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RESEARCH

Analysis of Renewal Batch Arrival Queues with Multiple Vacations and Geometric Abandonment

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Abstract
We investigate the dynamics of customer geometric abandonment within a queueing framework characterized by renewal input batch arrivals and multiple vacations. Customers' impatience becomes evident when confronted with server vacations, triggering instances of abandonment. This phenomenon reduces the number of customers within the system during abandonment epochs following a geometric distribution. The probability of customers leaving the queue escalates with prolonged waiting times. We derive concise and closed-form expressions for system-length distributions at pre-arrival and arbitrary epochs by harnessing the power of supplementary variable and difference operator methods. Furthermore, we elucidate specific instances of our model, shedding light on its versatility. To substantiate our theoretical framework, we provide a series of illustrative numerical experiments presented through meticulously crafted tables and graphs, thereby showcasing the robustness and applicability of our methodology.

Keywords Renewal input · Batch arrival · Geometric abandonment · Difference operator · Roots · Vacations

Mathematics Subject Classification 60K25 · 68M20 · 90B22

1 Introduction

Numerous articles have investigated server vacations in queueing systems, analyzing situations where servers may become temporarily unavailable to serve jobs during specific time intervals. These scenarios are prevalent in real-life applications. For further insights into vacation queues and related topics, one may refer to Doshi (1986), Takagi (1991) and Tian

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and Zhang (2006). Another area of research focuses on impatient queueing systems, where impatience can arise due to server unavailability (vacations). Customer satisfaction decreases when customers experience longer waiting times during server vacations, often leading to abandonment and reluctance to return to the system, resulting in customers leaving without being served. The literature, along with references such as Barrer (1957), Daley (1965), Baccelli and Hebuterne (1981), Baccelli et al. (1984), De Kok and Tijms (1985), Goswami (2015), and Goswami and Mund (2020), delves into customer impatience within the queueing system. This queueing model plays a significant role in various real-life scenarios, including call centers, emergency rooms of hospitals, and inventory systems storing perishable goods. The literature explores multiple types of impatience, considering customer abandonments such as independent, binomial, and geometric.

Several researchers have extensively studied independent abandonments in queueing systems, as evidenced by the works of Altman and Yechiali (2006), Yechiali (2007), Perel and Yechiali (2010) and Dudin et al. (2023), and the references to these. These studies have provided a foundation for exploring more complex abandonment patterns, such as geometric and synchronized abandonment. Dimou and Economou (2013) analyzed the M/M/1 queue with geometric reneging and catastrophes, providing valuable insights into the impact of geometric abandonment and unexpected system failures. The concept of synchronized abandonment, involving various variations in simple Markovian queues, has been thoroughly discussed in the works of Economou (2004), Adan et al. (2009), Economou and Kapodistria (2010), Kapodistria (2011), Panda et al. (2016) and Panda and Goswami (2020). Adan et al. (2009) also examined the M/M/1 and M/G/1 multiple vacation queues with synchronized abandonment, considering both single and multiple abandonment epochs occurring during each vacation period. Goswami and Panda (2021) analyzed the renewal input multiple vacations queue with synchronized abandonment, contributing to a comprehensive understanding of abandonment patterns in complex queuing systems. Dimou et al. (2011) extended the study to encompass the M/M/1 queue with vacations and geometric abandonment, adding to the existing knowledge on the interplay between vacations and geometric abandonment in queueing systems. Furthermore, Sun et al. (2023) explored customers' strategic behavior in an observable N policy M/M/1 queue with geometric abandonments, shedding light on the influence of observable queue information on customer decision-making. These research contributions have significantly advanced our understanding of abandonment patterns in queueing systems, providing valuable insights that apply to real-life scenarios in diverse industries.

Queueing models that prioritize customers arriving in batches rather than individually find practical applications in various domains, including inventory and manufacturing systems. However, the current literature review highlights the need for more attention to the renewal batch arrival single server vacation queue with geometric abandonment. Fortunately, there is a simple and effective solution using the supplementary variable technique and the shift operator technique, which enables the derivation of analytical explicit results without the need for obtaining the Markov chain's transition probability matrix at arrival moments or inverting any probability-generating function. This method has gained popularity among researchers due to its ease of implementation and understanding (Barbhuiya and Gupta 2019; Barbhuiya et al. 2019; Yu and Tang 2022). A batch arrival single server vacation queue with geometric abandonment is a specific queuing system type encompassing multiple customer arrivals in batches, a single server responsible for handling these customers, and the possibility of customers abandoning the queue based on a geometric distribution. As customers arrive in batches, the system efficiently serves a single server. However, customers might leave the system during vacation if their waiting time surpasses a certain threshold. A geometric distribution accurately models the abandonment behavior, where the probability of customers leaving the queue increases as their waiting time escalates. This comprehensive model effectively captures real-world scenarios where customers may abandon the queue due to extended waiting times during server vacations.

A realistic understanding of customer behavior in batch arrival queueing models with vacations and geometric abandonment is essential due to their relevance in real-life applications. Travel, hospitality, and retail industries experience fluctuations in customer demands and service requirements during vacation. These peak vacation periods often witness a surge in customer traffic and longer service times due to increased demand and limited staff availability. In such scenarios, a geometric abandonment queueing system becomes relevant, where number of customers leaving the queue sequentially based on a geometric distribution if they have to wait beyond a certain threshold during server's absence. Implementing a vacation queueing model enables organizations to effectively manage their resources and meet customer needs during peak vacation periods. Despite its practical significance, the literature on this topic remains limited due to the analytical complexity arising from geometric abandonment. This paper addresses this gap by analyzing the abandonment issue in a renewal batch arrival and vacation queue setting. By delving into the dynamics of customer behavior in such systems, the study provides valuable insights for organizations to optimize their operations and enhance customer satisfaction during peak vacation periods. For instance, consider a customer support call center that operates year-round but experiences a surge in call volume during vacations. The call center has a limited number of operators available to handle the calls, which leads to longer average time between arrivals during vacation periods. To maintain a high service level, the call center aims to keep the average waiting time for customers minimal. If a customer waits beyond a certain threshold, there is a probability that they may abandon the queue. In such a setting, the call center can use a geometric abandonment queue model to analyze the system's performance during vacations. By employing the geometric abandonment queueing model, the call center can optimize staffing levels and other operational parameters to achieve the desired service level targets and minimize customer abandonment rates during peak vacation periods. This approach allows the call center to efficiently manage customer demands and enhance overall service quality during increased call volume.

The rest of the paper is organized into the following sections. Section 3 describes the model and governing equations. Section 4 delves into analyzing steady-state queue-length distributions at various epochs using the shift operator and the theory of difference equation method. Different performance descriptors of the system are presented in Section 5, while Section 6 explores several specific cases of the proposed model. In Section 7, various numerical examples illustrate the findings. Finally, Section 8 concludes the paper.

2 Call Center Application

A renewal batch arrival queue with vacations and geometric abandonment is a queuing system that incorporates several features to model real-world scenarios more accurately. Let us take an example of the call center. Call centers (Fig. 1) are one of the primary channels of communication between businesses and their clients in various industries.

Operations must be increasingly customized and cater to customers' needs as an essential requirement to meet their expectations. The quality of services can be measured using stochastic models in queuing theory. In a call center queueing model, users call to get information or resolve a particular issue, the servers are the system's agents providing services

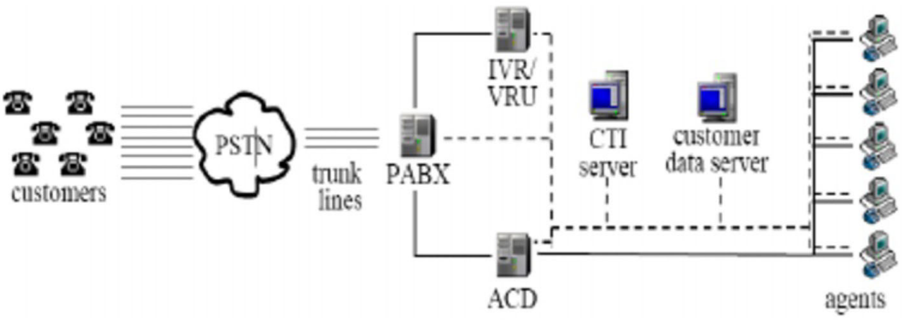

Fig. 1 Schematic diagram of a call center

to users, and the queues are the users waiting for assistance. Call Centers frequently use a queueing model to measure operational service quality in terms of performance metrics and congestion. Whitt (2005) examined the queueing model in call centers without considering the abandonment behavior of users, while literature (Baccelli and Hebuterne 1981; Garnett et al. 2002; Mandelbaum and Zeltyn 2005; Brown et al. 2005) examined models that included abandonment. The call center sector has increased, and due to this growth, managing call centers has become a complex business. Call center managers should choose a more appropriate set of actions to maintain strategic and operating decisions.

In the existing literature, researchers discussed the arrival of calls single at a time that follows Poisson distribution. There needs to be more discussion on the issue that multiple calls arrive at the queue simultaneously as a group or batch following a general distribution. However, as call centers expand, managing them becomes increasingly complex, necessitating data-driven decisions to improve strategic and operational aspects (Fig. 2). This study proposes a $GI^X/M/1 + M$ queue model with multiple vacations and geometric abandonment, where the impatience timer follows an exponential distribution. This approach effectively assesses congestion issues and supports design and operational decisions in call centers. The model considers incoming calls arriving in batches during specific time intervals, with call center agents taking scheduled breaks (vacations) during which no service is provided. Arriving callers may either wait until the planned vacation ends or abandon the queue due to extended waiting times. The exponential distribution helps to model abandonment patterns, where the probability of abandonment decreases with each attempt to get service. In other

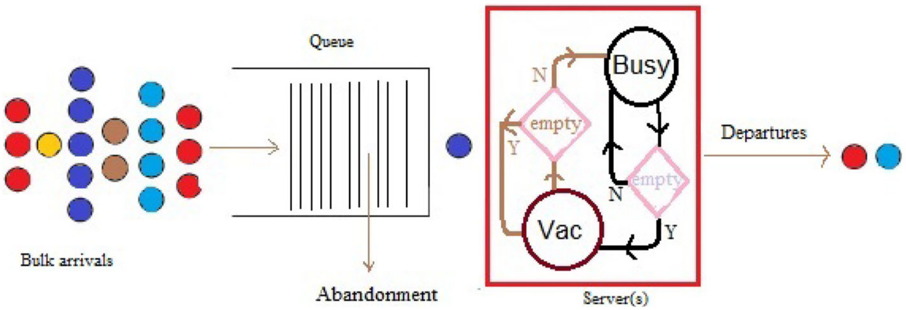

Fig. 2 A batch queueing model representation of a call center

words, the longer an entity waits in the queue, the more likely it is to abandon it. Applying this model enables businesses and organizations to assess call center performance, identify bottlenecks, and optimize resource allocation to enhance customer satisfaction and overall efficiency. Organizations can make data-driven decisions that improve service quality and customer experience by understanding call center dynamics through queuing models.

