



Asymptotic Results for the Absorption Time of Telegraph Processes with Elastic Boundary at the Origin

Claudio Macci¹ · Barbara Martinucci² · Enrica Pirozzi³

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Abstract

We consider a telegraph process with elastic boundary at the origin studied recently in the literature (see e.g. Di Crescenzo et al. (Methodol Comput Appl Probab 20:333–352 2018)). It is a particular random motion with finite velocity which starts at $x \geq 0$, and its dynamics is determined by upward and downward switching rates λ and μ , with $\lambda > \mu$, and an absorption probability (at the origin) $\alpha \in (0, 1]$. Our aim is to study the asymptotic behavior of the absorption time at the origin with respect to two different scalings: $x \rightarrow \infty$ in the first case; $\mu \rightarrow \infty$, with $\lambda = \beta\mu$ for some $\beta > 1$ and $x > 0$, in the second case. We prove several large and moderate deviation results. We also present numerical estimates of β based on an asymptotic Normality result for the case of the second scaling.

Keywords Finite velocity · Random motion · Large deviations · Moderate deviations

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✉ Claudio Macci
macci@mat.uniroma2.it

Barbara Martinucci
bmartinucci@unisa.it

Enrica Pirozzi
enrica.pirozzi@unina.it

¹ Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, 00133, Rome, Italy

² Dipartimento di Matematica, Università degli Studi di Salerno, Via Giovanni Paolo II n. 132, 84084, Fisciano, SA, Italy

³ Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Cintia, Complesso Monte S. Angelo, 80126, Naples, Italy

1 Introduction

The (integrated) telegraph process is an alternating random motion with finite velocity, and has several applications in different fields (for instance in physics, finance and mathematical biology). The literature on the telegraph process and its generalizations is quite large. In this paper we refer to a recent model with elastic barrier studied in Di Crescenzo et al. (2018), where it is possible to find several references. Here we only recall some of them.

We start with some references that studied the solution of the telegraph equation; see e.g. Goldstein (1951) and Kac (1974). Among the first references that studied some probabilistic aspects, we recall (Orsingher 1990) and (Foong 1992). Moreover we also cite Orsingher (1995) and Ratanov (1997) where the telegraph process in the presence of reflecting and absorbing barriers was investigated. Among the more recent references with some generalizations, we recall (Stadje and Zacks 2004) for a model with random velocities, Di Crescenzo and Martinucci (2010) for a damped telegraph process, Crimaldi et al. (2013) for a model driven by certain random trials, Di Crescenzo and Zacks (2015) for a telegraph process perturbed by a Brownian motion, De Gregorio and Orsingher (2011) and Garra and Orsingher (2014) for certain multivariate extensions, and Ratanov (2015) for a model with jumps having some interest in finance. Finally, since in this paper we prove results on large deviations, we also recall (Macci 2016) (see also some references cited therein) and the previous paper (Macci 2011).

We also cite some references on stochastic processes with elastic barriers: Dominé (1995) and Dominé (1996) for the Wiener process, Giorno et al. (2006) for some diffusion processes and, more recently, Jacob (2012) and Jacob (2013) for the Langevin process.

Now we describe the stochastic process studied in Di Crescenzo et al. (2018). It represents a random motion of a particle on the half-line $[0, \infty)$. The particle moves up and down in an alternating way; moreover it has velocity 1 for the upward periods, and it has velocity -1 for the downward periods. Initially the motion proceeds upward for a positive random time U_1 and, after that, the particle moves downward for a positive random time D_1 ; moreover the motion alternates the random times $U_2, D_2, U_3, D_3, \dots$, where $\{U_n : n \geq 1\}$ and $\{D_n : n \geq 1\}$ are independent sequences of i.i.d. positive random variables. Furthermore, when the particle hits the origin, it can be either absorbed or reflected upwards with probabilities α and $1 - \alpha$, respectively (here $\alpha \in (0, 1)$ but, actually, the case $\alpha = 1$ is also allowed). More precisely, if during a downward period D_j , say, the particle reaches the origin and it is not absorbed, then instantaneously the particle starts an upward period for an independent random time U_{j+1} . We also remark that, here, we restrict our attention on the case in which the random variables $\{U_n : n \geq 1\}$ and $\{D_n : n \geq 1\}$ are exponentially distributed with parameters λ and μ , respectively; moreover we assume that $\lambda > \mu$ and this guarantees that $\mathbb{E}[D_1] > \mathbb{E}[U_1]$.

In particular we are interested in the random variable $A_x = A_x(\lambda, \mu)$, i.e. the absorption time of the particle when it starts at x ; see equation (2) in Di Crescenzo et al. (2018). The aim is to study the asymptotic behavior of that random variable with respect two different scalings:

- **Scaling 1:** $x \rightarrow \infty$;
- **Scaling 2:** $\mu \rightarrow \infty$, and $\lambda = \beta\mu$ for some fixed $\beta > 1$ and $x > 0$.

This will be done by referring to the theory of large deviations (see e.g. Dembo and Zeitouni (1998) as a reference on this topic). This theory allows to give an evaluation of probabilities of rare events on an exponential scale. Some preliminaries on this topic will be recalled in the next Section 2.

Some of these asymptotic results concern moderate deviations; this term is used in the literature when one has a class of large deviation principles which fills the gap between the convergence to a constant (typically governed by a large deviation principle) and an asymptotic Normality result (see Remarks 3.1 and 4.2). Interestingly in this paper we can also present a non-central moderate deviation result as $\mu \rightarrow \infty$ stated in Proposition 4.3; this means that we have a class of large deviation principles that fills the gap between the convergence to a constant and the weak convergence to a non-Gaussian limit (see Remark 4.4).

In this paper we also present some numerical estimates for β (approximate confidence intervals and point estimations) obtained by simulations and based on an asymptotic Normality result as $\mu \rightarrow \infty$; moreover, as far as the scaling 1 is concerned, we also study the case in which the particle starts at some independent random point $Y(x)$.

We conclude with the outline of the paper. We start with some preliminaries in Section 2. The results are proved in Section 3 (for the scaling 1) and in Section 4 (for the scaling 2). Finally, in Section 5, we present the numerical estimates.

2 Preliminaries

We start with some preliminaries on large deviations. Then we conclude with some details on the model studied in Di Crescenzo et al. (2018); actually we recall some preliminaries on the absorption time $A_x = A_x(\lambda, \mu)$.

2.1 Preliminaries on Large Deviations

We start with some basic definitions (see e.g. Dembo and Zeitouni (1998), pages 4-5). Let \mathcal{Z} be a topological space equipped with its completed Borel σ -field. A family of \mathcal{Z} -valued random variables $\{Z_r : r > 0\}$ (defined on the same probability space (Ω, \mathcal{F}, P)) satisfies the large deviation principle (LDP for short) with speed function v_r and rate function I if: $\lim_{r \rightarrow \infty} v_r = \infty$; the function $I : \mathcal{Z} \rightarrow [0, \infty]$ is lower semi-continuous;

$$\limsup_{r \rightarrow \infty} \frac{1}{v_r} \log P(Z_r \in F) \leq - \inf_{z \in F} I(z) \text{ for all closed sets } F; \tag{1}$$

$$\liminf_{r \rightarrow \infty} \frac{1}{v_r} \log P(Z_r \in G) \geq - \inf_{z \in G} I(z) \text{ for all open sets } G. \tag{2}$$

A rate function I is said to be *good* if its level sets $\{z \in \mathcal{Z} : I(z) \leq \eta\} : \eta \geq 0\}$ are compact.

Throughout this paper we prove LDPs with $\mathcal{Z} = \mathbb{R}$. We recall the following known result for future use.

Theorem 2.1 (Gärtner Ellis Theorem (on \mathbb{R}); see e.g. Theorem 2.3.6 in Dembo and Zeitouni (1998)) *Let $\{Z_r : r > 0\}$ be a family of real valued random variables (defined on the same probability space (Ω, \mathcal{F}, P)). Assume that the function $\Lambda : \mathbb{R} \rightarrow (-\infty, \infty]$ defined by*

$$\Lambda(s) := \lim_{r \rightarrow \infty} \frac{1}{v_r} \log \mathbb{E} \left[e^{v_r s Z_r} \right] \text{ (for all } s \in \mathbb{R} \text{)}$$

exists, and it is finite in a neighborhood of the origin $s = 0$. Moreover let $\Lambda^ : \mathbb{R} \rightarrow [0, \infty]$ defined by*

$$\Lambda^*(z) := \sup_{s \in \mathbb{R}} \{sz - \Lambda(s)\}.$$

Then: Eq. 1 holds with $I = \Lambda^*$; a weak form of Eq. 2 with $I = \Lambda^*$ holds, i.e.

$$\liminf_{r \rightarrow \infty} \frac{1}{v_r} \log P(Z_r \in G) \geq - \inf_{z \in G \cap \mathcal{E}} \Lambda^*(z) \text{ for all open sets } G$$

where \mathcal{E} is the set of exposed points of I (namely the points in which I is finite and strictly convex); if Λ is essentially smooth and lower semi-continuous, then the LDP holds with good rate function $I = \Lambda^*$.

We also recall that Λ in the above statement is essentially smooth (see e.g. Definition 2.3.5 in Dembo and Zeitouni (1998)) if the interior of the set $\mathcal{D}_\Lambda := \{s \in \mathbb{R} : \Lambda(s) < \infty\}$ is non-empty, if it is differentiable throughout the interior of that set, and if it is a steep function (namely $|\Lambda'(s)|$ tends to infinity when s in the interior of \mathcal{D}_Λ approaches any finite point of its boundary).

