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A Discrete-Time $GI^X/Geo/1$ Queue with Multiple Working Vacations Under Late and Early Arrival System

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Abstract
This paper studies a discrete-time batch arrival $GI/Geo/1$ queue where the server may take multiple vacations depending on the state of the queue/system. However, during the vacation period, the server does not remain idle and serves the customers with a rate lower than the usual service rate. The vacation time and the service time during working vacations are geometrically distributed. Keeping note of the specific nature of the arrivals and departures in a discrete-time queue, we study the model under late arrival system with delayed access and early arrival system independently. We formulate the system using supplementary variable technique and apply the theory of difference equation to obtain closed-form expressions of steady-state system content distribution at pre-arrival and arbitrary epochs simultaneously, in terms of roots of the associated characteristic equations. We discuss the stability conditions of the system and develop few performance measures as well. Through some numerical examples, we illustrate the feasibility of our theoretical work and highlight the asymptotic behavior of the probability distributions at pre-arrival epochs. We further discuss the impact of various parameters on the performance of the system. The model considered in this paper covers a wide class of vacation and non-vacation queueing models which have been studied in the literature.

Keywords Bulk arrival · Difference equation method · Discrete-time · $GI/Geo/1$ queue · Multiple working vacations · Supplementary variable technique

Mathematics Subject Classification (2010) 60K25 · 90B22

1 Introduction

Since last few decades, discrete-time queueing models with various vacation policies have drawn the attention of researchers because of its potential application in modeling

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computer networks and digital telecommunication systems. These systems are intended to serve real-time applications where the processing of data takes place within a defined time constraint in slots or units of equal lengths. The stochastic processes involved in these systems occurs near the slot boundaries which eventually gives rise to two variations in the modeling of discrete-time queues: late arrival system with delayed access (LAS-DA or LAS) and early arrival system (EAS). A detailed study of discrete-time queues with vacations can be found in Takagi (1993). For further reference, one may also see the survey papers by Doshi (1986) and Ke et al. (2010). The discrete-time $Geo/G/1$ queue and $GI/Geo/1$ queue with multiple vacations was respectively studied by Zhang and Tian (2001) and Tian and Zhang (2002) where they assumed that the server takes a random maximum number of vacations after serving the customers present in the system. Fiems and Bruneel (2002) analyzed the discrete-time $GI/G/1$ queue subject to server vacations which are governed by timers. Various single and multiple server queueing models in both discrete and continuous-time set-up are addressed by Tian and Zhang (2006) under different vacation policies. Further, Samanta et al. (2007a, b) investigated discrete-time finite buffer $Geo^X/G^{(a,b)}/1/N$ queue with single and multiple vacations, and $GI/Geo/1/N$ queue with multiple vacations, respectively. In all the works mentioned above, it is assumed that the server does not perform any service during the vacation period.

While the introduction of vacation policies in the classical queueing models captures many real-world systems, the assumption that whether or not the server takes up some kind of service during the vacation phase considerably impacts the performance of the system. Servi and Finn (2002) was the first to introduce the concept of ‘working server’ during a vacation period and studied the $M/M/1$ queue with multiple working vacations (MWV). Baba (2005) extended the work done in Servi and Finn (2002) and investigated the infinite buffer renewal input $GI/M/1$ queue with MWV, whereas Ye and Liu (2016) carried out the analysis of $GI/M/1$ queue with single working vacation and vacations. Banik et al. (2007) considered the same model as in Baba (2005) under the assumption of finite buffer i.e., $GI/M/1/N$ queue with MWV. Subsequently, Yu et al. (2009) addressed the analytical as well as computational aspects of a finite-buffer bulk-arrival bulk-service $GI^X/M^b/1/L$ queueing system with MWV and partial batch rejection. Recently, Guha and Banik (2013) generalized the model considered in Baba (2005) into a batch arrival renewal input queue under both single and multiple working vacation policy and modeled it for end user system in an ethernet passive optical network (EPON). The work done in Baba (2005) and Banik et al. (2007) was carried forward in discrete-time set up by Li et al. (2007) and Goswami and Mund (2010), respectively. The former analyzed the model under the consideration of both LAS-DA and EAS disciplines whereas the latter addressed only the EAS policy. Li et al. (2010) presented the steady-state analysis of a discrete-time batch arrival $Geo^X/G/1$ queue under working vacations. In recent years Goswami and Mund (2011) investigated a discrete-time batch service renewal input $GI/Geo^{(1,b)}/1$ queue with MWV and studied the effect of various parameters on the performance of the system. The analysis of finite buffer $GI^X/Geo/1/N$ queue with negative customers and MWV was presented by Gao et al. (2013) where they adopted the partial batch rejection policy. It may be mentioned that the analysis carried out in Li et al. (2010), Goswami and Mund (2011) and Gao et al. (2013) were based on EAS discipline. Meanwhile, the memoryless property of geometric interarrival time distribution makes the analytical analysis of the model quite simpler, but it does not represent well the situations arising in many real-world systems. Thus, the assumption of general uncorrelated interarrival time distribution is preferable in modeling of computer networks, manufacturing systems, etc.

From the literature survey, we came across two important observations related to the analysis of working vacation models in discrete-time queues: (i) in most of the cases the arrivals are assumed to occur individually, otherwise, whenever batch arrival is considered the waiting space is restricted to finite capacity; (ii) except for Li et al. (2007), all the authors have addressed only the EAS policy. The importance of the model under LAS-DA policy has somehow been ignored which may be due to the fact that its analysis is relatively more complicated (see Chaudhry and Gupta 1997). Thus, with an aim to cover up the gap in the literature on discrete-time queues, we analyze an infinite buffer $GI^X/Geo/1$ queue with multiple working vacations. The server may take a random number of vacations depending on the state of the queue, but during vacation period the server serves the customers with a rate lower than the usual service rate. We first formulate the governing equations of the system via supplementary variable technique (SVT) and then using the displacement operator method we simultaneously obtain the steady-state distributions of the number of customers in the system with respect to the state of the server at pre-arrival and arbitrary epochs in terms of roots of the associated characteristic equations. For the convenience of the readers, we distinguish the analysis of LAS-DA and EAS in two different sections.

The methodology used in this paper to carry out the analysis is in many ways different from the well-known methods available in the literature. Li et al. (2007) and Ye and Liu (2016) used the matrix-geometric method (MGM) (developed by Neuts 1994) which is considered to be one of the most powerful methods to derive the analytical results for $GI/M/1-$type queueing models, but its numerical implementation is relatively difficult because of the large number of iterations involved in the computation of the rate matrix **R**. In this connection, one may refer to Chaudhry et al. (2016). Eventually, embedded Markov chain technique (EMCT) is widely used by researchers (e.g., Samanta et al. 2007b; Gao et al. 2013; Goswami and Mund 2010) to analyze such type of queues. It requires the construction of a transition probability matrix (tpm) associated with the Markov chain, using which an expression of probability generating function (pgf) at pre-arrival epoch is obtained and it is further inverted via roots method (e.g., Chaudhry et al. 2012; Chaudhry and Gupta 1997) in order to extract the probabilities. But a major challenge in this procedure is the construction of the tpm which becomes more involved with the increase in the complexity of the queueing model. The methods described above enables one to obtain the distribution at pre-arrival epoch and subsequently, a relation between the probabilities at pre-arrival and arbitrary epochs is established to obtain the latter one. Keeping a note of all these discussions we ascertain few major contributions of the present work. Firstly, the approach used to accomplish the analysis of the model is theoretically tractable and practically very easy to implement, as we obtain the explicit and closed-form expression for distributions at pre-arrival and arbitrary epochs simultaneously in terms of finite number of roots and the corresponding constants. The procedure is based on the theory of difference equation (see Elaydi 2005) which completely bypasses the complexity involved in the construction of any rate matrix or tpm at the embedded points, and also does not require the inversion of any generating functions. Secondly, the model considered in this paper is a generalization of the already studied models in the literature and hence the procedure developed throughout presents an alternative approach to their solution. This has been thoroughly discussed in Section 4. We also evaluate some quantitative measures of the system and highlight the impact of several parameters on its performance. Finally, we numerically obtain the estimation of the tail probabilities at pre-arrival epochs based on the unique largest root of the characteristic equations.

The queueing model considered in this paper may find application in cloud computing services which provides service through a network of servers hosted over the internet rather than local networks or personal computers. It allows an enormous storage capacity meant for processing huge data at a very reduced cost. But a major issue in using the cloud platform is its high energy consumption. The data transmitted into the cloud environment in the form of packets often arrive in random size. The compute nodes (server) have to be powered on all the time in order to accept the incoming jobs, which sometimes results in a huge wastage of energy. At this instance, the working vacation queueing model can be efficiently used to model the task schedule of the servers. When no task arrives, the server may switch to a working vacation phase keeping the service rate lower than the usual rate. This may reduce the power consumption and save operational ability of the system to a large extent. For a better insight along this direction one may see Vilaplana et al. (2014), Cheng et al. (2015) and the references therein.

The remaining portion of the paper is organized as follows: in Section 2 we give a comprehensive description of the model; in Section 3 we provide the theoretical analysis of the model under LAS-DA and EAS policies and also discuss the stability condition of the system. In Section 4 we present some special cases of our model. In Section 5 we evaluate several system performances and implement the derived results through some numerical examples which are followed by the conclusion in Section 6.

2 Model Description

In discrete-time set-up, we assume that the time axis is divided into intervals of equal length termed as slots, and are separated by slot boundaries $0, 1, 2, \ldots, m, \ldots$. The arrival and departure of customers and the server vacation takes place around these slot boundaries. The case when arrival and departure occur just after and just before the slot boundary respectively is termed as early arrival system (EAS). On the other hand, when arrival and departure occur just before and just after the slot boundary respectively is termed as late arrival system with delayed access (LAS-DA). Under EAS policy an arrival may be departed in the same interval provided the server is empty and it gets served, but the same is not possible in case of LAS-DA policy. Meanwhile, for both the systems the server begins or ends a working vacation just before a potential batch arrival. A much detailed concepts on both EAS and LAS-DA discipline can be found in Hunter (1983), Gravey and Hebuterne (1992) and Chaudhry (2000).

Customers arrive into the system in batches of size X, which is a random variable with probability mass function (p.m.f) $P(X = i) = g_i$, $i = 1, 2, \ldots$. For mathematical convenience and from a more realistic point of view we assume that the maximum acceptable size of the arriving batch is b. However, for batch size distribution with infinite support, our methodology can be used by considering the maximum batch size to be a sufficiently larger value. Consequently, the probability generating function (pgf) and the average size of the arriving batch are denoted by $G(z) = \sum_{i=1}^{b} g_i z^i$, $|z| \leq 1$ and $\overline{g} = \sum_{i=1}^{b} i g_i$, respectively. Further, the inter-arrival times T between the batches are independent and identically distributed (i.i.d) random variables with p.m.f $P(T = n) = a_n,\ n \geq 1$, pgf $A(z) = \sum_{n=1}^{\infty} a_n z^n$ and mean $a = \frac{1}{\lambda} = A'(1) = \sum_{n=1}^{\infty} n a_n$, where λ is the arrival rate of the batches. The customers on arrival are lined up according to first-come first-served (FCFS) basis and are served individually by a single server.