3 Description of the Model and Governing Equations

We consider a continuous-time $\mathrm{GI}^X/\mathrm{M}/1 + \mathrm{M}$ queue with multiple vacations and geometric abandonment. The key assumptions and description of the model are as follows:

- Customers arrive in batches of random size X with probability mass function (pmf) g_i for $i = 1, 2, \ldots$. The probability generating function (pgf) is denoted as $G(z) = \sum_{i=1}^{\infty} g_i z^i$, and the mean batch size is $\bar{g} = \sum_{i=1}^{\infty} i g_i$. Since real-world scenarios often involve finite batch sizes, we assume the maximum batch size is $\hbar$ for practical and computational reasons. Therefore, the pgf and the mean batch size become $G(z) = \sum_{i=1}^{\hbar} g_i z^i$ and $\bar{g} = \sum_{i=1}^{\hbar} i g_i$, respectively.
- The inter-arrival times of successive batch arrivals are independent and identically distributed (iid) random variables with cumulative distribution function $A(u)$, probability density function $a(u)$ for $u \geq 0$, Laplace-Stieltjes transform (LST) denoted as $A^*(s)$, and mean inter-arrival time $1/\lambda = -A^{*(1)}(s)$, where $A^{*(1)}(0)$ represents the first derivative of $A^*(s)$ evaluated at $s = 0$.
- The service times are exponentially distributed random variables with rate μ. Customers upon arrival join a single queue with infinite capacity and are served on a first-come-first-served basis.
- After a service completion, if the system becomes empty the server goes on vacation. Upon returning from vacation, if the server finds an empty system, it takes another vacation. Otherwise, they end the vacation and return to active mode to serve customers in the queue. Vacation times are exponentially distributed with rate η.
- During server vacations, customers may abandon the queue according to a Poisson process with rate φ. At each abandonment opportunity, customers are considered one by one sequentially. They either abandon the system with probability p or choose to stay in the system with probability q, where $p + q = 1$. Alternatively, we can also assume that at an abandonment opportunity epoch, the number of customers in the system decreases based on a geometric distribution.
- The traffic intensity of the system is given by $\rho = \lambda\bar{g}/\mu$, where $\rho < 1$ ensures the stability of the system under steady-state conditions.
  In summary, the $\mathrm{GI}^X/\mathrm{M}/1 + \mathrm{M}$ model accounts for the complexities of real-world scenarios, such as batch arrivals, server vacations, and customer abandonment, which are vital considerations for understanding and optimizing the performance of queueing systems in various practical applications.

We study steady-state system using the supplementary variable technique. The system state at time t can be described by a Markov process $\{(N(t), I(t), V(t)), t \geq 0\}$, with state space $\{(n, i) : n \geq i = 0, 1\} \times [0, \infty)$, where

- $N(t)$ is the number of customers in the system at time t,
- $I(t)$ is the server's state at time t, 0 if the server is in vacation, and 1 if the server is busy,
- $V(t)$ is the remaining inter-arrival time of the next arrival at time t.

Let us define the joint probabilities as

$$\pi_{k,0}(v,t)dv = P\left\{N(t)=k, v \leq V(t) < v+dv, I(t)=0\right\}, v \geq 0, \;\; k \geq 0,$$
$$\pi_{k,1}(v,t)dv = P\left\{N(t)=k, v \leq V(t) < v+dv, I(t)=1\right\}, \; v \geq 0, \; k \geq 1.$$

In steady-state, we have

$$\pi_{k,0}(v) = \lim_{t\to\infty} \pi_{k,0}(v,t), \; v \geq 0, \; k \geq 0,$$
$$\pi_{k,1}(v) = \lim_{t\to\infty} \pi_{k,1}(v,t), \; v \geq 0, \; k \geq 1.$$

By relating the states of the system at time t and $t+\Delta t$, using supplementary variable technique and taking $\lim_{t\to\infty}$ after simplification, we obtain the following set of difference-differential equations.

$$-\frac{d}{dv}\pi_{0,0}(v) = \mu\pi_{1,1}(v) + \varphi\sum_{j=1}^{\infty} p^j \pi_{j,0}(v), \tag{1}$$

$$\begin{aligned} -\frac{d}{dv}\pi_{k,0}(v) =& -(\varphi+\eta)\pi_{k,0}(v) + a(v)\sum_{i=1}^{k} g_i\pi_{k-i,0}(0) \\ &+ \varphi q\sum_{j=k}^{\infty} p^{j-k}\pi_{j,0}(v), \; 1 \leq k \leq \hbar-1, \end{aligned} \tag{2}$$

$$\begin{aligned} -\frac{d}{dv}\pi_{k,0}(v) =& -(\varphi+\eta)\pi_{k,0}(v) + a(v)\sum_{i=1}^{\hbar} g_i\pi_{k-i,0}(0) \\ &+ \varphi q\sum_{j=k}^{\infty} p^{j-k}\pi_{j,0}(v), \; k \geq \hbar, \end{aligned} \tag{3}$$

$$-\frac{d}{dv}\pi_{1,1}(v) = -\mu\pi_{1,1}(v) + \eta\pi_{1,0}(v) + \mu\pi_{2,1}(v), \tag{4}$$

$$\begin{aligned} -\frac{d}{du}\pi_{k,1}(v) =& -\mu\pi_{k,1}(v) + \mu\pi_{k+1,1}(v) \\ &+ a(v)\sum_{i=1}^{k-1} g_i\pi_{k-i,1}(0) + \eta\pi_{k,0}(v), \;\; 2 \leq k \leq \hbar, \end{aligned} \tag{5}$$

$$\begin{aligned} -\frac{d}{du}\pi_{k,1}(v) =& -\mu\pi_{k,1}(v) + \mu\pi_{k+1,1}(v) \\ &+ a(v)\sum_{i=1}^{\hbar} g_i\pi_{k-i,1}(0) + \eta\pi_{k,0}(v), \;\; k \geq \hbar+1. \end{aligned} \tag{6}$$

To obtain the steady-state probabilities $\pi_{k,i},\ k \geq i,\ i = 0, 1$, we introduce the following Laplace-Stieltjes transforms (LSTs).

$$\pi^*_{k,i}(s) = \int_0^\infty e^{-sv}\pi_{k,i}(v)dv,\ \ k \geq i,\ i = 0, 1,$$

$$\int_0^\infty e^{-sv}\frac{d}{dv}\pi_{k,i}(v)dv = s\pi^*_{k,i}(s) - \pi_{k,i}(0),\ \ n \geq i,\ i = 0, 1.,$$

$$\pi_{k,i} = \pi^*_{k,i}(0) = \int_0^\infty \pi_{k,i}(v)dv,\ \ k \geq i,\ i = 0, 1.$$

Multiplying (1) to (4) by $e^{-\theta v}$ and integrating with respect to v from 0 to ∞, we have

$$(\varphi - s)\pi^*_{0,0}(s) = \mu\pi^*_{1,1}(s) + \varphi\sum_{j=0}^{\infty} p^j\pi^*_{j,0}(s) - \pi_{0,0}(0), \tag{7}$$

$$\begin{aligned}(\varphi + \eta - s)\pi^*_{k,0}(s) &= A^*(s)\sum_{i=1}^{k} g_i\pi_{k-i,0}(0)\\ &\quad + \varphi\sum_{j=k}^{\infty} qp^{j-k}\pi^*_{j,0}(s) - \pi_{k,0}(0),\ 1 \leq k \leq \hbar - 1,\end{aligned} \tag{8}$$

$$\begin{aligned}(\varphi + \eta - s)\pi^*_{k,0}(s) &= A^*(s)\sum_{i=1}^{\hbar} g_i\pi_{k-i,0}(0)\\ &\quad + \varphi\sum_{j=k}^{\infty} qp^{j-k}\pi^*_{j,0}(s) - \pi_{k,0}(0),\ k \geq \hbar,\end{aligned} \tag{9}$$

$$(\mu - s)\pi^*_{1,1}(s) = \mu\pi^*_{2,1}(s) + \eta\pi^*_{1,0}(s) - \pi_{1,1}(0) \tag{10}$$

$$\begin{aligned}(\mu - s)\pi^*_{k,1}(s) &= \mu\pi^*_{k+1,1}(s) + \eta\pi^*_{k,0}(s)\\ &\quad + A^*(s)\sum_{i=1}^{k-1} g_i\pi_{k-i,1}(0) - \pi_{k,1}(0),\ \ 2 \leq k \leq \hbar,\end{aligned} \tag{11}$$

$$\begin{aligned}(\mu - s)\pi^*_{k,1}(s) &= \mu\pi^*_{k+1,1}(s) + \eta\pi^*_{k,0}(s)\\ &\quad + A^*(s)\sum_{i=1}^{\hbar} g_i\pi_{k-i,1}(0) - \pi_{k,1}(0),\ \ k \geq \hbar + 1.\end{aligned} \tag{12}$$

Adding Eqs. (7) to (12) for all values of n, and simplifying gives

$$\sum_{k=0}^{\infty}\pi^*_{k,0}(s) + \sum_{k=1}^{\infty}\pi^*_{k,1}(s) = \frac{1 - A^*(s)}{s}\left\{\sum_{k=0}^{\infty}\pi_{k,0}(0) + \sum_{k=1}^{\infty}\pi_{k,1}(0)\right\}. \tag{13}$$

Applying the normalization condition $\sum\limits_{k=0}^{\infty}\pi_{k,0} + \sum\limits_{k=1}^{\infty}\pi_{k,1} = 1$ in (13) and taking limit as $s \to 0$, we obtain the relation

$$\sum_{k=0}^{\infty}\pi_{k,0}(0) + \sum_{k=1}^{\infty}\pi_{k,1}(0) = \lambda. \tag{14}$$

The left-hand side of (14) refers to the average number of system entries per unit of time and is equal to the average arrival rate λ.

