundary conditions (4) which become

$$\begin{aligned} &f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1, \\ &f'(\infty) = \frac{a}{c}, \quad \theta(\infty) = 0. \end{aligned} \quad (8)$$

Here primes denote differentiation with respect to η, Pr is the Prandtl number and the constant $\lambda(\geqslant 0)$ is the buoyancy parameter defined as

$$\lambda = \frac{Gr_x}{\mathrm{Re}_x^2}, \quad (9)$$

with $Gr_x = g\,\beta\,(T_w - T_\infty)\,x^3/\nu^2$ is the local Grashof number and $\mathrm{Re}_x = u_w\,x/\nu$ is the local Reynolds number. When $\lambda = 0$ and $a/c = 1$, the solution of Eq. 6 subject to boundary condition (8) is given by

$$f(\eta) = \eta. \quad (10)$$

The physical quantities of interest are the skin friction coefficient and the local Nusselt number, which are defined as

$$C_f = \frac{\tau_w}{\rho\, u_w^2}, \qquad Nu_x = \frac{x\, q_w}{k\,(T_w - T_\infty)}, \quad (11)$$

where the skin friction τ_w and the heat transfer from the plate q_w are given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \qquad q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}, \quad (12)$$

with μ and k being the dynamic viscosity and thermal conductivity, respectively. Using the non-dimensional variables (5), we get

$$C_f \mathrm{Re}_x^{1/2} = f''(0), \qquad Nu_x/\mathrm{Re}_x^{1/2} = -\theta'(0). \quad (13)$$

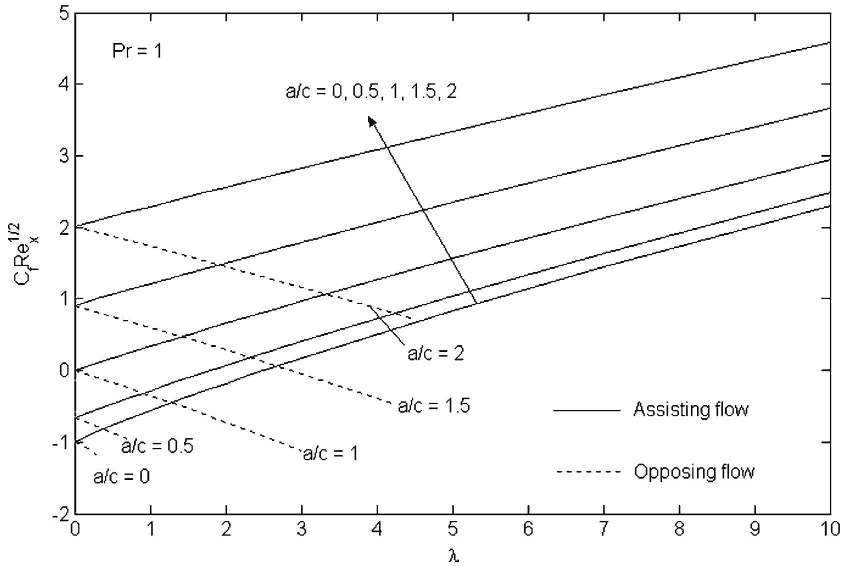

Fig. 2 Variation with λ of the skin friction coefficient for some values of a/c when Pr=1

3. Results and discussion

Equations (6) and (7) subject to boundary conditions (8) have been solved numerically using the Keller-box method, which is described in the book by Cebeci and Bradshaw [27]. The results are given to carry out a parametric study showing influences of several of these non-dimensional parameters. For the validation of the numerical method used in this study, the case when the buoyancy term $\lambda\,\theta$ in Eq. 6 is absent has been also considered and compared with the results reported by Mahapatra and Gupta [25] and Nazar et al. [11]. This comparison is shown in Table 1. It is seen that the present values of $C_f\,\mathrm{Re}_x^{1/2}$ are in very good agreement with those obtained by Mahapatra and Gupta [25] and Nazar et al. [11]. Therefore, it can be concluded that the present numerical method can be used with great confidence to study the problem discussed in this paper.