When the server is in normal busy period the service times S of the customers are independent and geometrically distributed with parameter μ and p.m.f $P(S = n) = \overline{\mu}^{n-1}\mu$,

$0 < \mu < 1, n \geq 1$. At the epoch when the system (queue + server) becomes empty, the server takes a working vacation such that the working vacation times V are independent and geometrically distributed with parameter ϕ and p.m.f $P(V = n) = \overline{\phi}^{n-1}\phi$, $0 < \phi < 1$, $n \geq 1$. The working vacation time is the period when the server remains active, but serves the arriving customers with a rate lower than the normal service rate. When the server is in working vacation period, the service times S_v of the customers are i.i.d geometric random variables with parameter η and p.m.f $P(S_v = n) = \overline{\eta}^{n-1}\eta$, $0 < \eta < 1$, $n \geq 1$ such that $\eta \leq \mu$. As soon as the working vacation time terminates, the server becomes active and serves with its usual service rate μ, provided it finds the queue non-empty, otherwise it goes for another working vacation, and the process continues. The working vacation time, arrival process and the service process are independent of each other. The traffic intensity ρ is given by $\rho = \frac{\overline{g}}{a\mu}$ and $\rho < 1$ ensures the stability of the system. Such a queueing model can be mathematically denoted by $GI^X/Geo/1 - MWV$ queue.

3 Analysis of the Model

This section is devoted to the complete analytical analysis of the aforementioned queueing model under both LAS-DA and EAS policies. At first the mathematical formulation of the model is done using the supplementary variable technique (SVT) and then the theory of difference equation is applied in order to obtain the steady-state distribution of system-content at pre-arrival and arbitrary epochs. The section is divided into two subsections, one for the analysis with LAS-DA policy and the other with EAS policy, respectively. It is mainly done with an aim to provide the readers a clearer perception of the steps involved and the results obtained throughout the analysis.

3.1 Modeling with LAS-DA Policy

As mentioned in the previous section, in LAS-DA policy the arrival of batches takes place in the interval $(m-, m)$, the departure of customers occurs in the interval $(m, m+)$ and the working vacation time begins or ends at the instant $m-$. We determine the governing equations of the model by taking the remaining inter-arrival time of the next batch to be the supplementary variable for which we define the following random variables at the instant just before a potential batch arrival i.e., $m-$.

- N_{m-}= Number of customers in the system (queue + server).
- U_{m-}= Remaining inter-arrival time of the next batch.
- Y_{m-}= State of the server which takes the values 0 and 1 corresponding to whether it is in working vacation period or in normal busy period.

Further we define the joint probabilities as

$$\hat{p}_{n,0}(m-, u) = P\{N_{m-} = n, U_{m-} = u, Y_{m-} = 0\}, \quad u \geq 0, \; n \geq 0,$$

$$\hat{p}_{n,1}(m-, u) = P\{N_{m-} = n, U_{m-} = u, Y_{m-} = 1\}, \quad u \geq 0, \; n \geq 1.$$

Thus in steady-state we have

$$p_{n,0}(u) = \lim_{m \longrightarrow \infty} \hat{p}_{n,0}(m-, u), n \geq 0 \text{ and } p_{n,1}(u) = \lim_{m \longrightarrow \infty} \hat{p}_{n,1}(m-, u), n \geq 1.$$

Relating the states of the system at two consecutive time epochs $m-$ and $(m+1)-$ and using probabilistic arguments, we obtain (for $u \geq 1$) the following set of governing equations in steady-state

$$p_{0,0}(u-1) = p_{0,0}(u) + \eta p_{1,0}(u) + \mu p_{1,1}(u) \quad (1)$$

$$p_{1,0}(u-1) = \overline{\phi}\left\{\eta\left[p_{2,0}(u) + a_u g_1 p_{1,0}(0)\right] + \overline{\eta} p_{1,0}(u) + a_u g_1 p_{0,0}(0)\right\} \quad (2)$$

$$p_{n,0}(u-1) = \overline{\phi}\left\{\eta\left[p_{n+1,0}(u) + a_u \sum_{i=1}^{n} g_i p_{n-i+1,0}(0)\right] + \overline{\eta}\left[p_{n,0}(u) + a_u \sum_{i=1}^{n-1} g_i p_{n-i,0}(0)\right] + a_u g_n p_{0,0}(0)\right\}, \quad 2 \leq n \leq b \quad (3)$$

$$p_{n,0}(u-1) = \overline{\phi}\left\{\eta\left[p_{n+1,0}(u) + a_u \sum_{i=1}^{b} g_i p_{n-i+1,0}(0)\right] + \overline{\eta}\left[p_{n,0}(u) + a_u \sum_{i=1}^{b} g_i p_{n-i,0}(0)\right]\right\}, \; n \geq b+1 \quad (4)$$

$$p_{1,1}(u-1) = \phi\left\{\eta\left[p_{2,0}(u) + a_u g_1 p_{1,0}(0)\right] + \overline{\eta} p_{1,0}(u) + a_u g_1 p_{0,0}(0)\right\} + \overline{\mu} p_{1,1}(u) + \mu p_{2,1}(u) + a_u g_1 \mu p_{1,1}(0) \quad (5)$$

$$p_{n,1}(u-1) = \phi\left\{\overline{\eta}\left[p_{n,0}(u) + a_u \sum_{i=1}^{n-1} g_i p_{n-i,0}(0)\right] + \eta\left[p_{n+1,0}(u) + a_u \sum_{i=1}^{n} g_i p_{n-i+1,0}(0)\right] + a_u g_n p_{0,0}(0)\right\} + \overline{\mu}\left[p_{n,1}(u) + a_u \sum_{i=1}^{n-1} g_i p_{n-i,1}(0)\right] + \mu\left[p_{n+1,1}(u) + a_u \sum_{i=1}^{n} g_i p_{n-i+1,1}(0)\right], \quad 2 \leq n \leq b \quad (6)$$

$$p_{n,1}(u-1) = \phi\left\{\overline{\eta}\left[p_{n,0}(u) + a_u \sum_{i=1}^{b} g_i p_{n-i,0}(0)\right] + \eta\left[p_{n+1,0}(u) + a_u \sum_{i=1}^{b} g_i p_{n-i+1,0}(0)\right]\right\} + \overline{\mu}\left[p_{n,1}(u) + a_u \sum_{i=1}^{b} g_i p_{n-i,1}(0)\right] + \mu\left[p_{n+1,1}(u) + a_u \sum_{i=1}^{b} g_i p_{n-i+1,1}(0)\right], \quad n \geq b+1 \quad (7)$$

In order to obtain the steady-state probabilities $p_{n,0}$, $n \geq 0$ and $p_{n,1}$, $n \geq 1$ from the set of Eqs. 1–7, we introduce the transform $p^*_{n,0}(z) = \sum_{u=0}^{\infty} p_{n,0}(u) z^u$ and $p^*_{n,1}(z) =$

$\sum_{u=0}^{\infty} p_{n,1}(u)z^u$ so that $p_{n,0} = p_{n,0}^*(1)$ and $p_{n,1} = p_{n,1}^*(1)$. Thus multiplying (1)–(7) by z^u and summing over u from 1 to ∞ we obtain the following transformed equations

$$(z-1)p_{0,0}^*(z) = \eta\left[p_{1,0}^*(z) - p_{1,0}(0)\right] + \mu\left[p_{1,1}^*(z) - p_{1,1}(0)\right] - p_{0,0}(0) \tag{8}$$

$$\begin{aligned}(z-\overline{\eta}\overline{\phi})p_{1,0}^*(z) &= \overline{\phi}\left\{\eta\left[p_{2,0}^*(z) - p_{2,0}(0) + A(z)g_1 p_{1,0}(0)\right]\right.\\ &\quad \left. -\overline{\eta}p_{1,0}(0) + A(z)g_1 p_{0,0}(0)\right\}\end{aligned} \tag{9}$$

$$\begin{aligned}(z-\overline{\eta}\overline{\phi})p_{n,0}^*(z) &= \overline{\phi}\left\{\eta\left[p_{n+1,0}^*(z) - p_{n+1,0}(0) + A(z)\sum_{i=1}^{n} g_i p_{n-i+1,0}(0)\right]\right.\\ &\quad +\overline{\eta}\left[A(z)\sum_{i=1}^{n-1} g_i p_{n-i,0}(0) - p_{n,0}(0)\right]\\ &\quad \left.+A(z)g_n p_{0,0}(0)\right\}, \quad 2 \le n \le b\end{aligned} \tag{10}$$

$$\begin{aligned}(z-\overline{\eta}\overline{\phi})p_{n,0}^*(z) &= \overline{\phi}\left\{\eta\left[p_{n+1,0}^*(z) - p_{n+1,0}(0) + A(z)\sum_{i=1}^{b} g_i p_{n-i+1,0}(0)\right]\right.\\ &\quad \left.+\overline{\eta}\left[A(z)\sum_{i=1}^{b} g_i p_{n-i,0}(0) - p_{n,0}(0)\right]\right\}, \quad n \ge b+1\end{aligned} \tag{11}$$

$$\begin{aligned}(z-\overline{\mu})p_{1,1}^*(z) &= \phi\left\{\eta\left[p_{2,0}^*(z) - p_{2,0}(0) + A(z)g_1 p_{1,0}(0)\right] + \overline{\eta}\left[p_{1,0}^*(z) - p_{1,0}(0)\right]\right.\\ &\quad \left.+A(z)g_1 p_{0,0}(0)\right\} + \mu\left[p_{2,1}^*(z) - p_{2,1}(0)\right.\\ &\quad \left.+A(z)g_1 p_{1,1}(0)\right] - \overline{\mu}p_{1,1}(0)\end{aligned} \tag{12}$$

$$\begin{aligned}(z-\overline{\mu})p_{n,1}^*(z) &= \phi\left\{\eta\left[p_{n+1,0}^*(z) - p_{n+1,0}(0) + A(z)\sum_{i=1}^{n} g_i p_{n-i+1,0}(0)\right]\right.\\ &\quad \left.+\overline{\eta}\left[p_{n,0}^*(z) - p_{n,0}(0) + A(z)\sum_{i=1}^{n-1} g_i p_{n-i,0}(0)\right] + A(z)g_n p_{0,0}(0)\right\}\\ &\quad +\mu\left[p_{n+1,1}^*(z) - p_{n+1,1}(0) + A(z)\sum_{i=1}^{n} g_i p_{n-i+1,1}(0)\right]\\ &\quad +\overline{\mu}\left[A(z)\sum_{i=1}^{n-1} g_i p_{n-i,1}(0) - p_{n,1}(0)\right], \quad 2 \le n \le b\end{aligned} \tag{13}$$

$$\begin{aligned}(z-\overline{\mu})p_{n,1}^*(z) &= \phi\left\{\eta\left[p_{n+1,0}^*(z) - p_{n+1,0}(0) + A(z)\sum_{i=1}^{b} g_i p_{n-i+1,0}(0)\right]\right.\\ &\quad \left.+\overline{\eta}\left[p_{n,0}^*(z) - p_{n,0}(0) + A(z)\sum_{i=1}^{b} g_i p_{n-i,0}(0)\right]\right\}\\ &\quad +\mu\left[p_{n+1,1}^*(z) - p_{n+1,1}(0) + A(z)\sum_{i=1}^{b} g_i p_{n-i+1,1}(0)\right]\end{aligned}$$

$$+\overline{\mu}\left[A(z)\sum_{i=1}^{b}g_i p_{n-i,1}(0)-p_{n,1}(0)\right],\ n\geq b+1. \tag{14}$$

