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Stochastic Square of the Brennan-Schwartz Diffusion Process: Statistical Computation and Application

Ahmed Nafidi[1] · **Ghizlane Moutabir**[1] · **Ramón Gutiérrez-Sánchez**[2] · **Eva Ramos-Ábalos**[2]

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Abstract
In this paper, we study a new one-dimensional homogeneous stochastic process, termed the Square of the Brennan-Schwartz model, which is used in various contexts. We first establish the probabilistic characteristics of the model, such as the analytical expression solution to Itô's stochastic differential equation, after which we determine the trend functions (conditional and non-conditional) and the likelihood approach in order to estimate the parameters in the drift. Then, in the diffusion coefficient, we consider the problem of parameter estimation, doing so by a numerical approximation. Finally, we present an application to population growth by the use of real data, namely the growth of the total population aged 65 and over, resident in the Arab Maghreb, to illustrate the research methodology presented.

Keywords Brennan Schwartz diffusion Process · Stochastic differential equation · Statistical inference in diffusion process · Stationary distribution · Trend function · Application to population growth

Mathematics Subject Classification (2010) 62M86 · 60H30 · 65C30

1 Introduction

In recent years, considerable progress has been made in discovering, understanding and controlling diffusion processes, making use of this new understanding to model phenomena that evolve randomly and continuously in time. Diffusion models have extensive areas and domains of application. For example, in mathematical finance, the pricing and hedging of products that largely depend on interest rates requires the use of mathematical models. These models have been the object of particular attention in the field of stochastic finance.

✉ Ghizlane Moutabir
ghizlane.moutabir@gmail.com

[1] Univ.Hassan 1, LAMSAD, École Nationale des Sciences Appliquées de Berrechid, Avenue de l'université, BP 280, Berrechid, Morocco

[2] Department of Statistics and Operational Research, University of Granada, Facultad de Ciencias, Campus de Fuentenueva 18071 Granada, Spain

Among other work in this field, Vasicek (1977) proposed a general form of the term structure of interest rates, and Brennan and Schwartz (1979) developed an no-arbitrage model of the term structure of interest rates. In addition, Bezborodov et al. (2016) consider the problem related to payoffs of polynomial growth and Mrkvicka et al. (2017) study goodness-of-fit tests and multiple Monte Carlo testing with applications in spatial point processes.

Another field in which diffusion processes are often analysed is that of biology, where microscopic studies may be performed, for example to identify sequences on a strand of DNA or to characterise the evolution of cancerous tumours. On the other hand, macroscopic phenomena may also be examined, concerning the behaviour patterns of large groups of individuals and their interactions (the extinction of populations, the balance of ecosystems, the predator-prey balance, meta populations), or problems of population genetics. In population dynamics, processes of life and death and of branching constitute fundamental models; see, for example, (Haccou et al. 2005; Allen 2010). An important analysis of the spatial dynamics of a population, i.e. the evolution of its spatial distribution over time, has been performed by Phillips et al. (2006).

Various stochastic models have been proposed with respect to population dynamics, in order to study population growth under fluctuating conditions. In a stochastic differential equation (SDE), the growth rate or any other parameter included is considered to be a random process. In consequence, stochastic models enable us to understand the growth of populations in processes such as the evolution of seasonal infectious disease and stochastic epidemics. In this respect, see papers such as Lin and Ludkovski (2014) on sequential Bayesian inference in hidden Markov stochastic kinetic models with application to detection and response to seasonal epidemics, and Campillo et al. (2016) on an analysis and approximation of stochastic growth model with extinction.

The process examined in the present study is termed the Stochastic Square of the Brennan-Schwartz Diffusion Process (SBSDP), which is an extension of the homogeneous lognormal diffusion process (see, for example, Tintner and Sengupta (1972)). The term adopted for the process we study, i.e. "square of the Brennan-Schwartz process" can be proved by stochastic calculus. However, estimating continuous time processes remains difficult due to the unattainability of a continuous sample of observations. Instead, the model is discretized, after which estimation methods can be applied. Although there exists a class of continuous time stochastic processes for which the transition probability density functions (tpdf) are not known, the estimation can be achieved by alternative techniques (see for example Gutierrez et al. (2006, 2007)).

This article focuses on the latter problem. As the tpdf of the process cannot be obtained, we propose a method for calculating the trend functions, after which we estimate the parameters in the drift and the diffusion coefficient by applying, respectively, the likelihood approach and a numerical approximation. Finally, to illustrate the results obtained, this method is applied to analyse population growth in the Arab Maghreb region.

2 Probabilistic Characteristics of the Model

2.1 The Proposed Model

Let $\{x(t); t \in [t_0, T]; t_0 \geq 0\}$ be the one-dimensional homogenous diffusion process that takes values on $(0, \infty)$ and which satisfies the following non-linear Itô's stochastic differential equation SDE:

$$dx(t) = \left(\alpha x(t) + \beta\sqrt{x(t)}\right) dt + \sigma x(t) dw(t), \quad x(t_0) = x_{t_0}, \tag{1}$$

where $\sigma > 0$, α and β are real parameters, $w(t)$ is a one-dimensional standard Wiener process and $x_{t_0} > 0$ is a fixed real value.

Following the method described in Kloeden and Platen (1992) (p.113) the nonlinear SDE (1) can be reduced to a linear SDE, by the appropriate transformation $y(t) = \sqrt{x(t)}$, then

$$dy(t) = (ay(t) + b)\, dt + cy(t) dw(t), \quad y(t_0) = \sqrt{x_{t_0}}, \tag{2}$$

where $a = \dfrac{\alpha}{2} - \dfrac{\sigma^2}{8}$, $b = \dfrac{\beta}{2}$ and $c = \dfrac{\sigma}{2}$.

2.2 The Analytical Expression of the Model

The SDE (2) has a unique solution $y(t)$ which is known, especially in the field of stochastic finance, as the Brenann-Schwartz diffusion process (see for example Kloeden and Platen (1992) and Gutiérrez et al. (2005)). The analytical expression of this solution is given by:

$$\begin{aligned} y(t) \;=\; & \left(y(t_0) + b\int_{t_0}^{t} \exp\left[-\left(a - \frac{c^2}{2}\right)(\tau - t_0) - c\left(w(\tau) - w(t_0)\right)\right] d\tau\right) \\ & \left(\exp\left[\left(a - \frac{c^2}{2}\right)(t - t_0) + c\left(w(t) - w(t_0)\right)\right]\right), \end{aligned}$$

Then, by the continuous mapping theorem, it can be deduced that the SDE (1) has a unique solution, namely $x(t) = y^2(t)$ which is given analytically by the following expression:

$$\begin{aligned} x(t) \;=\; & \left(\sqrt{x_{t_0}} + b\int_{t_0}^{t} \exp\left[-\left(a - \frac{c^2}{2}\right)(\tau - t_0) - c\left(w(\tau) - w(t_0)\right)\right] d\tau\right)^2 \\ & \left(\exp\left[2\left(a - \frac{c^2}{2}\right)(t - t_0) + 2c\left(w(t) - w(t_0)\right)\right]\right), \end{aligned}$$