4 System-Length Distribution

We define the shift operator E for the sequences $\left\{\pi_{k,\ell}(0)\ k \geq \ell\right\}$ and $\left\{\pi^*_{k,\ell}(s)\ k \geq \ell\right\}$ $(\ell = 0, 1)$ by $E\pi^*_{k,\ell}(s) = \pi^*_{k+1,\ell}(s)$ and $E\pi_{k,\ell}(0) = \pi_{k+1,\ell}(0),\ \ell = 0, 1$. Applying the shift operator to Eq. (8) and simplifying, we get

$$\left(s - \eta - \varphi + \varphi q \sum_{\ell=0}^{\infty} p^{\ell} E^{\ell}\right) \pi^*_{k,0}(s) = \left(E^{\hbar} - A^*(s) \sum_{i=1}^{\hbar} g_i E^{\hbar-i}\right) \pi_{k-\hbar,0}(0),\ k \geq \hbar. \tag{15}$$

Setting $s = \eta + \varphi - \varphi q \sum_{\ell=0}^{\infty} p^{\ell} E^{\ell}$ in (15), we have

$$\left[E^{\hbar} - A^*\left(\eta + \varphi - \frac{\varphi\, q}{1 - pE}\right) \sum_{i=1}^{\hbar} g_i E^{\hbar-i}\right] \pi_{k,0}(0) = 0,\ k \geq 0. \tag{16}$$

The characteristic equation representing to (16) is

$$z^{\hbar} - A^*\left(\eta + \varphi - \frac{\varphi\, q}{1 - pz}\right) \sum_{i=1}^{\hbar} g_i z^{\hbar-i} = 0,\ \text{since } |pz| < 1. \tag{17}$$

Theorem 1 *The characteristic equation* $z^{\hbar} - A^*\left(\eta + \frac{\varphi\, p(1-z)}{1-pz}\right) \sum_{i=1}^{\hbar} g_i z^{\hbar-i} = 0$ *has precisely* $\hbar$ *roots inside the unit circle.*

Proof Let the functions $g(z) = z^{\hbar}$ and $k(z) = -A^*\left(\eta + \frac{\varphi\, p(1-z)}{1-pz}\right) \sum_{i=1}^{\hbar} g_i z^{\hbar-i}$ be analytic in the circle $|z| < 1$. Consider $H(z) = A^*\left(\eta + \frac{\varphi\, p(1-z)}{1-pz}\right)$. For a sufficiently small $\epsilon > 0$, $H(z)$ is holomorphic on and inside the closed disk $|z| = 1 + \epsilon$. There exists a power series $\sum_{\ell=0}^{\infty} h_k (z-1)^{\ell}$ which converges to $H(z)$ for complex analytic function in conformity with Taylor's theorem, where the coefficients $h_k = \frac{H^{\ell}(1)}{\ell!}$. Applying the Taylor series expansion for $H(z)$, on the simple closed curve $|z| = 1 - \delta$, where $\delta > 0$ and is sufficiently small, we have

$$\begin{aligned}
|g(z)| &= (1-\delta)^{\hbar} = 1 - \hbar\delta + o(\delta),\\
|k(z)| &= |H(z)| \left|\sum_{i=1}^{\hbar} g_i z^{\hbar-i}\right| \leq H(|z|) \sum_{i=1}^{\hbar} g_i |z|^{\hbar-i} = H(1-\delta) \sum_{i=1}^{\hbar} g_i (1-\delta)^{\hbar-i}\\
&= \left\{H(1) + \frac{H^{'}(1)}{1!}(1-\delta-1) + \sum_{\ell=2}^{\infty} \frac{H^{(\ell)}(1)}{\ell!}(1-\delta-1)^{\ell}\right\} \sum_{i=1}^{\hbar} g_i\, [1 - \hbar\delta + o(\delta)]\\
&= 1 - \hbar\delta + \left(\bar{g} - \frac{\mu\, p}{\lambda\, q}\right)\delta + o(\delta) \leq |g(z)|
\end{aligned}$$

So, from Rouché's theorem, one can state that $g(z)$ and $g(z) + k(z)$ have precisely $\hbar$ zeros inside the unit disk. □

Considering the roots of (17) have $\hbar$ distinct roots inside the unit circle of the complex plane and denoted by $\omega_1, \omega_2, \ldots, \omega_\hbar$. Thus, the general solution of (16) leads to the form

$$\pi_{k,0}(0) = \sum_{j=1}^{\hbar} c_j \omega_j^k, \quad k \geq 0, \tag{18}$$

where $c_j,\ j = 1, 2, \ldots, \hbar$ are arbitrary constants which are to be found. Putting (18) in (15), we have

$$\left(s - \eta - \varphi + \varphi q \sum_{\ell=0}^{\infty} p^\ell E^\ell \right) \pi^*_{k,0}(s) = \left(E^\hbar - A^*(s) \sum_{i=1}^{\hbar} g_i E^{\hbar-i}\right) \sum_{j=1}^{\hbar} c_j \omega_j^{k-\hbar}, \ k \geq \hbar. \tag{19}$$

Equation (19) is a non-homogeneous difference equation. The corresponding homogeneous part of (19) is $\pi^{*(hom)}_{k,0}(s) = C_1 \left(\frac{s-\eta-\varphi p}{p(s-\eta-\varphi)}\right)^k$, where C_1 is an arbitrary constant. The particular solution of (19) is

$$\pi^{*(par)}_{k,0}(s) = \sum_{j=1}^{\hbar} c_j \left\{ \frac{\omega_j^\hbar - A^*(s) \sum_{i=1}^{\hbar} g_i \omega_j^{\hbar-i}}{s - \eta - \varphi + \frac{\varphi\, q}{1-pz}} \right\} \omega_j^{k-\hbar}, \quad k \geq \hbar, \tag{20}$$

and thus, the general solution is of the form

$$\pi^*_{k,0}(s) = C_1 \left(\frac{s - \eta - \varphi p}{p(s - \eta - \varphi)}\right)^k + \sum_{j=1}^{\hbar} c_j \left\{ \frac{\omega_j^\hbar - A^*(s) \sum_{i=1}^{\hbar} g_i \omega_j^{\hbar-i}}{s - \eta - \varphi + \frac{\varphi\, q}{1-pz}} \right\} \omega_j^{k-\hbar}, \quad k \geq \hbar, \tag{21}$$

The undetermined constant $C_1 = 0$, as $s \to 0$ and $\sum_{k=\hbar}^{\infty} \pi^*_{k,0}(s) = \sum_{k=\hbar}^{\infty} \pi_{k,0} < 1$, otherwise $\sum_{k=\hbar}^{\infty} \pi_{k,0}$ will diverge. Therefore, (21) reduces to

$$\pi^*_{k,0}(s) = \sum_{j=1}^{\hbar} c_j \left\{ \frac{\omega_j^\hbar - A^*(s) \sum_{i=1}^{\hbar} g_i \omega_j^{\hbar-i}}{s - \eta - \varphi + \frac{\varphi\, q}{1-pz}} \right\} \omega_j^{k-\hbar}, \quad k \geq \hbar. \tag{22}$$

Now, we try to find the solution under which $\pi^*_{k,0}(s)$ has the same type as given in (22) for $1 \leq k \leq \hbar - 1$. For this, putting (18) into (8) and (9), and equating the first term of the right hand side of the Eqs. (8) and (9), we observe that c_j the unknown constants meet the relationship

$$\begin{aligned} A^*(s) \sum_{i=1}^{k} g_i \sum_{j=1}^{\hbar} c_j \omega_j^{k-i} &= A^*(s) \sum_{i=1}^{\hbar} g_i \sum_{j=1}^{\hbar} c_j \omega_j^{k-i} \\ \Rightarrow A^*(s) \sum_{j=1}^{\hbar} c_j \sum_{i=k+1}^{\hbar} g_i \omega_j^{k-i} &= 0, \quad 1 \leq k \leq \hbar - 1. \end{aligned} \tag{23}$$

Substituting $k = \hbar - 1, \hbar - 2, \ldots, 1$, recursively in (23), and remarking that $g_\hbar \neq 0$, we obtain the following set of $\hbar - 1$ equations.

$$\sum_{j=1}^{\hbar} \frac{c_j}{\omega_j} = \sum_{j=1}^{\hbar} \frac{c_j}{\omega_j^2} = \cdots = \sum_{j=1}^{\hbar} \frac{c_j}{\omega_j^{\hbar-2}} = \sum_{j=1}^{\hbar} \frac{c_j}{\omega_j^{\hbar-1}} = 0. \tag{24}$$

If the above results hold true, for any $k \geq 1$,

$$\pi^*_{k,0}(s) = \sum_{j=1}^{\hbar} c_j \left\{ \frac{\omega_j^k - A^*(s) \sum_{i=1}^{\hbar} g_i \omega_j^{k-i}}{s - \eta - \varphi + \frac{\varphi q}{1-pz}} \right\}. \tag{25}$$

We again apply the displacement operator E in (12), which yields

$$(s - \mu + \mu E)\, \pi^*_{k,1}(s) = \left(E^{\hbar} - A^*(s) \sum_{i=1}^{\hbar} g_i E^{\hbar-i} \right) \pi_{k-\hbar,1}(0) - \eta \pi^*_{k,0}(s),\ k \geq \hbar + 1. \tag{26}$$

Setting $s = \mu - \mu E$ in (26) and putting (25) into (26) gives

$$\begin{aligned} &\left[E^{\hbar} - A^*\left(\mu - \mu E\right) \sum_{i=1}^{\hbar} g_i E^{\hbar-i} \right] \pi_{k,1}(0) = \eta \pi^*_{k,0}(\mu - \mu E) \\ &= \eta \sum_{j=1}^{\hbar} c_j \left\{ \frac{\omega_j^{\hbar} - A^*(\mu - \mu E) \sum_{i=1}^{\hbar} g_i \omega_j^{\hbar-i}}{\mu - \mu E - \eta - \varphi + \frac{\varphi q}{1-p\omega_j}} \right\} \omega_j^k,\ k \geq 1. \end{aligned} \tag{27}$$

Employing Rouché's theorem as used earlier, the following Theorem 2 expresses that under specific conditions, the characteristic equation of the above difference equation also has exactly $\hbar$ roots inside the unit disk, let us denote $\hbar$ roots by $\psi_1, \psi_2, \ldots, \psi_{\hbar}$.