Adding (8)–(14) for all values of n we obtain

$$\sum_{n=0}^{\infty}p_{n,0}^*(z)+\sum_{n=1}^{\infty}p_{n,1}^*(z)=\frac{A(z)-1}{z-1}\left\{\sum_{n=0}^{\infty}p_{n,0}(0)+\sum_{n=1}^{\infty}p_{n,1}(0)\right\}.$$

Taking limit as $z\to 1$ and using the normalizing condition $\sum_{n=0}^{\infty}p_{n,0}+\sum_{n=1}^{\infty}p_{n,1}=1$ we have the following relation

$$\sum_{n=0}^{\infty}p_{n,0}(0)+\sum_{n=1}^{\infty}p_{n,1}(0)=\frac{1}{a}. \tag{15}$$

Now let us denote $p_{n,0}^-$, $n\geq 0$ and $p_{n,1}^-$, $n\geq 1$ as the probability that there are n customers in the system at pre-arrival epoch depending on whether the server is in working vacation period or normal busy period respectively. Applying Bayes' theorem we have

$$\begin{aligned}p_{n,j}^- &= P\{n \text{ customers in the system prior to an arrival of a batch when the server is in state } j \\ &\quad \mid \text{the server is either in working vacation period or in busy period at pre-arrival epoch}\}\\ &= \frac{p_{n,j}(0)}{\sum_{i=0}^{\infty}p_{i,0}(0)+\sum_{i=1}^{\infty}p_{i,1}(0)},\quad n\geq 0,\ j=0 \text{ or } n\geq 1,\ j=1.\end{aligned} \tag{16}$$

Using (15) in Eq. 16 we have

$$p_{n,j}^-=ap_{n,j}(0),\quad n\geq 0,\ j=0 \text{ or } n\geq 1,\ j=1. \tag{17}$$

Now using the right shift operator D on the sequence $\{p_{n,0}^*(z)\}$ and $\{p_{n,0}(0)\}$ defined by $Dp_{n,0}^*(z)=p_{n+1,0}^*(z)$ and $Dp_{n,0}(0)=p_{n+1,0}(0)$ for all n, Eq. 11 can be re-written in the form

$$\begin{aligned}\left[z-\overline{\phi}(\overline{\eta}+\eta D)\right]p_{n,0}^*(z) &= \overline{\phi}\left[\eta\left\{A(z)\sum_{i=1}^{b}g_i D^{b-i+1}-D^{b+1}\right\}\right.\\ &\quad\left.+\overline{\eta}\left\{A(z)\sum_{i=1}^{b}g_i D^{b-i}-D^{b}\right\}\right]p_{n-b,0}(0),\\ &\quad n\geq b+1.\end{aligned} \tag{18}$$

Substituting $z=\overline{\phi}(\overline{\eta}+\eta D)$ in Eq. 18 we obtain the following homogeneous difference equation with constant coefficient

$$\left[\overline{\phi}(\overline{\eta}+\eta D)\left\{A(\overline{\phi}(\overline{\eta}+\eta D))\sum_{i=1}^{b}g_i D^{b-i}-D^{b}\right\}\right]p_{n,0}(0)=0,\quad \forall\, n\geq 1. \tag{19}$$

The characteristic equation (c.e.) corresponding to Eq. 19 is

$$\overline{\phi}(\overline{\eta}+\eta s)\left\{A(\overline{\phi}(\overline{\eta}+\eta s))\sum_{i=1}^{b}g_i s^{b-i}-s^{b}\right\}=0 \tag{20}$$

and hence the general solution of Eq. 19 is given by

$$p_{n,0}(0)=\sum_{j=1}^{b}c_j r_j^n,\quad \forall\, n\geq 1 \tag{21}$$

where $r_1, r_2, ..., r_b$ are the roots of the c.e. (20) lying inside the unit circle $|s|=1$ (see Section 3.3), and $c_1, c_2, ..., c_b$ are the corresponding arbitrary constants independent of n. Using Eq. 21 in Eq. 18 we obtain

$$\left[z-\overline{\phi}(\overline{\eta}+\eta D)\right]p_{n,0}^{*}(z)=\overline{\phi}\sum_{j=1}^{b}c_j(\overline{\eta}+\eta r_j)(A(z)G(r_j^{-1})-1)r_j^n,\quad n\geq b+1. \tag{22}$$

Equation 22 is a non-homogeneous difference equation with constant coefficient and the corresponding solution is given by

$$p_{n,0}^{*}(z)=B\left(1-\frac{1}{\eta}+\frac{z}{\overline{\phi}\eta}\right)^{n}+\overline{\phi}\sum_{j=1}^{b}\frac{c_j(\overline{\eta}+\eta r_j)(A(z)G(r_j^{-1})-1)}{z-\overline{\phi}(\overline{\eta}+\eta r_j)}r_j^n,\quad n\geq b+1. \tag{23}$$

The first term in the RHS of Eq. 23 is the solution corresponding to the homogeneous part of Eq. 22 such that B is an arbitrary constant, whereas the second term represents a particular solution of Eq. 22. Now summing over the range of n and taking limit as $z\to 1$ in Eq. 23, we must have its convergence, since $\sum_{n=b+1}^{\infty}p_{n,0}^{*}(1)=\sum_{n=b+1}^{\infty}p_{n,0}\leq 1$. But in that case the first term in the R.H.S takes the form $B\sum_{n=b+1}^{\infty}\left(1+\frac{\phi}{\eta\overline{\phi}}\right)^{n}$ which tends towards infinity. Thus in order to ensure the convergence of Eq. 23 we must have $B=0$ and consequently the solution of Eq. 22 takes the form

$$p_{n,0}^{*}(z)=\overline{\phi}\sum_{j=1}^{b}\frac{c_j(\overline{\eta}+\eta r_j)(A(z)G(r_j^{-1})-1)}{z-\overline{\phi}(\overline{\eta}+\eta r_j)}r_j^n,\quad n\geq b+1. \tag{24}$$

We now find the conditions under which $p_{n,0}^{*}(z)$ holds the same expression as in Eq. 24 for $1\leq n\leq b$. Substituting the respective values in Eq. 10 we obtain

$$\sum_{j=1}^{b}c_j r_j^n(\overline{\eta}+\eta r_j)\sum_{i=n}^{b}g_i r_j^{-i}=g_n p_{0,0}(0)+\eta g_n\sum_{j=1}^{b}c_j r_j,\quad 2\leq n\leq b. \tag{25}$$

Setting $n=b$ in Eq. 25 and considering the fact that $g_b\neq 0$ we have

$$p_{0,0}(0)=\overline{\eta}\sum_{j=1}^{b}c_j. \tag{26}$$

Now using (26), considering the fact that $g_b\neq 0$ and setting $n=b-1,\ b-2, ..., 2$ in Eq. 25 we obtain the following condition

$$\sum_{j=1}^{b}\frac{c_j}{r_j^{b-n}}(\overline{\eta}+\eta r_j)=0,\quad n=2,3,...,b-1. \tag{27}$$

Similarly substituting the respective values in Eq. 9 we obtain

$$\sum_{j=1}^{b}\frac{c_j}{r_j^{b-1}}(\overline{\eta}+\eta r_j)=0. \tag{28}$$

Thus Eqs. 27 and 28 can be combined together in the form

$$\sum_{j=1}^{b} \frac{c_j}{r_j^n}(\overline{\eta} + \eta r_j) = 0, \quad n = 1, 2, ..., b-1. \tag{29}$$

Again, using the shift operator D over the sequence $\{p^*_{n,1}(z)\}$ and $\{p_{n,1}(0)\}$ as defined previously, Eq. 14 can be re-written in the form

$$\begin{aligned}[z - \overline{\mu} - \mu D]\, p^*_{n,1}(z) \;=\; & \left[\overline{\mu}A(z)\sum_{i=1}^{b} g_i D^{b-i} + \mu A(z)\sum_{i=1}^{b} g_i D^{b-i+1} - \overline{\mu}D^b - \mu D^{b+1}\right] p_{n-b,1}(0) \\ & + \phi\left\{\eta\left[p^*_{n+1,0}(z) - p_{n+1,0}(0) + A(z)\sum_{i=1}^{b} g_i\, p_{n-i+1,0}(0)\right]\right. \\ & \left. + \overline{\eta}\left[p^*_{n,0}(z) - p_{n,0}(0) + A(z)\sum_{i=1}^{b} g_i\, p_{n-i,0}(0)\right]\right\}, \quad n \geq b+1 \end{aligned} \tag{30}$$

Substituting $z = \overline{\mu} + \mu D$ in Eq. 30 we obtain

$$\begin{aligned} & \left[(\overline{\mu} + \mu D)(D^b - A(\overline{\mu} + \mu D)\sum_{i=1}^{b} g_i D^{b-i})\right] p_{n,1}(0) = \phi\left\{\eta\left[p^*_{n+b+1,0}(z) - p_{n+b+1,0}(0)\right.\right. \\ & \quad \left. + A(z)\sum_{i=1}^{b} g_i\, p_{n+b-i+1,0}(0)\right] + \overline{\eta}\left[p^*_{n+b,0}(z) - p_{n+b,0}(0)\right. \\ & \quad \left.\left. + A(z)\sum_{i=1}^{b} g_i\, p_{n+b-i,0}(0)\right]\right\}\Bigg|_{z=\overline{\mu}+\mu D}, \\ & \qquad n \geq 1. \end{aligned} \tag{31}$$

Equation 31 is a non-homogeneous difference equation with constant coefficient and the subsequent solution is given by

$$p_{n,1}(0) = \sum_{j=1}^{b} k_j \xi_j^n + \phi \sum_{j=1}^{b} \frac{c_j(\overline{\eta} + \eta r_j)}{\overline{\phi}(\overline{\eta} + \eta r_j) - (\overline{\mu} + \mu r_j)} r_j^n, \quad n \geq 1. \tag{32}$$

Here one may note that the first term in the R.H.S of Eq. 32 is the solution corresponding to the homogeneous equation of Eq. 31 where $\xi_1, \xi_2, ..., \xi_b$ are the roots of the c.e.

$$(\overline{\mu} + \mu s)(s^b - A(\overline{\mu} + \mu s)\sum_{i=1}^{b} g_i s^{b-i}) = 0 \tag{33}$$

lying inside the unit circle $|s| = 1$ (see Section 3.3), and $k_1, k_2, ..., k_b$ are the corresponding arbitrary constants independent of n. On the other hand, the second term in the R.H.S of

