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Discrete-Time Model of Company Capital Dynamics with Investment of a Certain Part of Surplus in a Non-Risky Asset for a Fixed Period

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Abstract
A periodic-review insurance model is studied under the following assumptions. One-period insurance claims form a sequence of independent identically distributed nonnegative random variables with a finite mean. At the beginning of each period a quota δ of the company surplus is invested in a non-risky asset for m periods. Theoretical expressions for finite-time and ultimate ruin probabilities, in terms of multiple integrals, are presented and applied to the particular case where claims are exponential. Dividend problems are also considered. Numerical results obtained by virtue of simulation are provided and other algorithmic approaches are discussed. Sensitivity analysis of ruin probability is carried out for the case of exponential claims.

Keywords Discrete-time insurance model · Investment · Finite-time ruin · Dividends · Simulation · Sensitivity

Mathematics Subject Classification (2010) 91B30 · 90C59

1 Introduction

Risk is a keyword in all definitions of actuarial sciences. Risk is present whenever the outcome is uncertain, whether favorable or unfavorable. Methods for transferring or distributing risk were practiced by Chinese and Babylonian traders as long ago as the 3rd and 2nd millennia BC, respectively. However actuarial science emerged significantly

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later (in the 17th century). It has an interesting history consisting of 4 periods, see, e.g., Bulinskaya (2017).

Not only insurance, but other applied probability research domains such as inventory and dams, finance, queueing theory, reliability and some others can be considered as special cases of decision making under uncertainty (or risk management) aimed at the systems performance optimization, thus eliminating or minimizing risk. For correct decision making one needs an appropriate mathematical model. Constructing an insurance company model one has to take into account its twofold nature. Originally all insurance societies were designed for risk sharing. Hence, their primary task is policyholders indemnification. Nowadays, for the most part they are joint stock companies. Thus, the secondary but very important task is dividend payments to shareholders.

The 20th century, belonging to the second (stochastic) period in actuarial sciences, is known for emergence of collective risk theory. The study of ruin probability for various modifications of (continuous-time) Cramér-Lundberg model has dominated. This allowed insurance companies to increase their solvency and thus provide a solution to the first task.

The modern period is characterized by investigation of complex systems, including dividends payment, reinsurance, tax, bank loans and investment. Interplay of actuarial and finance methods, in particular, unification of reliability and cost approaches is another feature of the last twenty years, see, e.g., Bulinskaya (2003). Furthermore, discrete-time models turned out to be more appropriate for description of some aspects of insurance company performance. On the other hand, discrete-time models can be used for approximation of continuous-time ones, see, e.g. Dickson and Waters (2004). Therefore we concentrate on such type of models although there exist a lot of simulation methods treating continuous-time insurance models (see, e.g., Dutang et al. (2008) and further modifications available at the site http://CRAN.R-project.org/). It is interesting to underline that discretization is widely used in many programs.

A first important discrete-time model giving start to investigation of dividends was introduced in a seminal paper Andersen (1957). The discrete model treating the ruin probability was firstly proposed in Gerber (1988). The paper Li et al. (2009) is a review of discrete-time models considered until 2009. Let us also mention three papers, not included in review: Chan and Zhang (2006) treating direct derivation of finite-time ruin probabilities, Alfa and Drekic (2007) presenting algorithmic analysis of the Sparre Andersen model in discrete time and Wei and Hu (2008) considering the models with stochastic rates of interest, see also Kordzakhia et al. (2012) and references therein. The model with dependence of finance and insurance risks and heavy tailed losses is considered in Tsitsiashvili (2010). Recurrent algorithms for ruin probability calculation are provided there.

The papers Blaževičius et al. (2010), Castañer et al. (2013) and Damarackas and Šiaulys (2014) investigate discrete-time models with different types of non-homogeneity. Inequalities for the ruin probability in a controlled discrete-time risk process are dealt with in Diasparra and Romera (2010). Sharp approximations of ruin probabilities in the discrete-time models are provided in Gajek and Rudz (2013). Quantitative analysis of ruin probability for a discrete-time insurance model with proportional reinsurance and surplus investment according to random interest rate is carried out in Jasulewicz and Kordecki (2015). Two cases of one period loss distributions were compared, namely, light-tailed (exponential) and heavy-tailed (Pareto).

Not only ruin probabilities were studied. Thus, discrete-time models with dividends and reinsurance are treated in Bulinskaya and Yartseva (2010). The discounted factorial moments of the deficit in discrete-time renewal risk model are dealt with in Bao and He

(2012). Optimization of discrete-time insurance model with capital injections and reinsurance is carried out in Bulinskaya et al. (2015). Asymptotic analysis of insurance models with bank loans one can find in Bulinskaya (2014) and (2018). A substantial review of recent results on discrete-time models is supplied, as well, in Section 5 of Bulinskaya (2017). The author's results concerning a discrete-time model with reinsurance and capital injections are given there in Section 6.2.

It is necessary to note that compound binomial models are still popular, see, e.g., Wat et al. (2018) and references therein.

In our paper we study the discrete-time insurance model under following assumptions. The premium obtained by the company each period is equal to a constant $c > 0$. The claims form a sequence of independent identically distributed nonnegative random variables. No assumptions are made about the claims distribution, except existence of density and finiteness of expectation. There exist investment possibility in a non-risky asset. More precisely, in contrast to above mentioned papers, we assume that only a certain part δ of surplus can be placed in a bank for $m \geqslant 1$ periods under a fixed interest rate β per period. At the end of the term the deposit is returned to the company along with interest and can be used for claims indemnification.

The paper is organized as follows. In Section 2 we give the model description. The ultimate and finite-time ruin probabilities are considered in Section 3. We obtain their expressions in terms of multi-dimensional integrals. For the particular case of exponentially distributed claims an explicit form of such integrals is calculated. The problem of dividends payment is dealt with in Section 4. Simulation results and sensitivity analysis of ruin probability for the case of exponential claims are presented in Section 5. In conclusion (Section 6) we discuss the obtained results and further research directions.