The skin friction coefficient, local Nusselt number, velocity and temperature fields are shown in Figs. 2–11. Figures 2 and 4 suggest that an assisting buoyancy flow produces an increase in the skin friction coefficient, while an opposing buoyant flow gives rise to a decrease in the skin friction coefficient. This is because, the fluid velocity increases when the buoyancy force increases, and hence increases the wall shear stress, which increases the skin friction coefficient. The values of the skin friction coefficient and the local Nusselt number for various Pr when $a/c = 1$ and $\lambda = 1$ are tabulated in Table 2, for both cases of assisting and opposing flows. It is worth mentioning that the

Table 1 Values of $C_f\mathrm{Re}_x^{1/2}$ for different values of a/c when the buoyancy force term $\lambda\,\theta$ in Eq. 6 is absent

a/c	Mahapatra and Gupta [25]	Nazar et al. [11]	Present
0.1	−0.9694	−0.9694	−0.9694
0.2	−0.9181	−0.9181	−0.9181
0.5	−0.6673	−0.6673	−0.6673
2	2.0175	2.0176	2.0175
3	4.7293	4.7296	4.7294

Table 2 Values of $C_f\mathrm{Re}_x^{1/2}$ and $\mathrm{Nu}_x/\mathrm{Re}_x^{1/2}$ for $a/c = 1, \lambda = 1$ and various Pr

Pr	Buoyancy assisting flow		Buoyancy opposing flow	
	$C_f\mathrm{Re}_x^{1/2}$	$\mathrm{Nu}_x/\mathrm{Re}_x^{1/2}$	$C_f\mathrm{Re}_x^{1/2}$	$\mathrm{Nu}_x/\mathrm{Re}_x^{1/2}$
0.72	0.3645	1.0931	−0.3852	1.0293
6.8	0.1804	3.2902	−0.1832	3.2466
20	0.1175	5.6230	−0.1183	5.5923
40	0.0873	7.9463	−0.0876	7.9227
60	0.0729	9.7327	−0.0731	9.7126
80	0.0640	11.2413	−0.0642	11.2235
100	0.0578	12.5726	−0.0579	12.5564

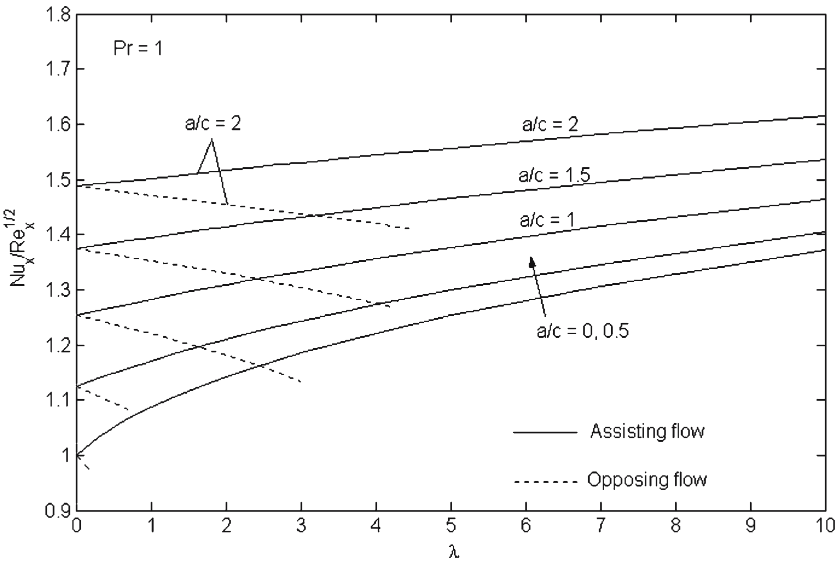

Fig. 3 Variation with λ of the local Nusselt number for some values of a/c when Pr = 1