Eq. 32 represents a particular solution of Eq. 31. Now using expression (32) in Eq. 30 we have

$$[z-\overline{\mu}-\mu D]\, p^{*}_{n,1}(z) = \sum_{j=1}^{b} k_j(\overline{\mu}+\mu\xi_j)(A(z)G(\xi_j^{-1})-1)\xi_j^n$$
$$+\phi\overline{\phi}\sum_{j=1}^{b}\frac{c_j(\overline{\eta}+\eta r_j)^2(A(z)G(r_j^{-1})-1)}{\overline{\phi}(\overline{\eta}+\eta r_j)-(\overline{\mu}+\mu r_j)}r_j^n$$
$$+\phi\overline{\phi}\sum_{j=1}^{b}\frac{c_j(\overline{\eta}+\eta r_j)^2(A(z)G(r_j^{-1})-1)}{z-\overline{\phi}(\overline{\eta}+\eta r_j)}r_j^n,\quad n\geq b+1 \quad (34)$$

which is also a non-homogeneous difference equation with constant coefficient and the corresponding solution is of the form

$$p^{*}_{n,1}(z) = K\left(1+\frac{(z-1)}{\mu}\right)^n + \sum_{j=1}^{b}\frac{k_j(\overline{\mu}+\mu\xi_j)(A(z)G(\xi_j^{-1})-1)}{z-(\overline{\mu}+\mu\xi_j)}\xi_j^n$$
$$+\phi\overline{\phi}\sum_{j=1}^{b}\frac{c_j(\overline{\eta}+\eta r_j)^2(A(z)G(r_j^{-1})-1)}{(z-\overline{\phi}(\overline{\eta}+\eta r_j))(\overline{\phi}(\overline{\eta}+\eta r_j)-(\overline{\mu}+\mu r_j))}r_j^n,\quad n\geq b+1. \quad (35)$$

The first term in the RHS of Eq. 35 is the solution corresponding to the homogeneous part of Eq. 34 such that K is an arbitrary constant, whereas the second and third term together represents a particular solution of Eq. 34. Now summing over the range of n and taking limit as $z\to 1$ in Eq. 35, we must have its convergence, since $\sum_{n=b+1}^{\infty} p^{*}_{n,1}(1) = \sum_{n=b+1}^{\infty} p_{n,1} \leq 1$. But in that case the first term in the R.H.S tends towards infinity. Thus in order to ensure the convergence of Eq. 35 we must have $K = 0$ and consequently the solution of Eq. 34 takes the form

$$p^{*}_{n,1}(z) = \sum_{j=1}^{b}\frac{k_j(\overline{\mu}+\mu\xi_j)(A(z)G(\xi_j^{-1})-1)}{z-(\overline{\mu}+\mu\xi_j)}\xi_j^n$$
$$+\phi\overline{\phi}\sum_{j=1}^{b}\frac{c_j(\overline{\eta}+\eta r_j)^2(A(z)G(r_j^{-1})-1)}{(z-\overline{\phi}(\overline{\eta}+\eta r_j))(\overline{\phi}(\overline{\eta}+\eta r_j)-(\overline{\mu}+\mu r_j))}r_j^n,$$
$$n\geq b+1. \quad (36)$$

We now find the condition under which $p^{*}_{n,1}(z)$ holds the expression in Eq. 36 for $1\leq n\leq b$. Substituting the respective values in Eq. 13 we obtain the following relation

$$\sum_{j=1}^{b}k_j\xi_j^n(\overline{\mu}+\mu\xi_j)\sum_{i=n}^{b}g_i\xi_j^{-i}+\phi\overline{\phi}\sum_{j=1}^{b}\frac{c_j(\overline{\eta}+\eta r_j)^2 r_j^n\sum_{i=n}^{b}g_i r_j^{-i}}{\overline{\phi}(\overline{\eta}+\eta r_j)-(\overline{\mu}+\mu r_j)}$$
$$-g_n\left(\mu\sum_{j=1}^{b}k_j\xi_j+\phi\sum_{j=1}^{b}\frac{c_j(\overline{\eta}+\eta r_j)(\overline{\phi}(\overline{\eta}+\eta r_j)-\overline{\mu})}{\overline{\phi}(\overline{\eta}+\eta r_j)-(\overline{\mu}+\mu r_j)}\right)=0,\quad 2\leq n\leq b. \quad (37)$$

Setting $n = b$ in Eq. 37 and considering the fact that $g_b \neq 0$ we obtain

$$\sum_{j=1}^{b} k_j + \phi \sum_{j=1}^{b} \frac{c_j(\overline{\eta} + \eta r_j)}{\overline{\phi}(\overline{\eta} + \eta r_j) - (\overline{\mu} + \mu r_j)} = 0. \tag{38}$$

Again taking $n = b-1,\ b-2, ..., 2$ in Eq. 37 we obtain the following condition

$$\sum_{j=1}^{b} \frac{k_j}{\xi_j^{b-n}}(\overline{\mu}+\mu\xi_j)+\phi\overline{\phi}\sum_{j=1}^{b} \frac{c_j(\overline{\eta}+\eta r_j)^2}{r_j^{b-n}(\overline{\phi}(\overline{\eta}+\eta r_j)-(\overline{\mu}+\mu r_j))}=0, \quad n=2, 3, ..., b-1. \tag{39}$$

In a similar manner substituting the respective values in Eq. 12 we obtain

$$\sum_{j=1}^{b} \frac{k_j}{\xi_j^{b-1}}(\overline{\mu} + \mu\xi_j) + \phi\overline{\phi}\sum_{j=1}^{b} \frac{c_j(\overline{\eta} + \eta r_j)^2}{r_j^{b-1}(\overline{\phi}(\overline{\eta} + \eta r_j) - (\overline{\mu} + \mu r_j))} = 0. \tag{40}$$

Now Eqs. 39 and 40 can be combined in the form

$$\sum_{j=1}^{b} \frac{k_j}{\xi_j^{n}}(\overline{\mu} + \mu\xi_j) + \phi\overline{\phi}\sum_{j=1}^{b} \frac{c_j(\overline{\eta} + \eta r_j)^2}{r_j^{n}(\overline{\phi}(\overline{\eta}+\eta r_j)-(\overline{\mu} +\mu r_j))} = 0, \quad n=1, 2, ..., b-1. \tag{41}$$

Moreover, using Eqs. 21, 26 and 32 over (15) we obtain the relation

$$\sum_{j=1}^{b} \frac{k_j\xi_j}{1-\xi_j}+\sum_{j=1}^{b} \frac{c_j r_j}{1-r_j}+\overline{\eta}\sum_{j=1}^{b} c_j+\phi\sum_{j=1}^{b} \frac{c_j(\overline{\eta}+\eta r_j)r_j}{(1-r_j)(\overline{\phi}(\overline{\eta}+\eta r_j)-(\overline{\mu}+\mu r_j))}=\frac{1}{a}. \tag{42}$$

One may observe that Eqs. 29, 38, 41 and 42 together constitutes a system of $2b$ equations in $2b$ unknowns which can be solved in order to obtain the constants c_j's and k_j's, $j = 1, 2, ...b$. Once the arbitrary constants are determined, the system-content distributions can be obtained in explicit form as

$$p_{0,0}^{-} = ap_{0,0}(0) = a\overline{\eta}\sum_{j=1}^{b} c_j \tag{43}$$

$$p_{n,0}^{-} = ap_{n,0}(0) = a\sum_{j=1}^{b} c_j r_j^n, \quad n \geq 1 \tag{44}$$

$$p_{n,0} = p_{n,0}^{*}(1) = \overline{\phi}\sum_{j=1}^{b} \frac{c_j(\overline{\eta} + \eta r_j)(G(r_j^{-1}) - 1)}{1 - \overline{\phi}(\overline{\eta} + \eta r_j)} r_j^n, \quad n \geq 1 \tag{45}$$

$$p_{n,1}^{-} = ap_{n,1}(0) = a\sum_{j=1}^{b} k_j\xi_j^n + a\phi\sum_{j=1}^{b} \frac{c_j(\overline{\eta} + \eta r_j)}{\overline{\phi}(\overline{\eta} + \eta r_j) - (\overline{\mu} + \mu r_j)} r_j^n, \quad n \geq 1 \tag{46}$$

$$\begin{aligned} p_{n,1} = p_{n,1}^{*}(1) &= \sum_{j=1}^{b} \frac{k_j(\overline{\mu} + \mu\xi_j)(G(\xi_j^{-1}) - 1)}{1 - (\overline{\mu} + \mu\xi_j)}\xi_j^n \\ &+\phi\overline{\phi}\sum_{j=1}^{b} \frac{c_j(\overline{\eta} + \eta r_j)^2(G(r_j^{-1}) - 1)}{(1 - \overline{\phi}(\overline{\eta} + \eta r_j))(\overline{\phi}(\overline{\eta} + \eta r_j) - (\overline{\mu} + \mu r_j))} r_j^n, \quad n \geq 1. \end{aligned} \tag{47}$$

Using the normalizing condition $\sum_{n=0}^{\infty} p_{n,0} + \sum_{n=1}^{\infty} p_{n,1} = 1$, we further have

$$p_{0,0} = 1 - \sum_{j=1}^{b} \frac{k_j(\overline{\mu} + \mu\xi_j)(G(\xi_j^{-1}) - 1)\xi_j}{\left\{1 - \xi_j\right\}\left\{1 - (\overline{\mu} + \mu\xi_j)\right\}}$$
$$-\overline{\phi} \sum_{j=1}^{b} \frac{c_j(\overline{\eta} + \eta r_j)(G(r_j^{-1}) - 1)r_j}{\left\{1 - r_j\right\}\left\{1 - \overline{\phi}(\overline{\eta} + \eta r_j)\right\}} \left[1 + \frac{\phi(\overline{\eta} + \eta r_j)}{\overline{\phi}(\overline{\eta} + \eta r_j) - (\overline{\mu} + \mu r_j)}\right] \quad (48)$$

This completes the analysis of $GI^X/Geo/1 - MWV$ queue under LAS-DA policy.