By substituting, we then have:

$$\begin{aligned} x(t) \;=\; & \left(\sqrt{x_{t_0}} + \frac{\beta}{2}\int_{t_0}^{t} \exp\left\{-\frac{1}{2}\left[\left(\alpha - \frac{\sigma^2}{2}\right)(\tau - t_0) + \sigma\left(w(\tau) - w(t_0)\right)\right]\right\} d\tau\right)^2 \\ & \left(\exp\left[\left(\alpha - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma\left(w(t) - w(t_0)\right)\right]\right). \end{aligned} \tag{3}$$

2.3 The Trend Functions

Since the closed form of the ptdf of the process is not available, we propose a method for obtaining the conditional and non conditional trend functions of the process from those corresponding to the Brennan-Schwartz process. This method can be summarised as follows:

On the one hand, obtaining the conditional form with respect to $x(s)$, taking the expectations in the SDE (1) and making use of the fact that $y(t) = \sqrt{x(t)}$, we have:

$$\begin{aligned} \frac{d}{dt}\left[\mathbb{E}\left(x(t) \mid x(s) = x_s\right)\right] \;=\; & \alpha\mathbb{E}\left(x(t) \mid x(s) = x_s\right) \\ & + \beta\mathbb{E}\left(y(t) \mid y(s) = \sqrt{x_s}\right), \end{aligned}$$

Furthermore, using the explicit form of the conditional trend function of the Brennan-Schwartz diffusion process (see Gutiérrez et al. (2005)), that is

$$\mathbb{E}(y(t) \mid y(s) = y_s) = \left(y_s + \frac{b}{a}\right) e^{a(t-s)} - \frac{b}{a},$$

It can be deduced that the conditional trend function of the proposed process $\varphi(t) = \mathbb{E}(x(t) \mid x(s) = x_s)$ solves the following non homogeneous ordinary differential equation (ODE):

$$\varphi'(t) = \alpha\varphi(t) + \beta\left(y_s + \frac{b}{a}\right)e^{a(t-s)} - \frac{b\beta}{a}\ ,\ \varphi(s) = x_s,$$

The unique solution of the latter ODE has the following form:

$$\begin{aligned}\varphi(t) \;&=\; x_s e^{\alpha(t-s)} + \frac{\beta}{a-\alpha}\left(\sqrt{x_s} + \frac{b}{a}\right)\left(e^{a(t-s)} - e^{\alpha(t-s)}\right)\\ &+\frac{b\beta}{a\alpha}\left(1 - e^{\alpha(t-s)}\right),\end{aligned}$$

Finally, the conditional trend function of the process is seen to be

$$\begin{aligned}\mathbb{E}(x(t) \mid x(s) = x_s) \;&=\; x_s e^{\alpha(t-s)} + \frac{\beta^2}{\alpha(\alpha - \sigma^2/4)}\left(1 - e^{\alpha(t-s)}\right)\\ &+\frac{2\beta}{\alpha + \sigma^2/4}\left(\sqrt{x_s} + \frac{\beta}{\alpha - \sigma^2/4}\right)\left(e^{\alpha(t-s)} - e^{\left(\frac{\alpha}{2} - \frac{\sigma^2}{8}\right)(t-s)}\right). \qquad (4)\end{aligned}$$

By assuming the initial conditional $\mathrm{P}\left(x(t_0) = x_{t_0}\right) = 1,$ the trend function of the process is:

$$\begin{aligned}\mathbb{E}(x(t)) \;&=\; x_{t_0} e^{\alpha(t-t_0)} + \frac{\beta^2}{\alpha(\alpha - \sigma^2/4)}\left(1 - e^{\alpha(t-t_0)}\right)\\ &+\frac{2\beta}{\alpha + \sigma^2/4}\left(\sqrt{x_0} + \frac{\beta}{\alpha - \sigma^2/4}\right)\left(e^{\alpha(t-t_0)} - e^{\left(\frac{\alpha}{2} - \frac{\sigma^2}{8}\right)(t-t_0)}\right). \qquad (5)\end{aligned}$$

2.4 Ergodicity and Stationary Distribution

We show (see the Appendix A) that for $\alpha < \dfrac{\sigma^2}{2}$ and $\beta > 0,$ the process is ergodic and its stationary density function is given by the following expression:

$$f(x) = \frac{\mu^\lambda}{2\Gamma(\lambda)} x^{-\left(\frac{\lambda}{2}+1\right)} e^{-\frac{\mu}{\sqrt{x}}}. \qquad (6)$$

where $\lambda = 2 - \dfrac{4\alpha}{\sigma^2}$, $\mu = \dfrac{4\beta}{\sigma^2}$ and $\Gamma(.)$ is the Gamma function. The ergodicity conditions of the process, in terms of λ and μ are equivalent to $\lambda > 0$ and $\mu > 0$.

Let X be a random variable with the stationary density function f given by (6). This expression can be used to calculate the asymptotic moment of order k, $(k \in \mathbb{N}^*)$, and thus we have for $\lambda > 2k$, (ie: $\alpha < (1-k)\frac{\sigma^2}{2}$)

$$\mathbb{E}[X^k] = \int_0^\infty x^k f(x)dx = \frac{\mu^{2k}\Gamma(\lambda - 2k)}{\Gamma(\lambda)}.$$

From the properties of the Euler function, the asymptotic trend function of the process is $(k = 1)$, for $\lambda > 2$, (ie: $\alpha < 0$)

$$\mathbb{E}[X] = \frac{\mu^2}{(\lambda-1)(\lambda-2)} = \frac{\beta^2}{\alpha(\alpha - \frac{\sigma^2}{4})}.$$

By taking the limit when t tends to ∞ in (5), we have for $\alpha < 0$

$$\lim_{t\to\infty} \mathbb{E}(x(t)) = \mathbb{E}(X).$$

This means that the limit of the trend function in (5) (when t tends to ∞) coincides with the asymptotic trend function.

3 Statistical Inference in the Model

Let us now determine the estimators of the parameters of the proposed model. The estimators of the drift parameters (α and β) are obtained by the maximum likelihood method, with continuous sampling.

3.1 Likelihood Estimation of Drift Parameters

Consider the one dimensional diffusion process defined by the following vectorial form:

$$dx(t) = A_t(x(t))\theta + B_t(x(t))dw(t), \quad t_0 \leq t \leq T,$$

where $\theta \in \mathbb{R}^k$, A_t is a k dimensional vector and B_t is $\mathbb{R}$-valued depending only on the sample path up to given instant. Assume that the previous equation has a unique solution for every θ. The maximum likelihood estimator of the vector θ is: (see, Florenz-Zmrou (1989); Yoshida (1992); Kloeden et al. (1996); Skiadas and Giovanis (1997) and Giovanis and Skiadas (1999)).

$$\hat{\theta} = S_T^{-1} H_T.$$

where H_T is the following k-dimensional vector

$$H_T = \int_{t_0}^{T} A_t^*(x(t))(B_t(x(t))B_t(x(t)))^{-1}dx(t),$$

and S_T is the $k \times k$ matrix

$$S_T = \int_{t_0}^{T} A_t^*(x(t))(B_t(x(t))B_t(x(t)))^{-1}A_t(x(t))dt,$$

and $*$ denote the transposition.