Theorem 2 *The characteristic equation $z^{\hbar} - A^*\left(\mu - \mu z\right) \sum_{i=1}^{\hbar} g_i z^{\hbar-i} = 0$ has precisely $\hbar$ roots inside the unit circle, if $\frac{\lambda \bar{g}}{\mu} < 1$.*

By similar procedure to find $\pi^*_{k,0}(s)$, the general solution of (27) is given by

$$\pi_{k,1}(0) = \sum_{j=1}^{\hbar} d_j \psi_j^k + \eta \sum_{j=1}^{\hbar} \frac{c_j \omega_j^k}{\mu(1 - \omega_j) - \eta - \varphi + \frac{\varphi q}{1-p\omega_j}},\ k \geq 1, \tag{28}$$

where $d_1, d_2, \ldots, d_{\hbar}$ in the first part of right hand side (28) are the arbitrary constants affiliated with the solution of homogeneous Eq. (27). Furthermore, the second part of (27) is a particular solution. Putting (25) and (28) into the right hand side (26), we get

$$\begin{aligned} (s - \mu + \mu E)\, \pi^*_{k,1}(s) = &\left(E^{\hbar} - A^*(s) \sum_{i=1}^{\hbar} g_i E^{\hbar-i} \right) \left\{ \sum_{j=1}^{\hbar} d_j \psi_j^{k-\hbar} \right. \\ &\left. + \eta \sum_{j=1}^{\hbar} \frac{c_j \omega_j^{k-\hbar}}{\mu(1 - \omega_j) - \eta - \varphi + \frac{\varphi q}{1-p\omega_j}} \right\} \\ &- \eta \sum_{j=1}^{\hbar} c_j \left\{ \frac{\omega_j^{\hbar} - A^*(s) \sum_{i=1}^{\hbar} g_i \omega_j^{\hbar-i}}{s - \eta - \varphi + \frac{\varphi q}{1-p\omega_j}} \right\} \omega_j^{k-\hbar} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\hbar} d_j \left(\psi_j^k - A^*(s) \sum_{i=1}^{\hbar} g_i \psi_j^{k-i} \right) \\
&\quad + \eta \sum_{j=1}^{\hbar} c_j \left\{ \frac{\omega_j^k - A^*(s) \sum_{i=1}^{\hbar} g_i \omega_j^{k-i}}{\mu(1-\omega_j) - \eta - \varphi + \frac{\varphi\, q}{1-p\omega_j}} \right\} \\
&\quad - \eta \sum_{j=1}^{\hbar} c_j \left\{ \frac{\omega_j^k - A^*(s) \sum_{i=1}^{\hbar} g_i \omega_j^{k-i}}{s - \eta - \varphi + \frac{\varphi\, q}{1-p\omega_j}} \right\}, \ k \geq \hbar + 1. \qquad (29)
\end{aligned}
$$

The general solution of the associated homogeneous equation of (29) has the form $\pi_{k,1}^{*(hom)}(s) = C_2 \left(1 - \frac{s}{\mu})\right)^k$, where C_2 is an arbitrary constant. For more details about difference equations, one may refer Eladyi (2005). The particular solution of (29) after simplification is

$$
\begin{aligned}
\pi_{k,1}^{*(par)}(s) &= \sum_{j=1}^{\hbar} \frac{d_j \left(\psi_j^k - A^*(s) \sum_{i=1}^{\hbar} g_i \psi_j^{k-i} \right)}{s - \mu + \mu\psi_j} \\
&\quad + \sum_{j=1}^{\hbar} \frac{\eta c_j \left(\omega_j^k - A^*(s) \sum_{i=1}^{\hbar} g_i \omega_j^{k-i} \right)}{\left(\left(\mu - \frac{\varphi\, p}{1-p\omega_j} \right)(1-\omega_j) - \eta \right) \left(s - \eta - \frac{\varphi\, p(1-\omega_j)}{1-p\omega_j} \right)}
\end{aligned} \qquad (30)
$$

Thus, for $k \geq \hbar + 1$, the general solution of (29) may be rewritten as $\pi_{k,1}^*(s) = \pi_{k,1}^{*(hom)}(s) + \pi_{k,1}^{*(par)}(s)$. Adding over all k from $\hbar$ to ∞ and assuming the limit as $k \to 0$, $\sum_{k=\hbar}^{\infty} \pi_{k,1}^*(0) = \sum_{k=\hbar}^{\infty} \pi_{k,1} \leq 1$ clearly holds. Thus, $C_2 = 0$. Thus, the solution of (29) has the following form

$$
\begin{aligned}
\pi_{k,1}^*(s) &= \sum_{j=1}^{\hbar} \frac{d_j \left(\psi_j^k - A^*(s) \sum_{i=1}^{\hbar} g_i \psi_j^{k-i} \right)}{s - \mu + \mu\psi_j} \\
&\quad + \sum_{j=1}^{\hbar} \frac{\eta c_j \left(\omega_j^k - A^*(s) \sum_{i=1}^{\hbar} g_i \omega_j^{k-i} \right)}{\left(\left(\mu - \frac{\varphi\, p}{1-p\omega_j} \right)(1-\omega_j) - \eta \right) \left(s - \eta - \frac{\varphi\, p(1-\omega_j)}{1-p\omega_j} \right)}, \ k \geq \hbar + 1.
\end{aligned} \qquad (31)
$$

Now, we obtain the condition under which the expression for $\pi_{k,1}^*(s)$ given in (11) also true when $2 \leq k \leq \hbar$. Putting Eqs. (25) and (29) into Eqs. (11) and (12), respectively. Comparing the third term of the right hand side of Eqs. (11) and (12), we get

$$
\sum_{j=1}^{\hbar} d_j \sum_{i=k}^{\hbar} g_i \psi_j^{k-i} + \sum_{j=1}^{\hbar} \frac{\eta c_j}{\left(\mu - \frac{\varphi\, p}{1-p\omega_j} \right)(1-\omega_j) - \eta} \sum_{i=k}^{\hbar} g_i \omega_j^{k-i} = 0, \ \ 2 \leq k \leq \hbar. \qquad (32)
$$

Substituting $k = \hbar, \hbar - 1, \ldots, 2$ in (32) and letting that $g_\hbar \neq 0$. Now, (32) may be written in linear equation with variables d_j and c_j $1 \leq j \leq \hbar$ as

$$\begin{cases} \sum_{j=1}^{\hbar} d_j + \sum_{j=1}^{\hbar} \frac{\eta c_j}{\left[\left(\mu - \frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta\right]} & = 0 \\ \sum_{j=1}^{\hbar} \frac{d_j}{\psi_j} + \sum_{j=1}^{\hbar} \frac{\eta c_j}{\left[\left(\mu - \frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta\right]\omega_j} & = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \sum_{j=1}^{\hbar} \frac{d_j}{\psi_j^{\hbar-3}} + \sum_{j=1}^{\hbar} \frac{\eta c_j}{\left[\left(\mu - \frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta\right]\omega_j^{\hbar-3}} & = 0 \\ \sum_{j=1}^{\hbar} \frac{d_j}{\psi_j^{\hbar-2}} + \sum_{j=1}^{\hbar} \frac{\eta c_j}{\left[\left(\mu - \frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta\right]\omega_j^{\hbar-2}} & = 0 \end{cases} \tag{33}$$

Therefore, for $k \geq 2$ has the following expression

$$\begin{aligned} \pi_{k,1}^*(s) = & \sum_{j=1}^{\hbar} \frac{d_j\left(1 - A^*(s)G(\psi_j^{-1})\right)\psi_j^k}{s - \mu + \mu\psi_j} \\ & + \sum_{j=1}^{\hbar} \frac{\eta c_j\left(1 - A^*(s)G(\omega_j^{-1})\right)\omega_j^k}{\left(\left(\mu - \frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta\right)\left(s - \eta - \frac{\varphi\, p(1-\omega_j)}{1-p\omega_j}\right)}, \ k \geq 2. \end{aligned} \tag{34}$$

Setting $s = \mu$ in (10), we have

$$\pi_{11}(0) - \mu\pi_{21}^*(\mu) - \eta\pi_{10}^*(\mu) = 0 \tag{35}$$

Using (28) and (34) in (35) reduces to

$$\sum_{j=1}^{\hbar} d_j\psi_j G(\psi_j^{-1}) + \sum_{j=1}^{\hbar} \frac{\eta c_j\omega_j G(\omega_j^{-1})}{\left(\mu - \frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j) - \eta} = 0. \tag{36}$$

Using (14), we have after simplification

$$\sum_{j=1}^{\hbar} \frac{d_j\psi_j}{1-\psi_j} + \sum_{j=1}^{\hbar} \frac{c_j\left[(\mu-\eta)(1-\omega_j) - \varphi + \frac{\varphi\, q}{1-p\omega_j}\right]}{\left(\left(\mu - \frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta\right)(1-\omega_j)} = \lambda. \tag{37}$$

4.1 Pre-arrival and Arbitrary Epoch Probabilities

Let $\{\pi_{k,j}^-\},\ k \geq j,\ j = 0, 1$ denote the pre-arrival epoch probability, that is, an arrival sees k customers in the system and the server is in state j at arrival epoch. Applying Bayes' theorem, we have

$$\pi_{i,j}^- = \lim_{t\to\infty} \frac{P[N(t) = k, I(t) = i, V(t) = 0]}{P[V(t) = 0]}, \quad j = 0, 1.$$

Further, using (14) in the above expression, we obtain

$$\pi_{k,j}^- = \frac{1}{\lambda}\pi_{k,j}(0),\ n \geq j,\ j = 0, 1. \tag{38}$$

We obtain pre-arrival epoch probability from the Eqs. (18), and (28) as

$$\pi^-_{k,i} = \begin{cases} \frac{1}{\lambda}\sum\limits_{j=1}^{\hbar} c_j\omega_j^k, \ k \geq 0, & \text{for } i = 0 \\ \frac{1}{\lambda}\left[\sum\limits_{j=1}^{\hbar} d_j\psi_j^k + \eta\sum\limits_{j=1}^{\hbar}\frac{c_j\omega_j^k}{\left(\mu - \frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta}\right], \ k \geq 1 & \text{for } i = 1. \end{cases} \tag{39}$$