2.2 Preliminaries on the Model

We start with a slight correction of Proposition 9 in Di Crescenzo et al. (2018) for the moment generating function of the absorption time $A_x = A_x(\lambda, \mu)$. In what follows we consider the function

$$\Lambda(s; \lambda, \mu) := \frac{1}{2} \left(\lambda - \mu - \sqrt{(\lambda + \mu - 2s)^2 - 4\lambda\mu} \right) \tag{3}$$

for $s \leq \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2}$. Throughout this paper we use the symbols $\Lambda'(s; \lambda, \mu)$ and $\Lambda''(s; \lambda, \mu)$ for the first and the second derivatives of $\Lambda(s; \lambda, \mu)$ with respect to s .

Remark 2.1 (The moment generating function of $A_x = A_x(\lambda, \mu)$) Proposition 9 in Di Crescenzo et al. (2018) provides an expression of the moment generating function $\mathbb{E}[e^{sA_x(\lambda, \mu)}]$ when $s \leq \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2}$ (it is equal to infinity otherwise). Actually a possible further restriction on s is needed, i.e. $(1 - \alpha)\mathbb{E}[e^{sC_0}] < 1$, where $\mathbb{E}[e^{sC_0}]$ is the moment generating function of the renewal cycles $\{C_{0,i} : i \geq 1\}$ introduced in Di Crescenzo et al. (2018). Then, since $\mathbb{E}[e^{sC_0}] \uparrow \sqrt{\frac{\lambda}{\mu}}$ as $s \uparrow \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2}$, we distinguish two cases:

- if $(1 - \alpha)\sqrt{\frac{\lambda}{\mu}} < 1$ or, equivalently, if $\alpha > 1 - \sqrt{\frac{\mu}{\lambda}}$, then we recover the expression in Proposition 9 in Di Crescenzo et al. (2018), i.e.

$$\mathbb{E}\left[e^{sA_x(\lambda, \mu)}\right] = \begin{cases} \frac{2\alpha\lambda e^{x\Lambda(s; \lambda, \mu)}}{2\lambda(\alpha-1) + \lambda + \mu - 2s + \sqrt{(\lambda + \mu - 2s)^2 - 4\lambda\mu}} & \text{for } s \leq \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2} \\ \infty & \text{otherwise;} \end{cases}$$

- if $(1 - \alpha)\sqrt{\frac{\lambda}{\mu}} \geq 1$ or, equivalently, if $\alpha \leq 1 - \sqrt{\frac{\mu}{\lambda}}$, then we have

$$\mathbb{E}\left[e^{sA_x(\lambda, \mu)}\right] = \begin{cases} \frac{2\alpha\lambda e^{x\Lambda(s; \lambda, \mu)}}{2\lambda(\alpha-1) + \lambda + \mu - 2s + \sqrt{(\lambda + \mu - 2s)^2 - 4\lambda\mu}} & \text{for } s < \hat{s}(\lambda, \mu, \alpha) \\ \infty & \text{otherwise,} \end{cases}$$

where $\hat{s}(\lambda, \mu, \alpha) := \frac{\alpha(\lambda(1-\alpha) - \mu)}{2(1-\alpha)} \in (0, \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2}]$; in particular we have $\hat{s}(\lambda, \mu, \alpha) = \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2}$ if $(1 - \alpha)\sqrt{\frac{\lambda}{\mu}} = 1$ or, equivalently, if $\alpha = 1 - \sqrt{\frac{\mu}{\lambda}}$.

Now we discuss some technical details on the function $\Lambda(\cdot) = \Lambda(\cdot; \lambda, \mu)$ (in particular the concept of steepness in the definition of essentially smooth function).

Remark 2.2 (Some properties of $\Lambda(\cdot; \lambda, \mu)$) The function $\Lambda(\cdot) = \Lambda(\cdot; \lambda, \mu)$ plays a crucial role in some applications of the Gärtner Ellis Theorem in this paper. In particular, by referring to Remark 2.1, it is a lower semi-continuous function if $\alpha > 1 - \sqrt{\frac{\mu}{\lambda}}$, and it is an essentially smooth function if $\alpha \geq 1 - \sqrt{\frac{\mu}{\lambda}}$.

In view of some results presented below, we compute the following Legendre transforms:

$$\Lambda^*(z; \lambda, \mu) := \sup_{s \in \mathbb{R}} \{sz - \Lambda(s; \lambda, \mu)\} = \sup_{s \leq \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2}}$$

and, if we consider $\hat{s}(\lambda, \mu, \alpha)$ in Remark 2.1 for $\alpha \leq 1 - \sqrt{\frac{\mu}{\lambda}}$,

$$\Lambda^*(z; \lambda, \mu, \alpha) := \sup_{s \leq \hat{s}(\lambda, \mu, \alpha)} \{sz - \Lambda(s; \lambda, \mu)\}.$$

Lemma 2.1 (Computation of Legendre transforms) *We have*

$$\Lambda^*(z; \lambda, \mu) = \begin{cases} \frac{1}{2} (\sqrt{(z-1)\lambda} - \sqrt{(z+1)\mu})^2 & \text{if } z \geq 1 \\ \infty & \text{otherwise.} \end{cases}$$

Moreover, for $\alpha \leq 1 - \sqrt{\frac{\mu}{\lambda}}$, if we set $\tilde{z}(\lambda, \mu, \alpha) := \frac{\lambda + \mu - 2\hat{s}(\lambda, \mu, \alpha)}{\sqrt{(\lambda + \mu - 2\hat{s}(\lambda, \mu, \alpha))^2 - 4\lambda\mu}}$ for $\hat{s}(\lambda, \mu, \alpha)$ as in Remark 2.1, then

$$\begin{aligned} \Lambda^*(z; \lambda, \mu, \alpha) &= \begin{cases} \Lambda^*(z; \lambda, \mu) & \text{if } z \leq \tilde{z}(\lambda, \mu, \alpha) \\ \hat{s}(\lambda, \mu, \alpha)z - \Lambda(\hat{s}(\lambda, \mu, \alpha); \lambda, \mu) & \text{otherwise} \end{cases} \\ &= \begin{cases} \infty & \text{if } z < 1 \\ \frac{1}{2} (\sqrt{(z-1)\lambda} - \sqrt{(z+1)\mu})^2 & \text{if } 1 \leq z \leq \tilde{z}(\lambda, \mu, \alpha) \\ \hat{s}(\lambda, \mu, \alpha)z - \Lambda(\hat{s}(\lambda, \mu, \alpha); \lambda, \mu) & \text{if } z > \tilde{z}(\lambda, \mu, \alpha). \end{cases} \end{aligned}$$

Proof We start with the first statement concerning $\Lambda^*(z; \lambda, \mu)$. For $z > 1$ one can check that the equation $z = \Lambda'(s; \lambda, \mu)$ has solution $s = s_z := \frac{1}{2} \left(\lambda + \mu - 2z\sqrt{\frac{\lambda\mu}{z^2-1}} \right)$, and

$$\Lambda^*(z; \lambda, \mu) = s_z z - \Lambda(s_z; \lambda, \mu)$$

yields the desired expression; for $z \leq 1$ one can check that

$$\Lambda^*(z; \lambda, \mu) = \lim_{s \rightarrow -\infty} sz - \Lambda(s; \lambda, \mu),$$

which yields again the desired expression (one has to distinguish the cases $z = 1$ and $z < 1$).

For $\Lambda^*(z; \lambda, \mu, \alpha)$ we proceed in the same way, and we omit some details. Some computations coincide with the ones presented above but, if s_z above is larger than $\hat{s}(\lambda, \mu, \alpha)$, then the supremum is attained at $s = \hat{s}(\lambda, \mu, \alpha)$. Moreover one can check that $s_z > \hat{s}(\lambda, \mu, \alpha)$ if and only if $z > \tilde{z}(\lambda, \mu, \alpha)$. The desired expression can be checked with straightforward computations. □

Finally, in the next Remark 2.3, we recall some formulas already presented in Di Crescenzo et al. (2018); actually we give the corrected expression of the variance.