The vector form of the SDE of the proposed model can be written with:

$$A_t(x(t)) = (x(t), \sqrt{x(t)}), \quad \theta^* = (\alpha, \beta), \quad B_t = \sigma x(t),$$

The corresponding vector H_T in this case is 2-dimensional and is given by:

$$H_T^* = \frac{1}{\sigma^2}\left(\int_{t_0}^{T} \frac{dx(t)}{x(t)}, \quad \int_{t_0}^{T} \frac{dx(t)}{x(t)\sqrt{x(t)}}\right),$$

and S_T is the following square matrix:

$$S_T = \frac{1}{\sigma^2}\begin{pmatrix} T - t_0 & \int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}} \\ \int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}} & \int_{t_0}^{T} \frac{dt}{x(t)} \end{pmatrix},$$

After some calculation (not shown), we obtain the expressions of the estimators

$$\hat{\alpha} = \frac{\int_{t_0}^{T} \frac{dt}{x(t)} \int_{t_0}^{T} \frac{dx(t)}{x(t)} - \int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}} \int_{t_0}^{T} \frac{dx(t)}{x(t)\sqrt{x(t)}}}{(T - t_0) \int_{t_0}^{T} \frac{dt}{x(t)} - \left(\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}\right)^2},$$

$$\hat{\beta} = \frac{(T - t_0) \int_{t_0}^{(T)} \frac{dx(t)}{x(t)\sqrt{x(t)}} - \int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}} \int_{t_0}^{T} \frac{dx(t)}{x(t)}}{(T - t_0) \int_{t_0}^{T} \frac{dt}{x(t)} - \left(\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}\right)^2},$$

The stochastic integrals in the latter expressions can be transformed into Riemann integrals by using Itô's formula and thus:

$$\int_{t_0}^{T} \frac{dx(t)}{x(t)} = \log(x_T) - \log(x_{t_0}) + \frac{\sigma^2}{2}(T - t_0),$$

$$\int_{t_0}^{T} \frac{dx(t)}{x(t)\sqrt{x(t)}} = 2\left(\frac{1}{\sqrt{x_{t_0}}} - \frac{1}{\sqrt{x_T}}\right) + \frac{3\sigma^2}{4}\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}.$$

Therefore, the expressions of the Maximum Likelihood estimators are:

$$\hat{\alpha} = \frac{\int_{t_0}^{T} \frac{dt}{x(t)} \left(\log(x_T/x_{t_0}) + \frac{\sigma^2}{2}(T - t_0)\right)}{(T - t_0)\int_{t_0}^{T} \frac{dt}{x(t)} - \left(\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}\right)^2} - \frac{2\left(\frac{1}{\sqrt{x_{t_0}}} - \frac{1}{\sqrt{x_T}} + \frac{3\sigma^2}{8}\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}\right)\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}}{(T - t_0)\int_{t_0}^{T} \frac{dt}{x(t)} - \left(\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}\right)^2}, \tag{7}$$

$$\hat{\beta} = \frac{2(T - t_0)\left(\frac{1}{\sqrt{x_{t_0}}} - \frac{1}{\sqrt{x_T}} + \frac{3\sigma^2}{8}\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}\right)}{(T - t_0)\int_{t_0}^{T} \frac{dt}{x(t)} - \left(\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}\right)^2} - \frac{\left(\log(x_T/x_{t_0}) + \frac{\sigma^2}{2}(T - t_0)\right)\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}}{(T - t_0)\int_{t_0}^{T} \frac{dt}{x(t)} - \left(\int_{t_0}^{T} \frac{dt}{\sqrt{x(t)}}\right)^2}. \tag{8}$$

3.2 Approximate Estimator of the Diffusion Coefficient

The estimators of the coefficient diffusion parameter can be approximated using a method similar to that discussed in Chesney and Elliott (1995), Skiadas and Giovanis (1997), Giovanis and Skiadas (1999), and Gutiérrez et al. (2008) (see Appendix B). This method can be summarised in the following steps:

By applying the Itô formula, we have:

$$d\left(\frac{1}{x(t)}\right) = -\frac{dx(t)}{x^2(t)} + \frac{\sigma^2}{x(t)}dt, \tag{9}$$

The differentials shown in the latter equation can be approximated by consecutive observations of a sample path of the process in $t-1$ and t, as follows:

$$d\left(\frac{1}{x(t)}\right) \simeq \frac{1}{x(t)} - \frac{1}{x(t-1)} \quad \text{and} \quad d(x(t)) \simeq x(t) - x(t-1),$$

By inserting these approximations in (9), an approximate estimator of the σ parameter between the latter observations is found to be

$$\hat{\sigma}_{(t-1,t)} = \mid x(t) - x(t-1) \mid / \sqrt{x(t)x(t-1)},$$

For n observations of a sample path of the process, the resulting approximate estimator has the following expression:

$$\hat{\sigma} = \frac{1}{n-1}\sum_{t=1}^{n} \frac{\mid x(t) - x(t-1) \mid}{\sqrt{x(t)x(t-1)}}. \tag{10}$$

3.3 Asymptotic Normality of Likelihood Estimators

As shown above, for $\lambda > 0$ (i.e., $\alpha < \dfrac{\sigma^2}{2}$) and $\beta > 0$, the conditions of ergodicity are confirmed (see for example Kutoyants (2004) and Gutiérrez et al. (2009)) and the process is shown to have ergodic properties. Then, we have, for a known σ and for $\theta = (\alpha, \beta) \in (\alpha_1, \alpha_2) \times (\beta_1, \beta_2)$, with $\alpha_2 < \dfrac{\sigma^2}{2}$ and $\beta_1 > 0$,

$$\mathcal{L}_\theta\left(\sqrt{T}(\hat{\theta} - \theta)\right) \to \mathcal{N}_2\left(0, \mathbb{I}^{-1}(\theta)\right) \quad , \quad \text{when} \quad T \to \infty, \tag{11}$$

where

$\mathbb{I}(\theta) = \mathbb{E}_\theta\left(\dfrac{\dot{a}(X)\dot{a}^*(X)}{b^2(X)}\right)$ and $\dot{a}(x) = \left(\dfrac{\partial a(x,\theta)}{\partial \alpha}, \dfrac{\partial a(x,\theta)}{\partial \beta}\right)^*$,

Then, by calculation, we obtain:

$$\mathbb{I}(\theta) = \frac{1}{\sigma^2}\mathbb{E}_\theta\begin{pmatrix} 1 & \frac{1}{\sqrt{X}} \\ \frac{1}{\sqrt{X}} & \frac{1}{X} \end{pmatrix},$$

It can then be shown straightforwardly that the random variable $\dfrac{1}{\sqrt{X}}$ has a Gamma distribution $\Gamma\left(\lambda, \dfrac{1}{\mu}\right)$ with parameters λ and $\dfrac{1}{\mu}$. Then, we have

$$\mathbb{E}\left(\frac{1}{\sqrt{X}}\right) = \frac{\lambda}{\mu},$$

Moreover, by simpler integration, we can show that

$$\mathbb{E}\left(\frac{1}{X}\right) = \frac{\lambda(\lambda+1)}{\mu^2},$$

From which the information matrix $\mathbb{I}(\theta)$ provides

$$\mathbb{I}(\theta) = \frac{1}{\sigma^2}\begin{pmatrix} 1 & \frac{\lambda}{\mu} \\ \frac{\lambda}{\mu} & \frac{\lambda(\lambda+1)}{\mu^2} \end{pmatrix},$$

and the inverse is:

$$\mathbb{I}^{-1}(\theta) = \sigma^2\begin{pmatrix} \lambda+1 & -\mu \\ -\mu & \frac{\mu^2}{\lambda} \end{pmatrix}, \tag{12}$$