We obtain the probabilities at arbitrary epoch, $\pi_{k,j}, k \geq j, j = 0, 1$ from the corresponding expressions of $\pi^*_{k,j}(0)$. Setting $s = 0$ in Eqs. (25) and (34), we obtain the arbitrary epoch probabilities as

$$\pi_{k,i} = \begin{cases} \sum\limits_{j=1}^{\hbar}\frac{c_j\left(G(\omega_j^{-1})-1\right)\omega_j^k}{\eta+\varphi-\frac{\varphi\, q}{1-p\omega_j}}, \ k \geq 1, & \text{for } i = 0 \\ \sum\limits_{j=1}^{\hbar}\frac{d_j\left(G(\psi_j^{-1})-1\right)\psi_j^k}{\mu-\mu\psi_j} + \sum\limits_{j=1}^{\hbar}\frac{\eta c_j\left(G(\omega_j^{-1})-1\right)\omega_j^k}{\left(\left(\mu-\frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta\right)\left(\eta+\frac{\varphi\, p(1-\omega_j)}{1-p\omega_j}\right)}, \ k \geq 1 & \text{for } i = 1 \end{cases} \tag{40}$$

Applying the normalizing condition, we obtain

$$\pi_{0,0} = 1 - \sum_{j=1}^{\hbar}\frac{d_j\psi_j\left(\psi_j G(\psi_j^{-1})(2-\psi_j)-1\right)}{\mu(1-\psi_j)^2} - \sum_{j=1}^{\hbar}\frac{c_j\omega_j}{(1-\omega_j)\left(\eta+\frac{\varphi\, p(1-\omega_j)}{1-p\omega_j}\right)}$$
$$\times\left(G(\omega_j^{-1}) - 1 + \frac{\eta\left[\left(\mu G(\omega_j^{-1})-1\right) - \left(\eta+\frac{\varphi\, p(1-\omega_j)}{1-p\omega_j}\right)G(\omega_j^{-1})(1-\omega_j)\right]}{\mu\left(\left(\mu-\frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta\right)}\right) \tag{41}$$

5 Performance Indices

The probability that the server is in a vacation $P\{I = 0\}$ and the probability that the server is in a busy mode $P\{I = 1\}$ are respectively,

$$P\{I = 0\} = \sum_{k=0}^{\infty}\pi_{k,0} = 1 - \sum_{j=1}^{\hbar}\frac{d_j\psi_j\left(\psi_j G(\psi_j^{-1})(2-\psi_j)-1\right)}{\mu(1-\psi_j)^2}$$
$$-\sum_{j=1}^{\hbar}\frac{\eta c_j\omega_j\left[\left(\mu G(\omega_j^{-1})-1\right)-\left(\eta+\frac{\varphi\, p(1-\omega_j)}{1-p\omega_j}\right)G(\omega_j^{-1})(1-\omega_j)\right]}{\mu(1-\omega_j)\left(\left(\mu-\frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta\right)\left(\eta+\frac{\varphi\, p(1-\omega_j)}{1-p\omega_j}\right)}$$

$$P\{I = 1\} = \sum_{k=1}^{\infty}\pi_{k,1} = \sum_{j=1}^{\hbar}\frac{d_j\psi_j\left(\psi_j G(\psi_j^{-1})(2-\psi_j)-1\right)}{\mu(1-\psi_j)^2}$$
$$+\sum_{j=1}^{\hbar}\frac{\eta c_j\omega_j\left[\left(\mu G(\omega_j^{-1})-1\right)-\left(\eta+\frac{\varphi\, p(1-\omega_j)}{1-p\omega_j}\right)G(\omega_j^{-1})(1-\omega_j)\right]}{\mu(1-\omega_j)\left(\left(\mu-\frac{\varphi\, p}{1-p\omega_j}\right)(1-\omega_j)-\eta\right)\left(\eta+\frac{\varphi\, p(1-\omega_j)}{1-p\omega_j}\right)}.$$

The mean system length (L) is

$$L = \sum_{j=1}^{\hbar} \frac{d_j \psi_j \left[G(\psi_j^{-1})(\psi_j^3 - 3\psi_j^2 + 3\psi_j) - 1 \right]}{\mu (1-\psi_j)^3} + \sum_{j=1}^{\hbar} \frac{c_j \omega_j}{(1-\omega_j)^2 \left(\eta + \frac{\varphi\, p(1-\omega_j)}{1-p\omega_j} \right)}$$
$$\times \left\{ \left(G(\omega_j^{-1}) - 1 \right) + \frac{\eta \left[\left(\mu G(\omega_j^{-1}) - 1 \right) - \left(\eta + \frac{\varphi\, p(1-\omega_j)}{1-p\omega_j} \right) G(\omega_j^{-1})(1-\omega_j)^2 \right]}{\mu \left(\left(\mu - \frac{\varphi\, p}{1-p\omega_j} \right)(1-\omega_j) - \eta \right)} \right\}.$$

The mean queue length (L_q) is

$$L_q = \sum_{j=1}^{\hbar} \frac{d_j \psi_j^2 \left(G(\psi_j^{-1} - 1 \right)}{\mu (1-\psi_j)^3}$$
$$+ \sum_{j=1}^{\hbar} \frac{c_j \omega_j^2 \left(G(\omega_j^{-1} - 1 \right) \left(\mu - \frac{\varphi\, p}{1-p\omega_j} \right)}{(1-\omega_j) \left(\eta + \frac{\varphi\, p(1-\omega_j)}{1-p\omega_j} \right) \left(\left(\mu - \frac{\varphi\, p}{1-p\omega_j} \right)(1-\omega_j) - \eta \right)}.$$

The mean abandonment rate (AR) is

$$AR = \varphi \sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} q p^{\ell-k} \pi_{k.0} = \sum_{j=1}^{\hbar} \frac{c_j \varphi\, q \left(G(\omega_j^{-1}) - 1 \right)}{(1-\omega_j) \left[\eta(1-p\omega_j) + \varphi p(1-\omega_j) \right]}.$$

The average sojourn time in the system (W) and in the queue (W_q) using Little's formula is given by $W = \frac{L}{\lambda}$ and $W_q = \frac{L_q}{\lambda}$, respectively.

6 Particular Cases

In this section, we find some particular cases from our model by assuming set of values for the parameters $\hbar$, g_i, φ, η.

6.1 GI/M/1 + M Queue with Geometric Abandonment and Multiple Vacations

Taking $\hbar = 1, g_1 = 1$ and $g_j = 0,\ j \geq 2$, the model reduces to GI/M/1 + M queue with geometric abandonment and multiple vacations. In this case, the characteristic equations are

$$z - A^* \left(\eta + \frac{\varphi\, p(1-z)}{1-pz} \right) = 0 \ \text{ and } \ z - A^* (\mu - \mu z) = 0,$$

respectively, and both have only one root inside the unit disk, say ω_1 and ψ_1. Solving (36) and (37), we can find the associated arbitrary constants c_1 and d_1 as

$$c_1 = \frac{\lambda(1-\psi_1) \left[\left(\mu - \frac{\varphi\, p}{1-p\omega_j} \right)(1-\omega_1) - \eta \right]}{(1-\psi_1) \left(\mu - \eta - \frac{\varphi\, p}{1-p\omega_j} \right) - \eta\omega_1}$$

$$d_1 = \frac{-c_1 \eta \omega_1}{\psi_1 \left\{ \left(\mu - \frac{\varphi\, p}{1-p\omega_1} \right)(1-\omega_1) - \eta \right\}}$$

Thus, from (39), we have pre-arrival epoch probabilities as

$$\pi^-_{k,0} = \frac{1}{\lambda} c_1 \omega_1^k, \quad k \geq 0,$$
$$\pi^-_{k,1} = \frac{1}{\lambda} \left[d_1 \psi_1^k + \frac{c_1 \eta \omega_1^k}{\left(\mu - \frac{\varphi p}{1-p\omega_1}\right)(1-\omega_1) - \eta} \right], \ k \geq 1.$$

We obtain the arbitrary epoch probabilities from (40) and (41) as

$$\pi_{k,0} = \frac{c_1(1-\omega_1)\omega_1^{k-1}}{\eta + \frac{\varphi p}{1-p\omega_1}}, \ k \geq 1,$$
$$\pi_{k,1} = \frac{c_1 \eta \omega_1}{\left(\mu - \frac{\varphi p}{1-p\omega_1}\right)(1-\omega_1) - \eta} \left[\frac{(1-\omega_1)\omega_1^{k-2}}{\eta + \frac{\varphi p}{1-p\omega_1}} - \frac{\psi_1^{k-2}}{\mu} \right], \ k \geq 2,$$
$$\pi_{1,1} = \frac{d_1 \eta}{\mu \left(\eta + \frac{\varphi p}{1-p\omega_1}\right)}.$$

6.2 M/M/1 + M Queue with Geometric Abandonment and Multiple Vacations

Assuming $\hbar = 1, g_1 = 1, g_j = 0, \ j \geq 2$, and exponential inter-arrival time, the model reduces to M/M/1 + M queue with geometric abandonment and multiple vacations. So, $A^*(s) = \frac{\lambda}{\lambda+s}$. Therefore, the single root inside the unit disk, say ω_1 and ψ_1 are

$$\omega_1 = \frac{\lambda + \lambda p + \eta + \varphi p - \sqrt{(\lambda + \lambda p + \eta + \varphi p)^2 - 4\lambda p(\lambda + \eta + \varphi)}}{2\lambda} \text{ and } \psi_1 = \frac{\lambda}{\mu},$$

respectively. The ω_1 matches with the results of Dimou et al. (2011). From (39)–(41), we get

$$\pi^-_{k,0} = \pi_{k,0}, \ k \geq 0, \ \ \pi^-_{k,1} = \pi_{k,1}, \ \ k \geq 1.$$

We may also obtain the above from subsection 6.1 directly.