2 Model Description

We consider the following generalization of the model introduced in Bulinskaya and Kolesnik (2018). For certainty, we proceed in terms of insurance company performance. During the ith period (year, month, week or day) the company gets a fixed premium amount c and pays a random indemnity X_i, $i = 1, 2, \dots$. It is supposed that $\{X_i\}_{i\geqslant 1}$ is a sequence of independent identically distributed nonnegative random variables (i.i.d. r.v.'s) with a known distribution function $F(x)$, possessing a density $p_X(x)$ and a finite mean. Let the initial capital $S_0 = x$ be fixed and positive. It is possible to place a quota $\delta \in [0, 1]$ of this amount in a bank for m periods ($m \geqslant 1$), the interest rate being β per period. Thus, the surplus at the end of the first period has the form

$$S_1 = (1 - \delta)S_0 + c - X_1.$$

The same procedure is repeated each period (investment of the part δ of available surplus, acquirement of premium and return of invested m periods earlier capital along with interest, payment of indemnity). For simplicity sake, we put $u_m = (1 + \beta)^m$ getting the following recurrent formula

$$S_n = \min[(1 - \delta)S_{n-1}, S_{n-1}] + c + u_m \delta S^+_{n-(m+1)} - X_n. \tag{1}$$

Here and further on it is assumed that $S_k = 0$ for $k < 0$. That means, nothing is returned if $n < m + 1$. The expression (1) is useful if it is permitted to delay the company insolvency, that is, the surplus can stay negative for some time. It takes into account the fact that negative surplus cannot be invested. Thus, for negative S_{n-1} we start from S_{n-1}, whereas for positive values of S_{n-1} the amount $(1-\delta)S_{n-1}$ is left after investment. In the same way, we write $u_m\delta S^+_{n-(m+1)}$ (where $S^+ = \max(S, 0)$), because it was impossible to invest negative $S_{n-(m+1)}$. Moreover, if the surplus at the previous step S_{n-1} was negative it is possible to obtain $S_n > 0$ due to arrival of previous investment (if any) and/or small indemnity. However in this case it is necessary to have $c - X_n + u_m\delta S^+_{n-(m+1)} > |S_{n-1}|$.

Hence, expression (1) is appropriate for the study of Parisian ruin τ^d which occurs if the process S_n stays below or at zero at least for a fixed time period $d \in \{1, 2, \ldots\}$. More precisely, see, e.g., Czarna et al. (2017),

$$\tau^d = \inf\{n > 0 : S_n \leqslant 0, n - \sup\{k < n : S_k > 0\} > d\}.$$

If we use the classical notion of ruin, then the ruin time τ is defined as follows

$$\tau = \inf\{n > 0 : S_n \leqslant 0\}, \tag{2}$$

that is, the delay $d = 0$. To calculate the ultimate ruin probability

$$P(\tau < \infty) = \sum_{n=1}^{\infty} P(S_1 > 0, \ldots, S_{n-1} > 0, S_n \leqslant 0)$$

one can use instead of Eq. 1 the relation

$$S_n = (1-\delta)S_{n-1} + c - X_n + u_m\delta S_{n-(m+1)}, \tag{3}$$

treated in Bulinskaya and Kolesnik (2018) only for $m = 1$. Clearly, expression (3) is also used for calculation of finite-time ruin probability.

There is no need to find the explicit form of S_n (although it is possible to obtain it). For further investigation we need only the following lemma.

Lemma 1 *Put* $S_0 = x$ *and*

$$S_n = f_n - \sum_{i=1}^{n} g_{n,i}X_i, \quad n \geqslant 1, \tag{4}$$

then $f_0 = x$, *whereas*

$$f_n = (1-\delta)f_{n-1} + c,\ n = \overline{1, m}, \quad f_n = (1-\delta)f_{n-1} + u_m\delta f_{n-(m+1)} + c,\ n > m, \tag{5}$$

and

$$g_{n,i} = (1-\delta)g_{n-1,i} + u_m\delta g_{n-(m+1),i}, \quad i = \overline{1, n-(m+1)},$$
$$g_{n,i} = (1-\delta)g_{n-1,i}, \quad i = \overline{n-m, n-1}, \quad g_{n,n} = 1. \tag{6}$$

Proof Obviously, combination of Eqs. 3 and 4 provides the desired recursive relations (5) and (6). □

3 Ruin Probability: Theoretical Results

3.1 General Claim Distribution

Our next aim is investigation of ultimate and finite-time ruin probabilities. To this end, we reformulate Lemma 1 as follows. The company surplus $S_n, n \geqslant 1$, has the form $S_n = f_n - Y_n$ with f_n defined by Eq. 5 and vector $\mathbf{Y}_n = (Y_1, \ldots, Y_n)$ given by $\mathbf{Y}_n = \mathbf{G}_n \cdot \mathbf{X}_n$. Here vector

$\mathbf{X}_n = (X_1, \dots, X_n)$ and matrix $\mathbf{G}_n = (g_{k,i})_{k,i=1,\dots,n}$, where $g_{k,i} = 0$ for $i > k$, the others being specified by Eq. 6.

Now we prove the following result.

Theorem 1 *The ultimate ruin probability is represented by*

$$\varphi(x) = \sum_{n=1}^{\infty} \int_0^{f_1} \cdots \int_0^{f_{n-1}} \int_{f_n}^{+\infty} \prod_{k=1}^{n} p_X(v_k(y_1, \dots, y_n)) dy_1 \dots dy_n,$$

where x is the initial surplus, f_n, $n \geqslant 1$, and $g_{n,i}$, $n \geqslant 1$, $i = \overline{1,n}$, are given by Eqs. 5 and 6, respectively. The function $v_k(y_1, \dots, y_n)$, $k = \overline{1,n}$, is the kth component of vector $\mathbf{G}_n^{-1}\mathbf{y}_n$ and $\mathbf{y_n} = (y_1, \dots, y_n)$.

Proof The ruin time is defined by Eq. 2. So, we calculate the probability of ruin at the nth step obtaining the distribution of the ruin time. To this end, we use Lemma 1. Hence, the probability under consideration $P(\tau = n)$ is given by $P(U_n)$ with

$$U_n = \left\{ g_{1,1}X_1 < f_1, ..., \sum_{i=1}^{n-1} g_{n-1,i}X_i < f_{n-1}, \sum_{i=1}^{n} g_{n,i}X_i \geqslant f_n \right\}.$$

In other words, $P(\tau = n)$ is a multiple integral over the domain

$$U_n' = \left\{ g_{1,1}x_1 < f_1, \dots, \sum_{i=1}^{n-1} g_{n-1,i}x_i < f_{n-1}, \sum_{i=1}^{n} g_{n,i}x_i \geqslant f_n \right\}$$

of $p_{X_1,\dots,X_n}(x_1, \dots, x_n)$. Performing the change of variables $\mathbf{y}_n = \mathbf{G}_n \cdot \mathbf{x}_n$, we see that the integral transforms into

$$\int_0^{f_1} \cdots \int_0^{f_{n-1}} \int_{f_n}^{+\infty} \frac{p_{X_1,\dots,X_n}(v_1(y_1, \dots, y_n), \dots, v_n(y_1, \dots, y_n))}{|\det G_n(v_1(y_1, \dots, y_n), \dots, v_n(y_1, \dots, y_n))|} dy_1 \dots dy_n.$$

Obviously, $\mathbf{x}_n = \mathbf{G}_n^{-1} \cdot \mathbf{y}_n$, so $v_i(y_1, \dots, y_n) = (\mathbf{G}_n^{-1} \cdot \mathbf{y}_n)_i$.