3.2 Modeling with EAS Policy

Under EAS policy the arrival of batches and the departure of individual customer takes place in the interval $(m, m+)$ and $(m-, m)$ respectively, whereas the server vacation starts or ends exactly at the slot boundary i.e., at the instant m. In order to formulate the set of equations governing the system we consider the remaining inter-arrival time of the next batch to be the supplementary variable and define the random variables N_m, U_m and Y_m as done in Section 3.1 at the instant just before a potential batch arrival i.e., at m. We define the joint probabilities as

$$\hat{q}_{n,0}(m, u) = P\{N_m = n, U_m = u, Y_m = 0\}, \;\; u \geq 0, \;\; n \geq 0,$$

$$\hat{q}_{n,1}(m, u) = P\{N_m = n, U_m = u, Y_m = 1\}, \;\; u \geq 0, \;\; n \geq 1.$$

In steady-state we have

$$q_{n,0}(u) = \lim_{m\to\infty} \hat{q}_{n,0}(m, u), n \geq 0 \;\text{ and }\; q_{n,1}(u) = \lim_{m\to\infty} \hat{q}_{n,1}(m, u), n \geq 1.$$

Relating the states of the system at two consecutive time epochs m and $(m + 1)$ and using probabilistic arguments, we obtain (for $u \geq 1$) the following equations in steady-state

$$q_{0,0}(u-1) = q_{0,0}(u) + \eta q_{1,0}(u) + \mu q_{1,1}(u) + a_u \eta g_1 q_{0,0}(0), \quad (49)$$

$$q_{n,0}(u-1) = \overline{\phi}\left\{\overline{\eta}\left[q_{n,0}(u) + a_u \sum_{i=1}^{n} g_i q_{n-i,0}(0)\right] + \eta\left[q_{n+1,0}(u) + a_u \sum_{i=1}^{n+1} g_i q_{n-i+1,0}(0)\right]\right\},$$
$$1 \leq n \leq b-1, \quad (50)$$

$$q_{n,0}(u-1) = \overline{\phi}\left\{\overline{\eta}\left[q_{n,0}(u) + a_u \sum_{i=1}^{b} g_i q_{n-i,0}(0)\right] + \eta\left[q_{n+1,0}(u) + a_u \sum_{i=1}^{b} g_i q_{n-i+1,0}(0)\right]\right\},$$
$$n \geq b, \quad (51)$$

$$q_{1,1}(u-1) = \phi\left\{\overline{\eta}\left[q_{1,0}(u) + a_u g_1 q_{0,0}(0)\right] + \eta\left[q_{2,0}(u) + a_u \sum_{i=1}^{2} g_i q_{2-i,0}(0)\right]\right\} + \overline{\mu} q_{1,1}(u)$$
$$+\mu q_{2,1}(u) + a_u g_1 \mu q_{1,1}(0), \quad (52)$$

$$q_{n,1}(u-1) = \phi\left\{\overline{\eta}\left[q_{n,0}(u)+a_u\sum_{i=1}^{n}g_iq_{n-i,0}(0)\right]+\eta\left[q_{n+1,0}(u)+a_u\sum_{i=1}^{n+1}g_iq_{n+1-i,0}(0)\right]\right\}$$
$$+\overline{\mu}\left[q_{n,1}(u)+a_u\sum_{i=1}^{n-1}g_iq_{n-i,1}(0)\right]+\mu\left[q_{n+1,1}(u)+a_u\sum_{i=1}^{n}g_iq_{n-i+1,1}(0)\right],\quad 2\leq n\leq b, \tag{53}$$

$$q_{n,1}(u-1) = \phi\left\{\overline{\eta}\left[q_{n,0}(u)+a_u\sum_{i=1}^{b}g_iq_{n-i,0}(0)\right]+\eta\left[q_{n+1,0}(u)+a_u\sum_{i=1}^{b}g_iq_{n+1-i,0}(0)\right]\right\}$$
$$+\overline{\mu}\left[q_{n,1}(u)+a_u\sum_{i=1}^{b}g_iq_{n-i,1}(0)\right]+\mu\left[q_{n+1,1}(u)+a_u\sum_{i=1}^{b}g_iq_{n-i+1,1}(0)\right],\quad n\geq b+1. \tag{54}$$

We introduce the transforms $q_{n,0}^*(z)=\sum_{u=0}^{\infty}q_{n,0}(u)z^u$ and $q_{n,1}^*(z)=\sum_{u=0}^{\infty}q_{n,1}(u)z^u$ so that the steady-state probabilities $q_{n,0}=q_{n,0}^*(1)$, $n\geq 0$ and $q_{n,1}=q_{n,1}^*(1)$, $n\geq 1$. Thus multiplying (49)–(54) by z^u and summing over u from 1 to ∞ we obtain the following set of transformed equations

$$(z-1)q_{0,0}^*(z) = \eta q_{1,0}^*(z)-q_{0,0}(0)-\eta q_{1,0}(0)$$
$$+\mu q_{1,1}^*(z)-\mu q_{1,1}(0)+\eta A(z)g_1q_{0,0}(0), \tag{55}$$

$$(z-\overline{\eta}\overline{\phi})q_{n,0}^*(z) = \overline{\phi}\left\{\eta\left[q_{n+1,0}^*(z)-q_{n+1,0}(0)+A(z)\sum_{i=1}^{n+1}g_iq_{n-i+1,0}(0)\right]\right.$$
$$\left.+\overline{\eta}\left[A(z)\sum_{i=1}^{n}g_iq_{n-i,0}(0)-q_{n,0}(0)\right]\right\},\quad 1\leq n\leq b-1, \tag{56}$$

$$(z-\overline{\eta}\overline{\phi})q_{n,0}^*(z) = \overline{\phi}\left\{\eta\left[q_{n+1,0}^*(z)-q_{n+1,0}(0)+A(z)\sum_{i=1}^{b}g_iq_{n-i+1,0}(0)\right]\right.$$
$$\left.+\overline{\eta}\left[A(z)\sum_{i=1}^{b}g_iq_{n-i,0}(0)-q_{n,0}(0)\right]\right\},\quad n\geq b, \tag{57}$$

$$(z-\overline{\mu})q_{1,1}^*(z) = \phi\left\{\eta\left[q_{2,0}^*(z)-q_{2,0}(0)+A(z)\sum_{i=1}^{2}g_iq_{2-i,0}(0)\right]+\overline{\eta}\left[q_{1,0}^*(z)-q_{1,0}(0)\right.\right.$$
$$\left.\left.+A(z)g_1q_{0,0}(0)\right]\right\}+\mu\left[q_{2,1}^*(z)-q_{2,1}(0)+A(z)g_1q_{1,1}(0)\right]-\overline{\mu}q_{1,1}(0), \tag{58}$$

$$(z-\overline{\mu})q_{n,1}^*(z) = \phi\left\{\eta\left[q_{n+1,0}^*(z)-q_{n+1,0}(0)+A(z)\sum_{i=1}^{n+1}g_iq_{n-i+1,0}(0)\right]\right.$$
$$\left.+\overline{\eta}\left[q_{n,0}^*(z)-q_{n,0}(0)+A(z)\sum_{i=1}^{n}g_iq_{n-i,0}(0)\right]\right\}$$
$$+\mu\left[q_{n+1,1}^*(z)-q_{n+1,1}(0)+A(z)\sum_{i=1}^{n}g_iq_{n-i+1,1}(0)\right]$$
$$+\overline{\mu}\left[A(z)\sum_{i=1}^{n-1}g_iq_{n-i,1}(0)-p_{n,1}(0)\right],\quad 2\leq n\leq b, \tag{59}$$

$$
\begin{aligned}
(z-\overline{\mu})q_{n,1}^{*}(z) = \phi \Bigg\{ \eta \left[q_{n+1,0}^{*}(z) - q_{n+1,0}(0) + A(z)\sum_{i=1}^{b} g_i q_{n-i+1,0}(0) \right] \\
+\overline{\eta} \left[q_{n,0}^{*}(z) - q_{n,0}(0) + A(z)\sum_{i=1}^{b} g_i q_{n-i,0}(0) \right] \Bigg\} \\
+\mu \left[q_{n+1,1}^{*}(z) - q_{n+1,1}(0) + A(z)\sum_{i=1}^{b} g_i q_{n-i+1,1}(0) \right] \\
+\overline{\mu} \left[A(z)\sum_{i=1}^{b} g_i q_{n-i,1}(0) - p_{n,1}(0) \right], n \geq b+1. \qquad (60)
\end{aligned}
$$

Adding (55)–(60) for all values of n, taking limit as $z \to 1$ and using the normalizing condition $\sum_{n=0}^{\infty} q_{n,0} + \sum_{n=1}^{\infty} q_{n,1} = 1$ we have the following relation

$$
\sum_{n=0}^{\infty} q_{n,0}(0) + \sum_{n=1}^{\infty} q_{n,1}(0) = \frac{1}{a}. \qquad (61)
$$

Let us denote $q_{n,0}^{-}$, $n \geq 0$ and $q_{n,1}^{-}$, $n \geq 1$ as the probability that there are n customers in the system at pre-arrival epoch depending on whether the server is in working vacation period or normal busy period, respectively. Applying the arguments of Bayes' theorem as in Section 3.1 and using (61) we have the following relation

$$
q_{n,j}^{-} = a q_{n,j}(0), \;\; n \geq 0, \;\; j = 0 \text{ or } n \geq 1, \;\; j = 1. \qquad (62)
$$

Now using the right shift operator D on the sequence $\{q_{n,0}^{*}(z)\}$ and $\{q_{n,0}(0)\}$ such that $Dq_{n,0}^{*}(z) = q_{n+1,0}^{*}(z)$ and $Dq_{n,0}(0) = q_{n+1,0}(0)$ for all n, Eq. 57 can be re-written in the form

$$
\begin{aligned}
\left[z - \overline{\phi}(\overline{\eta} + \eta D)\right] q_{n,0}^{*}(z) = \overline{\phi} \Bigg[\eta \left\{ A(z)\sum_{i=1}^{b} g_i D^{b-i+1} - D^{b+1} \right\} \\
+\overline{\eta} \left\{ A(z)\sum_{i=1}^{b} g_i D^{b-i} - D^{b} \right\} \Bigg] q_{n-b,0}(0), \\
n \geq b. \qquad (63)
\end{aligned}
$$

Substituting $z = \overline{\phi}(\overline{\eta} + \eta D)$ we obtain

$$
\left[\overline{\phi}(\overline{\eta} + \eta D) \left(A(\overline{\phi}(\overline{\eta} + \eta D)) \sum_{i=1}^{b} g_i D^{b-i} - D^{b} \right) \right] q_{n,0}(0) = 0, \;\; n \geq 0, \qquad (64)
$$

which is a homogeneous difference equation with constant coefficient and the corresponding c.e. is same as Eq. 20. Thus the general solution of Eq. 64 is

$$
q_{n,0}(0) = \sum_{i=1}^{b} e_i \alpha_i^{n}, \;\; n \geq 0, \qquad (65)
$$

where $\alpha_1, \alpha_2, ..., \alpha_b$ are the roots of the c.e. lying inside the unit circle $|s| = 1$ and, e_1, $e_2, ..., e_b$ are the arbitrary constants corresponding to each root α_i which are independent of n . Now using Eq. 65 in Eq. 63 we have

$$\left[z - \overline{\phi}(\overline{\eta} + \eta D)\right] q^*_{n,0}(z) = \overline{\phi} \sum_{j=1}^{b} e_j(\eta\alpha_j + \overline{\eta})(A(z)G(\alpha_j^{-1}) - 1)\alpha_j^n, \quad n \geq b, \tag{66}$$

which is a non-homogeneous difference equation with constant coefficient. Now using similar argument as in Section 3.1 we have the general solution of Eq. 66 as

$$q^*_{n,0}(z) = \overline{\phi} \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)(A(z)G(\alpha_j^{-1}) - 1)}{z - \overline{\phi}(\overline{\eta} + \eta\alpha_j)} \alpha_j^n, \quad n \geq b. \tag{67}$$