Remark 2.3 (A correction of a variance formula in Di Crescenzo et al. (2018)) Here we give the correct version of some formulas in Di Crescenzo et al. (2018). More precisely we mean the n -th moments of C_x and A_x , i.e.

$$\begin{aligned} \mathbb{E}[C_x^n] &= \frac{\lambda}{\lambda + \mu} e^{\frac{x}{2}(\lambda - \mu)} \frac{2^n n!}{(\lambda + \mu)^n} \sum_{h=0}^n \left(-\frac{\lambda + \mu}{(\sqrt{\lambda} - \sqrt{\mu})^2} \right)^h \\ &\times {}_2F_1 \left(\frac{1 + n - h}{2}, \frac{2 + n - h}{2}; 2; \frac{4\lambda\mu}{(\lambda + \mu)^2} \right) \\ &\times \sum_{j=0}^{+\infty} \left[-\frac{(\lambda - \mu)x}{2} \right]^j \frac{1}{j!} \binom{j/2}{h} {}_2F_1 \left(-h, -\frac{j}{2}; \frac{j}{2} + 1 - h; \left(\frac{\sqrt{\lambda} - \sqrt{\mu}}{\sqrt{\lambda} + \sqrt{\mu}} \right)^2 \right), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[A_x^n] &= 2\alpha\lambda n! e^{\frac{x}{2}(\lambda - \mu)} \sum_{h=0}^n \left(-\frac{2}{(\sqrt{\lambda} - \sqrt{\mu})^2} \right)^h \frac{(8\lambda(\alpha - 1))^{n-h}}{(4\lambda\alpha(\mu + \lambda(\alpha - 1)))^{n-h+1}} \\ &\times \left[2\mu + 2\lambda(\alpha - 1) + \frac{2\lambda\mu}{\lambda + \mu} \sum_{m=1}^{n-h} \left(\frac{\alpha\mu + \alpha\lambda(\alpha - 1)}{(\alpha - 1)(\lambda + \mu)} \right)^m \right. \\ &\times \left. {}_2F_1 \left(\frac{m + 1}{2}, \frac{m + 2}{2}; 2; \frac{4\lambda\mu}{(\lambda + \mu)^2} \right) \right] \\ &\times \sum_{j=0}^{+\infty} \left[-\frac{(\lambda - \mu)x}{2} \right]^j \frac{1}{j!} \binom{j/2}{h} {}_2F_1 \left(-h, -\frac{j}{2}; \frac{j}{2} + 1 - h; \left(\frac{\sqrt{\lambda} - \sqrt{\mu}}{\sqrt{\lambda} + \sqrt{\mu}} \right)^2 \right). \end{aligned}$$

In particular we also recall the correct expressions in Proposition 12 in Di Crescenzo et al. (2018) (actually only the variances should be corrected):

$$\begin{aligned} \mathbb{E}[C_x] &= \frac{2 + (\lambda + \mu)x}{\lambda - \mu} \text{ and } \text{Var}[C_x] = \frac{4(\lambda + \mu + 2\lambda\mu x)}{(\lambda - \mu)^3}; \\ \mathbb{E}[A_x] &= \frac{2 + \alpha(\lambda + \mu)x}{\alpha(\lambda - \mu)} \text{ and } \text{Var}[A_x] = \frac{4(\lambda + 2\lambda\mu x\alpha^2 + \mu(2\alpha - 1))}{(\lambda - \mu)^3\alpha^2}. \end{aligned} \tag{4}$$

3 Asymptotic Results Under the Scaling 1

We start with the standard large deviation result.

Proposition 3.1 (LD as $x \rightarrow \infty$) *Assume that $\alpha \geq 1 - \sqrt{\frac{\mu}{\lambda}}$. Then the family $\left\{ \frac{A_x}{x} : x > 0 \right\}$ satisfies the LDP with speed x , and good rate function I_1 defined by $I_1(z) := \Lambda^*(z; \lambda, \mu)$.*

Proof It is easy to check (by taking into account Remark 2.1) that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \mathbb{E} \left[e^{sA_x(\lambda, \mu)} \right] = \Lambda(s; \lambda, \mu) \text{ (for all } s \in \mathbb{R} \text{)}.$$

Then, by taking into account Remark 2.2, the desired LDP holds by a straightforward application of the Gärtner Ellis Theorem. □

Note that $\frac{A_x}{x}$ converges to $z_1 := \frac{\lambda + \mu}{\lambda - \mu}$ almost surely as $x \rightarrow \infty$ (this can be checked in a standard way noting that the rate function I_1 uniquely vanishes at z_1). Moreover $z_1 = \Lambda'(0; \lambda, \mu) = \lim_{x \rightarrow \infty} \frac{\mathbb{E}[A_x]}{x}$. Finally z_1 can be seen as the abscissa of the intersection (in the $\tilde{x}\tilde{y}$ plane) of the lines $\tilde{y} = 0$ and $\tilde{y} = 1 + \frac{\frac{1}{\lambda} - \frac{1}{\mu}}{\frac{1}{\lambda} + \frac{1}{\mu}} \tilde{x}$. A version of Proposition 3.1 concerning the case $\alpha < 1 - \sqrt{\frac{\mu}{\lambda}}$ will be illustrated in Remark 3.2 (case $r = 1$).

Now we present the moderate deviation result. As it typically happens, we have a class of LDPs governed by the same quadratic rate function (i.e. \tilde{I}_1). Moreover this class of LDPs fills the gap between a convergence to zero and a weak convergence to a centered Normal distribution; see Remark 3.1 for some details and comments.

Proposition 3.2 (MD as $x \rightarrow \infty$) *For every family of positive numbers $\{\varepsilon_x : x > 0\}$ such that*

$$\varepsilon_x \rightarrow 0 \text{ and } x\varepsilon_x \rightarrow \infty, \tag{5}$$

the family $\left\{ \frac{A_x - \mathbb{E}[A_x]}{\sqrt{x/\varepsilon_x}} : x > 0 \right\}$ satisfies the LDP with speed $1/\varepsilon_x$, and good rate function \tilde{I}_1 defined by $\tilde{I}_1(z) := \frac{z^2}{2\Lambda''(0; \lambda, \mu)}$, where $\Lambda''(0; \lambda, \mu) = \frac{8\lambda\mu}{(\lambda - \mu)^3}$.

Proof It suffices to show that

$$\lim_{x \rightarrow \infty} \frac{1}{1/\varepsilon_x} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_x} \frac{A_x - \mathbb{E}[A_x]}{\sqrt{x/\varepsilon_x}}} \right] = \frac{\Lambda''(0; \lambda, \mu)}{2} s^2 \text{ (for all } s \in \mathbb{R}\text{);}$$

in fact the limit is a finite and differentiable function (with respect to $s \in \mathbb{R}$) and, noting that

$$\tilde{I}_1(z) = \sup_{s \in \mathbb{R}} \left\{ sz - \frac{\Lambda''(0; \lambda, \mu)}{2} s^2 \right\} \text{ (for all } z \in \mathbb{R}\text{),}$$

the desired LDP is a straightforward application of the Gärtner Ellis Theorem.

We remark that

$$\frac{1}{1/\varepsilon_x} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_x} \frac{A_x - \mathbb{E}[A_x]}{\sqrt{x/\varepsilon_x}}} \right] = \varepsilon_x \left(\log \mathbb{E} \left[e^{\frac{s}{\sqrt{x\varepsilon_x}} A_x} \right] - \frac{s}{\sqrt{x\varepsilon_x}} \mathbb{E}[A_x] \right)$$

and, since $\frac{s}{\sqrt{x\varepsilon_x}}$ is close to zero for x large enough, it is easy to check (by the expressions of the moment generating function in Remark 2.1 and by Eq. 3) that

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{1}{1/\varepsilon_x} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_x} \frac{A_x - \mathbb{E}[A_x]}{\sqrt{x/\varepsilon_x}}} \right] \\ &= \lim_{x \rightarrow \infty} \varepsilon_x \left(x\Lambda \left(\frac{s}{\sqrt{x\varepsilon_x}}; \lambda, \mu \right) - \frac{s}{\sqrt{x\varepsilon_x}} \mathbb{E}[A_x] \right) \\ &= \lim_{x \rightarrow \infty} x\varepsilon_x \left(\Lambda \left(\frac{s}{\sqrt{x\varepsilon_x}}; \lambda, \mu \right) - \frac{s}{\sqrt{x\varepsilon_x}} \frac{\mathbb{E}[A_x]}{x} \right). \end{aligned}$$

Now we take into account the Mac Laurin formula of order 2 for the function $\Lambda(\cdot; \lambda, \mu)$, and we have

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{1}{1/\varepsilon_x} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_x} \frac{A_x - \mathbb{E}[A_x]}{\sqrt{x/\varepsilon_x}}} \right] \\ &= \lim_{x \rightarrow \infty} x\varepsilon_x \left(\left(\Lambda'(0; \lambda, \mu) - \frac{\mathbb{E}[A_x]}{x} \right) \frac{s}{\sqrt{x\varepsilon_x}} + \frac{\Lambda''(0; \lambda, \mu)}{2} \frac{s^2}{x\varepsilon_x} + o \left(\frac{s^2}{x\varepsilon_x} \right) \right) \end{aligned}$$

where, by the mean value in Eq. 4,

$$\Lambda'(0; \lambda, \mu) = \frac{\mathbb{E}[A_x]}{x} = \frac{\lambda + \mu}{\lambda - \mu} - \frac{2 + \alpha(\lambda + \mu)x}{\alpha(\lambda - \mu)x} = -\frac{2}{\alpha(\lambda - \mu)x};$$

thus

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1}{1/\varepsilon_x} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_x} \frac{A_x - \mathbb{E}[A_x]}{\sqrt{x/\varepsilon_x}}} \right] &= \lim_{x \rightarrow \infty} -\frac{2s\sqrt{x\varepsilon_x}}{\alpha(\lambda - \mu)x} + \frac{\Lambda''(0; \lambda, \mu)}{2} s^2 + x\varepsilon_x o\left(\frac{s^2}{x\varepsilon_x}\right) \\ &= \frac{\Lambda''(0; \lambda, \mu)}{2} s^2 + \lim_{x \rightarrow \infty} -\frac{2s\sqrt{\varepsilon_x}}{\alpha(\lambda - \mu)\sqrt{x}} \\ &\quad + x\varepsilon_x o\left(\frac{s^2}{x\varepsilon_x}\right) = \frac{\Lambda''(0; \lambda, \mu)}{2} s^2. \end{aligned}$$

□

Remark 3.1 (Typical features on MD in Proposition 3.2) The class of LDPs in Proposition 3.2 fills the gap between the two following asymptotic regimes as $x \rightarrow \infty$:

- the convergence to zero of $\frac{A_x - \mathbb{E}[A_x]}{x}$ (case $\varepsilon_\mu = 1/x$);
- the weak convergence of $\frac{A_x - \mathbb{E}[A_x]}{\sqrt{x}}$ to the centered Normal distribution with variance $\Lambda''(0; \lambda, \mu)$ (case $\varepsilon_x = 1$).