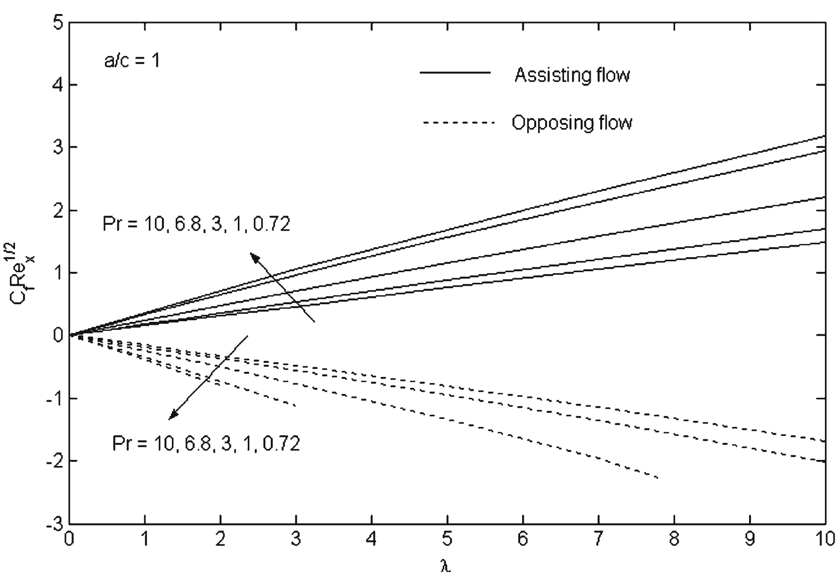

Fig. 4 Variation with λ of the skin friction coefficient for some values of Pr when $a/c = 1$

values of $Nu_x/\mathrm{Re}_x^{1/2} = -\theta'(0)$ are positive in all cases discussed in this study. This follows from the integral relationship $Nu_x/\mathrm{Re}_x^{1/2} = -\theta'(0) = 2\mathrm{Pr}\int_0^\infty f'(\eta)\theta(\eta)\,\mathrm{d}\eta$, which is obtained from Eqs. 7 and 8. Also, the effects of λ on the skin friction coefficient are found to be more significant for fluids having smaller Pr since the viscosity is less than the fluids with larger Pr. Figure 4 shows that all curves intersect at a point where $\lambda = 0$; that is when the buoyancy force is zero. This is because, Eqs. 6 and 7 are uncoupled when $\lambda = 0$, in other words, the solutions to the flow field are not affected by the thermal field in which the buoyancy force is lacking. Also in this case, the value of $C_f\mathrm{Re}_x^{1/2}$ remains constant, namely zero. This value agreed with the exact solution (10), which implies $f''(\eta) = 0$, for all η. Moreover, for assisting flow, it can be seen that $C_f\mathrm{Re}_x^{1/2}$ decreases when Pr increases for a