We now find the condition under which Eq. 67 holds for $q^*_{n,0}(z)$, $1 \leq n \leq b-1$ as well. Substituting the respective values in Eq. 56 we obtain the following condition

$$\eta \sum_{j=1}^{b} e_j\alpha_j^{n+1} \left(\sum_{i=1}^{b} g_i\alpha_j^{-i} - \sum_{i=1}^{n+1} g_i\alpha_j^{-i} \right) + \overline{\eta} \sum_{j=1}^{b} e_j\alpha_j^{n} \left(\sum_{i=1}^{b} g_i\alpha_j^{-i} - \sum_{i=1}^{n} g_i\alpha_j^{-i} \right) = 0, \quad 1 \leq n \leq b-1. \tag{68}$$

Setting $n = b-1,\ b-2, ..., 1$ in Eq. 68 and using the fact that $g_b \neq 0$ we have

$$\sum_{j=1}^{b} \frac{e_j}{\alpha_j^n} = 0, \quad n = 1, 2, ..., b-1. \tag{69}$$

Now using the definition of the operator D over the sequence $\{q^*_{n,1}(z)\}$ and $\{q_{n,1}(0)\}$, for all n as done above, and further using the expression of Eqs. 65 and 67, 60 can be re-written in the form:

$$\begin{aligned}
[z - \overline{\mu} - \mu D]\, q^*_{n,1}(z) = & \left[\overline{\mu} A(z) \sum_{i=1}^{b} g_i D^{b-i} + \mu A(z) \sum_{i=1}^{b} g_i D^{b-i+1} - \overline{\mu} D^b - \mu D^{b+1} \right] q_{n-b,1}(0) \\
& + \phi \sum_{j=1}^{b} e_j(\overline{\eta} + \eta\alpha_j)(A(z)G(\alpha_j^{-1}) - 1)\alpha_j^n \\
& + \phi\overline{\phi} \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)^2(A(z)G(\alpha_j^{-1}) - 1)}{z - \overline{\phi}(\overline{\eta} + \eta\alpha_j)} \alpha_j^n, \quad n \geq b+1.
\end{aligned} \tag{70}$$

Substituting $z = \overline{\mu} + \mu D$ in Eq. 70 we obtain:

$$\begin{aligned}
& \left[(\overline{\mu} + \mu D) \left(D^b - A(\overline{\mu} + \mu D) \sum_{i=1}^{b} g_i D^{b-i} \right) \right] q_{n,1}(0) = \phi \Bigg[\sum_{j=1}^{b} e_j(\overline{\eta} + \eta\alpha_j)(A(z)G(\alpha_j^{-1}) - 1)\alpha_j^{n+b} \\
& \quad + \overline{\phi} \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)^2(A(z)G(\alpha_j^{-1}) - 1)}{z - \overline{\phi}(\overline{\eta} + \eta\alpha_j)} \alpha_j^{n+b} \Bigg] \Bigg|_{z=\overline{\mu}+\mu D}, \\
& \qquad n \geq 1.
\end{aligned} \tag{71}$$

Equation 71 is a non-homogeneous difference equation with constant coefficient and the general solution is given by

$$q_{n,1}(0) = \sum_{j=1}^{b} l_j\beta_j^n + \phi\sum_{j=1}^{b}\frac{e_j(\overline{\eta}+\eta\alpha_j)}{\overline{\phi}(\overline{\eta}+\eta\alpha_j)-(\overline{\mu}+\mu\alpha_j)}\alpha_j^n,\quad n\geq 1. \tag{72}$$

The first term in the R.H.S of Eq. 72 is the solution corresponding to the homogeneous part of Eq. 71. $\beta_1,\ \beta_2, ...,\ \beta_b$ are the roots of the c.e. (which is same as Eq. 33) of the homogeneous equation lying inside the unit circle $|s| = 1$, and $l_1,\ l_2, ..., l_b$ are the corresponding arbitrary constants independent of n. On the other hand, the second term in the R.H.S of Eq. 72 represents a particular solution of Eq. 71. Now using expression (72) on (70) we have

$$\begin{aligned}[z-\overline{\mu}-\mu D]\,q^*_{n,1}(z) &= \sum_{j=1}^{b} l_j(\overline{\mu}+\mu\beta_j)(A(z)G(\beta_j^{-1})-1)\beta_j^n\\ &+\phi\overline{\phi}\sum_{j=1}^{b}\frac{e_j(\overline{\eta}+\eta\alpha_j)^2(A(z)G(\alpha_j^{-1})-1)}{z-\overline{\phi}(\overline{\eta}+\eta\alpha_j)}\alpha_j^n\\ &+\phi\overline{\phi}\sum_{j=1}^{b}\frac{e_j(\overline{\eta}+\eta\alpha_j)^2(A(z)G(\alpha_j^{-1})-1)}{\overline{\phi}(\overline{\eta}+\eta\alpha_j)-(\overline{\mu}+\mu\alpha_j)}\alpha_j^n,\quad n\geq b+1. \end{aligned}\tag{73}$$

Equation 73 is a non-homogeneous difference equation with constant coefficient. Using a similar argument as in Section 3.1 we obtain the general solution of Eq. 73 as

$$\begin{aligned} q^*_{n,1}(z) &= \sum_{j=1}^{b}\frac{l_j(\overline{\mu}+\mu\beta_j)(A(z)G(\beta_j^{-1})-1)}{z-(\overline{\mu}+\mu\beta_j)}\beta_j^n\\ &+\phi\overline{\phi}\sum_{j=1}^{b}\frac{e_j(\overline{\eta}+\eta\alpha_j)^2(A(z)G(\alpha_j^{-1})-1)}{(z-\overline{\phi}(\overline{\eta}+\eta\alpha_j))(\overline{\phi}(\overline{\eta}+\eta\alpha_j)-(\overline{\mu}+\mu\alpha_j))}\alpha_j^n\\ &\qquad n\geq b+1. \end{aligned}\tag{74}$$

We now find the condition under which (74) holds true for $q^*_{n,1}(z)$, $1\leq n\leq b$ as well. Substituting the respective values in Eq. 59 we obtain the following condition

$$\begin{aligned} &\overline{\mu}\sum_{j=1}^{b} l_j\beta_j^n\sum_{i=n}^{b} g_i\beta_j^{-i} + \mu\sum_{j=1}^{b} l_j\beta_j^{n+1}\sum_{i=n+1}^{b} g_i\beta_j^{-i} - \mu g_n\phi\sum_{j=1}^{b}\frac{e_j(\overline{\eta}+\eta\alpha_j)\alpha_j}{\overline{\phi}(\overline{\eta}+\eta\alpha_j)-(\overline{\mu}+\mu\alpha_j)}\\ &\quad -\eta\phi g_{n+1}\sum_{j=1}^{b} e_j - \phi g_n\sum_{j=1}^{b} e_j(\overline{\eta}+\eta\alpha_j)\\ &\quad +\phi\overline{\phi}\sum_{j=1}^{b}\frac{e_j(\overline{\eta}+\eta\alpha_j)^2\alpha_j^n\sum_{i=n}^{b} g_i\alpha_j^{-i}}{\overline{\phi}(\overline{\eta}+\eta\alpha_j)-(\overline{\mu}+\mu\alpha_j)} = 0,\quad 2\leq n\leq b. \end{aligned}\tag{75}$$

Setting $n = b$ and $n = b - 1$ in Eq. 75 and using the fact that $g_b \neq 0$ we respectively obtain

$$\sum_{j=1}^{b} l_j + \phi \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)}{\overline{\phi}(\overline{\eta} + \eta\alpha_j) - (\overline{\mu} + \mu\alpha_j)} = 0, \quad (76)$$

$$\sum_{j=1}^{b} \frac{l_j}{\beta_j}(\overline{\mu} + \mu\beta_j) - \eta\phi \sum_{j=1}^{b} e_j + \phi\overline{\phi} \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)^2}{\alpha_j \left\{\overline{\phi}(\overline{\eta} + \eta\alpha_j) - (\overline{\mu} + \mu\alpha_j)\right\}} = 0. \quad (77)$$

Again setting $n = b - 2,\ b - 3, ...,\ 2$ in Eq. 75 we have

$$\sum_{j=1}^{b} \frac{l_j}{\beta_j^{b-n}}(\overline{\mu} + \mu\beta_j) + \phi\overline{\phi} \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)^2}{\alpha_j^{b-n} \left\{\overline{\phi}(\overline{\eta} + \eta\alpha_j) - (\overline{\mu} + \mu\alpha_j)\right\}} = 0, \quad n = 2, 3, ..., b - 2. \quad (78)$$

In a similar manner setting the respective values in Eq. 58 we obtain the following condition

$$\sum_{j=1}^{b} \frac{l_j}{\beta_j^{b-1}}(\overline{\mu} + \mu\beta_j) + \phi\overline{\phi} \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)^2}{\alpha_j^{b-1} \left\{\overline{\phi}(\overline{\eta} + \eta\alpha_j) - (\overline{\mu} + \mu\alpha_j)\right\}} = 0. \quad (79)$$

Thus (78) and (79) can be combined together in the form

$$\sum_{j=1}^{b} \frac{l_j}{\beta_j^{n}}(\overline{\mu} + \mu\beta_j) + \phi\overline{\phi} \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)^2}{\alpha_j^{n} \left\{\overline{\phi}(\overline{\eta} + \eta\alpha_j) - (\overline{\mu} + \mu\alpha_j)\right\}} = 0, \quad n = 2, 3, ...b - 1. \quad (80)$$

Also, using Eqs. 65 and 72 in Eq. 61 we have the following relation

$$\sum_{j=1}^{b} \frac{l_j\beta_j}{1 - \beta_j} + \sum_{j=1}^{b} \frac{e_j}{1 - \alpha_j} + \phi \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)\alpha_j}{(1 - \alpha_j)\left\{\overline{\phi}(\overline{\eta} + \eta\alpha_j) - (\overline{\mu} + \mu\alpha_j)\right\}} = \frac{1}{a}. \quad (81)$$

It can be seen that Eqs. 69, 76, 77, 80 and 81 together constitutes a system of $2b$ equations which can be solved in order to obtain the $2b$ unknowns namely $e_1,\ e_2, \ldots,\ e_b$ and $l_1,\ l_2, \ldots,\ l_b$. Thus we have the closed-form expressions of the system-content distributions at pre-arrival and arbitrary epochs as

$$q^-_{n,0} = aq_{n,0}(0) = a \sum_{j=1}^{b} e_j\alpha_j^n, \quad n \geq 0 \quad (82)$$

$$q_{n,0} = q^*_{n,0}(1) = \overline{\phi} \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)(G(\alpha_j^{-1}) - 1)}{1 - \overline{\phi}(\overline{\eta} + \eta\alpha_j)}\alpha_j^n, \quad n \geq 1 \quad (83)$$

$$q^-_{n,1} = aq_{n,1}(0) = a \sum_{j=1}^{b} l_j\beta_j^n + a\phi \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)}{\overline{\phi}(\overline{\eta} + \eta\alpha_j) - (\overline{\mu} + \mu\alpha_j)}\alpha_j^n, \quad n \geq 1 \quad (84)$$

$$\begin{aligned} q_{n,1} = q^*_{n,1}(1) &= \sum_{j=1}^{b} \frac{l_j(\overline{\mu} + \mu\beta_j)(G(\beta_j^{-1}) - 1)}{1 - (\overline{\mu} + \mu\beta_j)}\beta_j^n \\ &+ \phi\overline{\phi} \sum_{j=1}^{b} \frac{e_j(\overline{\eta} + \eta\alpha_j)^2(G(\alpha_j^{-1}) - 1)}{(1 - \overline{\phi}(\overline{\eta} + \eta\alpha_j))(\overline{\phi}(\overline{\eta} + \eta\alpha_j) - (\overline{\mu} + \mu\alpha_j))}\alpha_j^n, \quad n \geq 1. \end{aligned} \quad (85)$$

Further, using the normalizing condition we have

$$q_{0,0} = 1 - \sum_{j=1}^{b} \frac{l_j(\overline{\mu}+\mu\beta_j)(G(\beta_j^{-1})-1)\beta_j}{\{1-\beta_j\}\{1-(\overline{\mu}+\mu\beta_j)\}}$$
$$-\overline{\phi}\sum_{j=1}^{b} \frac{e_j(\overline{\eta}+\eta\alpha_j)(G(\alpha_j^{-1})-1)\alpha_j}{\{1-\alpha_j\}\{1-\overline{\phi}(\overline{\eta}+\eta\alpha_j)\}}\left[1+\frac{\phi(\overline{\eta}+\eta\alpha_j)}{\overline{\phi}(\overline{\eta}+\eta\alpha_j)-(\overline{\mu}+\mu\alpha_j)}\right]. \quad (86)$$

This completes the analysis of $GI^X/Geo/1 - MWV$ queue under EAS policy.