In both cases one condition in Eq. 5 holds, and the other one fails. We also note that, by taking into account the variance expression in Eq. 4, we have $\Lambda''(0; \lambda, \mu) = \lim_{x \rightarrow \infty} \frac{\text{Var}[A_x]}{x}$.

We conclude this section by considering a generalization of Proposition 3.1 with an independent random perturbation $Y(x)$ of the initial state x under suitable hypotheses collected in Condition 1 below; this generalization will be given in Proposition 3.4, and it will be followed by some remarks and comments. We start with the following slight generalization of Proposition 3.1 where the initial state is modified in a deterministic way; we recover the case in that proposition by setting $r = 1$.

Proposition 3.3 (Slight extension of Proposition 3.1) *Assume that $\alpha \geq 1 - \sqrt{\frac{\mu}{\lambda}}$ and let $r > 0$ be arbitrarily fixed. Then the family $\left\{ \frac{A_{rx}}{x} : x > 0 \right\}$ satisfies the LDP with speed x , and good rate function $I_1(\cdot; r)$ defined by $I_1(z; r) := r\Lambda^*(z/r; \lambda, \mu)$.*

Proof It is easy to check that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \mathbb{E} \left[e^{sA_{rx}(\lambda, \mu)} \right] = r\Lambda(s; \lambda, \mu) \text{ (for all } s \in \mathbb{R} \text{)}$$

(it is a slight modification of the analogue limit in the proof of Proposition 3.1 where $r = 1$). Then, by taking into account Remark 2.2, the desired LDP holds by a straightforward application of the Gärtner–Ellis Theorem. In fact the governing rate function $I_1(\cdot; r)$ is defined by

$$I_1(z; r) := \sup_{s \in \mathbb{R}} \{sz - r\Lambda(s; \lambda, \mu)\} = r \sup_{s \in \mathbb{R}} \{sz/r - \Lambda(s; \lambda, \mu)\},$$

and this coincides with the rate function in the statement of the proposition. □

Arguing as we did just after Proposition 3.1, we can say that $\frac{A_{rx}}{x}$ converges to $rz_1 = r \frac{\lambda + \mu}{\lambda - \mu}$ almost surely as $x \rightarrow \infty$ (and the rate function $I_1(\cdot; r)$ uniquely vanishes at rz_1).

Remark 3.2 (Versions of Propositions 3.1 and 3.3 with exposed points) Here we discuss what happens when we consider the inequality $\alpha < 1 - \sqrt{\frac{\mu}{\lambda}}$ in Proposition 3.3 (and therefore in Proposition 3.1 for the case $r = 1$). We have to consider some items in the second part of Lemma 2.1 and, by the Gärtner Ellis Theorem, we have

$$\limsup_{x \rightarrow \infty} \frac{1}{x} \log P \left(\frac{A_{rx}}{x} \in F \right) \leq - \inf_{z \in F} r \Lambda^*(z/r; \lambda, \mu, \alpha) \text{ for all closed sets } F$$

and

$$\liminf_{x \rightarrow \infty} \frac{1}{x} \log P \left(\frac{A_{rx}}{x} \in G \right) \geq - \inf_{z \in G \cap \mathcal{E}} r \Lambda^*(z/r; \lambda, \mu, \alpha) \text{ for all open sets } G$$

where $\mathcal{E} = (r\tilde{z}(\lambda, \mu, \alpha), \infty)$ is the set of exposed points of $r \Lambda^*(\cdot/r; \lambda, \mu, \alpha)$. Note that $r\tilde{z}(\lambda, \mu, \alpha) > rz_1$, and therefore both $r \Lambda^*(\cdot/r; \lambda, \mu, \alpha)$ and $I_1(\cdot; r)$ uniquely vanish at rz_1 .

Now we introduce the condition on the random perturbation of the initial state.

Condition 1 Let $\{Y(x) : x \geq 0\}$ be a family of nonnegative random variables and assume that there exists the function $\Psi_Y : \mathbb{R} \rightarrow (-\infty, \infty]$ such that

$$\Psi_Y(s) := \lim_{x \rightarrow \infty} \frac{1}{x} \log \mathbb{E} \left[e^{sY(x)} \right]$$

for all $s \in \mathbb{R}$. The function Ψ_Y is nondecreasing by construction; so we consider the set

$$\mathcal{D}_Y := \{s \in \mathbb{R} : \Psi_Y(s) < \infty\},$$

and we assume that either $\mathcal{D}_Y = \mathbb{R}$ or, for some $\bar{s} > 0$, $\mathcal{D}_Y = (-\infty, \bar{s})$ or $\mathcal{D}_Y = (-\infty, \bar{s}]$ (note that $(-\infty, 0] \subset \mathcal{D}_Y$).

We remark that Condition 1 holds when $\{Y(x) : x \geq 0\}$ belongs to a wide class of nondecreasing (with respect to x) Lévy processes, also called subordinators; in this case we have $\Psi_Y(s) := \log \mathbb{E}[e^{sY(1)}]$. For instance we recall the following examples of infinitely divisible distributions concerning the random variable $Y(1)$.

Distribution	parameters	\mathcal{D}_Y	$\Psi_Y(s)$ for $s \in \mathcal{D}_Y$	$\Psi'_Y(0)$	$\Psi''_Y(0)$
Poisson	$\lambda > 0$	\mathbb{R}	$\lambda(e^s - 1)$	λ	λ
Gamma	$\lambda, \theta > 0$	$(-\infty, \bar{s})$, $\bar{s} = \theta$	$\gamma \log \frac{\theta}{\theta - s}$	$\frac{\lambda}{\theta}$	$\frac{\lambda}{\theta^2}$
Inverse Gaussian	$\xi > 0$	$(-\infty, \bar{s}]$, $\bar{s} = \frac{\xi^2}{2}$	$\xi - \sqrt{\xi^2 - 2s}$	ξ^{-1}	ξ^{-3}

So now we are ready to state the main generalization of Proposition 3.1.

Proposition 3.4 (Extension of Proposition 3.1) *Assume that $\alpha \geq 1 - \sqrt{\frac{\mu}{\lambda}}$ and that a process $\{Y(x) : x \geq 0\}$, independent of $\{A_x : x \geq 0\}$, satisfies Condition 1. Moreover let Λ_Y be the function defined by*

$$\Lambda_Y(s) := \begin{cases} \Psi_Y(\Lambda(s; \lambda, \mu)) & \text{for } \Lambda(s; \lambda, \mu) \in \mathcal{D}_Y \text{ and } s \leq \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2} \\ \infty & \text{otherwise,} \end{cases}$$

and assume that it is essentially smooth. Then the family $\left\{ \frac{A_Y(x)}{x} : x > 0 \right\}$ satisfies the LDP with speed x , and good rate function Λ_Y^* defined by $\Lambda_Y^*(z) := \sup_{s \in \mathbb{R}} \{sz - \Lambda_Y(s)\}$.

Proof We compute the moment generating function of $A_{Y(x)}$ by considering the well-known equality $\mathbb{E} \left[e^{sA_{Y(x)}(\lambda, \mu)} \right] = \mathbb{E} \left[\mathbb{E} \left[e^{sA_r(\lambda, \mu)} \right] \Big|_{r=Y(x)} \right]$. Moreover, by the expression of the moment generating function in Remark 2.1, we get

$$\mathbb{E} \left[e^{sA_{Y(x)}(\lambda, \mu)} \right] = \begin{cases} \frac{2\alpha\lambda\mathbb{E} \left[e^{Y(x)\Lambda(s; \lambda, \mu)} \right]}{2\lambda(\alpha-1) + \lambda + \mu - 2s + \sqrt{(\lambda + \mu - 2s)^2 - 4\lambda\mu}} & \text{for } s \leq \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2} \\ \infty & \text{otherwise.} \end{cases}$$

So, by Condition 1, we get

$$\lim_{x \rightarrow \infty} \frac{1}{x} \log \mathbb{E} \left[e^{sA_{Y(x)}(\lambda, \mu)} \right] = \Lambda_Y(s) \text{ (for all } s \in \mathbb{R}\text{),}$$

where Λ_Y is the function in the statement of the proposition. In conclusion, since Λ_Y is an essentially smooth function, the desired LDP holds by a straightforward application of the Gärtner Ellis Theorem. \square

Now we present some remarks and comments on Proposition 3.4. In what follows we assume that the function Ψ_Y is differentiable in the interior of \mathcal{D}_Y .