6.3 GI^X/M/1 Queue with Multiple Vacations

Taking $\varphi = 0$, the model reduces to GI^X/M/1 queue with multiple vacations but not geometric abandonment. In this case, (17) reduces to

$$z^{\hbar} - A^*(\eta) \sum_{i=1}^{\hbar} g_i z^{\hbar - i} = 0.$$

Thus, $\omega_j, j = 1, 2, \ldots, \hbar$ are the roots of the above characteristic equation. But the roots $\psi_j, j = 1, 2, \ldots, \hbar$ remains same. The unknowns c_j and d_j, $j = 1, 2, \ldots, \hbar$ can be found using (24) and the below $\hbar + 1$ equations.

$$\sum_{j=1}^{\hbar}\frac{d_j}{\psi_j^{\ell-1}}+\sum_{j=1}^{\hbar}\frac{\eta c_j}{\left[\mu(1-\omega_j)-\eta\right]\omega_j^{\ell-1}}=0,\ \ell=1,2,\ldots,\hbar-1,$$

$$\sum_{j=1}^{\hbar}d_j\psi_jG(\psi_j^{-1})+\sum_{j=1}^{\hbar}\frac{\eta c_j\omega_jG(\omega_j^{-1})}{\mu(1-\omega_j)-\eta}=0.$$

$$\sum_{j=1}^{\hbar}\frac{d_j\psi_j}{1-\psi_j}+\sum_{j=1}^{\hbar}\frac{c_j(\mu-\eta)}{\mu(1-\omega_j)-\eta}=\lambda.$$

From (39)–(41), we get pre-arrival and arbitrary epoch probabilities.

6.4 GI/M/1 Queue with Multiple Vacations

Taking $\varphi=0, \hbar=1, g_1=1, g_j=0,\ \ j\geq 2$, the model reduces to GI/M/1 queue with multiple vacations. Here the single roots inside the unit disk are ω_1 and ψ_1.

$$\pi^-_{k,0}=\frac{(1-\psi)(\eta-\mu(1-\omega_1))\omega_1^k}{\eta-\mu(1-\psi_1)},\ \ k\geq 0,$$

$$\pi^-_{k,1}=\frac{\eta(1-\psi)}{\eta-\mu(1-\psi_1)}\left(\psi_1^k-\omega_1^k\right),\ \ k\geq 1.$$

We obtain the arbitrary epoch probabilities from (40) and (41) as

$$\pi_{k,0}=\frac{\lambda(1-\psi)(1-\omega_1)(\eta-\mu(1-\omega_1))\omega_1^{k-1}}{\eta(\eta-\mu(1-\psi_1))},\ k\geq 1,$$

$$\pi_{k,1}=\frac{\lambda(1-\psi)}{\mu(\eta-\mu(1-\psi_1))}\left(\eta\psi_1^{k-1}-\mu(1-\omega_1)\omega_1^{k-1}\right),\ k\geq 1.$$

The results analytically matches with Tian et al. (1989).

6.5 GI^X/M/1 Queue without Vacation and Abandonment

Taking $\eta\to\infty, \varphi=0$, the model reduces to GI^X/M/1 queue without vacation and abandonment. To find expressions among pre-arrival and arbitrary epoch probabilities specified in Barbhuiya and Gupta (2019), let us assume $y_1=1$ and $y_i=0$ for $i\geq 2$. Then, we obtain the pre-arrival and arbitrary epoch probabilities as

$$\pi^-_{k,1}=\frac{1}{\lambda}\sum_{j=1}^{\hbar}d_j\psi_j^k,\ \ k\geq 1,$$

$$\pi_{k,1}=\sum_{j=1}^{\hbar}d_j\left\{\frac{\sum_{i=1}^{\hbar}g_i\psi_j^{k-i}-\psi_j^k}{\mu(1-\psi_j)}\right\}\ \ k\geq 1,\ \ \text{and}$$

$$\pi_{0,0}=1-\sum_{j=1}^{\hbar}\frac{d_j\psi_j\left(G(\psi_j^{-1})-1\right)}{\mu(1-\psi_j)^2}.$$

6.6 GI/M/1 Queue without Vacation and Abandonment

Taking $g_1 = 1, g_j = 0,\ j \geq 2, \eta \to \infty, \varphi = 0$, the model reduces to GI/M/1 queue without batch arrival, vacation and abandonment. Here, the single root inside the unit disk is ψ_1 and the corresponding constant $d_1 = \lambda(1 - \psi_1)$. Accordingly, the pre-arrival and arbitrary epoch probabilities matches exactly with the results available in the literature (Stewart 2009, 548).

7 Numerical Results

In this section, we present the numerical results obtained from analytical calculations performed to evaluate the proposed continuous-time GI^X/M/1 + M queue with multiple vacations and geometric abandonment. This section outlines the numerical computation setup, present the key performance metrics, discuss the interpretation of the results, and provide insights into the practical implications of the findings. We used the following algorithm to compute the steady state probability distributions using the linear difference techniques and generating function method.

Heavy-tailed distributions play a significant role in Internet communication and financial applications. Internet traffic statistics have suggested that many relevant quantities like, file sizes, packet lengths, interarrival times, connection times, etc., should be modeled with heavy-tailed distributions. A difficulty in analyzing queues with heavy-tailed distributions is that many of them do not have a closed-form, analytic Laplace transforms. This makes analytical methods more complex and intractable. There are several approximation methods such as transform approximation method (TAM), Padé-Laplace approximation (PLA), etc., to resolve this problems.

We consider four probability distributions like Phase-type, Matrix exponential, inverse Gaussian and Weibull distribution as the customer inter-batch arrival distribution. The computation of the characteristic roots and associated unknown coefficients are tabulated for each of these four distributions. Also, we present the steady-state queue-length distribution at prearrival and arbitrary epochs. All the computations are performed on a PC having Intel core i7 8th Gen CPU @ 1.8 GHz, 8 GB RAM using Maple software. All the numerical values were run with corrected up to 30 decimal places, but are presented up to six decimal places to reduce space.

Phase-Type (PH) Arrival

A PH distribution is the distribution of the time to absorption for an absorbing finite state Markov chain in continuous time. One important property of PH distributions is that they can be used to approximate any kind of distribution. For our numerical experiment, we consider a continuous PH distribution with representation PH($\boldsymbol{\alpha}, T$), where $\boldsymbol{\alpha} := [0.5, 0.2, 0.3]$, $T = \begin{bmatrix} -7 & 1 & 2 \\ 1 & -2 & 0 \\ 1 & 3 & -10 \end{bmatrix}$. The mean arrival rate is $\lambda = 2.5$ and the mean service rate is $\mu = 25$. The LST of the inter batch arrival times is $A^*(s) = \frac{4s^2+53s+120}{(s+10.39)(s+6.94)(s+1.66)}$, and the batch size distribution is $G(z) = 0.3z + 0.15z^2 + 0.2z^5 + 0.1z^6 + 0.15z^8 + 0.1z^{10}$ with a maximum batch size of 10. Other queueing parameters are set to $p = 0.7, \varphi = 0.1, \eta = 0.3$. The characteristic roots are presented in Table 1. The steady-state probabilities of system-length

Table 1 Characteristic roots for PH and Weibull arrivals with parameters $p = 0.7$, $\varphi = 0.1$, $\eta := 0.3$, $\hbar = 10$, $g_1 = 0.3$, $g_2 = 0.15$, $g_5 = 0.2$, $g_6 = 0.1$, $g_8 = 0.15$, $g_{10} = 0.1$

	PH^X/M/1 + M with $\lambda = 2.5$, $\mu = 25$		**Wb^X/M/1 + M** with $\lambda = 5$, $\mu = 40$	
j	ω_j	ψ_j	ω_j	ψ_j
1	0.975671	0.868753	0.988109	0.949571
2	0.598934 + 0.508957 I	0.489212 + 0.454001 I	0.603557 + 0.512286 I	0.522827 + 0.483102 I
3	0.598934 − 0.508957 I	0.489212 − 0.454001 I	0.603557 − 0.512286 I	0.522827 − 0.483102 I
4	0.254595 + 0.723067 I	0.151613 + 0.613885 I	0.258965 + 0.726775 I	0.170915 + 0.659199 I
5	0.254595 − 0.723067 I	0.151613 − 0.613885 I	0.258965 − 0.726775 I	0.170915 − 0.659199 I
6	−0.173484 + 0.645216 I	−0.180367 + 0.562091 I	−0.173462 + 0.647195 I	−0.182850 + 0.593289 I
7	−0.173484 − 0.645216 I	−0.180367 − 0.562091 I	−0.173462 − 0.647195 I	−0.182850 − 0.593289 I
8	−0.625976 + 0.537224 I	−0.509902 + 0.395031 I	−0.630106 + 0.541867 I	−0.547136 + 0.431787 I
9	−0.625976 − 0.537224 I	−0.509902 − 0.395031 I	−0.630106 − 0.541867 I	−0.547136 − 0.431787 I
10	−0.817567	−0.649498	−0.822706	−0.699026

Algorithm 1 Computation of the steady-state probabilities

Step 1: Find the inside roots of the unit circle of the two characteristic equations

$$z^{\hbar} - A^*\left(\eta + \varphi - \frac{\varphi q}{1-pz}\right)\sum_{i=1}^{\hbar} g_i z^{\hbar-i} = 0 \text{ and } z^{\hbar} - A^*(\mu - \mu z)\sum_{i=1}^{\hbar} g_i z^{\hbar-i} = 0$$

Let these roots are ω_j and ψ_j, $j = 1, 2, \ldots, \hbar$, respectively.
Step 2: Using these roots, solve the following system of $2\hbar$ linear equations to compute the unknowns c_j and d_j, $j = 1, 2, \ldots, \hbar$.

$$\sum_{j=1}^{\hbar} \frac{c_j}{\omega_j} = 0 \qquad 0 = \sum_{j=1}^{\hbar} d_j + \sum_{j=1}^{\hbar} \frac{\eta c_j}{\left[\left(\mu - \frac{\varphi p}{1-p\omega_j}\right)(1-\omega_j) - \eta\right]}$$

$$\sum_{j=1}^{\hbar} \frac{c_j}{\omega_j^2} = 0 \qquad 0 = \sum_{j=1}^{\hbar} \frac{d_j}{\psi_j} + \sum_{j=1}^{\hbar} \frac{\eta c_j}{\left[\left(\mu - \frac{\varphi p}{1-p\omega_j}\right)(1-\omega_j) - \eta\right]\omega_j}$$

$$\vdots \qquad\qquad \vdots$$

$$\sum_{j=1}^{\hbar} \frac{c_j}{\omega_j^{\hbar-2}} = 0 \qquad 0 = \sum_{j=1}^{\hbar} \frac{d_j}{\psi_j^{\hbar-3}} + \sum_{j=1}^{\hbar} \frac{\eta c_j}{\left[\left(\mu - \frac{\varphi p}{1-p\omega_j}\right)(1-\omega_j) - \eta\right]\omega_j^{\hbar-3}}$$

$$\sum_{j=1}^{\hbar} \frac{c_j}{\omega_j^{\hbar-1}} = 0 \qquad 0 = \sum_{j=1}^{\hbar} \frac{d_j}{\psi_j^{\hbar-2}} + \sum_{j=1}^{\hbar} \frac{\eta c_j}{\left[\left(\mu - \frac{\varphi p}{1-p\omega_j}\right)(1-\omega_j) - \eta\right]\omega_j^{\hbar-2}}$$

$$\sum_{j=1}^{\hbar} d_j \psi_j G(\psi_j^{-1}) + \sum_{j=1}^{\hbar} \frac{\eta c_j \omega_j G(\omega_j^{-1})}{\left(\mu - \frac{\varphi p}{1-p\omega_j}\right)(1-\omega_j) - \eta} = 0 \text{ and}$$

$$\sum_{j=1}^{\hbar} \frac{d_j \psi_j}{1-\psi_j} + \sum_{j=1}^{\hbar} \frac{c_j\left[(\mu-\eta)(1-\omega_j) - \varphi + \frac{\varphi q}{1-p\omega_j}\right]}{\left(\left(\mu - \frac{\varphi p}{1-p\omega_j}\right)(1-\omega_j) - \eta\right)(1-\omega_j)} = \lambda.$$

Step 3: Putting the values of c_j and d_j in (39)–(41), we get the prearrival epoch probabilities $\pi_{k,0}^-$, $\pi_{k,1}^-$ and the arbitrary epoch probabilities $\pi_{k,0}$, $\pi_{k,1}$ of the number of customers in the system.

at prearrival and arbitrary epochs are given in Table 2. The average system-lengths at prearrival and arbitrary epochs are given in the last row.