Using the fact that the determinant is a product of diagonal elements $g_{i,i} = 1$ and the sequence X_i consists of i.i.d. r.v.'s we establish the desired form of probability

$$\int_0^{f_1} \cdots \int_0^{f_{n-1}} \int_{f_n}^{+\infty} \prod_{k=1}^{n} p_X(v_k(y_1, \dots, y_n)) dy_1 \dots dy_n. \tag{7}$$

Summing these expressions over all $n \geqslant 1$ we obtain the ultimate ruin probability. According to Lemma 1 integration limits f_k, $k \geqslant 1$, depend on x, so the ultimate ruin probability is also a function of the initial surplus x, that is, $\varphi(x) = P(\tau < \infty | S_0 = x)$. □

Another interesting problem is calculation of finite-time ruin probability. In other words, we want to get the expression of probability

$$\varphi_N(x) = P(\tau \leqslant N | S_0 = x)$$

for a fixed number N.

Corollary 1 *The following relation holds*

$$\varphi_N(x) = 1 - \int_0^{f_1} \dots \int_0^{f_{N-1}} \int_0^{f_N} \prod_{k=1}^{N} p_X(v_k(y_1, \dots, y_N)) dy_1 \dots dy_N. \tag{8}$$

Proof Clearly, using Eq. 7 which provides $P(\tau = n)$ we get $\varphi_N(x)$ as a sum of corresponding summands for $n \leqslant N$. On the other hand, it can be written as $1 - P(\tau > N) = 1 - P(S_1 > 0, \dots, S_N > 0)$. Thus, proceeding along the same lines as in the proof of Theorem 1 we immediately obtain expression (8). □

3.2 Exponential Claim Distribution

Theorem 1 gives an explicit formula for finding the probability of ultimate ruin. However, as it often happens in actuarial and financial mathematics, it contains multi-dimensional integrals that are difficult to calculate. As usual, we can apply the formula to the simplest particular case of the model and obtain analytical expressions. Specifically, in Theorem 2 we assume that X has an exponential distribution (which is a common assumption) and give an expression for ruin probability $P(\tau = n)$ containing no integrals. This result can be used for further theoretical analysis of the model.

Let E_n denote an $n \times n$ identity matrix. We first formulate a useful lemma.

Lemma 2 *Let* λ, $(a_i)_{i=1}^n$, $(f_i)_{i=1}^n$, $(l_{ij}, 1 \leqslant j < i \leqslant n)$ *be real numbers,* $\lambda > 0$, $f_i > 0$ *for all* i. *Let*

$$\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n-1} \\ l_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ l_{21} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{n-1\,1} & l_{n-1\,2} & \dots & 0 & 0 \\ l_{n1} & l_{n2} & \dots & l_{n\,n-1} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}$$

be a vector of linear combinations of variables $(y_i)_{i=1}^n$ *and denote*

$$I_n := \int_{l_1}^{f_1} \cdots \int_{l_n}^{f_n} \exp\left(-\lambda \sum_{i=1}^{n} a_i y_i\right) dy_1 \dots dy_n. \tag{9}$$

Then

$$I_n = \frac{1}{\lambda^n} \sum_{k=0}^{2^n-1} s_k, \tag{10}$$

$$s_k := (-1)^{k_1+\dots k_n} \left(\prod_{p=1}^{n} a_p^k\right)^{-1} \exp\left(-\lambda \sum_{p=1}^{n-1} k_p a_p^k f_p\right), \tag{11}$$

where $\overline{k_n k_{n-1} \dots k_2 k_1}$ *is the binary representation of* k, $0 \leqslant k \leqslant 2^n - 1$, *that is,* k_i *is either 0 or 1, and*

$$\begin{pmatrix} a_1^k \\ a_2^k \\ \vdots \\ a_{n-1}^k \\ a_n^k \end{pmatrix} = (E_n + (1-k_1)L_1)\dots(E_n + (1-k_n)L_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}, \tag{12}$$

$$(L_i)_{st} = \begin{cases} l_{ts} & \text{if } t = i, s < i, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Proof We will argue by induction over n. If $n = 1$, the result is easily verified.

Assume the formula is true for all dimensions not higher than $n - 1$. Taking the integral with respect to y_n in Eq. 9 gives $I_n = I'_{n-1} + I''_{n-1}$, where

$$I'_{n-1} := \frac{1}{\lambda a_n} \int_{l_1}^{f_1} \cdots \int_{l_{n-1}}^{f_{n-1}} \exp\left(-\lambda \sum_{i=1}^{n-1} (a_i + a_n l_{ni}) y_i\right) dy_1 \dots dy_{n-1},$$

$$I''_{n-1} := -\frac{1}{\lambda a_n} e^{-\lambda a_n f_n} \int_{l_1}^{f_1} \cdots \int_{l_{n-1}}^{f_{n-1}} \exp\left(-\lambda \sum_{i=1}^{n-1} a_i y_i\right) dy_1 \dots dy_{n-1}.$$