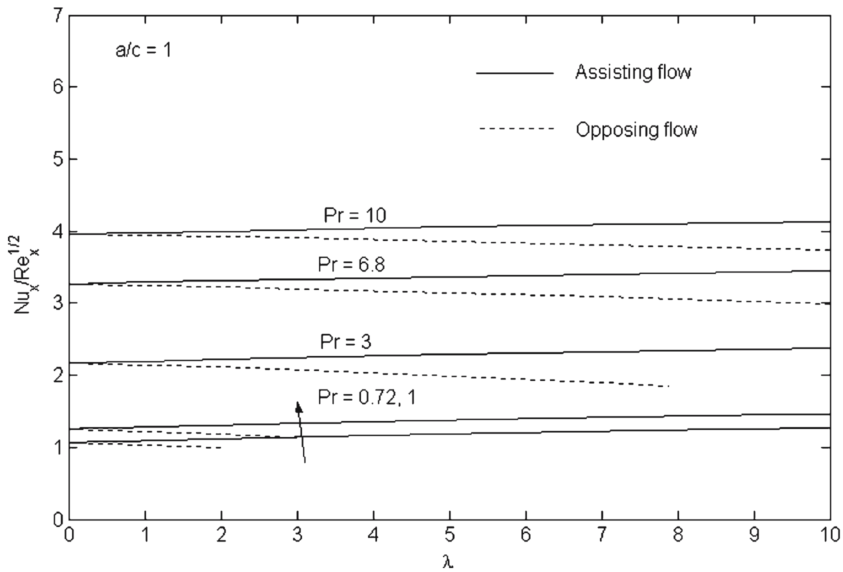

Fig. 5 Variation with λ of the local Nusselt number for some values of Pr when $a/c = 1$

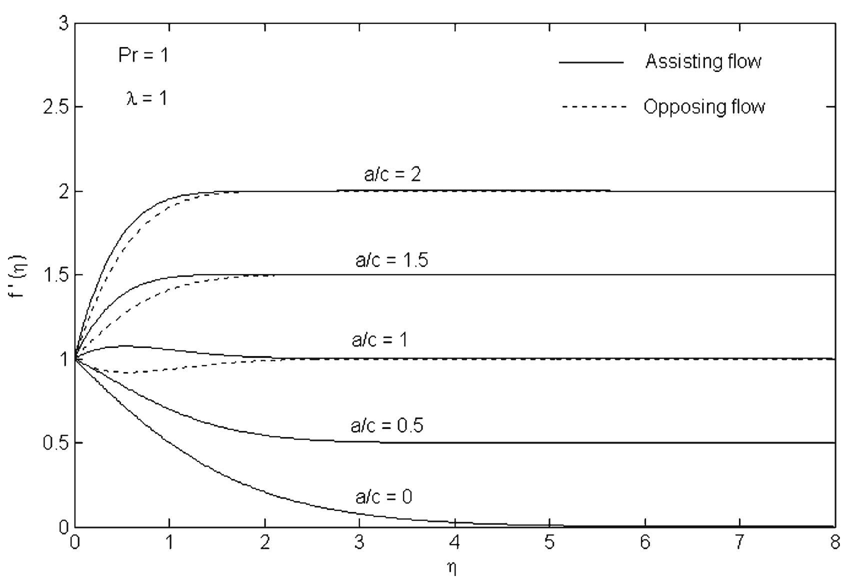

Fig. 6 Velocity profiles for some values of a/c when Pr = 1 and $\lambda = 1$

fixed value of λ. This is because, when Pr increases, the viscosity increases and slows down the flow, hence reduces the surface shear stress, thus reduces the skin friction coefficient $C_f \mathrm{Re}_x^{1/2}$. The opposite trends can be observed for opposing flow.

In Fig. 5, it is observed that for a particular value of Pr, the local Nusselt number is slightly increased as the buoyancy parameter λ is increased, for the case of assisting flow. The opposite trend occurs for opposing flow. This is clear from the fact that assisting buoyant flow produces a favorable pressure gradient that enhances the momentum transport, which in turn increases the surface heat transfer rate. On the other hand, opposing buoyant flow leads to an adverse pressure gradient, which slows down the fluid motion, and hence gives rise to a decrease in the local Nusselt number. In addition, the effects of Pr can be examined, that is, increasing