Remark 3.3 (Extension of some parts in the proof of Lemma 2.1) We consider $\Psi'_Y(-\infty) := \lim_{s \rightarrow -\infty} \Psi'_Y(s)$ (this limit is well-defined because Ψ'_Y is monotonic by the convexity of Ψ_Y). Then, for $z > \Psi'_Y(-\infty)$, one can check that the equation $z = \Lambda'_Y(s)$ has solution $s = \tilde{s}_z$, and we have

$$\Lambda_Y^*(z) = z\tilde{s}_z - \Lambda_Y(\tilde{s}_z).$$

On the other hand, for $z \leq \Psi'_Y(-\infty)$ one can check that

$$\Lambda_Y^*(z) = \lim_{s \rightarrow -\infty} sz - \Lambda_Y(s),$$

which is finite for $z = \Psi'_Y(-\infty)$ and infinite for $z < \Psi'_Y(-\infty)$.

Remark 3.4 (On the essential smoothness of Λ_Y) In general, if s belongs to the interior of the set where $\Lambda_Y(s) < \infty$, we have

$$\Lambda'_Y(s) = \Psi'_Y(\Lambda(s; \lambda, \mu))\Lambda'(s; \lambda, \mu).$$

Then, if we refer to Condition 1, we have two cases.

1. If $\mathcal{D}_Y = \mathbb{R}$, then we have to check that $\Lambda'_Y(s) \uparrow \infty$ as $s \uparrow \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2}$. This statement is true because $\Lambda'(s; \lambda, \mu) \uparrow \infty$ and $\Psi'_Y(\Lambda(s; \lambda, \mu))$ tends to a positive limit.
2. If we have $\mathcal{D}_Y = (-\infty, \bar{s})$ or $\mathcal{D}_Y = (-\infty, \bar{s}]$ for some $\bar{s} \in (0, \infty)$, then we take $s_0 := \Lambda^{-1}(\bar{s}; \lambda, \mu) \wedge \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2}$ and we have to check that $\Lambda'_Y(s) \uparrow \infty$ as $s \uparrow s_0$. If $s_0 = \frac{(\sqrt{\lambda} - \sqrt{\mu})^2}{2}$, then we can conclude following the lines of the previous case ($\mathcal{D}_Y = \mathbb{R}$). If $s_0 = \Lambda^{-1}(\bar{s}; \lambda, \mu)$, we also require the condition $\Psi'_Y(s) \uparrow \infty$ as $s \uparrow \bar{s}$, and then we have $\Lambda'_Y(s) \uparrow \infty$ because $\Psi'_Y(\Lambda(s; \lambda, \mu)) \uparrow \infty$ and $\Lambda'(s; \lambda, \mu)$ tends to a positive limit.

We continue with some further comments and, from now on, we assume that $\Psi'_Y(0) > 0$; note that this condition holds for the examples tabulated above. Moreover we assume to

have the hypotheses of Propositions 3.3 and 3.4 that guarantee the validity of the LDPs stated in those propositions. It is known that $\Lambda_Y^*(z) = 0$ if and only if $z = \hat{z} := \Lambda'_Y(0) = \Psi'_Y(0)\Lambda'(0; \lambda, \mu)$, and $I_1(z; r) = 0$ if and only if $z = z_r^* := r\Lambda'(0; \lambda, \mu)$. So, if we take $r = \Psi'_Y(0)$, we have $\hat{z} = z_r^*$, both rate functions Λ_Y^* and $I_1(\cdot; r)$ uniquely vanish at \hat{z} , and therefore both $\frac{A_Y(x)}{x}$ and $\frac{A_{rx}}{x}$ converge to same limit \hat{z} (as $x \rightarrow \infty$). Thus, in this case, it is interesting to find inequalities between rate functions (when z belongs to a neighborhood of \hat{z} , except $z = \hat{z}$) to say that we have a faster convergence in the case governed by the (locally) larger rate function.

We start noting that $\Psi_Y(s) \geq \Psi'_Y(0)s$ by the convexity of Ψ_Y and by $\Psi_Y(0) = 0$; thus we obtain

$$\Lambda_Y(s) \geq \Psi'_Y(0)\Lambda(s; \lambda, \mu) \text{ for all } s \in \mathbb{R}$$

(in fact, if $s > \frac{(\sqrt{\lambda}-\sqrt{\mu})^2}{2}$, we have $\Lambda_Y(s) = \Psi'_Y(0)\Lambda(s; \lambda, \mu) = \infty$). So we easily obtain the following inequality between rate functions:

$$\Lambda_Y^*(z) = \sup_{s \in \mathbb{R}}\{sz - \Lambda_Y(s)\} \leq \sup_{s \in \mathbb{R}}\{sz - \Psi'_Y(0)\Lambda(s; \lambda, \mu)\} = I_1(z; \Psi'_Y(0)).$$

In conclusion the rate function which governs the LDP of $\left\{ \frac{A_{\Psi'_Y(0)x}}{x} : x > 0 \right\}$ cannot be smaller than the one for the LDP of $\left\{ \frac{A_Y(x)}{x} : x > 0 \right\}$; this is not surprising because we expect to have a faster convergence (to \hat{z} as $x \rightarrow \infty$) when the perturbation of the initial position is deterministic.

We also remark that, under suitable conditions (for instance if Ψ_Y is strictly convex, as happens for the tabulated examples above), we have the strict inequality $\Lambda_Y^*(z) < I_1(z; \Psi'_Y(0))$ except for the cases in which both $\Lambda_Y^*(z)$ and $I_1(z; \Psi'_Y(0))$ are equal to zero (i.e. if $z = \hat{z}$) or to infinity (i.e. if $z < \Psi'_Y(-\infty)$).

As a final comment we also briefly discuss the comparison between the second derivatives of the rate functions at $z = \hat{z}$; indeed a larger second derivative corresponds to a locally larger rate function in a neighborhood of \hat{z} , except $z = \hat{z}$. We have

$$(\Lambda_Y^*)''(\Lambda'_Y(0)) = \frac{1}{\Lambda''_Y(0)} = \frac{1}{\Psi''_Y(0)(\Lambda'(0; \lambda, \mu))^2 + \Psi'_Y(0)\Lambda''(0; \lambda, \mu)}$$

and

$$I''_1(r\Lambda'(0; \lambda, \mu); r) = \frac{1}{r\Lambda''(0; \lambda, \mu)};$$

thus, if we set $r = \Psi'_Y(0)$ in the last equalities, we get

$$(\Lambda_Y^*)''(\hat{z}) \leq I''_1(\hat{z}; \Psi'_Y(0))$$

by the convexity of the function Ψ_Y which yields $\Psi''_Y(0) \geq 0$. Actually in several common cases we have the strict inequality $(\Lambda_Y^*)''(\hat{z}) < I''_1(\hat{z}; \Psi'_Y(0))$ because $\Lambda'(0; \lambda, \mu) > 0$ and, as happens for the tabulated examples above, $\Psi''_Y(0) > 0$.

4 Asymptotic Results Under the Scaling 2

Throughout this section we set $\lambda = \beta\mu$ for some $\beta > 1$ and $x > 0$. We start with the standard large deviation result.

Proposition 4.1 (LD as $\mu \rightarrow \infty$) *Assume that $\alpha \geq 1 - \sqrt{\frac{1}{\beta}}$. Then the family $\{A_x(\beta\mu, \mu) : \mu > 0\}$ satisfies the LDP with speed μ , and good rate function I_2 defined by $I_2(z) := x\Lambda^*(z/x; \beta, 1)$.*

Proof It is easy to check (by taking into account Remark 2.1) that

$$\lim_{\mu \rightarrow \infty} \frac{1}{\mu} \log \mathbb{E} \left[e^{\mu s A_x(\beta\mu, \mu)} \right] = x\Lambda(s; \beta, 1) \text{ (for all } s \in \mathbb{R}\text{)}.$$

Then, by taking into account Remark 2.2, the desired LDP holds by a straightforward application of the Gärtner Ellis Theorem. In fact the governing rate function I_2 is defined by

$$I_2(z) := \sup_{s \in \mathbb{R}} \{sz - x\Lambda(s; \beta, 1)\} = x \sup_{s \in \mathbb{R}} \{sz/x - \Lambda(s; \beta, 1)\},$$

and this coincides with the rate function in the statement of the proposition. □

Note that $A_x(\beta\mu, \mu)$ converges to $z_2 := x \frac{\beta+1}{\beta-1}$ almost surely as $\mu \rightarrow \infty$ (in fact the rate function I_2 uniquely vanishes at z_2). Moreover $z_2 = x\Lambda'(0; \beta, 1) = \lim_{\mu \rightarrow \infty} \mathbb{E}[A_x(\beta\mu, \mu)]$. Finally z_2 can be seen as the abscissa of the intersection (in the $\tilde{x}\tilde{y}$ plane) of the lines $\tilde{y} = 0$ and $\tilde{y} = x + \frac{\frac{1}{\beta}\mu - \frac{1}{\mu}}{\frac{1}{\beta}\mu + \frac{1}{\mu}}\tilde{x}$.