An approximate and asymptotic confidence region of θ and an approximate and asymptotic marginal confidence intervals of $\hat{\alpha}$ and $\hat{\beta}$ can be obtained by substitution of (11) and (12). The above-mentioned region is given, for a large T, by:

$$\mathrm{P}\left[T\left(\theta - \hat{\theta}\right)^* \hat{\mathbb{I}}(\theta)\left(\theta - \hat{\theta}\right) \le \chi^2_{2,\gamma}\right] = 1 - \gamma. \tag{13}$$

where $\widehat{\mathbb{I}}(\theta)$ is obtained by replacing the parameters by their estimators in the expression (12) and $\chi^2_{2,\gamma}$ is the is the upper 100γ per cent points of the chi squared distribution with two degrees of freedom.

The $\gamma\%$ confidence (marginal) intervals for the parameters α and β are given, for a large T, by

$$\mathrm{P}\left(\alpha \in \left[\hat{\alpha} \pm \xi_\gamma \sigma \sqrt{\left(\hat{\lambda}+1\right)/T}\right]\right) = 1-\gamma, \tag{14}$$

$$\mathrm{P}\left(\beta \in \left[\hat{\beta} \pm \xi_\gamma \sigma \hat{\mu} \sqrt{1/\hat{\lambda}T}\right]\right) = 1-\gamma. \tag{15}$$

where ξ_γ is the 100γ per cent points of the normal standard distribution.

In expressions (13), (14) and (15) it is assumed that σ is known with value $\sigma = \hat{\sigma}$.

4 Computational Aspects

4.1 Approximate Likelihood Estimators

In order to use the above expressions, (7) and (8), to estimate the parameters, we must have continuous observations. In practice, continuous sample paths are not able to observed. Rather, the state of the diffusion process is observed at a finite number of time instances $(0 = t_0 < t_1 < \cdots < t_n = T)$, then an alternative estimation procedure that is frequently utilised (see for example Giovanis and Skiadas (1999) and Gutiérrez et al. (2008)) for such data is to use the continuous time maximum likelihood estimators with suitable approximations of the integrals that appear in the expressions (7) and (8); specifically, the Riemann-Stieljes integrals are approximated by means of the trapezoidal formula.

An approximation of the standard error of the estimator of $\hat{\sigma}$ is given by:

$$es(\hat{\sigma}) = \frac{1}{n-1}\sum_{t=1}^{n}\left(\hat{\sigma}_{(t-1,t)} - \hat{\sigma}\right)^2.$$

4.2 Estimated Trend Functions

By using Zehna's theorem see Zehna and et al (1966), the estimated trend function (ETF) and estimated conditional trend function (ECTF) of the process are obtained by replacing the parameters in (4) and (5) by their estimators given in (7), (8) and (10). Then the ECTF and ETF are given by the following expression:

$$\widehat{\mathbb{E}}(x(t)/x(s)=x_s) = x_s e^{\hat{\alpha}(t-s)} + \frac{\hat{\beta}^2}{\hat{\alpha}(\hat{\alpha}-\hat{\sigma}^2/4)}\left(1-e^{\hat{\alpha}(t-s)}\right)$$
$$+\frac{2\hat{\beta}}{\hat{\alpha}+\hat{\sigma}^2/4}\left(\sqrt{x_s}+\frac{\hat{\beta}}{\hat{\alpha}-\hat{\sigma}^2/4}\right)\left(e^{\hat{\alpha}(t-s)} - e^{\left(\frac{\hat{\alpha}}{2}-\frac{\hat{\sigma}^2}{8}\right)(t-s)}\right), \tag{16}$$

$$\widehat{\mathbb{E}}(x(t)) = x_{t_0} e^{\hat{\alpha}(t-t_0)} + \frac{\hat{\beta}^2}{\hat{\alpha}(\hat{\alpha}-\hat{\sigma}^2/4)}\left(1-e^{\hat{\alpha}(t-t_0)}\right)$$
$$+\frac{2\hat{\beta}}{\hat{\alpha}+\hat{\sigma}^2/4}\left(\sqrt{x_{t_0}}+\frac{\hat{\beta}}{\hat{\alpha}-\hat{\sigma}^2/4}\right)\left(e^{\hat{\alpha}(t-t_0)} - e^{\left(\frac{\hat{\alpha}}{2}-\frac{\hat{\sigma}^2}{8}\right)(t-t_0)}\right). \tag{17}$$

4.3 Approximate Asymptotic Confidence Interval of the Trend Functions

Asymptotic and approximate confidence intervals of the ETF of the model can be obtained by replacing in (3) and (4) the parameters α and β by the extreme values of those confidence intervals: the lower limit of α and β (α_{ll} and β_{ll} respectively) and the upper limit of α and β (α_{ul} and β_{ul} respectively) which are given in expression (14) and (15). Then, the lower limit of the ETF (ETF_{ll}) is given by:

$$
\begin{aligned}
\widehat{\mathbb{E}}_{ll}\left[x(t)\right] = {} & e^{\hat{\alpha}_{ll}(t-t_0)} + \frac{\hat{\beta}_{ll}^2}{\hat{\alpha}_{ll}(\hat{\alpha}_{ll}-\hat{\sigma}^2/4)}\left(1-e^{\hat{\alpha}_{ll}(t-t_0)}\right) \\
& + \frac{2\hat{\beta}_{ll}}{\hat{\alpha}_{ll}+\hat{\sigma}^2/4}\left(\sqrt{x_{t_0}}+\frac{\hat{\beta}_{ll}}{\hat{\alpha}_{ll}-\hat{\sigma}^2/4}\right)\left(e^{\hat{\alpha}_{ll}(t-t_0)}-e^{\left(\frac{\hat{\alpha}_{ll}}{2}-\frac{\hat{\sigma}^2}{8}\right)(t-t_0)}\right), \qquad (18)
\end{aligned}
$$

and the upper limit of the ETF (ETF_{ul}) is:

$$
\begin{aligned}
\widehat{\mathbb{E}}_{ul}\left[x(t)\right] = {} & e^{\hat{\alpha}_{ul}(t-t_0)} + \frac{\hat{\beta}_{ul}^2}{\hat{\alpha}_{ul}(\hat{\alpha}_{ul}-\hat{\sigma}^2/4)}\left(1-e^{\hat{\alpha}_{ul}(t-t_0)}\right) \\
& + \frac{2\hat{\beta}_{ul}}{\hat{\alpha}_{ul}+\hat{\sigma}^2/4}\left(\sqrt{x_{t_0}}+\frac{\hat{\beta}_{ul}}{\hat{\alpha}_{ul}-\hat{\sigma}^2/4}\right)\left(e^{\hat{\alpha}_{ul}(t-t_0)}-e^{\left(\frac{\hat{\alpha}_{ul}}{2}-\frac{\hat{\sigma}^2}{8}\right)(t-t_0)}\right). \qquad (19)
\end{aligned}
$$

These functions are utilised in the last section to fit and predict the future evolution of the stochastic diffusion process under consideration.