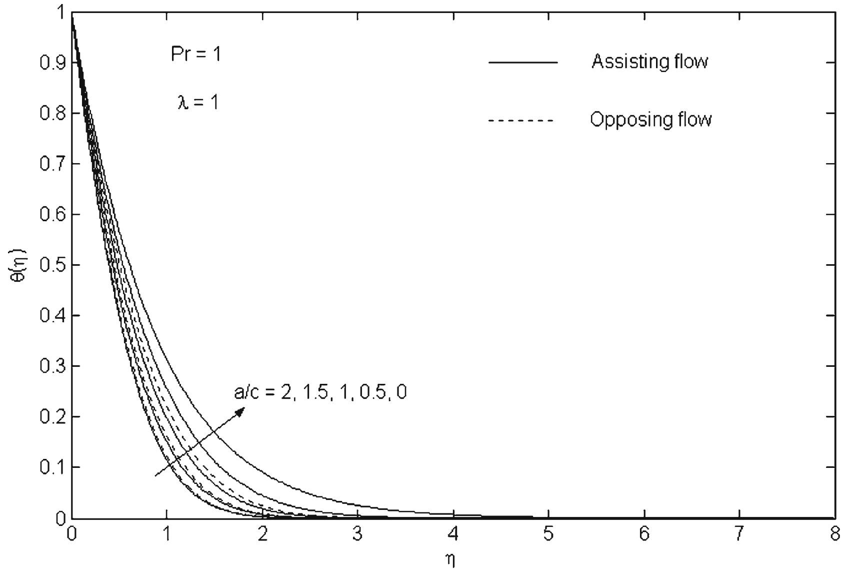

Fig. 7 Temperature profiles for some values of a/c when Pr = 1 and $\lambda = 1$

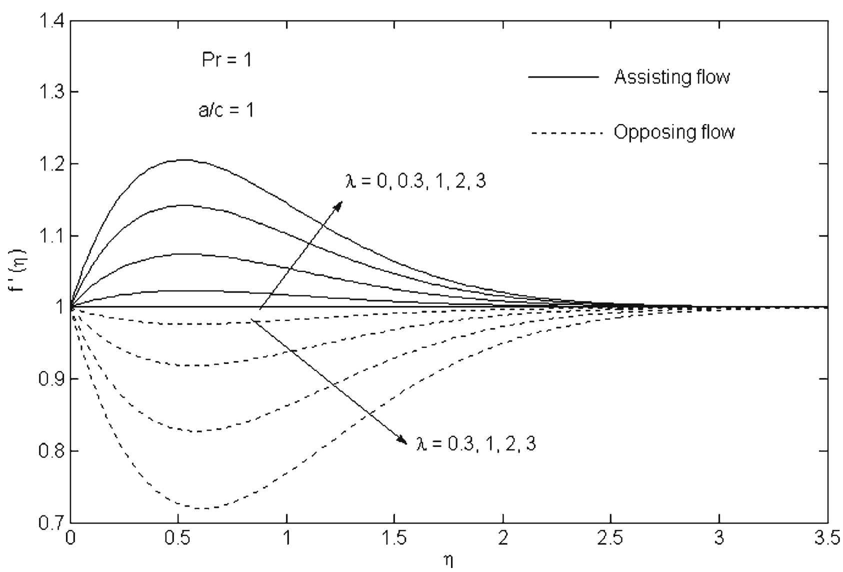

Fig. 8 Velocity profiles for some values of λ when Pr = 1 and $a/c = 1$

Pr enhances the rate of heat transfer since increasing of Pr will cause the increasing of viscosity, then reduces the thermal conductivity, and thus $-\theta'(0)$ increases.

The resulting profiles of dimensionless velocity $f'(\eta)$ and dimensionless temperature $\theta(\eta)$ are shown in Figs. 6–11 for various values of $a/c, \lambda$, and Pr. It is evident from Fig. 11 that an increase in Pr results in a decrease in the thermal boundary layer thickness, associated with an increase in the wall temperature gradient, and hence produces an increase in the surface heat transfer rate. For both dimensionless temperature and velocity profile, the effects of buoyancy force are found to be more pronounced for a fluid with a small Pr. Thus, fluid with smaller Pr is more susceptible to the buoyancy force effects. From Fig. 6, it can be seen that when $a/c > 1$, the flow has a boundary layer