Remark 4.1 (A version of Proposition 4.1 with exposed points) Here we discuss what happens when we consider the inequality $\alpha < 1 - \sqrt{\frac{1}{\beta}}$ in Proposition 4.1. In this case we still have to consider some items in the second part of Lemma 2.1 (as in Remark 3.2) and, by the Gärtner Ellis Theorem, we have

$$\limsup_{\mu \rightarrow \infty} \frac{1}{\mu} \log P(A_x(\beta\mu, \mu) \in F) \leq - \inf_{z \in F} x\Lambda^*(z/x; \beta, 1, \alpha) \text{ for all closed sets } F$$

and

$$\liminf_{\mu \rightarrow \infty} \frac{1}{\mu} \log P(A_x(\beta\mu, \mu) \in G) \geq - \inf_{z \in G \cap \mathcal{E}} x\Lambda^*(z/x; \beta, 1, \alpha) \text{ for all open sets } G$$

where $\mathcal{E} = (x\tilde{z}(\beta, 1, \alpha), \infty)$ is the set of exposed points of $x\Lambda^*(\cdot/x; \beta, 1, \alpha)$. Note that $x\tilde{z}(\beta, 1, \alpha) > z_2$, and therefore both $x\Lambda^*(\cdot/x; \beta, 1, \alpha)$ and I_2 uniquely vanish at z_2 .

Now we present the moderate deviation result. As it typically happens, we have a class of LDPs governed by the same quadratic rate function (i.e. \tilde{I}_2). Moreover this class of LDPs fills the gap between a convergence to zero and a weak convergence to a centered Normal distribution; see Remark 4.2 for some details and comments.

Proposition 4.2 (MD as $\mu \rightarrow \infty$) *For every family of positive numbers $\{\varepsilon_\mu : \mu > 0\}$ such that*

$$\varepsilon_\mu \rightarrow 0 \text{ and } \mu\varepsilon_\mu \rightarrow \infty, \tag{6}$$

the family $\{\sqrt{\mu\varepsilon_\mu}(A_x(\beta\mu, \mu) - \mathbb{E}[A_x(\beta\mu, \mu)]) : \mu > 0\}$ satisfies the LDP with speed $1/\varepsilon_\mu$, and good rate function \tilde{I}_2 defined by $\tilde{I}_2(z) := \frac{z^2}{2x\Lambda''(0; \beta, 1)}$, where $\Lambda''(0; \beta, 1) = \frac{8\beta}{(\beta-1)^3}$.

Proof It suffices to show that

$$\lim_{\mu \rightarrow \infty} \frac{1}{1/\varepsilon_\mu} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_\mu} \sqrt{\mu \varepsilon_\mu} (A_x(\beta\mu, \mu) - \mathbb{E}[A_x(\beta\mu, \mu)])} \right] = \frac{x\Lambda''(0; \beta, 1)}{2} s^2 \text{ (for all } s \in \mathbb{R}\text{);}$$

in fact the limit is a finite and differentiable function (with respect to $s \in \mathbb{R}$) and, noting that

$$\tilde{I}_2(z) = \sup_{s \in \mathbb{R}} \left\{ sz - \frac{x\Lambda''(0; \beta, 1)}{2} s^2 \right\} \text{ (for all } z \in \mathbb{R}\text{),}$$

the desired LDP is a straightforward application of the Gärtner Ellis Theorem.

We remark that

$$\begin{aligned} & \frac{1}{1/\varepsilon_\mu} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_\mu} \sqrt{\mu \varepsilon_\mu} (A_x(\beta\mu, \mu) - \mathbb{E}[A_x(\beta\mu, \mu)])} \right] \\ &= \varepsilon_\mu \left(\log \mathbb{E} \left[e^{s \frac{\sqrt{\mu}}{\sqrt{\varepsilon_\mu}} A_x(\beta\mu, \mu)} \right] - \frac{s\sqrt{\mu}}{\sqrt{\varepsilon_\mu}} \mathbb{E}[A_x(\beta\mu, \mu)] \right); \end{aligned}$$

moreover

$$\log \mathbb{E} \left[e^{s \frac{\sqrt{\mu}}{\sqrt{\varepsilon_\mu}} A_x(\beta\mu, \mu)} \right] = \log \mathbb{E} \left[e^{\frac{s\mu}{\sqrt{\mu \varepsilon_\mu}} A_x(\beta\mu, \mu)} \right] = \log \mathbb{E} \left[e^{\frac{s}{\sqrt{\mu \varepsilon_\mu}} A_{x\mu}(\beta, 1)} \right]$$

where the last equality holds by the expressions of the moment generating function in Remark 2.1 and by Eq. 3; then, since $\frac{s}{\sqrt{\mu \varepsilon_\mu}}$ is close to zero for μ large enough, it is easy to check that

$$\begin{aligned} & \lim_{\mu \rightarrow \infty} \frac{1}{1/\varepsilon_\mu} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_\mu} \sqrt{\mu \varepsilon_\mu} (A_x(\beta\mu, \mu) - \mathbb{E}[A_x(\beta\mu, \mu)])} \right] \\ &= \lim_{\mu \rightarrow \infty} \varepsilon_\mu \left(x\mu \Lambda \left(\frac{s}{\sqrt{\mu \varepsilon_\mu}}; \beta, 1 \right) - \frac{s\mu}{\sqrt{\mu \varepsilon_\mu}} \mathbb{E}[A_x(\beta\mu, \mu)] \right) \\ &= \lim_{\mu \rightarrow \infty} \mu \varepsilon_\mu \left(x\Lambda \left(\frac{s}{\sqrt{\mu \varepsilon_\mu}}; \beta, 1 \right) - \frac{s}{\sqrt{\mu \varepsilon_\mu}} \mathbb{E}[A_x(\beta\mu, \mu)] \right). \end{aligned}$$

Now we take into account the Mac Laurin formula of order 2 for the function $\Lambda(\cdot; \beta, 1)$, and we have

$$\begin{aligned} & \lim_{\mu \rightarrow \infty} \frac{1}{1/\varepsilon_\mu} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_\mu} \sqrt{\mu \varepsilon_\mu} (A_x(\beta\mu, \mu) - \mathbb{E}[A_x(\beta\mu, \mu)])} \right] \\ &= \lim_{\mu \rightarrow \infty} \mu \varepsilon_\mu \left((x\Lambda'(0; \beta, 1) - \mathbb{E}[A_x(\beta\mu, \mu)]) \frac{s}{\sqrt{\mu \varepsilon_\mu}} + \frac{x\Lambda''(0; \beta, 1)}{2} \frac{s^2}{\mu \varepsilon_\mu} + o\left(\frac{s^2}{\mu \varepsilon_\mu}\right) \right) \end{aligned}$$

where, by the mean value in Eq. 4,

$$x\Lambda'(0; \beta, 1) - \mathbb{E}[A_x(\beta\mu, \mu)] = x \frac{\beta + 1}{\beta - 1} - \frac{2 + \alpha(\beta + 1)\mu x}{\alpha(\beta - 1)\mu} = -\frac{2}{\alpha(\beta - 1)\mu};$$

thus

$$\begin{aligned} & \lim_{\mu \rightarrow \infty} \frac{1}{1/\varepsilon_\mu} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_\mu} \sqrt{\mu \varepsilon_\mu} (A_x(\beta\mu, \mu) - \mathbb{E}[A_x(\beta\mu, \mu)])} \right] \\ &= \frac{x\Lambda''(0; \beta, 1)}{2} s^2 + \lim_{\mu \rightarrow \infty} -\frac{2\sqrt{\varepsilon_\mu}}{\alpha(\beta - 1)\sqrt{\mu}} s + \mu \varepsilon_\mu o\left(\frac{s^2}{\mu \varepsilon_\mu}\right) = \frac{x\Lambda''(0; \beta, 1)}{2} s^2. \end{aligned}$$

□

Remark 4.2 (Typical features on MD in Proposition 4.2) The class of LDPs in Proposition 4.2 fills the gap between the two following asymptotic regimes as $\mu \rightarrow \infty$:

- the convergence to zero of $A_x(\beta\mu, \mu) - \mathbb{E}[A_x(\beta\mu, \mu)]$ (case $\varepsilon_\mu = 1/\mu$);
- the weak convergence of $\sqrt{\mu}(A_x(\beta\mu, \mu) - \mathbb{E}[A_x(\beta\mu, \mu)])$ to the centered Normal distribution with variance $x\Lambda''(0; \beta, 1)$ (case $\varepsilon_\mu = 1$).

In both cases one condition in Eq. 6 holds, and the other one fails. We also note that, by taking into account the variance expression in Eq. 4, we have $x\Lambda''(0; \beta, 1) = \lim_{\mu \rightarrow \infty} \mu \text{Var}[A_x(\beta\mu, \mu)]$.

We conclude this section with another moderate deviation result, which will be stated in Proposition 4.3. Namely we mean a class of LDPs that fills the gap between two asymptotic regimes, as $\mu \rightarrow \infty$, as in Proposition 4.2; more precisely the convergence to a constant, and the weak convergence to a suitable non degenerate limit law (this will be explained in Remark 4.4 below). In some sense we have a non-central moderate deviation result because the limit law is non-Gaussian; actually, as shown in the next Lemma 4.1, we deal with a family of equally distributed random variables and therefore the weak convergence trivially holds.

Lemma 4.1 (A weak convergence result as $\mu \rightarrow \infty$) *The random variables $\{\mu A_{x/\mu}(\beta\mu, \mu) : \mu > 0\}$ are equally distributed.*

Proof The result can be easily proved by taking the moment generating function of the involved random variables, and by referring to the formulas presented in Remark 2.1. One can easily check (we omit the details) that, under every condition on α stated in Remark 2.1, we have the same moment generating function for every random variables of the family $\{\mu A_{x/\mu}(\beta\mu, \mu) : \mu > 0\}$ (in fact it does not depend on μ). □

Now we can prove the non-central moderate deviation result.