5 Application to Real Data

5.1 Dynamic of Population Aging

Fertility rates have fallen to very low levels in most world regions, and at the same time, people tend to live longer. In consequence, the world population is rapidly aging. The world's population aged 65 years and older has increased from 562 million to 627 million in recent years. Most studies of the dynamics of population aging, therefore, analyse declining rates of fertility by regions and countries, and observe indicators of population aging, such as dependency ratios and median age.

Populations where fertility rates remain high have a population distribution with larger numbers of young people and a lower proportion of the elderly. In contrast, where fertility rates are low, the society is older. The population of Africa is the youngest in the world and will remain so in the coming decades. The Arab Maghreb is composed of five countries to the south of the Mediterranean - Morocco, Algeria, Tunisia, Libya and Mauritania - and their political union is designed to boost cooperation and integration between these countries. This region has abundant natural and energetic resources that are currently being employed to promote the industrialisation of these countries. The population is young, with high rates of fertility, and therefore the region does not experience the type of problems encountered in countries with older populations. Although in recent years fertility rates have decreased in the region, they are expected to remain relatively high for some time. In consequence, the Arab Maghreb has a young population that has grown considerably and will continue to do so.

Because of low fertility rates and emigration elsewhere, however, the number of children and young adults in the world has already begun to fall. In contrast, the number of people aged 65 years and over will soon have grown fivefold, making aging an important social, economic and demographic issue. Studying the growth of the older population in the Arab Maghreb is a major challenge, aimed at answering questions such as the following, which are of vital importance to economic development. How many years can older people expect to live in good health?

How long can they live independently? How many are still working? Will they have sufficient economic resources to last their lifetimes?

5.2 Data and Results

In our study, the SBSDP model incorporating the above-described statistical methodology was applied to the total population aged 65 years and over living in the Arab Maghreb, in accordance with the de facto definition of population, which counts all residents regardless of legal status or citizenship. The de facto population, thus, is a concept under which individuals are attributed to a given geographical area at a specified time. In the case in question, the time period taken was from 1960 to 2016. The data, available by year and country were accessed at https://data.worldbank.org/.

Note that these values correspond to observations of the stochastic process in a time discretization at equal-amplitude intervals of one year. The values observed for the period 1960 to 2011 were used to estimate the drift parameters given in the equations presented in Section 3.1, together with the approximation of the estimator of the diffusion coefficient, obtained by the Matlab package. Tables 1 and 2 show the values observed and those adjusted by the conditioned trends of the stochastic SBSDP. The estimators calculated and the upper and lower limits of the 95% confidence intervals for the parameters of the drift coefficient of the process are shown in Table 3. Note that these values correspond to observations.

The data from 2012 to 2016, which were not used for the statistical fit, were used to make forecasts of the future values of the process, with the trend and conditional trend functions and the confidence interval of 95% are shown in Table 4.

The original data and the corresponding data fitted by the ETF and the ECTF together with the corresponding confidence intervals for the respective ETF are shown in Figs. 1 and 2.

5.3 Goodness of Fit of the Model

In evaluating a forecast, the measure of goodness of fit describes the deviation between the observed values and those expected under the model. Among many measures of forecasting accuracy that have been proposed, the most common are the mean absolute percentage error (MAPE), the symmetric mean absolute percentage (SMAPE) and the relative root mean square error (RRMSE). The actual value is denoted by y_i, the forecast value by $\hat{y}_i$ and the total number of predictions by n. These three measures of forecasting accuracy are defined as follows:

- **MAPE** is the most commonly used measure of forecasting accuracy. It is expressed as a percentage and provides reliability, ease of interpretation and independence of the units. It is defined by the formula:

$$MAPE = \frac{1}{n}\sum_{t=1}^{n}\frac{|\hat{y}_i - y_i|}{y_i} \times 100.$$

  According to Lewis see Lewis (1982), the following are typical MAPE values and their interpretation (see Table 5.)

- **SMAPE** is based on percentage errors. It illustrates the fact that the geometric-mean combination of different forecasts provides a better forecast. This measure is defined as follows:

$$SMAPE = \frac{100}{n}\sum_{t=1}^{n}\frac{|\hat{y}_i - y_i|}{(|\hat{y}_i| + |y_i|)/2}$$

Table 1 Fit from 1960 to 1985

Years Observed values	Data	ETF	ECTF
1960	954708	954708	954708
1961	995722	991081	991081
1962	1036593	1028602	1033390
1963	1076290	1067304	1075548
1964	1113783	1107217	1116490
1965	1148428	1148377	1155154
1966	1188187	1190816	1190878
1967	1225668	1234572	1231872
1968	1261765	1279680	1270513
1969	1297156	1326177	1307724
1970	1332256	1374101	1344205
1971	1377365	1423492	1380384
1972	1421442	1474390	1426875
1973	1463249	1526836	1472298
1974	1501302	1580872	1515379
1975	1535344	1636543	1554589
1976	1571462	1693892	1589664
1976	1571462	1693892	1589664
1977	1604262	1752966	1626876
1978	1636079	1813811	1660667
1979	1669599	1876476	1693445
1980	1706579	1941010	1727975
1981	1759496	2007464	1766067
1982	1816762	2075891	1820572
1983	1878054	2146342	1879553
1984	1943340	2218874	1942676
1985	2013345	2293543	2009907

- **RRMSE** is used to estimate the accuracy of data involving large magnitudes, for example distances of 10000 m and over. It is defined as follows:

$$RRMSE = \sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(\frac{|\hat{y}_i - y_i|}{|y_i|}\right)^2}.$$

Our calculation of these three measures of error (as shown in Table 6.) shows that the SBSDP is accurate and efficient.

5.4 The Comparison of the SBSDP with an Exponential Growth Model

In order to evaluate the results obtained using the SBSDP in studying our data series, we compared it with an exponential growth model, namely the Stochastic Lognormal Diffusion Process (SLDP) (see Appendix C).