Matrix Exponential (ME) Arrival

The ME distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{s})$ is given by $P(X \le x) = 1 + \boldsymbol{\alpha} e^{\boldsymbol{T}x}\boldsymbol{T}^{-1}\boldsymbol{s}$, with density $f(x) = \boldsymbol{\alpha} e^{\boldsymbol{T}x}\boldsymbol{s}$. It is a generalization of the PH distribution. ME distributions have rational Laplace-Stieltjes transforms. Any distribution with a rational Laplace transform is identical to the ME distribution. We consider an ME distribution $(\boldsymbol{\alpha}, \boldsymbol{T}, \boldsymbol{s})$ where $\boldsymbol{\alpha} := [1, 0, 0]$, $\boldsymbol{T} = \begin{bmatrix} 0 & -4\pi^2 - 1 & 4\pi^2 + 1 \\ 3 & 2 & -6 \\ 2 & 2 & -5 \end{bmatrix}$ and $\boldsymbol{s} = [0, 1, 1]^{tr}$ with tr denoting the transpose in matrix. For this system, the mean rates of arrival and service are $\lambda = 0.952917$ and $\mu = 10$

Table 2 Steady-state probabilities at prearrival and arbitrary epochs for PH and Weibull arrivals with parameters $p = 0.7, \varphi = 0.1, \eta := 0.3, \hbar = 10, g_1 = 0.3, g_2 = 0.15, g_5 = 0.2, g_6 = 0.1, g_8 = 0.15, g_{10} = 0.1$

n	**PH^X/M/1 + M** with $\lambda = 2.5, \mu = 25$				n	**Wb^X/M/1 + M** with $\lambda = 5, \mu = 40$			
	$\pi^-_{n,0}$	$\pi^-_{n,1}$	$\pi_{n,0}$	$\pi_{n,1}$		$\pi^-_{n,0}$	$\pi^-_{n,1}$	$\pi_{n,0}$	$\pi_{n,1}$
0	0.058787	-	0.165707	-	0	0.022404	-	0.081914	-
1	0.015651	0.004877	0.015053	0.005780	1	0.006347	0.001341	0.005705	0.002776
2	0.012011	0.005302	0.011559	0.006087	2	0.004976	0.001552	0.004478	0.002901
3	0.005410	0.005626	0.005244	0.006332	3	0.002334	0.001715	0.002138	0.003012
4	0.003250	0.005958	0.003192	0.006599	4	0.001408	0.001866	0.001331	0.003126
5	0.012103	0.006355	0.011667	0.006909	5	0.004978	0.002055	0.004501	0.003256
6	0.011727	0.006703	0.011303	0.007141	6	0.004940	0.002273	0.004465	0.003377
7	0.008391	0.006949	0.008114	0.007298	7	0.003672	0.002458	0.003343	0.003482
8	0.013686	0.007174	0.013175	0.007444	8	0.005833	0.002633	0.005257	0.003585
9	0.008030	0.007347	0.007764	0.007529	9	0.003585	0.002817	0.003264	0.003678
10	0.013242	0.007484	0.012746	0.007603	10	0.005743	0.002978	0.005174	0.003767
50	0.003698	0.003655	0.003566	0.003526	50	0.002728	0.004144	0.002468	0.003885
100	0.001079	0.001069	0.001041	0.001031	100	0.001500	0.002558	0.001357	0.002325
150	0.000315	0.000312	0.000304	0.000301	200	0.000453	0.000786	0.000410	0.000712
200	0.000092	0.000091	0.000089	0.000088	300	0.000137	0.000238	0.000124	0.000215
250	0.000027	0.000027	0.000026	0.000026	500	0.000013	0.000022	0.000011	0.000020
300	0.000008	0.000008	0.000008	0.000007	700	0.000001	0.000002	0.000001	0.000002
412	0.000000	0.000000	0.000000	0.000000	816	0.000000	0.000000	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.558246	0.441754	0.647356	0.435735	Sum	0.431499	0.568494	0.452107	0.547893
	$L^- = 41.06849039$		$L = 39.68099042$			$L^- = 91.878018$		$L = 83.808826$	

respectively. The LST of the inter batch arrival times is $A^*(s) = \frac{4\pi^2+1}{(s+1)(4\pi^2+1+2s+s^2)}$, and the PGF of the batch size is $G(z) = 0.3z + 0.15z^2 + 0.2z^5 + 0.1z^6 + 0.15z^8 + 0.1z^{10}$ with a maximum batch size of 10. We tabulate the characteristic roots in Table 3 and the system-length probabilities in Table 4.

Inverse-Gaussian (IG) Arrival

The inverse Gaussian distribution is a two-parameter family of continuous probability distributions with infinite support. It is a right-skewed distribution bounded at zero and is a well-known competitor of the Weibull, gamma and lognormal distributions in modeling asymmetric data. Its pdf is $f(x; \alpha, \beta) = \sqrt{\frac{\alpha}{2\pi x^3}}\, e^{-\frac{\alpha(x-\beta)^2}{2\beta^2 x}}$ for $x > 0$, where $\beta > 0$ is the mean and $\alpha > 0$ is the shape parameter. In our numerical experiment, we consider the parameters $\alpha = 0.5625$, $\beta = 0.75$. The pdf is $a(t) = \sqrt{\frac{\alpha}{2\pi t^3}}\, e^{-\frac{\alpha(t-\beta)^2}{2\beta^2 t}}$ with LST $A^*(s) = e^{0.75(1-\sqrt{1+2s})}$ and batch size distribution $G(z) = 0.3z + 0.15z^2 + 0.2z^5 + 0.1z^6 + 0.15z^8 + 0.1z^{10}$. Since $A^*(s)$ is a transcendental function, computation of the roots of the associated characteristic equations is intractable using Maple program. To resolve this, we approximate $A^*(s)$ by means of Padé rational approximation, $\frac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials of degree m and n respectively. Using the Padé rational approximation of degree (4, 5) to approximate $A^*(s)$

$$A^*(s) = \frac{P(s)}{Q(s)} = \frac{1 + \frac{108344}{29509}s + \frac{32654}{7971}s^2 + \frac{43552}{34173}s^3 - \frac{8517}{130727}s^4}{1 + \frac{28099}{6355}s + \frac{18398}{2723}s^2 + \frac{66589}{15981}s^3 + \frac{23027}{25599}s^4 + \frac{4149}{127505}s^5}.$$

Applying Algorithm 1, we compute the characteristic roots, and the steady-state system-length probabilities. The values are presented in Tables 3 and 4.

Weibull (Wb) Arrival

Weibull distribution plays significant role in modeling insurance problems, where claim sizes can take on extremely large values. It is a two parameter family of continuous distributions with infinite support. The Weibull pdf is $f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^\alpha}$, for $x \geq 0$, where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter of the distribution. In our numerical experiments, we consider the distribution parameters as $\alpha = 0.5$, $\beta = 0.1$ with pdf $a(t) = \frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1} e^{-\left(\frac{t}{\beta}\right)^\alpha}$. The LST of the Weibull distribution does not exist. Using the moments of all possible orders, we construct the LST $A^*(s) = \sum_{i=0}^{100}(-1)^i m_i s^i$. Then, we approximate $A^*(s)$ by means of Padé rational approximation, $\frac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials of degree m and n respectively. Using the Padé rational approximation of degree (4, 5) to approximate $A^*(s)$

$$A^*(s) = \frac{1 + 8.8s + 23.52s^2 + 21.12s^3 + 4.632s^4}{1 + 9s + 25.2s^2 + 25.2s^3 + 7.56s^4 + 0.3s^5}$$

$G(z) = 0.3z + 0.15z^2 + 0.2z^5 + 0.1z^6 + 0.15z^8 + 0.1z^{10}$ Using this approximated LST in Algorithm 1, we compute the roots, unknown constants, and the steady-state queue-length probabilities. The values are presented in Tables 1 and 2.