Proposition 4.3 (Non-central MD as $\mu \rightarrow \infty$) *Assume that $\alpha \geq 1 - \sqrt{\frac{1}{\beta}}$. Then, for every family of positive numbers $\{\varepsilon_\mu : \mu > 0\}$ such that Eq. 6 holds, the family $\{\mu \varepsilon_\mu A_{x/(\mu \varepsilon_\mu)}(\beta\mu, \mu) : \mu > 0\}$ satisfies the LDP with speed $1/\varepsilon_\mu$, and good rate function I_2 (presented in Proposition 4.1).*

Proof We want to apply the Gärtner Ellis Theorem. So we have

$$\begin{aligned} & \frac{1}{1/\varepsilon_\mu} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_\mu} \mu \varepsilon_\mu A_{x/(\mu \varepsilon_\mu)}(\beta\mu, \mu)} \right] = \varepsilon_\mu \log \mathbb{E} \left[e^{s \mu A_{x/(\mu \varepsilon_\mu)}(\beta\mu, \mu)} \right] \\ & = \begin{cases} \varepsilon_\mu \log \frac{2\alpha \beta \mu e^{x/(\mu \varepsilon_\mu) \Lambda(s\mu; \beta\mu, \mu)}}{2\beta\mu(\alpha-1) + \beta\mu + \mu - 2s\mu + \sqrt{(\beta\mu + \mu - 2s\mu)^2 - 4\beta\mu^2}} & \text{for } s\mu \leq \frac{(\sqrt{\beta\mu} - \sqrt{\mu})^2}{2} \\ \infty & \text{otherwise} \end{cases} \\ & = \begin{cases} \varepsilon_\mu \log \frac{2\alpha \beta e^{x\Lambda(s; \beta, 1)/\varepsilon_\mu}}{2\beta(\alpha-1) + \beta + 1 - 2s + \sqrt{(\beta + 1 - 2s)^2 - 4\beta}} & \text{for } s \leq \frac{(\sqrt{\beta} - 1)^2}{2} \\ \infty & \text{otherwise;} \end{cases} \end{aligned}$$

then

$$\lim_{\mu \rightarrow \infty} \frac{1}{1/\varepsilon_\mu} \log \mathbb{E} \left[e^{\frac{s}{\varepsilon_\mu} \mu \varepsilon_\mu A_{x/(\mu \varepsilon_\mu)}(\beta \mu, \mu)} \right] = x \Lambda(s; \beta, 1) \text{ (for all } s \in \mathbb{R}),$$

and, by Remark 2.2, the desired LDP is a straightforward application of the Gärtner Ellis Theorem. \square

In the next remark we follow the same lines of Remarks 3.2 and 4.1.

Remark 4.3 (A version of Proposition 4.3 with exposed points) Here we discuss what happens when we consider the inequality $\alpha < 1 - \sqrt{\frac{1}{\beta}}$ in Proposition 4.3. In this case we still have to consider some items in the second part of Lemma 2.1 (as in Remarks 3.2 and 4.1) and, by the Gärtner Ellis Theorem, we have

$$\limsup_{\mu \rightarrow \infty} \frac{1}{1/\varepsilon_\mu} \log P(\mu \varepsilon_\mu A_{x/(\mu \varepsilon_\mu)}(\beta \mu, \mu) \in F) \leq - \inf_{z \in F} x \Lambda^*(z/x; \beta, 1, \alpha) \text{ for all closed sets } F$$

and

$$\liminf_{\mu \rightarrow \infty} \frac{1}{1/\varepsilon_\mu} \log P(\mu \varepsilon_\mu A_{x/(\mu \varepsilon_\mu)}(\beta \mu, \mu) \in G) \geq - \inf_{z \in G \cap \mathcal{E}} x \Lambda^*(z/x; \beta, 1, \alpha) \text{ for all open sets } G$$

where $\mathcal{E} = (x\tilde{z}(\beta, 1, \alpha), \infty)$ is the set of exposed points of $x \Lambda^*(\cdot/x; \beta, 1, \alpha)$. Note that $x\tilde{z}(\beta, 1, \alpha) > z_2$, and therefore both $x \Lambda^*(\cdot/x; \beta, 1, \alpha)$ and I_2 uniquely vanish at z_2 .

We conclude with the analogue of Remark 4.2, where we also give some comments on the limit of the scaled variance.

Remark 4.4 (The analogue of Remark 4.2) The class of LDPs in Proposition 4.3 fills the gap between the two following asymptotic regimes as $\mu \rightarrow \infty$:

- the convergence of $A_x(\beta \mu, \mu)$ to $x \Lambda'(0; \beta, 1)$ (case $\varepsilon_\mu = 1/\mu$), which follows from the LDP in Proposition 4.1;
- the weak convergence of $\mu A_{x/\mu}(\beta \mu, \mu)$ to $A_x(\beta, 1)$ (case $\varepsilon_\mu = 1$) proved in Lemma 4.1.

In both cases one condition in Eq. 6 holds, and the other one fails. We can also provide the following limit for the scaled variance (where we take into account the variance expression in Eq. 4):

$$\lim_{\mu \rightarrow \infty} \frac{1}{\varepsilon_\mu} \text{Var} [\mu \varepsilon_\mu A_{x/(\mu \varepsilon_\mu)}(\beta \mu, \mu)] = x \Lambda''(0; \beta, 1).$$

Thus the variance of the equally distributed random variables in Lemma 4.1 (and therefore the variance of the weak limit $A_x(\beta, 1)$) can be expressed as

$$\text{Var}[A_x(\beta, 1)] = \frac{4(\beta + 2\beta x \alpha^2 + 2\alpha - 1)}{(\beta - 1)^3 \alpha^2} = \Delta(\alpha, \beta) + x \Lambda''(0; \beta, 1),$$

where $x\Lambda''(0; \beta, 1)$ is the limit value obtained above, and $\Delta(\alpha, \beta) := \frac{4(\beta+2\alpha-1)}{(\beta-1)^3\alpha^2} > 0$; moreover $\Delta(\alpha, \beta)$ tends to zero as $\beta \rightarrow \infty$.

5 Numerical Estimates by Simulations

In this section we refer to the asymptotic Normality result under the scaling 2 stated in Remark 4.2. We present numerical values obtained by simulations to estimate β ; actually we assume that $\beta > \beta_0$ for some known $\beta_0 > 1$. In the final part we also present some figures concerning sample paths for some $\beta > 1$.

We denote the standard Normal distribution by Φ . Then, for every $\delta > 0$, we have

$$\lim_{\mu \rightarrow \infty} P\left(A_x(\beta\mu, \mu) - \frac{\delta}{\sqrt{\mu}} \leq \mathbb{E}[A_x(\beta\mu, \mu)] \leq A_x(\beta\mu, \mu) + \frac{\delta}{\sqrt{\mu}}\right) = 2\Phi\left(\frac{\delta}{\sqrt{8\beta x/(\beta-1)^3}}\right) - 1;$$

so, if we choose $\delta = \sqrt{\frac{8\beta x}{(\beta-1)^3}} \Phi^{-1}\left(\frac{1+\ell}{2}\right)$ for some $\ell \in (0, 1)$, the above limit is equal to ℓ . Thus we can consider the following approximated confidence interval for $\mathbb{E}[A_x(\beta\mu, \mu)]$ at the level ℓ , when μ is large:

$$A_x(\beta\mu, \mu) \pm \sup_{\beta > \beta_0} \sqrt{\frac{8\beta x}{(\beta-1)^3}} \frac{\Phi^{-1}\left(\frac{1+\ell}{2}\right)}{\sqrt{\mu}}.$$

We already remarked just after Proposition 4.1 that

$$\lim_{\mu \rightarrow \infty} \mathbb{E}[A_x(\beta\mu, \mu)] = x\Lambda'(0; \beta, 1) = x \frac{\beta+1}{\beta-1};$$

thus, for μ large enough ($\mu > \mu_0$, say) the approximation $\mathbb{E}[A_x(\beta\mu, \mu)] \approx x \frac{\beta+1}{\beta-1}$ can be adopted. Moreover, since $\sup_{\beta > \beta_0} \sqrt{\frac{8\beta x}{(\beta-1)^3}} = \sqrt{\frac{8\beta_0 x}{(\beta_0-1)^3}}$, the above approximated confidence interval can be specified as follows:

$$A_x(\beta\mu, \mu) \pm \sqrt{\frac{8\beta_0 x}{(\beta_0-1)^3}} \frac{\Phi^{-1}\left(\frac{1+\ell}{2}\right)}{\sqrt{\mu}}.$$