3.3 Stability Analysis

The analysis done in Sections 3.1 and 3.2 are mainly based on the roots of the characteristic equations. As a result, it is significant to determine the conditions under which the system remains stable, which is discussed in the following theorem.

Theorem 1 *Under the conditions $\rho = \frac{\overline{g}}{a\mu} < 1$ and $0 < \phi \le 1$, $0 \le \eta \le \mu < 1$, the root equations $s^b - A(\overline{\mu}+\mu s)\sum_{i=1}^{b} g_i s^{b-i} = 0$ and $s^b - A(\overline{\phi}(\overline{\eta}+\eta s))\sum_{i=1}^{b} g_i s^{b-i} = 0$, respectively possesses exactly b roots inside the unit circle $|s| = 1$.*

Proof Let us first consider the equation $s^b - A(\overline{\mu}+\mu s)\sum_{i=1}^{b} g_i s^{b-i} = 0$ and assume $f_1(s) = s^b$ and $f_2(s) = -A(\overline{\mu}+\mu s)\sum_{i=1}^{b} g_i s^{b-i}$. Consider the circle $|s| = 1-\delta$ where $\delta > 0$ and is a sufficiently smaller quantity. Let $A(\overline{\mu}+\mu s) = K(s) = \sum_{i=0}^{\infty} k_i s^i$, where $k_i \ge 0$ for all i. This gives

$$|f_1(s)| = |s|^b = (1-\delta)^b = 1 - b\delta + o(\delta)$$
$$|f_2(s)| = |K(s)||\sum_{i=1}^{b} g_i s^{b-i}| \le K(1-\delta)\sum_{i=1}^{b} g_i(1-\delta)^{b-i} = 1 - b\delta + (\overline{g}-\mu a)\delta + o(\delta)$$

As δ is a very small quantity, we have $|f_2(s)| < |f_1(s)|$ under the condition that $\overline{g} < \mu a$, i.e., $\rho = \frac{\overline{g}}{a\mu} < 1$. Hence from Rouché's theorem we can conclude that $|f_1(s)|$ and $|f_1(s)| + |f_2(s)|$ have exactly b roots inside the unit circle. Here one may note that the condition $\rho < 1$ is necessary as well as sufficient for the stability of the system (see Abolnikov and Dukhovny 1991).

Similarly, for equation $s^b - A(\overline{\phi}(\overline{\eta}+\eta s))\sum_{i=1}^{b} g_i s^{b-i} = 0$ assume $h_1(s) = s^b$ and $h_2 = -A(\overline{\phi}(\overline{\eta}+\eta s))\sum_{i=1}^{b} g_i s^{b-i}$. As before, consider the circle $|s| = 1-\delta$ and let $A(\overline{\phi}(\overline{\eta}+\eta s)) = B(s) = \sum_{i=0}^{\infty} b_i s^i$ such that $b_i \ge 0$ for all i. This gives

$$|h_1(s)| = 1 - b\delta + o(\delta)$$

$$|h_2(s)| = |B(s)||\sum_{i=1}^{b} g_i s^{b-i}| \le B(1-\delta)\sum_{i=1}^{b} g_i(1-\delta)^{b-i}$$
$$= A(\overline{\phi}) - \delta(\eta\overline{\phi}A^{(1)}(\overline{\phi}) + bA(\overline{\phi}) - \overline{g}A(\overline{\phi})) + o(\delta) \quad (87)$$

It can observed that as δ is sufficiently small, $|h_2(s)| < |h_1(s)|$ under the condition that $A(\overline{\phi}) < 1$ and $\eta\overline{\phi}A^{(1)}(\overline{\phi}) + bA(\overline{\phi}) - \overline{g}A(\overline{\phi})) > 0$, which holds true provided $0 < \phi \le 1$ and $0 \le \eta \le \mu < 1$. Thus from Rouché's theorem we conclude that $|h_1(s)|$ and $|h_1(s)| + |h_2(s)|$ have exactly b roots inside the unit circle. □

4 Some Special Models

In this section we discuss in brief few special cases of our model which can be deduced by taking some fixed values of the parameters η, ϕ and/or $g_i's$. Based on the analysis done in Section 3, the pre-arrival and arbitrary epoch probabilities can be obtained in an explicit and readily computable form under both LAS-DA and EAS policies.

Case 1: Let $g_1 = 1$ and $g_i = 0$, $\forall\ i \geq 2$, i.e., the customers arrive into the system individually rather than in batches. Thus our model becomes $GI/Geo/1$ queue with multiple working vacations (Li et al. 2007). Correspondingly, the two root equations reduces to $(s - A(\overline{\phi}(\overline{\eta} + \eta s))) = 0$ and $(s - A(\overline{\mu} + \mu s)) = 0$ such that both possesses single root in the interval $0 < s < 1$. Solving a system of two equations given by Eqs. 38 and 42 for LAS-DA, and Eqs. 76 and 81 for EAS will give the constants and hence the steady-state probabilities can be evaluated using Eqs. 43–48 and Eqs. 82–86. Here it is worthy to mention that our analysis provides an alternative approach to study the queueing model considered in Li et al. (2007).

Case 2: Let $\eta = 0$, i.e., the server remains idle during vacation period. Thus our model becomes $GI^X/Geo/1$ queue with multiple vacations. Our analysis provides a methodology to derive the results of this queueing model which has not been investigated in the past. The two c.e.'s and the system of $2b$ equations should be solved by putting $\eta = 0$ and correspondingly, the steady-state probabilities can be obtained using (43)–(48) and (82)–(86). The model considered in this case may again reduce to $GI/Geo/1$ queue with multiple vacations (Tian and Zhang 2002) by assuming single arrivals i.e., $g_1 = 1$ and $g_i = 0,\ \forall\, i \geq 2$.

Case 3: Let $\phi = 1$ and $\eta = 0$, i.e., the server does not take any vacation and thus our model reduces to $GI^X/Geo/1$ queue without vacation (Chaudhry and Gupta 1997). Correspondingly we have only one root equation given by $(s^b - A(\overline{\mu}+\mu s)\sum_{i=1}^{b} g_i s^{b-i}) = 0$ which have b roots inside the unit circle. The associated b arbitrary constants (k_j's) for LAS-DA policy can be obtained by solving the following system of b equations which are deduced from Eqs. 26, 41 and 42 as

$$\sum_{j=1}^{b} \frac{k_j}{\xi_j^n}(\overline{\mu} + \mu\xi_j) = 0, \quad n = 1, 2, ..., b-1, \quad \text{and} \quad \sum_{j=1}^{b} k_j \left(\frac{\overline{\mu} + \mu\xi_j}{1 - \xi_j}\right) = \frac{1}{a},$$

and the state probabilities can be evaluated from Eqs. 43, 46, 47 and 48. Similarly, for EAS policy, the b arbitrary constants (l_j's) can be obtained by solving the following system of b equations deduced from Eqs. 76, 77, 80 and 81 as

$$\sum_{j=1}^{b} \frac{l_j}{\beta_j^n} = 0, \quad n = 1, 2, ..., b-1 \quad \text{and} \quad \sum_{j=1}^{b} \frac{l_j}{1 - \beta_j} = \frac{1}{a}$$

and the state probabilities can be evaluated from Eqs. 82, 84, 85, and 86. The model considered in this case can be further used to derive the results of the classical $GI/Geo/1$ queue (Chaudhry et al. 1996) by assuming single arrivals instead of batch arrivals, i.e., $g_1 = 1$ and $g_i = 0,\ \forall\, i \geq 2$.

5 Performance Measures

Once the steady-state probabilities at pre-arrival and arbitrary epochs are known, different performance measures of the system can be obtained as discussed below.

5.1 Working Vacation/Busy Period Probabilities

For an arriving customer it is important to know whether the server is in busy period or in working vacation. Thus the probability that the server is in working vacation period (P_{wv}) or the server is in normal busy period, (P_b) is given by

$$P_{wv} = \sum_{n=0}^{\infty} p_{n,0}^{-}, \quad P_b = \sum_{n=1}^{\infty} p_{n,1}^{-}.$$

One may note that the above performance measures are for LAS-DA policy. In case of EAS policy $p_{n,0}^{-}$ and $p_{n,1}^{-}$ should be replaced by $q_{n,0}^{-}$ and $q_{n,1}^{-}$, respectively.

5.2 System Length

The average system-length when the server is in working vacation period (L_{wv}) or in busy period (L_b), and the average system-length (L_s) at arbitrary epoch under LAS-DA policy is given by

$$L_{wv} = \sum_{n=0}^{\infty} np_{n,0}, \quad L_b = \sum_{n=1}^{\infty} np_{n,1}, \quad L_s = L_{wv} + L_b.$$

Further, the corresponding average system length L_{wv}^{-}, L_b^{-} and L_s^{-} at pre-arrival epoch can be obtained by replacing $p_{n,0}$ and $p_{n,1}$ by $p_{n,0}^{-}$ and $p_{n,1}^{-}$ respectively. Also under EAS policy $p_{n,0}$ and $p_{n,1}$ should be replaced by $q_{n,0}$ and $q_{n,1}$, respectively.