Table 2 Fit from 1986 to 2011

Years Observed values	Data	ETF	ECTF
1986	2089488	2370406	2081992
1987	2169740	2449523	2160391
1988	2256017	2530955	2243015
1989	2350239	2614763	2331835
1990	2453335	2701013	2428826
1991	2569496	2789769	2534943
1992	2689849	2881099	2654497
1993	2814228	2975073	2778355
1994	2943046	3071761	2906346
1995	3076966	3171236	3038894
1996	3219713	3273573	3176680
1997	3366092	3378848	3323538
1998	3514659	3487140	3474120
1999	3662726	3598529	3626942
2000	3809334	3713099	3779239
2001	3947773	3830935	3930026
2002	4081214	3952123	4072402
2003	4212039	4076753	4209632
2004	4342602	4204916	4344164
2005	4474910	4336707	4478421
2006	4603476	4472222	4614467
2007	4734289	4611560	4746659
2008	4866930	4754823	4881156
2009	4998913	4902115	5017527
2010	5130705	5053542	5153217
2011	5241221	5209215	5288705

Table 3 Estimation of the parameters and the limits of the 95% confidence intervals

Parameters estimation	Lower limit	Upper limit
$\hat{\alpha} = 0.024957390573$	0.018869360346	0.031045420800
$\hat{\beta} = 12.26390299011$	12.26390065421	12.26390532602
$\hat{\sigma^2} = 0.033392073701$	0.033333895322	0.033450252080

Table 4 Predictions from trend function and conditional trend function of the process

Years	Data	ETF_{ll}	ETF	ETF_{ul}	ECTF
2012	5349608	4025252	5369246	7178805	5402317
2013	5469842	4126771	5533750	7438632	5513737
2014	5623220	4230532	5702845	7707248	5637333
2015	5823530	4336580	5876654	7984939	5794994
2016	6053008	4444962	6055300	8272000	6000889

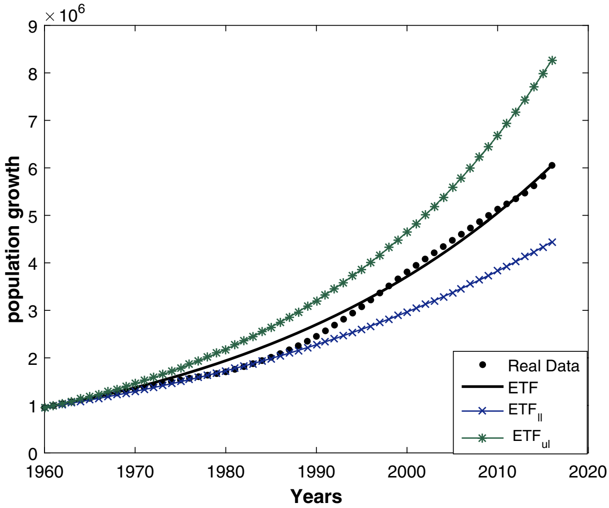

Fig. 1 It illustrates the data observed versus those fitted by ETF, the ETF_{ll} and the ETF_{ul}

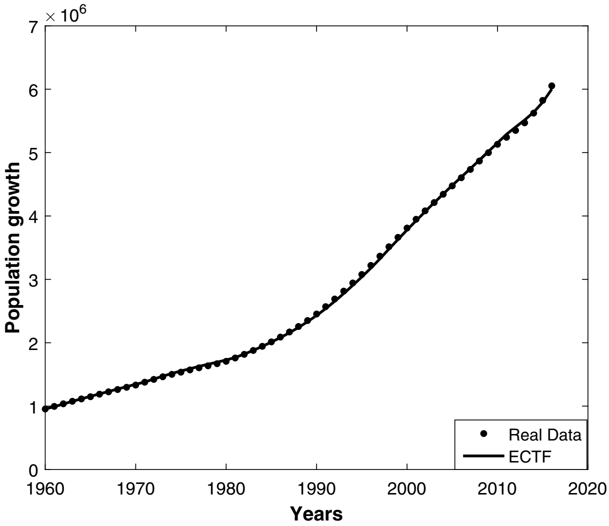

Fig. 2 It shows the real data versus $ECTF$: the conditional estimated trend function

Table 5 Interpretation of typical MAPE values

MAPE	Interpretation
< 10	Highly accurate forecasting
$20 - 30$	Good forecasting
$30 - 50$	Reasonable forecasting
> 50	Inaccurate forecasting

Table 6 Goodness of fit of the model

Measures of forecasting accuracy error	Values
MAPE	0.4178463999
SMAPE	0.4244362742
RRMSE	0.5912382771

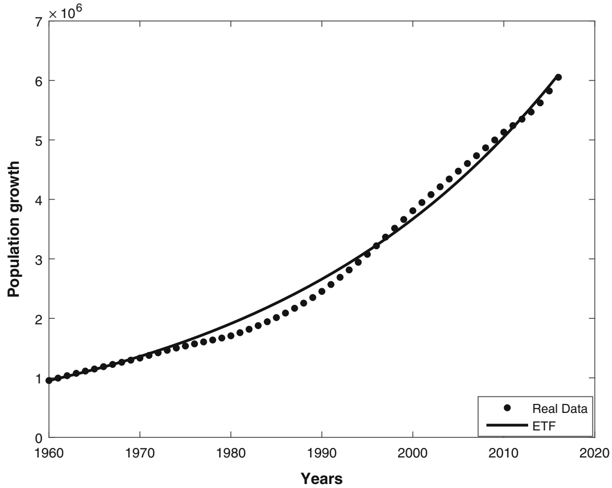

Fig. 3 It illustrates the data observed versus those fitted by SBSDP

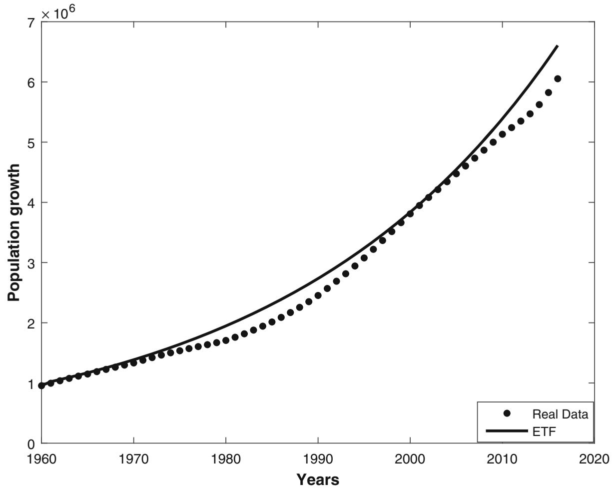

Fig. 4 It illustrates the real data versus those fitted by the SLDP

Table 7 Predictions from trend functions of the SBSDP and SLDP

Years	ETF (SBSDP)	ETF (SLDP)
2012	5369246	5767940
2013	5533750	5967047
2014	5702845	6173026
2015	5876654	6386116
2016	6055300	6606562

Table 8 Goodness of fit of the two models

Measures of forecasting accuracy error	Values of SBSDP	Values of SLDP
MAPE	0.4178463999	8.3366037
SMAPE	0.4244362742	8.62727416

As stated in the Introduction, the SBSDP is an extension of the SLDP, as well-known stochastic growth model that has been widely used to model exponential growth phenomena in biology, economics and other fields, see, for example (Tintner and Sengupta 1972 and Gutiérrez et al. 2009).

The results obtained using the SBSDP in the data series were compared with those obtained by the SLDP, as shown in Figs. 3 and 4. Table 7 shows that the forecasts obtained by the SBSDP for 2012 to 2016 are better than those obtained by the SLDP. Finally, in evaluating a forecast of the processes, Table 8 shows a comparison between the results obtained by some measures of errors (MAPE and SMAPE) of the SBSDP and the SLDP.