After applying the formula for dimension $n - 1$ and getting

$$I'_{n-1} = \frac{1}{\lambda^n a_n} \sum_{k=0}^{2^{n-1}-1} s'_k,$$

$$I''_{n-1} = -\frac{1}{\lambda^n a_n} e^{-\lambda a_n f_n} \sum_{k=0}^{2^{n-1}-1} s''_k$$

for certain s'_k and s''_k, it can be easily seen that

$$\frac{1}{a_n} s'_k = s_k,$$

$$-\frac{1}{a_n} e^{-\lambda a_n f_n} s''_k = s_{k+2^{n-1}},$$

which completes the proof. □

Theorem 2 *If the density function of* X *is exponential, namely,* $p_X(y) = e^{-\lambda y} \mathbf{1}\{y > 0\}$, *where* $\mathbf{1}\{\cdot\}$ *is the indicator function, then the probability of ruin at a particular time is given by*

$$P(\tau = n) = \frac{1}{\lambda^n} e^{-\lambda f_n} \sum_{k=0}^{2^{n-1}-1} s_k,$$

$$s_k := (-1)^{k_1 + \dots k_{n-1}} \left(\prod_{p=1}^{n-1} a_p^k\right)^{-1} \exp\left(-\lambda \sum_{p=1}^{n-1} k_p a_p^k f_p\right),$$

where $\overline{k_{n-1}k_{n-2}\dots k_2k_1}$ *is the binary representation of* k*,* $0 \leqslant k \leqslant 2^{n-1}-1$*, so* k_i *is either 0 or 1, and*

$$\begin{pmatrix} a_1^k \\ a_2^k \\ \vdots \\ a_{n-2}^k \\ a_{n-1}^k \end{pmatrix} = (E + (1-k_1)L_1)\dots(E + (1-k_{n-1})L_{n-1}) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{pmatrix},$$

$$a_i = \begin{cases} (1-u)\delta & \text{if } 1 \leqslant i \leqslant n-m-1, \\ \delta & \text{if } n-m \leqslant i \leqslant n-1, \end{cases}$$

$$(L_i)_{st} = \begin{cases} u\delta & \text{if } (s,t) = (i-m-1, i), \\ 1-\delta & \text{if } (s,t) = (i-1, i), \\ 0 & \text{otherwise.} \end{cases}$$

Proof Using the recurrence relations for $g_{n,i}$ it is easy to show that

$$\left(\mathbf{G}_n^{-1}\right)_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \delta - 1 & \text{if } i - j = 1, \\ -u\delta & \text{if } i - j = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} v_1(y_1,\dots,y_n) &= y_1, \\ v_2(y_1,\dots,y_n) &= y_2 - (1-\delta)y_1, \\ v_3(y_1,\dots,y_n) &= y_3 - (1-\delta)y_2, \\ &\dots \\ v_{m+1}(y_1,\dots,y_n) &= y_{m+1} - (1-\delta)y_m, \\ v_{m+2}(y_1,\dots,y_n) &= y_{m+2} - (1-\delta)y_{m+1} - u\delta y_1, \\ v_{m+3}(y_1,\dots,y_n) &= y_{m+3} - (1-\delta)y_{m+2} - u\delta y_2, \\ &\dots \\ v_n(y_1,\dots,y_n) &= y_n - (1-\delta)y_{n-1} - u\delta y_{n-m-1}. \end{aligned}$$

It follows that

$$\sum_{k=1}^{n} v_k(y_1,\dots,y_n) = \sum_{k=1}^{n-m-1} (1-u)\delta y_k + \sum_{k=n-m}^{n-1} \delta y_k + y_n. \tag{14}$$

The expression for the ruin probability at time n

$$P(\tau = n) = \int_0^{f_1} \cdots \int_0^{f_{n-1}} \int_{f_n}^{\infty} \prod_{k=1}^{\infty} p_X\left(v_k(y_1,\dots,y_n)\right) dy_1 \dots dy_n$$

can be rewritten in the form

$$P(\tau = n) = \int_{l_1}^{f_1} \cdots \int_{l_{n-1}}^{f_{n-1}} \int_{f_n}^{\infty} e^{-\lambda \sum_{k=1}^{n} v_k(y_1,\dots,y_n)} dy_1 \dots dy_n. \tag{15}$$

The lower limits can be found by solving the inequalities $v_k(y_1, \dots, y_n) \geq 0$:

$$\begin{aligned} l_1 &= 0, \\ l_2 &= (1-\delta)y_1, \\ l_3 &= (1-\delta)y_2, \\ &\dots \\ l_{m+1} &= (1-\delta)y_m, \\ l_{m+2} &= (1-\delta)y_{m+1} + u\delta y_1, \\ l_{m+3} &= (1-\delta)y_{m+2} + u\delta y_2, \\ &\dots \\ l_{n-1} &= (1-\delta)y_{n-2} + u\delta y_{n-m-2}. \end{aligned}$$

Using (14) and taking the integral in (15) with respect to y_n, we obtain

$$P(\tau = n) = \frac{1}{\lambda} e^{-\lambda f_n} \int_{l_1}^{f_1} \cdots \int_{l_{n-1}}^{f_{n-1}} \exp\left(-\lambda \sum_{p=1}^{n-1} a_p y_p\right) dy_1 \dots dy_{n-1}. \tag{16}$$

Using Lemma 2, we get

$$\int_{l_1}^{f_1} \cdots \int_{l_{n-1}}^{f_{n-1}} \exp\left(-\lambda \sum_{p=1}^{n-1} a_p y_p\right) dy_1 \dots dy_{n-1}$$

$$= \frac{1}{\lambda^{n-1}} \sum_{k=0}^{2^{n-1}-1} (-1)^{k_1+\dots+k_{n-1}} \left(\prod_{p=1}^{n-1} a_p^k\right)^{-1} \exp\left(-\lambda \sum_{p=1}^{n-1} k_p a_p^k f_p\right). \tag{17}$$

Combining (16) and (17) proves the desired result. □

Corollary 2 *The finite-time ruin probability is given by*

$$\phi_N = 1 - \lambda^N I_N$$

for I_N as found in Eqs. 10–13.

4 Dividends

Another characteristic important to any insurance company is dividends payment to its shareholders. The simplest and widely used dividends strategy is a so-called barrier strategy. It is specified by some barrier level $b > 0$. If the company capital crosses upwards this level the excess is immediately paid out as dividends. The first problem is to find the probability that dividends will be paid at least once before the ruin. To solve it we define a new random variable

$$t_b = \inf\{n > 0 : S_n \geqslant b\}.$$

Hence, we would like to calculate $P(t_b < \tau)$. It is not difficult to prove the following result.

Theorem 3 *The relation*

$$P(t_b < \tau) = \sum_{n=1}^{\infty} \int_{f_1-b}^{f_1} \cdots \int_{f_{n-1}-b}^{f_{n-1}} \int_{0}^{f_n-b} \prod_{k=1}^{n} p_X(v_k(y_1, \dots, y_n)) dy_1 \dots dy_n \tag{18}$$

is valid for $S_0 = x > 0$. The expressions of f_k and $v_k(y_1, \dots, y_n)$ are defined in Theorem 1.