Table 3 Characteristic roots for IG and ME arrivals with parameters $p = 0.7, \varphi = 0.1, \eta := 0.3, g_1 = 0.3, g_2 = 0.15, g_5 = 0.2, g_6 = 0.1, g_8 = 0.15, g_{10} = 0.1$

j	**IG^X/M/1 + M** with $\lambda = 1.33, \mu = 12$		**ME^X/M/1 + M** with $\lambda = 0.95, \mu = 10$	
	ω_j	ψ_j	ω_j	ψ_j
1	0.956769	0.846384	0.94035	0.787822
2	0.592002 − 0.503703 I	0.443441 − 0.446875 I	0.585923 + 0.498908 I	0.409546 + 0.423530 I
3	0.592002 + 0.503703 I	0.443441 + 0.446875 I	0.585923 − 0.498908 I	0.409546 − 0.423530 I
4	0.248127 − 0.717055 I	0.0982851 − 0.568071 I	0.242554 + 0.711383 I	0.0914193 + 0.521465 I
5	0.248127 + 0.717055 I	0.0982851 + 0.568071 I	0.242554 − 0.711383 I	0.0914193 − 0.521465 I
6	−0.619571 − 0.530088 I	−0.449678 − 0.292117 I	−0.173447 + 0.639031 I	−0.182916 + 0.464188 I
7	−0.619571 + 0.530088 I	−0.449678 + 0.292117 I	−0.173447 − 0.639031 I	−0.182916 − 0.464188 I
8	−0.173482 − 0.642036 I	−0.202719 − 0.496925 I	−0.613755 + 0.523657 I	−0.416668 + 0.291357 I
9	−0.173482 + 0.642036 I	−0.202719 + 0.496925 I	−0.613755 − 0.523657 I	−0.416668 − 0.291357 I
10	−0.809570	−0.519197	−0.802270	−0.509103

Table 4 Steady-state probabilities at prearrival and arbitrary epochs for IG and ME arrivals with parameters $p = 0.7, \varphi = 0.1, \eta := 0.3, g_1 = 0.3, g_2 = 0.15, g_5 = 0.2, g_6 = 0.1, g_8 = 0.15, g_{10} = 0.1$

	IG^X/M/1 + M with $\lambda = 1.33, \mu = 12$					**ME^X/M/1 + M** with $\lambda = 0.95, \mu = 10$			
n	$\pi^-_{n,0}$	$\pi^-_{n,1}$	$\pi_{n,0}$	$\pi_{n,1}$	n	$\pi^-_{n,0}$	$\pi^-_{n,1}$	$\pi_{n,0}$	$\pi_{n,1}$
0	0.093003	-	0.128659	-	0	0.147335	-	0.139161	-
1	0.022446	0.010362	0.021901	0.010044	1	0.032506	0.014686	0.033359	0.013526
2	0.016688	0.010877	0.016287	0.010648	2	0.023522	0.014811	0.024129	0.013925
3	0.007052	0.011368	0.0069	0.011104	3	0.009396	0.015075	0.009577	0.014193
4	0.004221	0.012092	0.004149	0.011659	4	0.005647	0.0157	0.005691	0.014709
5	0.017017	0.01291	0.016614	0.012339	5	0.024282	0.016343	0.024884	0.015391
6	0.01588	0.013279	0.015505	0.012766	6	0.021906	0.016251	0.022448	0.015538
7	0.010698	0.013512	0.010457	0.012991	7	0.013972	0.016039	0.014273	0.015441
8	0.018272	0.013793	0.017832	0.013217	8	0.024976	0.015918	0.025619	0.015421
9	0.009944	0.01375	0.009719	0.013258	9	0.012685	0.015314	0.012962	0.01505
10	0.017223	0.013788	0.016806	0.01329	10	0.023014	0.014935	0.023613	0.014784
50	0.002113	0.003019	0.002063	0.002948	50	0.001361	0.001436	0.001394	0.001471
100	0.000232	0.000332	0.000226	0.000324	100	0.000063	0.000066	0.000064	0.000068
150	0.000025	0.000036	0.000025	0.000036	150	0.000003	0.000003	0.000003	0.000003
200	0.000003	0.000004	0.000003	0.000004	200	0	0	0	0
248	0	0	0	0					
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.504438	0.495562	0.530433	0.481752	Sum	0.587527	0.412473	0.590039	0.409961
	$L^- = 23.158361$		$L = 22.592438$			$L^- = 15.553650$		$L = 15.855534$	

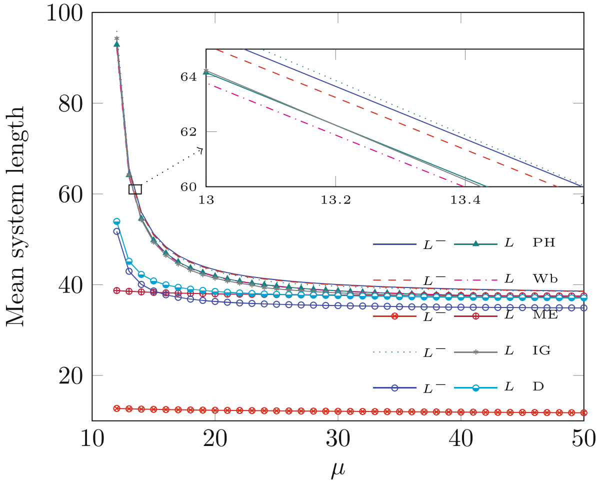

Fig. 3 Effect of μ on the Mean system length at prearrival and arbitary epochs for different arrival processes with $\lambda = 2.5$, $\varphi = 0.1$, $p = 0.7$, $\eta = 0.3$

In the next set of experiments, we present a comparative study of the different batch arrival distributions. We conducted a sensitivity analysis to assess the model's behavior under varying parameters. The effect of several system parameters on the mean system length at pre-arrival (L^-) and arbitrary arrival (L) epochs is presented for queues with inter-batch arrival distributions being Deterministic, Phase-type, Matrix exponential, Weibull and inverse Gaussian. We set the shape and scale parameters of these five distributions in such a way to get the same inter-batch arrival mean, but different variances. The following system parameters are fixed for all the experiment: $\lambda = 2.5$, $\mu = 20.0$, $\varphi = 0.1$, $\eta = 0.3$, $p = 0.7$, $q = 0.3$, $\hbar = 10$, and batch size distribution $G(z) = 0.3z + 0.15z^2 + 0.2z^5 + 0.1z^6 + 0.15z^8 + 0.1z^{10}$. The mean system length is always monotonic decreasing in service rate (Fig. 3), abandonment rate (Fig. 4), vacation rate (Fig. 5) and probability of abandonment (Fig. 6).

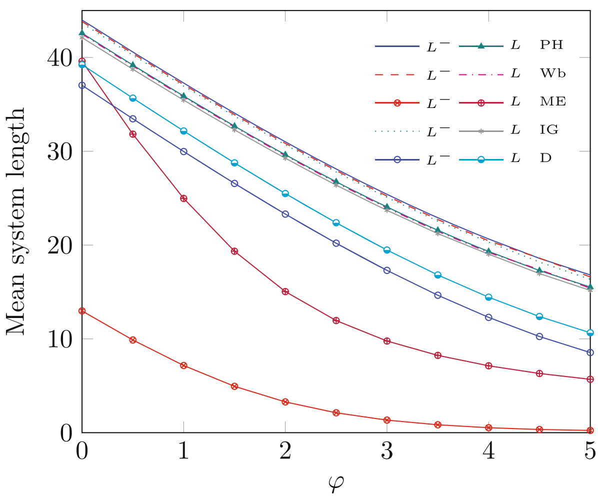

Fig. 4 Effect of φ on the Mean system length at prearrival and arbitary epochs for different arrival processes with $\lambda = 2.5$, $\mu = 20.0$, $p = 0.7$, $\eta = 0.3$

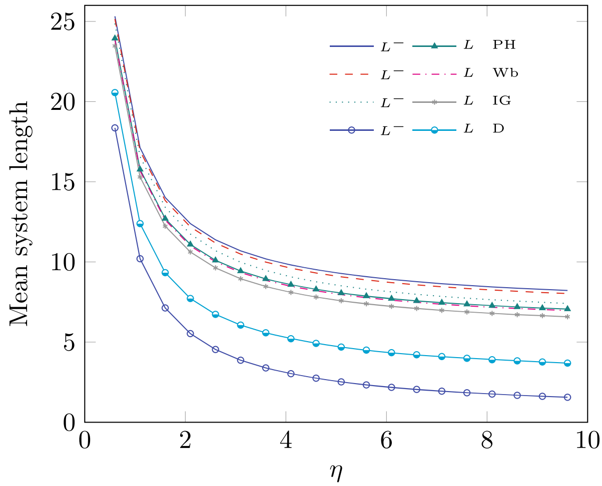

Fig. 5 Effect of η on the Mean system length at prearrival and arbitary epochs for different arrival processes with $\lambda = 2.5$, $\mu = 20.0$, $\varphi = 0.1$, $p = 0.7$

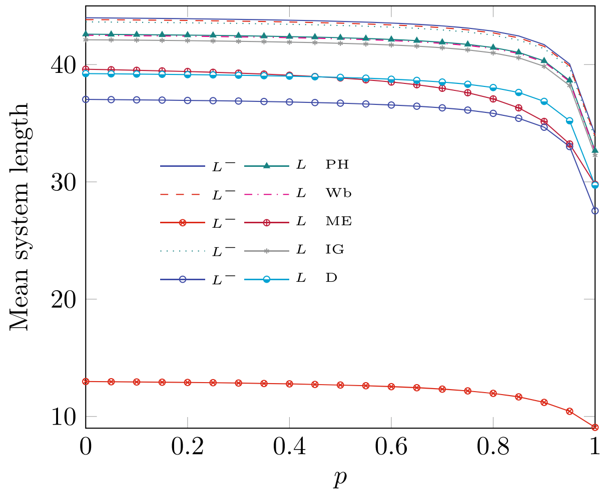

Fig. 6 Effect of p on the Mean system length at prearrival and arbitary epochs for different arrival processes with $\lambda = 2.5$, $\mu = 20.0$, $\varphi = 0.1$, $\eta = 0.3$

8 Conclusion

In this study, we delved into the dynamics of renewal input batch arrival queueing systems with multiple vacations and geometric abandonment, uncovering its practical applications in a spectrum of real-time scenarios such as customer service counters, call centers, public transportation, online customer support, hospital emergency rooms, internet servers, telecommunications systems, and retail checkout lines. The model's analysis emerges as both straightforward and explicit, readily lending itself to numerical tractability. Leveraging the potency of the supplementary variable and difference operator methods, we successfully derived closed-form expressions for system-length distributions at pre-arrival and arbitrary epochs. Our exploration unearthed several intriguing special cases within the model's realm. The inclusion of illustrative numerical examples, showcasing the model's behavior under different inter-arrival time distributions, underscored our study's viability and robustness. This empirical dimension added depth to our theoretical findings, affirming their relevance in practical contexts. As we traverse the contours of future research, an exciting avenue beckons - the extension of our work to encompass batch arrival and bulk services within queueing models featuring vacations and abandonment. The insights gleaned from this study serve as a solid foundation to build, promising a richer understanding of intricate queueing phenomena and their implications for real-world operational dynamics.

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Data Availability All data used in our experiments have been produced with Maple and no external datasets have been used.

Declarations

Competing Interests The authors declare no competing interests.

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