6 Conclusion

- In this paper, we evaluate the capability of the SBSDP for modelling real data in the field of population dynamics. This model produced a good fit to the real data for the total population aged 65 years and over, resident in the Arab Maghreb during the period 1960-2011.
- The ETF and the ECTF presented a reasonable description of the changing levels of this population. Furthermore, the forecasts and the real data for the period 2011 to 2016 were situated within the confidence interval of the ETF. However, the description and forecast obtained using the conditioned trend were better than those based on the trend alone. Moreover, the fit for the period 1975-1994 using ETF could be improved by the inclusion of exogenous factors in the model see for example Nafidi et al. (2016). Moreover, to evaluate the forecasting accuracy obtained, three different evaluation statistics were calculated. All showed the model to be highly accurate and reliable.
- In the same way, by fitting the SLDP to our data series, a reasonable description of the changing levels of this population is obtained. However, that of the SBSDP is considerably better than that based on the SLDP. In additon, for the period 2012 to 2016, the forecasts obtained using the SBSDP are more suitable than those obtained with the SLDP. Finally, the resulting values obtained by two differents measures of errors (MAPE and SMAPE) show that the SBSDP is more reliable than the SLDP.
- Taking into account these points, we conclude that the results obtained by the SBSDP are better than those obtained by the SLDP.

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Appendix A: Ergodicity and stationary distribution of the SBSDP

Here, we study the asymptotic behaviour of the process proposed in this paper, analyse the problem of ergodicity and the existence of the stationary distribution of the process and explicitly obtain its density function.

In general (see Nobile and Ricciardi (1984) and Nicolau (2005)), a diffusion process $\{x(t), t \geq 0\}$, with state space $I = (l, r)$, is governed by the following SDE:

$$dx(t) = a(x(t))dt + b(x(t))dW_t, \quad x_0 = x,$$

where W_t is a standard Wiener process and x is either a constant value or a random value independent of W_t. We assume that $a(x)$ and $b(x)$ have continuous derivatives.

Let $s(z) = \exp\left\{-\int_{z_0}^{z} \frac{2a(u)}{b^2(u)}du\right\}$ be the scale density function (z_0 is an arbitrary point inside I).

The speed density function is: $m(u) = \left(b^2(u)s(u)\right)^{-1}$. And we denote by:

$$S[x, y] = \int_x^y s(u)du, \quad S(l, y] = \lim_{x\to l} \int_x^y s(u)du \quad \text{and}$$

$$S[x, r) = \lim_{y\to r} \int_x^y s(u)du,$$

where, $l < x < y < r$, and then if:

$$S(l, x] = S[x, r) = \infty \quad \text{and} \quad \int_l^r m(u)du < \infty,$$

the process $\{x(t), t \geq 0\}$ is ergodic and has an invariant (stationary) density function which is given by:

$$f(x) = m(x)/\int_l^r m(u)du.$$

In our diffusion, the drift and diffusion coefficient are, respectively:

$$a(x) = \alpha x + \beta\sqrt{x} \quad \text{and} \quad b^2(x) = \sigma^2 x^2,$$

and $I = (0, \infty)$. In this case, it follows that:

$$s(z) = kz^{-\frac{2\alpha}{\sigma^2}} e^{\frac{4\beta}{\sigma^2\sqrt{z}}}, \quad \text{with} \quad k = z_0^{\frac{2\alpha}{\sigma^2}} e^{-\frac{4\beta}{\sigma^2\sqrt{z_0}}}.$$

and we have, for $0 < x < y < \infty$

$$S[x, y] = \int_x^y s(u)du = k\int_x^y u^{-\frac{2\alpha}{\sigma^2}} e^{\frac{4\beta}{\sigma^2\sqrt{u}}} du,$$

with the variable change $v = u^{-1/2}$, the latter expression can be written as

$$S[x, y] = 2k\int_{1/\sqrt{y}}^{1/\sqrt{x}} v^{\frac{4\alpha}{\sigma^2}-3} e^{\frac{4\beta}{\sigma^2}v} dv. \tag{20}$$

On the one hand, taking the limit as x tends to 0 in (20). We have, for $\beta > 0$, $S(0, y] = \infty$ (the case of $\beta = 0$ is excluded, because the process is lognormal, and this is not ergodic). On the other hand, taking the limit when y tends to ∞ in (20), we have, for $\alpha \leq \frac{\sigma^2}{2}$, $S[x, \infty) = \infty$. And therefore, for $\alpha \leq \frac{\sigma^2}{2}$ and $\beta > 0$,

$$S[x, \infty) = S(0, y] = \infty.$$

The speed density in this case is

$$m(x) = \frac{1}{k\sigma^2} x^{\frac{2\alpha}{\sigma^2}-2} e^{\frac{-4\beta}{\sigma^2\sqrt{x}}},$$

and we have:

$$\begin{aligned}\int_0^\infty m(x)dx &= \frac{1}{k\sigma^2}\int_0^\infty x^{\frac{2\alpha}{\sigma^2}-2} e^{\frac{-4\beta}{\sigma^2\sqrt{x}}} dx \\ &= \frac{2}{k\sigma^2}\int_0^\infty v^{\frac{-4\alpha}{\sigma^2}+1} e^{-\frac{4\beta}{\sigma^2}v} dv,\end{aligned}$$

and according to Gradshteyn and Ryzhik (1979) 3.18. p.317, for $\nu > 0$ and $\mu > 0$,

$$\int_0^\infty x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu}\Gamma(\nu),$$

we have, for $\alpha < \frac{\sigma^2}{2}$ and $\beta > 0$,

$$\int_0^\infty m(x)dx = \frac{2}{k\sigma^2}\left(\frac{4\beta}{\sigma^2}\right)^{\left(\frac{4\alpha}{\sigma^2}-2\right)} \Gamma\left(2 - \frac{4\alpha}{\sigma^2}\right) < \infty,$$

Then, by combining the two conditions, we deduce that for $\alpha < \frac{\sigma^2}{2}$ and $\beta > 0$, the process is ergodic and its stationary density function is given by the following expression:

$$f(x) = m(x)/\int_0^\infty m(u)du = \frac{\mu^\lambda}{2\Gamma(\lambda)} x^{-\left(\frac{\lambda}{2}+1\right)} e^{-\frac{\mu}{\sqrt{x}}}$$

where $\lambda = 2 - \frac{4\alpha}{\sigma^2}$ and $\mu = \frac{4\beta}{\sigma^2}$, and the conditions of ergodicity in terms of λ and μ are equivalent to $\lambda > 0$ and $\mu > 0$.

Appendix B: Approximate estimator of the diffusion coefficient of the SBSDP

To estimate the parameter σ, we used an extension of the method described by Chesney and Elliott (1995). This method has been used by Giovanis and Skiadas (1999) in a paper on a stochastic logistic innovation diffusion model studying electricity consumption in Greece and the United States, and in another on a stochastic Bass innovation diffusion model to study the growth of electricity consumption in Greece, see Skiadas and Giovanis (1997).