Proof We begin by establishing the form of $P(t_b = n, t_b < \tau)$. That means, we calculate the probability of crossing level b by surplus before ruin. Obviously, we can write

$$P(t_b = n, t_b < \tau) = P(S_1 \in (0,b), \dots, S_{n-1} \in (0,b), S_n \geqslant b).$$

Recalling that $S_k = f_k - Y_k$, $k \geqslant 1$, we rewrite this probability as

$$P(f_1 - b < Y_1 < f_1, \dots, f_{n-1} - b < Y_{n-1} < f_{n-1}, Y_n < f_n - b)$$
$$= \int_{f_1-b}^{f_1} \dots \int_{f_{n-1}-b}^{f_{n-1}} \int_0^{f_n-b} \prod_{k=1}^{n} p_X(v_k(y_1,\dots,y_n))dy_1 \dots dy_n.$$

The last expression is obtained along the same lines as Eq. 7. Since $P(t_b < \tau) = \sum_{n=1}^{\infty} P(t_b = n, t_b < \tau)$ we get immediately (18). □

Remark 1 Theorems 1, 3 and Corollary 1 can be extended to treat the case of independent non-identically distributed random variables X_i, $i \geqslant 1$. Instead of $\prod_{k=1}^{n} p_X(v_k(y_1,\dots,y_n))$ one has to write $\prod_{k=1}^{n} p_{X_k}(v_k(y_1,\dots,y_n))$, here $p_{X_k}(x)$ is the density of X_k.

5 Numerical Results

5.1 Simulation

Using the Python 3 programming language and the package `numpy`, we generated a large sample of pseudo-random variables to simulate the claim sizes X_i. The process S_n was modeled multiple times (100 000 times during this iteration) and the empirical probability of ruin was calculated. (It is equal to the number of attempts in which ruin occurred divided by the total number of attempts). In the same way, the empirical probability of the event that dividends are paid at least once before ruin was calculated as well. It is a very good numerical method for obtaining the values of these probabilities, and it is better than trying to calculate the multi-dimensional integrals found in explicit theoretical formulas above. Indeed, calculating those integrals by definition is unfeasible because the runtime grows exponentially as the number of dimensions increases. The simple Monte-Carlo method of calculating multi-dimensional integrals is also inappropriate in this case, because the measure of the domain grows very quickly, causing the standard error to grow so much that the answer contains no information (see Table 1).

Table 1 The probability of ruin calculated using the Monte-Carlo method

horizon	ruin	standard error	runtime
1	0.247	0.001	8.756
4	0.385	0.002	24.343
7	0.408	0.016	29.040
10	0.334	0.262	31.556

The standard error increases rapidly. Fixed parameters: $\delta = 0.8$, $u = 1.5$, $m = 1$, $c = 6.0$, $S_0 = 5.0$, X_i have the density $p_X(t) = \lambda e^{-\lambda t}\mathbf{1}_{\{t \geqslant 0\}}$ with parameter $\lambda = 0.2$

The reader may see the code of one simulation in Listing 1.

It is impossible to model the process S_n for all positive integers n, so we checked how quickly the probability of ruin over a finite horizon converges to the actual probability of ruin. Table 2 shows the results. It is clear from this table that it is enough to calculate S_n for only the first few values of n, the remaining steps do not change the probability much. We have verified this convergence quality for other values of parameters as well.

Taking that into account, we will confine ourselves to only considering the horizon 1000. Table 3 shows how the two empirical probabilities depend on the distribution of X_i.

It is clear from this table that the result for X_i distributed exponentially differs from that for X_i distributed uniformly with the same mean. More precisely, if the distribution of claims is uniform, the situation is better for the insurance company: indeed, the claim

```
import numpy as np

def div_and_ruin_for_Xs(
      horizon, delta, u, m, c, S_0, barrier, Xs,
):
   dividends_are_paid = False
   ruin_occured = False

   S = [None for _ in range(horizon + 1)]
   S[0] = S_0
   for i in range(1, horizon + 1):
      if i >= m + 1:
         S[i] = (1 - delta) * S[i - 1] + c +\
              u * delta * S[i - (m + 1)] - Xs[i]
      else:
         S[i] = (1 - delta) * S[i - 1] + c - Xs[i]
      if S[i] < 0:
         ruin_occured = True
         return (dividends_are_paid, ruin_occured)
      if S[i] >= barrier:
         dividends_are_paid = True

   return dividends_are_paid, ruin_occured

def one_simulation(
      horizon, delta, u, m, c, lambda_, S_0, barrier,
):
   Xs = [None] + list(
      np.random.exponential(
         scale = 1 / lambda_, size=horizon,
      )
   )
   return div_and_ruin_for_Xs(
      horizon, delta, u, m, c, S_0, barrier, Xs,
   )
```

Listing 1 The code of one simulation

Table 2 The empirical probability of ruin and at least one payment of dividends before ruin: premium rate c and horizon are varied

c	horizon	dividends are paid	ruin	runtime
2.0	3	0.000	0.939	5.239
2.0	10	0.001	0.996	5.338
2.0	32	0.002	1.000	6.812
2.0	100	0.002	1.000	11.694
2.0	316	0.002	1.000	25.094
2.0	1000	0.002	1.000	71.171
2.0	3162	0.002	1.000	202.847
6.0	3	0.000	0.582	4.709
6.0	10	0.346	0.730	6.381
6.0	32	0.346	0.753	10.327
6.0	100	0.348	0.752	23.776
6.0	316	0.346	0.753	64.307
6.0	1000	0.347	0.753	197.461
6.0	3162	0.347	0.752	588.802
10.0	3	0.449	0.281	4.889
10.0	10	0.753	0.341	7.776
10.0	32	0.754	0.342	16.328
10.0	100	0.754	0.342	43.908
10.0	316	0.753	0.342	131.265
10.0	1000	0.754	0.342	415.139
10.0	3162	0.753	0.342	1325.672

Fixed parameters: $\delta = 0.8$, $u = 1.5$, $m = 3$, $S_0 = x = 5.0$, $b = 10.0$, X_i have the density $p_X(t) = \lambda e^{-\lambda t}\mathbf{1}_{\{t \geqslant 0\}}$ with parameter $\lambda = 0.2$. The runtime is how long it took the algorithm to run 100 000 simulations before giving the answer, in seconds