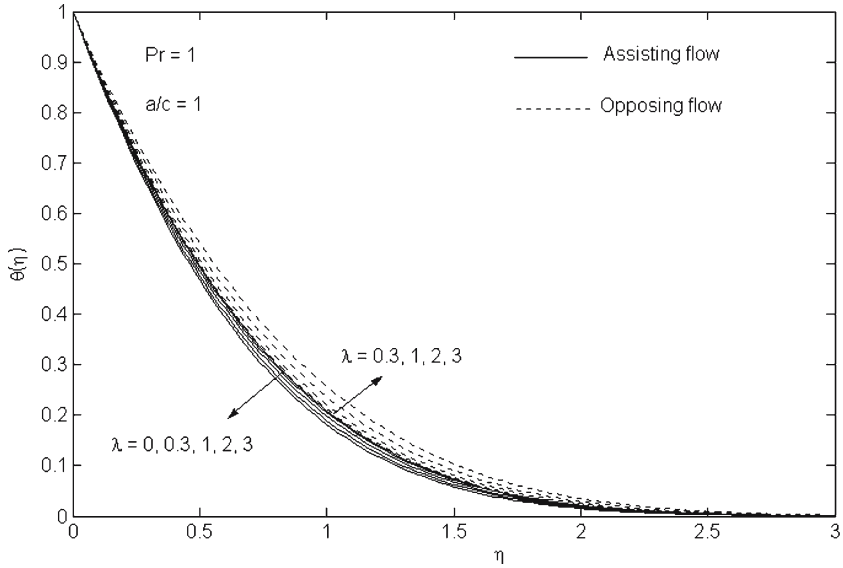

Fig. 9 Temperature profiles for some values of λ when Pr =1 and $a/c = 1$

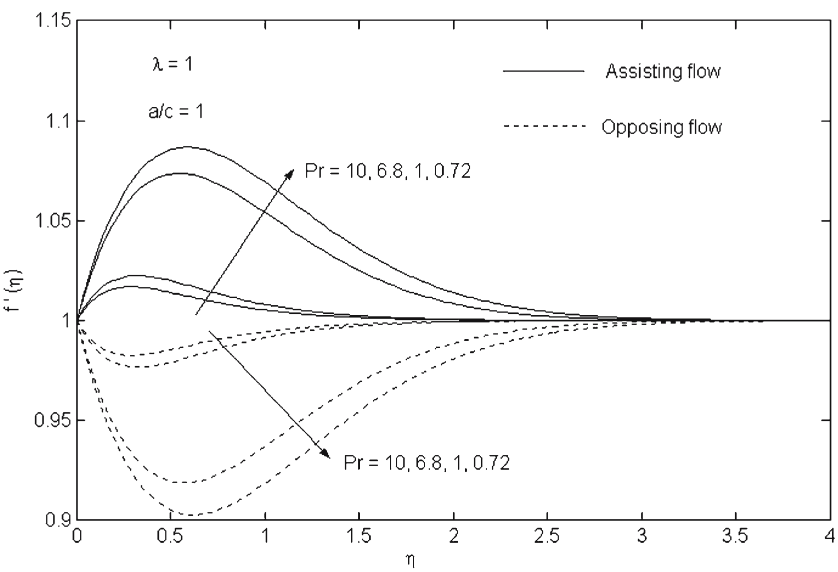

Fig. 10 Velocity profiles for some values of Pr when $\lambda = 1$ and $a/c = 1$

structure and the thickness of the boundary layer decreases with increase in a/c. According to Mahapatra and Gupta [25], it can be explained as follows: for a fixed value of c corresponding to the stretching of the surface, an increase in a in relation to c (such that $a/c > 1$) implies an increase in straining motion near the stagnation region resulting in increased acceleration of the external stream, and this leads to thinning of the boundary layer with an increase in a/c. Further, it is seen from Fig. 6 that when $a/c < 1$, the flow has an inverted boundary layer structure. It results from the fact that when $a/c < 1$, the stretching velocity cx of the surface exceeds the velocity ax of the external stream.