5.3 Numerical Results and Discussion

In this section we discuss the significance of the analytical results obtained in the previous sections through some numerical examples. The results are displayed in the form of self explanatory tables. In Tables 1 and 2, system-content distribution at pre-arrival and arbitrary epochs are obtained for geometric and deterministic inter-arrival time distributions, respectively. Some performance measures are also given at the bottom of each table. It may be remarked that all the results presented here are rounded off to eight decimal places. In Table 1 one may note that the distributions at pre-arrival and arbitrary epochs are exactly the same irrespective of the state of the server which is mainly due to Bernoulli arrivals i.e., BASTA property (see Takagi 1993). Moreover, a very significant trend can be observed in the fourth and seventh column of both the tables. As n becomes larger, the ratio of the pre-arrival epoch probabilities (working vacation/ busy period) converges to the unique largest real root of the c.e.'s (Eqs. 20 and 33) lying inside the unit circle. This suggests that the tail probabilities at pre-arrival epoch can be well approximated using the unique largest real root, i.e., $p_{n,0}^{-} = ac_b r_b^n$, $p_{n,1}^{-} = ak_b \xi_b^n$, $q_{n,0}^{-} = ae_b \alpha_b^n$, $q_{n,1}^{-} = al_b \beta_b^n$, where $r_b(\alpha_b)$ and $\xi_b(\beta_b)$ are respectively the largest root of Eqs. 20 and 33 for LAS(EAS) policy.

We also investigate the impact of different parameters on the system performance through some graphical representations. The graphs are plotted for both late (LAS) and

Table 1 System-content distributions at various epochs for $Geo^X/Geo/1/MVW$ queue in LAS-DA policy with parameters $g_2 = 0.3$, $g_3 = 0.25$, $g_4 = 0.35$, $g_6 = 0.1$, $a = 5$, $\mu = 0.8$, $\eta = 0.6$, $\phi = 0.5$, $\rho = 0.8375$

n	$p^-_{n,0}$	$p_{n,0}$	$p^-_{n+1,0}/p^-_{n,0}$	$p^-_{n,1}$	$p_{n,1}$	$p^-_{n+1,1}/p^-_{n,1}$
0	0.15500898	0.15500898	0.01394697			
1	0.00216191	0.00216191	3.50000000	0.04681888	0.04681888	1.24035007
2	0.00756667	0.00756667	0.91785227	0.05807180	0.05807180	1.04844168
3	0.00694509	0.00694509	1.05834838	0.06088489	0.06088489	1.00025312
4	0.00735032	0.00735032	0.19989220	0.06090030	0.06090030	0.89217738
5	0.00146927	0.00146927	1.84198875	0.05433387	0.05433387	0.96606833
6	0.00270638	0.00270638	0.25274103	0.05249023	0.05249023	0.89391814
7	0.00068401	0.00068401	0.65207976	0.04692197	0.04692197	0.91232969
8	0.00044603	0.00044603	0.64357200	0.04280831	0.04280831	0.90834132
⋮	⋮	⋮	⋮	⋮	⋮	⋮
23	0.00000007	0.00000007	0.55418535	0.00898905	0.00898905	0.90017282
24	0.00000004	0.00000004	0.55290267	0.00809170	0.00809170	0.90017148
25	0.00000002	0.00000002	0.55394196	0.00728392	0.00728392	0.90017061
26	0.00000001	0.00000001	0.55350573	0.00655677	0.00655677	0.90017009
27	0.00000001	0.00000001	0.55330098	0.00590221	0.00590221	0.90016978
28	0.00000000	0.00000000	0.55358438	0.00531299	0.00531299	0.90016958
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.18497306	0.18497306		0.81502693	0.81502693	

$L_{wv} = 0.10595350$, $L_b = 8.91166727$, $L_s = 9.01762077$, $P_{wv} = 0.18497306$, $P_b = 0.81502693$

early (EAS) arrival systems. Figure 1a demonstrates the effect of service rate during vacation (η) on the average system content (L_s) for different distributions of inter-arrival time namely, geometric, arbitrary and deterministic, i.e., $A(z) = \frac{\lambda z}{1-(1-\lambda)z}$, $A(z) = 0.25z + 0.2z^3 + 0.35z^5 + 0.15z^{11} + 0.05z^{15}$ and $A(z) = z^a$ respectively. Other parameters are taken as $g_2 = 0.3$, $g_3 = 0.25$, $g_4 = 0.35$, $g_6 = 0.1$, $\mu = 0.95$, $\phi = 0.5$, $a = 5$. Certainly, L_s decreases with the increase in η irrespective of any inter-arrival time distribution. However, for any fixed value of η, L_s decreases as one moves from geometric to arbitrary and then to deterministic inter-arrival time distribution for both LAS and EAS systems. It may be concluded that despite having the same mean inter-arrival time, the distribution of the inter-arrival time plays a major role in determining the performance of the system. In addition, if we compare LAS and EAS, we observe that for a very low value of η (say 0.1), L_s is almost the same for both the systems. But as η increases, the value of L_s for EAS goes on decreasing at a faster rate as compared to LAS, i.e., the difference in L_s between both the systems becomes significantly higher. This is mainly due to the fact that in EAS policy the customer may depart in the same slot in which it arrived if it finds the server idle. But this is not possible in case of LAS policy where the arriving customer(s) will have to wait for at least one slot before getting served. This leads to the accumulation of customers in the queue and as a result the average system length in LAS becomes more than that of EAS.

Figure 1b highlights the impact of arrival rate (λ) on L_s for $Geo^X/Geo/1$ queue under various vacation policies namely, multiple working vacations (MWV) ($\phi = 0.5$, $\eta = 0.6$), multiple vacations (MV) ($\phi = 0.5$, $\eta = 0$) and no vacation ($\phi = 1$, $\eta = 0$). Further

Table 2 System-content distributions at various epochs for $D^X/Geo/1/MVW$ queue in EAS policy with parameters $g_1 = 0.4$, $g_2 = 0.2$, $g_3 = 0.2$, $g_4 = 0.1$, $g_5 = 0.1$, $a = 4$, $\mu = 0.8$, $\eta = 0.5$, $\phi = 0.4$, $\rho = 0.71875$

n	$q^-_{n,0}$	$q_{n,0}$	$q^-_{n+1,0}/q^-_{n,0}$	$q^-_{n,1}$	$q_{n,1}$	$q^-_{n+1,1}/q^-_{n,1}$
0	0.52827341	0.31639294	0.02490078			
1	0.01315442	0.04606541	0.71070490	0.14928442	0.14265395	0.78495881
2	0.00934891	0.03255534	0.62328096	0.11718212	0.12796348	0.66358167
3	0.00582700	0.02402104	0.45633005	0.07775991	0.10468334	0.56918251
4	0.00265903	0.01524961	0.28491835	0.04425958	0.07469578	0.53894165
5	0.00075761	0.00696148	0.22939905	0.02385333	0.04753733	0.53999202
6	0.00017379	0.00085327	0.44075506	0.01288061	0.02682281	0.53464499
7	0.00007660	0.00042332	0.37367359	0.00688655	0.01528418	0.52598890
8	0.00002862	0.00018005	0.32122215	0.00362225	0.00829973	0.52325918
9	0.00000919	0.00006239	0.31107070	0.00189538	0.00440256	0.52207646
10	0.00000286	0.00001834	0.34488862	0.00098953	0.00231973	0.52097858
11	0.00000099	0.00000584	0.35603925	0.00051552	0.00121783	0.52009069
12	0.00000035	0.00000220	0.33380952	0.00026812	0.00063644	0.51958094
13	0.00000012	0.00000075	0.32968594	0.00013931	0.00033155	0.51929905
14	0.00000004	0.00000025	0.33818942	0.00007234	0.00017248	0.51910119
15	0.00000001	0.00000008	0.34257830	0.00003755	0.00008965	0.51896287
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.56031297	0.44279235		0.43968702	0.55720764	

$L_{wv} = 0.28941911$, $L_b = 1.67700126$, $L_s = 1.96642037$, $P_{wv} = 0.56031297$, $P_b = 0.43968702$

$g_1 = 0.75$, $g_2 = 0.2$, $g_3 = 0.05$ and $\mu = 0.98$. It may be observed that an increase in λ results in a significant increase in average system content regardless of any vacation policy. If the server takes multiple vacations then the average system content is more, whereas keeping some sort of service during vacation period will lead to a decrease in system length, which is reasonable. However, the system length further decreases if the server does not take any vacation at all. This behavior is experienced by both LAS and EAS systems. Furthermore, one may carefully observe that the curves for LAS and EAS under multiple vacation policy superimpose one another. It is because under MV policy the server does not serve the customers in the queue during vacation period. As a result if the arriving customer finds the server idle it will have to wait for at least one slot for the vacation period to end and get served, irrespective of LAS or EAS system. Hence the average system length remains the same for both the systems. Also for a fixed λ, L_s is lower for EAS as compared to LAS under MWV and no vacation policy. The reason remains the same as discussed in Fig. 1a. We finally conclude that the performance of the system is not only affected by the system parameters and vacation policy adopted, but also considerably differs under the assumption of late or early arrival system. Hence both the systems should be necessarily taken into account while dealing with discrete-time queues.

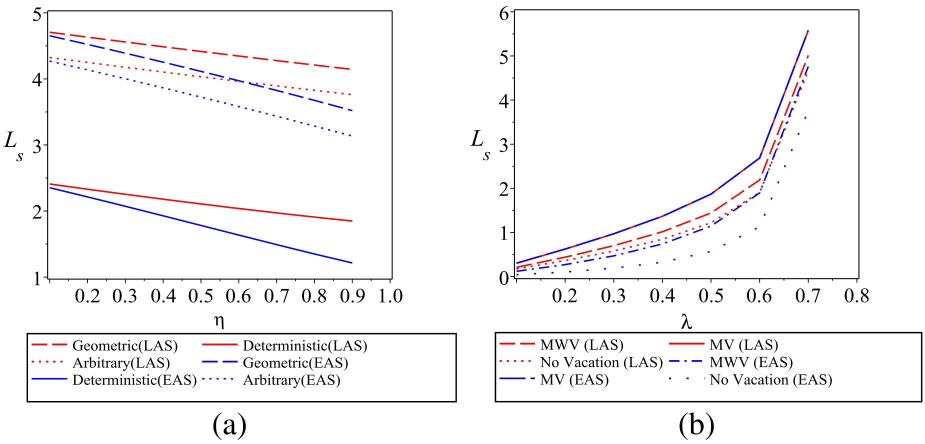

Fig. 1 **a** Effect of η on average system length (L_s) for different distributions of interarrival time. **b** Effect of arrival rate (λ) on average system length (L_s) under various vacation policies

6 Conclusion

In this paper, the steady-state analysis of a discrete-time infinite buffer $GI^X/Geo/1$ queue with multiple working vacations has been presented. The study is carried out under the assumption of both late arrival and early arrival systems independently. We have employed two important methods, the supplementary variable technique and the shift operator method, and obtained an explicit closed-form solution of the system-content distribution at pre-arrival and arbitrary epochs by considering different states of the server. Meanwhile, the numerical results suggests that the tail probabilities at pre-arrival epochs can be estimated by the unique largest root of the underlying characteristic equations present inside the unit circle. The investigations further concludes that in addition to the system parameters, the choice of arrival and departure policy, i.e., late arrival and early arrival systems significantly impacts the system characteristics. Hence both the systems should be taken into consideration while studying a discrete-time queue. Moreover, the methodology developed throughout the analysis is not only analytically tractable but is also easy to implement as illustrated by the numerical examples. It enables us to completely avoid the construction of any transition probability matrix and the inversion of generating functions in order to obtain the probabilities. We are of the opinion that the procedure used here and the results derived throughout the analysis will be of theoretical and practical importance to the researchers working along this area.

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