Table 3 The empirical probability of ruin and at least one payment of dividends before ruin: the distribution of X_i is varied

distribution of X_i	dividends are paid	ruin	runtime
exponential with $\lambda = 0.1$	0.052	0.996	14.377
exponential with $\lambda = 0.2$	0.347	0.752	40.779
exponential with $\lambda = 0.3$	0.644	0.396	78.986
exponential with $\lambda = 0.4$	0.820	0.192	97.866
exponential with $\lambda = 0.5$	0.910	0.093	115.296
exponential with $\lambda = 0.6$	0.956	0.044	122.637
exponential with $\lambda = 0.7$	0.979	0.021	127.139
uniform on [0.0, 2.0]	1.000	0.000	111.898
uniform on [0.0, 6.0]	1.000	0.000	121.689
uniform on [0.0, 10.0]	0.285	0.729	42.049
uniform on [0.0, 14.0]	0.059	0.971	15.218

Fixed parameters: $\delta = 0.8$, $u = 1.5$, $c = 6.0$, $m = 3$, $S_0 = x = 5.0$, $b = 10.0$. The runtime is how long it took the algorithm to run 100 000 simulations before giving the answer, in seconds

Table 4 The empirical probability of ruin and at least one payment of dividends before ruin: m and δ are varied

δ	m	dividends are paid	ruin	runtime
0.0	1	0.739	0.503	70.118
0.0	2	0.742	0.499	74.127
0.0	6	0.738	0.503	70.278
0.0	10	0.740	0.501	71.366
0.0	14	0.737	0.504	69.061
0.05	1	0.728	0.485	69.319
0.05	2	0.724	0.470	76.895
0.05	6	0.709	0.471	71.198
0.05	10	0.709	0.487	74.023
0.05	14	0.709	0.512	70.852
0.1	1	0.716	0.473	74.496
0.1	2	0.699	0.466	72.326
0.1	6	0.676	0.495	74.844
0.1	10	0.673	0.539	75.732
0.1	14	0.675	0.589	64.763
0.15	1	0.701	0.468	74.902
0.15	2	0.680	0.468	72.277
0.15	6	0.635	0.528	74.752
0.15	10	0.626	0.598	68.505
0.15	14	0.627	0.663	52.311
0.2	1	0.688	0.467	77.648
0.2	2	0.655	0.473	74.138
0.2	6	0.589	0.562	64.429
0.2	10	0.576	0.650	54.871
0.2	14	0.570	0.730	45.221
0.4	1	0.647	0.479	69.677
0.4	2	0.576	0.526	64.270
0.4	6	0.394	0.700	45.936
0.4	10	0.313	0.814	33.595
0.4	14	0.278	0.895	25.112
0.6	1	0.606	0.510	70.180
0.6	2	0.505	0.591	56.527
0.6	6	0.232	0.809	33.800
0.6	10	0.099	0.910	23.496
0.6	14	0.041	0.960	17.957

Fixed parameters: $\beta = 0.15$, $c = 6.0$, $S_0 = x = 5.0$, $b = 10.0$, X_i have the density $p_X(t) = \lambda e^{-\lambda t}\mathbf{1}_{\{t \geqslant 0\}}$ with parameter $\lambda = 0.2$. The runtime is how long it took the algorithm to run 100 000 simulations before giving the answer, in seconds amounts are bounded from above, so there is a smaller chance of losing too much in one period.

It is also very interesting how the performance of the company depends on the relative amount δ of the invested capital, and the number m of periods this fraction is kept in the bank. Table 4 shows our results. The data suggests that the probability of ruin is concave with respect to δ. Verifying this conjecture formally and finding the optimum is another theoretical problem.

A different problem is the complex behavior of the integrals in formulas (7), (8) and (18). In particular, it is impossible to give any simple expressions for the sequence f_n, because the characteristic polynomial $x^{m+1}-(1-\delta)x-u\delta$ of the recurrence relation $d_n = (1-\delta)d_{n-1}+u\delta d_{n-(m+1)}$ may not be solvable in radicals. Though it is easy to show (see Lemma 3) that the sequence f_n is monotonic if $c - \delta S_0 \geqslant 0$, we cannot say much about the other case. For an example of such a sequence see Table 5. Numerical data suggests that f_n is always monotonic for sufficiently large n.

5.2 Sensitivity Analysis of the Ruin Probability

In the exponential case, the explicit formula for the probability of ruin with finite horizon has been obtained in Corollary 2. If the horizon is small enough, this formula is much more convenient to use than other methods, because the value is precise and takes little time to compute. We fix a small value of horizon (5 for speed and simplicity) and choose m to be 1. The probability of ruin is then a function of five parameters: δ, u, c, λ, x. We are interested in discovering how sensitive the probability is to changes in each parameter.

We consider the parameters to be independent uniformly distributed random variables in segments [0.0, 1.0], [1.0, 1.4], [2.0, 10.0], [0.5, 1.5], [0.0, 16.0] respectively and simulate $N = 20\,000$ values of the target function. This way, we obtain five scatterplots shown in Fig. 1. It is immediately obvious that for the given segments the parameter c has the greatest influence, λ and x have a smaller effect on the function, δ still smaller but noticeable, and u does not seem to influence the probability of ruin at all (that may change if a much larger segment is chosen).

We now prove this conclusion quantitatively. The target function Y is a function of five parameters (which are viewed as random variables): $Y = h(Z_1, Z_2, Z_3, Z_4, Z_5)$. We are interested in the first-order sensitivity index of ith parameter S_i which represents the main effect contribution of this input factor:

$$S_i = \frac{V[E(Y|Z_i)]}{V(Y)}.$$

The total effect index accounts for the total contribution of the output variation due to factor Z_i, i.e. its first-order effect plus all higher-order effects due to interactions:

$$S_{T_i} = 1 - \frac{V[E(Y|Z_{\sim i})]}{V(Y)},$$

Table 5 The sequence f_n. Fixed parameters: $\delta = 0.5$, $\beta = 0.15$, $c = 2.0$, $S_0 = x = 10.0$, $m = 7$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
10.00	7.00	5.50	4.75	4.38	4.19	4.09	4.05	17.32	19.97
f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}	f_{17}	f_{18}	f_{19}
19.30	17.97	16.80	15.97	15.43	15.10	32.59	44.86	50.10	50.95