From Figs. 8–10, it is seen that for assisting flow, the velocity increases at the beginning until it achieves a certain value, then decreases until the value becomes constant, that is unity, at the outside

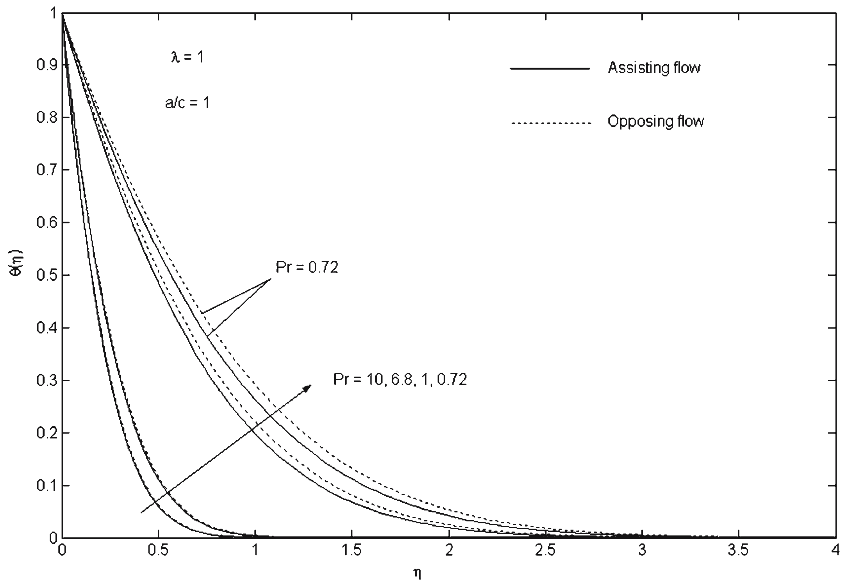

Fig. 11 Temperature profiles for some values of Pr when $\lambda = 1$ and $a/c = 1$

of the boundary layer. The effects of velocity are found to be more pronounced for larger λ. This is because, large value of λ produces large buoyancy force which produces large kinetic energy. Then the energy is used to overcome the resistant along the flow. As a result, it decreases and becomes constant far away from the surface. The opposite trend occurs for opposing flow. From Figs. 7, 9, and 11, it is observed that the temperature of the fluid decreases as the distance from the surface increases, for both cases of assisting and opposing flow, for all values of $a/c, \lambda$ and Pr until it achieves a constant value, namely zero. This is not surprising since the fluid receives the heat from the surface and then the heat energy is changed into other energy forms such as kinetic energy.

4. Conclusions

The steady two-dimensional stagnation-point flow of an incompressible fluid over a stretching vertical sheet in its own plane has been analyzed in detail. The problem is formulated in such a way that the stretching velocity and the surface temperature vary linearly with the distance from the stagnation-point. From an analytical investigation of the governing boundary layer equation, we have been able to deduce solutions for the non-dimensional velocity and temperature functions, skin friction coefficient and local Nusselt number. For the case of buoyancy force is absent ($\lambda = 0$), we have compared our present results with those of Mahapatra and Gupta [25], and Nazar et al. [11]. The agreement between the results is excellent. We are, therefore, confident that the present results are very accurate. Effects of the buoyancy parameter λ and Prandtl number Pr of the fluid on the flow and heat transfer characteristics have been examined and discussed in detail. It can be concluded that, for assisting flow, both the skin friction coefficient and the local Nusselt number increase as the parameter λ increases, while only the local Nusselt number increases but the skin friction coefficient decreases as Pr increases. For opposing flow, both the skin friction coefficient and the local Nusselt number decrease as the buoyancy parameter increases, but both are increase as Pr increases.

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