Then we can obtain numerical values for this confidence interval by performing simulations of $A_x(\beta\mu, \mu)$. Specifically, the validate simulations of $A_x(\beta\mu, \mu)$ are those performed for selected values of β , i.e. for chosen values $\beta = \beta_* > \beta_0 > 1$, for which the fraction of sample paths such that

$$x \frac{\beta_* + 1}{\beta_* - 1} \in \left(\bar{A}_x(\beta_*\mu, \mu) - \sqrt{\frac{8\beta_0 x}{(\beta_0-1)^3}} \frac{\Phi^{-1}\left(\frac{1+\ell}{2}\right)}{\sqrt{\mu}}, \bar{A}_x(\beta_*\mu, \mu) + \sqrt{\frac{8\beta_0 x}{(\beta_0-1)^3}} \frac{\Phi^{-1}\left(\frac{1+\ell}{2}\right)}{\sqrt{\mu}}\right),$$

where $\bar{A}_x(\beta_*\mu, \mu)$ is the simulated sample mean, is at least ℓ ; this is also equivalent to say that

$$x \frac{\beta_* + 1}{\beta_* - 1} - \sqrt{\frac{8\beta_0 x}{(\beta_0-1)^3}} \frac{\Phi^{-1}\left(\frac{1+\ell}{2}\right)}{\sqrt{\mu}} < \bar{A}_x(\beta_*\mu, \mu) < x \frac{\beta_* + 1}{\beta_* - 1} + \sqrt{\frac{8\beta_0 x}{(\beta_0-1)^3}} \frac{\Phi^{-1}\left(\frac{1+\ell}{2}\right)}{\sqrt{\mu}}. \tag{7}$$

Table 1 Numerical approximations for the confidence interval for β varying α

α	μ	β_*	$x \frac{\beta_*+1}{\beta_*-1}$	$\bar{A}_x(\beta_*\mu, \mu)$	Confidence interval (8)	Point estimation (9)
0.7	1000	1.75	$3\bar{6}$	5.15575	(1.349423,1.772864)	1.481261
0.8	1000	1.75	$3\bar{6}$	4.496909	(1.394876,2.03684)	1.571934
0.9	1000	1.75	$3\bar{6}$	4.03037	(1.434939,2.367616)	1.659985
0.925	1000	1.75	$3\bar{6}$	3.92666	(1.444975,2.472007)	1.683373

Thus, when μ is large, β can be estimated by the following items.

- The confidence interval at the level ℓ , when $x < \bar{A}_x(\beta_*\mu, \mu) - \sqrt{\frac{8\beta_0x}{(\beta_0-1)^3} \frac{\Phi^{-1}(\frac{1+\ell}{2})}{\sqrt{\mu}}}$:

$$\left(\frac{\bar{A}_x(\beta_*\mu, \mu) + \sqrt{\frac{8\beta_0x}{(\beta_0-1)^3} \frac{\Phi^{-1}(\frac{1+\ell}{2})}{\sqrt{\mu}}} + x}{\bar{A}_x(\beta_*\mu, \mu) + \sqrt{\frac{8\beta_0x}{(\beta_0-1)^3} \frac{\Phi^{-1}(\frac{1+\ell}{2})}{\sqrt{\mu}}} - x}, \frac{\bar{A}_x(\beta_*\mu, \mu) - \sqrt{\frac{8\beta_0x}{(\beta_0-1)^3} \frac{\Phi^{-1}(\frac{1+\ell}{2})}{\sqrt{\mu}}} + x}{\bar{A}_x(\beta_*\mu, \mu) - \sqrt{\frac{8\beta_0x}{(\beta_0-1)^3} \frac{\Phi^{-1}(\frac{1+\ell}{2})}{\sqrt{\mu}}} - x} \right). \tag{8}$$
- The point estimation:

$$\frac{\bar{A}_x(\beta_*\mu, \mu) + x}{\bar{A}_x(\beta_*\mu, \mu) - x}. \tag{9}$$

Moreover, in addition to these estimators, the meaningful information carried by these simulations concern both μ_0 and β_* for which the inequality (7) is satisfied.

Now we are ready to present some numerical values. In all cases we perform simulations by setting $x = 1$ and $\beta_0 = 1.25$; furthermore, the size of simulated sample paths is 10^3 and the confidence level is $\ell = 0.95$.

We conclude with some comments, and we also refer to the figures presented below. In Table 1, we have increasing values of α and we find decreasing values of the sample mean $\bar{A}_x(\beta_*\mu, \mu)$ (as expected) that tends to the asymptotic value $x \frac{\beta_*+1}{\beta_*-1}$. Furthermore, we find also increasing values of the point estimate (9) and wider confidence intervals. In Table 2, for increasing values of μ , we find quite stable values for $\bar{A}_x(\beta_*\mu, \mu)$ and the point estimation (9). Moreover we obtain more and more narrow confidence intervals as μ increases. In Table 3, it appears evident that, for increasing values of β_* , the values of the sample mean $\bar{A}_x(\beta_*\mu, \mu)$ become more accurate estimations for the corresponding values of $x \frac{\beta_*+1}{\beta_*-1}$, whereas the right endpoints of the confidence intervals are less accurate. We also

Table 2 Numerical approximations for the confidence interval for β varying μ

α	μ	β_*	$x \frac{\beta_*+1}{\beta_*-1}$	$\bar{A}_x(\beta_*\mu, \mu)$	Confidence interval (8)	Point estimation (9)
0.9	1000	2	3	3.297049	(1.517462,3.743193)	1.870682
0.9	5000	2	3	3.292031	(1.66817,2.257219)	1.872588
0.9	10000	2	3	3.292868	(1.717179,2.112946)	1.87227
0.9	20000	2	3	3.291491	(1.756974,2.030459)	1.872794

Table 3 Numerical approximations for the confidence interval for β varying β_*

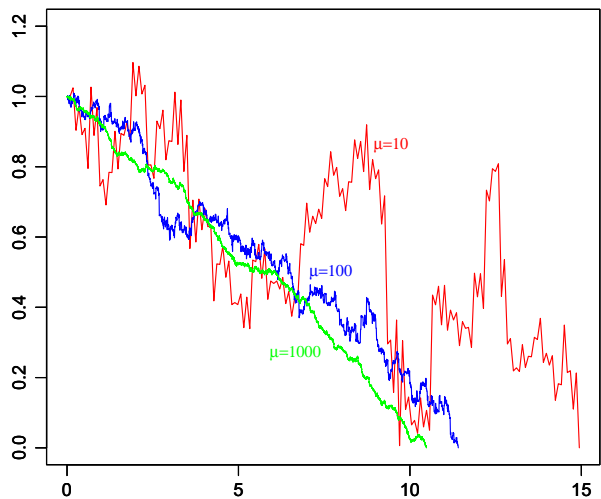
α	μ	β_*	$x \frac{\beta_*+1}{\beta_*-1}$	$\bar{A}_x(\beta_*\mu, \mu)$	confidence interval (8)	point estimation (9)
0.8	1000	1.5	5	6.128791	(1.298652,1.561668)	1.389955
0.8	1000	2	3	3.678367	(1.470994,2.80116)	1.746724
0.8	1000	2.5	$2.\bar{3}$	2.860925	(1.583278,7.827018)	2.074734
0.8	1000	2.75	$2.\overline{142857}$	2.626983	(1.625987,34.89173)	2.229269

remark that, in all tables, the estimated values based on the point estimation (9) are less than the corresponding set values of β_* .

From Tables 1–3, and all performed simulations results, we can say that the numerical strategy to obtain the above estimates is reliable for high values of α . This is easily understandable because the above estimates are reliable in a neighborhood of the asymptotic value $x \frac{\beta_*+1}{\beta_*-1}$ (for high value of μ , i.e. for high rate of downward steps) or, in some sense equivalently, for high value of absorbing probability α (compare Fig. 1 and left side of Fig. 2). Furthermore, as far as the value μ_0 is concerned (i.e. the value such that we can obtain reliable estimates, at the confidence level 0.95, when $\mu > \mu_0$), we can take $\mu_0 = 1000$. The results in Table 2 show that the approximation of the confidence interval improves as μ increases. Finally we also stress that all these numerical values provide indications on the true value of β under the scaling 2 for finite values of μ (instead of asymptotic results as $\mu \rightarrow \infty$).

We conclude with some brief comments on Figs. 1–2. They show that sample paths of the process for different choices of values for parameters μ, α, β . In Fig. 1 it is possible to observe how the paths change for different values of μ . In Fig. 2 we consider different values of α and β ; in particular, we set $\mu = 10$ because the effect of different values of α and β on the sample paths appears more evident.

Fig. 1 Sample paths for different values of μ , for $\alpha = 0.9$ and $\beta = 1.25$



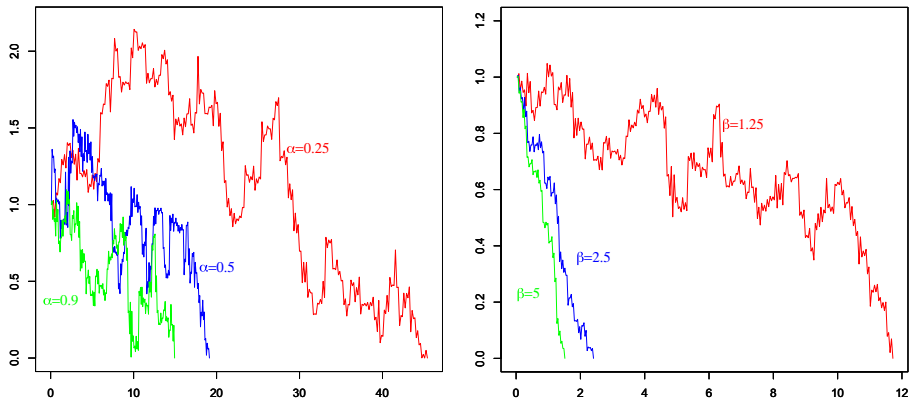


Fig. 2 Left: sample paths for different values of α , for $\mu = 10$ and $\beta = 1.25$. Right: the same varying β , for $\mu = 20$ and $\alpha = 0.9$

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