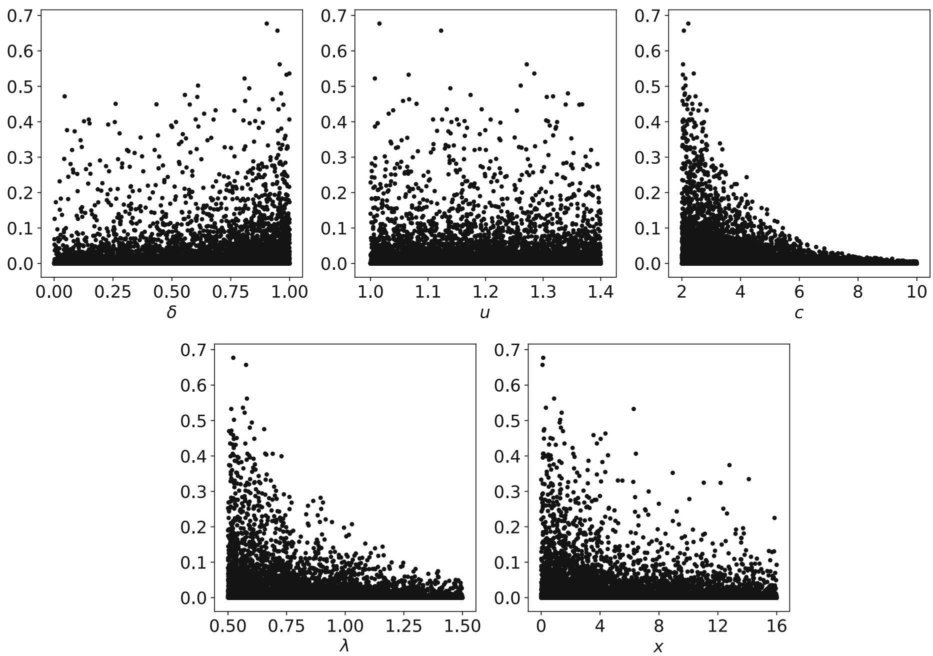

Fig. 1 The scatterplots versus δ, u, c, λ, x

where by $Z_{\sim i}$ we mean all parameters other than the ith one. See Saltelli et al. (2008) or Bulinskaya and Gusak (2016) for background on these sensitivity indices and an algorithm for calculating them. In our case, we get the following values:

$$\begin{aligned} S_\delta &= 0.031137, & S_{T_\delta} &= 0.156902,\\ S_u &= 0.000145, & S_{T_u} &= 0.001926,\\ S_c &= 0.174891, & S_{T_c} &= 0.682256,\\ S_\lambda &= 0.096090, & S_{T_\lambda} &= 0.481226,\\ S_x &= 0.066017, & S_{T_x} &= 0.444585. \end{aligned}$$

These numbers unequivocally confirm our previous observation about the effects of the five parameters on the probability of ruin with finite horizon.

6 Conclusion and Further Research Directions

We have considered a periodic-review insurance model with investment in a non-risky asset. Although many researchers use investment in risky assets (see, e.g. Hussain and Parvez 2017), it may be dangerous for insurance companies (see, Kabanov and Pergamenshchikov 2016 and references therein). So, we treated the investment in a bank for a given time providing a fixed interest. The formulas for calculation of finite-time and ultimate ruin probabilities are obtained in terms of multi-dimensional integrals. For the case of exponential claims we computed these integrals getting the formulas useful for numerical investigations. Dividends payment under the barrier strategy is also considered.

We provided the following numerical results obtained by modeling the surplus dynamics using the programming language Python 3. The code of one simulation is given by Listing 1. Empirical ruin probability was calculated as well as probability of at least one dividends payment before the ruin. It turned out that such a method is better than calculation of ruin probability by Monte-Carlo method (see Table 1). The above mentioned empirical probabilities were calculated for exponential distribution if premium rate c and horizon are varied. As Table 2 shows, the ruin probability over a finite horizon quickly converges to the ultimate one. Results for exponential and uniform claim distributions are compared in Table 3 (for fixed investment parameters). Parameters m and δ are varied for exponential distribution in Table 4. The data suggests that the probability of ruin is concave with respect to δ.

Verification of this conjecture formally and finding the optimal investment policy (minimizing the ruin probability) is the theoretical problem we plan to solve.

Although it was proved in Lemma 3 that the sequence f_n monotonically grows for $c - \delta S_0 \geqslant 0$, we cannot say much about the other case. An example of such a sequence is given in Table 5. Numerical data suggests that f_n is always monotonic for sufficiently large n. However it is necessary to prove this fact.

In Section 5.2 we carried out the sensitivity analysis of the model to small fluctuations of parameters for the case of exponential claim distribution using the methods presented in Saltelli et al. (2008), see Bulinskaya and Gusak (2016) as well. It turned out much more convenient to use the formula of Corollary 2 than other methods, since the result is precise and takes less time to compute. The methods of probability metrics, see, e.g., Rachev et al. (2013), will be useful for treatment of underlying processes perturbation.

Investigation of the asymptotic behavior of ruin probability for light- and heavy-tailed claim distributions is currently under development.

The results established for Parisian ruin and dividends payment with Parisian implementation delay will be published in a forthcoming paper.

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Appendix

For further investigation, the following results will be useful.

Lemma 3 *The sequence $\{f_n\}_{n\geqslant 0}$ defined by Eq. 5 is increasing if $c > \delta x$. If $c = \delta x$ then the sequence $\{f_n\}_{n\geqslant m}$ is increasing whereas $f_k = f_0 = x$ for $k = \overline{1, m}$.*

Proof Put $h_n = f_n - f_{n-1}$, $n \geqslant 1$, then, according to Eq. 5, $h_1 = f_1 - f_0 = c - \delta x$, $h_k = (1-\delta)^{k-1} h_1$, $k = \overline{2, m}$, $h_{m+1} = (1-\delta)^m h_1 + u_m \delta x$ and, for $k \geqslant 1$,

$$h_{m+k+1} = (1-\delta) h_{m+k} + u_m \delta h_k. \tag{19}$$

Thus,

$$h_{m+k} = [(1-\delta)^{m+k-1} + (k-1) u_m \delta (1-\delta)^{k-2}] h_1 + u_m \delta (1-\delta)^{k-1} x, \quad k = \overline{2, m+1},$$

whereas

$$h_{2m+2} = [(1-\delta)^{2m+1} + (m+1) u_m \delta (1-\delta)^m] h_1 + [u_m \delta (1-\delta)^{m+1} + (u_m \delta)^2] x.$$

Using Eq. 19 we conclude that $h_n = \varkappa_n h_1 + \gamma_n x$ with $\varkappa_n = (1-\delta)^{n-1}$, for $n = \overline{1, m+1}$, and $\gamma_n = 0, n = \overline{1, m}$, $\gamma_{m+1} = u_m\delta$. Moreover, $\varkappa_{m+2} = (1-\delta)^{m+1} + u_m\delta$, $\gamma_{m+2} = u_m\delta(1-\delta)$ and $\varkappa_{m+l+1} = (1-\delta)\varkappa_{m+l} + u_m\delta\varkappa_l$, for $l > 1$, while $\gamma_{m+l+1} = (1-\delta)\gamma_{m+l} + u_m\delta\gamma_l$. It follows immediately that $\varkappa_n > 0$ for $n \geqslant 1$, whereas all γ_n are non-negative. Thus, $h_n > 0$ for $n \geqslant 1$ if $h_1 = c - \delta x > 0$. If $h_1 = 0$ then $h_n = 0$ for $n = \overline{1, m}$ and $h_n > 0$ for $n > m$. Since $f_n = f_{n-1} + h_n$, it is clear that, for all $n > m$, $f_n > f_{n-1}$ if $h_1 = c - \delta x \geqslant 0$. □

It is possible to get an explicit form of coefficients $\varkappa_n$ and γ_n for any m and n.

Corollary 3 *The following relations hold*

$$\varkappa_n = \sum_{i=0}^{[\frac{n}{m+1}]} a_i^{(m,n)} (u_m\delta)^i (1-\delta)^{n-1-(m+1)i},$$

with $a_0^{(m,n)} = 1$, $a_i^{(m,n)} = a_i^{(m,n-1)} + a_{i-1}^{(m,n-m-1)}$ *for* $i > 1$.

Moreover,

$$\gamma_n = \sum_{k=1}^{[\frac{n}{m+1}]} a_{n,k}^{(m)} (u_m\delta)^k (1-\delta)^{n-(m+1)k}$$

with $a_{n,1}^{(m)} = 1$ *and* $a_{n,k}^{(m)} = a_{n-1,k}^{(m)} + a_{n-1-m,k-1}^{(m)}$ *for* $k > 1$. *The sum over empty set is equal to zero.*

Proof The results follow in a straightforward way from the expression $h_n = h_1\varkappa_n + x\gamma_n$ leading to the following recurrence relations:

$$\varkappa_n = (1-\delta)\varkappa_{n-1} + u_m\delta\varkappa_{n-1-m},$$
$$\gamma_n = (1-\delta)\gamma_{n-1} + u_m\delta\gamma_{n-1-m}.$$
□

Remark 2 It follows easily from definition of Y_n that it can be rewritten in a following form:

$$Y_n = \sum_{k=1}^{n} d_{n-k} X_k \quad \text{where} \quad d_{n-k} = g_{n,k}.$$

Hence, $d_k = (1-\delta)^k$, $k = \overline{0, m}$, and $d_l = (1-\delta)d_{l-1} + u_m\delta d_{l-1-m}$ for $l > m$. It is not difficult to obtain an explicit expression of d_n for any n :

$$d_n = \sum_{i=0}^{[\frac{n}{m+1}]} a_i^{(n)} (u_m\delta)^i (1-\delta)^{n-i(m+1)}$$

with $a_0^{(n)} = 1$, $a_1^{(n)} = 1 + a_1^{(n-1)}$ for all n and $a_i^{(n)} = a_i^{(n-1)} + a_{i-1}^{(n-m-1)}$ for $i > 1$.

Lemma 4 *Consider the sequence* $\{f_n\}_{n\geqslant 0}$ *for the case* $m = 1$, *given by the following recurrence relation:*

$$f_n = (1-\delta)f_{n-1} + \delta u_1 f_{n-2} + c, \quad n \geq 2,$$
$$f_0 = x, \quad f_1 = (1-\delta)x + c.$$

The solution can be written explicitly:

$$f_n = c_1\frac{x_1^{n+1}-1}{x_1-1} + c_2\frac{x_2^{n+1}-1}{x_2-1}, \quad c_1 = \frac{c - x\,(\delta + x_2)}{x_1 - x_2}, \quad c_2 = \frac{c - x\,(\delta + x_1)}{x_2 - x_1},$$

where $x_1 < x_2$ *are the roots of the quadratic equation* $y^2 - (1-\delta)y - u_1\delta = 0$.

Proof For convenience, let $f_{-1} = 0$, and put $h_n = f_n - f_{n-1}$ for $n \geqslant 0$. Then $\{h_n\}_{n \geqslant 0}$ satisfies:

$$h_n = (1-\delta)h_{n-1} + \delta u_1 h_{n-2}, \quad n \geqslant 2,$$
$$h_0 = x, \quad h_1 = c - \delta x.$$

It is a simple homogeneous recurrence relation of order 2. The characteristic polynomial $p(y) = y^2 - (1-\delta)y - u_1\delta$ has two different roots $x_1 < x_2$, therefore, the solution is

$$h_n = c_1 x_1^n + c_2 x_2^n.$$

The values of c_1 and c_2 are found from the initial conditions $h_0 = c_1 + c_2 = x$, $h_1 = c_1x_1 + c_2x_2 = c - \delta x$.

Recalling that

$$f_n = \sum_{i=0}^{n} h_i$$

completes the proof. □

Corollary 4 *In the case $m = 1$, $f_n \sim dx_2^n \to +\infty$ as $n \to +\infty$, where d is a positive constant. In particular, this sequence is strictly increasing for sufficiently large n.*

Proof Since the discriminant $D = (1-\delta)^2 + 4u_1\delta$ of $p(y)$ is greater than $(1+\delta)^2$, the following three inequalities take place:

$$x_1 = \frac{1-\delta-\sqrt{D}}{2} < -\delta,$$
$$x_2 = \frac{1-\delta+\sqrt{D}}{2} > 1,$$
$$|x_1| < |x_2|.$$

Then, obviously, the relation $f_n \sim dx_2^n$ with $d = \frac{c - x(\delta + x_1)}{(x_2 - x_1)(x_2 - 1)} x_2$ follows from Lemma 4. We only need to show $c - x(\delta + x_1) > 0$, or, equivalently, $x_1 < \frac{c}{x} - \delta$. It is true because $x_1 < -\delta < \frac{c}{x} - \delta$. □

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