In our study, an approximate estimator of the σ parameter between two observations has the general form:

$$\hat{\sigma}_{(t-1,t)} = \frac{|\, x(t) - x(t-1)\,|}{\sqrt{x(t)x(t-1)}},$$

Then, we have

$$\hat{\sigma}_{(0,1)} = \frac{|\, x(1) - x(0)\,|}{\sqrt{x(1)x(0)}},$$

$$\hat{\sigma}_{(1,2)} = \frac{\mid x(2) - x(1) \mid}{\sqrt{x(2)x(1)}},$$

$$\vdots$$

$$\hat{\sigma}_{(n-1,n)} = \frac{\mid x(n) - x(n-1) \mid}{\sqrt{x(n)x(n-1)}}.$$

The method described by Chesney and Elliot was then applied to each time interval, and the average of these estimators for n observations of a sample path of the process is the approximate estimator. It takes the following form:

$$\hat{\sigma} = \frac{1}{n-1} \sum_{t=1}^{n} \frac{\mid x(t) - x(t-1) \mid}{x(t)\sqrt{x(t-1)}}.$$

Other approximate estimators of σ can be obtained by the same procedure, for example:

– **Using Itô's lemma to the transformation** $y = ln(x(t))$ **in (1) as follows:**

$$d(lnx(t)) = \frac{dx(t)}{x(t)} - \frac{\sigma^2}{2}dt,$$

By substituting:

$$(d(lnx(t)))^2 = \sigma^2 dt,$$

Considering that

$$(d(lnx(t))) \simeq ln(x(t)) - ln(x(t-1)),$$

an approximate estimator of σ is:

$$\hat{\sigma} = \mid ln(x(t)) - ln(x(t-1)) \mid,$$

Then, for n observations of a sample path of the process, the resulting approximate estimator has the following expression:

$$\hat{\sigma} = \frac{1}{n-1} \sum_{t=1}^{n} \mid ln(x(t)) - ln(x(t-1)) \mid .$$

– **Alternatively, from the stochastic differential equation (1) of the variable** $x(t)$**, we obtain:**

$$\left(\frac{dx(t)}{x(t)}\right) = \left(\alpha + \frac{\beta}{\sqrt{x(t)}}\right) dt + \sigma dw(t),$$

and then:

$$\left(\frac{dx(t)}{x(t)}\right)^2 = \sigma^2 dt,$$

Considering that

$$d(x(t)) \simeq x(t) - x(t-1),$$

An approximate value of σ is:

$$\hat{\sigma} = \frac{\mid x(t) - x(t-1) \mid}{x(t)},$$

Table 9 Values obtained using the three expressions of $\hat{\sigma}$

$\hat{\sigma}=\frac{1}{n-1}\sum_{t=1^n}\frac{\mid x(t)-x(t-1)\mid}{\sqrt{x(t)x(t-1)}}$	$\hat{\sigma}=\frac{1}{n-1}\sum_{t=1^n}\mid ln(x(t))-ln(x(t-1))\mid$	$\hat{\sigma}=\frac{1}{n-1}\sum_{t=1^n}\frac{\mid x(t)-x(t-1)\mid}{x(t)}$
0.033392073701	0.033390279030	0.032810853600

Therefore, for n observations of a sample path of the process, the resulting approximate estimator has the following expression:

$$\hat{\sigma}=\frac{1}{n-1}\sum_{t=1}^{n}\frac{\mid x(t)-x(t-1)\mid}{x(t)}.$$

By applying the three expressions of $\hat{\sigma}$ to our real data, we obtain the following results:

Similar values are obtained by all three methods.

Appendix C: Probabilistic characteristics and statistical inference of the SLDP

The SLDP from the SBSDP

In the equation (1) when $\beta = 0$, the homogeneous lognormal diffusion process is obtained as a particular case in which the infinitesimal moments are given by:

$$A_1(x)=\alpha x \quad , \quad A_2(x)=\sigma^2x^2,$$

This satisfies the following Itô's SDE:

$$dx(t)=\alpha x(t)dt+\sigma x(t)dw(t) \quad , \quad x(t_0)=x_{t_0} \quad ,$$

where $\sigma > 0$ and α are real parameters, W_t is a standard Wiener process and x_{t_0} is fixed in $\mathbb{R}_+^*$.

- **The analytical expression of the SLDP**

  By taking $\beta = 0$ in equation (3) of the Section 2.2, the previous SDE has a unique solution which is given analytically by the following expression:

$$x(t) = x_{t_0}\left(\exp\left[\left(\alpha-\frac{\sigma^2}{2}\right)(t-t_0)+\sigma\left(w(t)-w(t_0)\right)\right]\right)$$

- **The trend functions of the SLDP**

  In the same way, by taking $\beta = 0$ in equations (4) and (5 of the Section 2.3, the conditional trend function of the process is:

$$\mathbb{E}(x(t)\mid x(s)=x_s)=x_se^{\alpha(t-s)}. \tag{21}$$

  and by assuming the initial condition $\mathrm{P}\left(x(t_0)=x_{t_0}\right) = 1$, the trend function of the process is:

$$\mathbb{E}(x(t))=x_{t_0}e^{\alpha(t-t_0)}. \tag{22}$$

Parameter estimation in the SLDP

We now determine the estimator of the parameter α of the SLDP using the method described. The estimator of the drift parameters α is obtained by the maximum likelihood method, with continuous sampling.

- **Likelihood estimation of the drift parameter:**

The vector form of the SDE of the SLDP can be written as:

$$A_t(x(t)) = x(t), \quad \theta^* = \alpha, \quad B_t(x(t)) = \sigma x(t),$$

The corresponding vector H_T in this case is one-dimensional and is given by:

$$H_T^* = \frac{1}{\sigma^2} \int_{t_0}^{T} \frac{dx(t)}{x(t)},$$

and S_T has the following form:

$$S_T = \frac{T - t_0}{\sigma^2}$$

Then the expression of the estimator is

$$\hat{\alpha} = \frac{\int_{t_0}^{T} \frac{dx(t)}{x(t)}}{T - t_0},$$

The stochastic integral in the latter expression can be transformed into Riemann integrals by using Itô's formula and thus:

$$\int_{t_0}^{T} \frac{dx(t)}{x(t)} = \log(x_T) - \log(x_{t_0}) + \frac{\sigma^2}{2}(T - t_0).$$

Therefore, the expression of the Maximum Likelihood estimator is:

$$\hat{\alpha} = \frac{\left(\log(x_T) - \log(x_{t_0}\right) + \frac{\sigma^2}{2}(T - t_0)\big)}{T - t_0}.$$

- **Approximate estimator of the diffusion coefficient**

The estimator of the coefficient diffusion parameter can be approximated using a method similar to that described Section 3.2. By following the same steps and for n observations of a sample path of the process, the resulting approximate estimator is:

$$\hat{\sigma} = \frac{1}{n-1} \sum_{t=1}^{n} \frac{\mid x(t) - x(t-1) \mid}{\sqrt{x(t)x(t-1)}}.$$

Computational aspects in SLDP

- **Approximated likelihood estimators**
  As for the SBSDP, in order to use the above expression to estimate the parameter α, we must have continuous observations. Therefore, we use an alternative estimation procedure based on continuous time maximum likelihood estimators with suitable approximations of the integrals that appear in the expression. The Riemann-Stieljes integrals are approximated by means of the trapezoidal formula.
- **Estimated trend functions**
  By applying Zehna's theorem and by taking $\beta = 0$ in the equations (21) and (22), the estimated trend function (ETF) and estimated conditional trend function (ECTF) of the process are obtained as:

$$\widehat{\mathbb{E}}(x(t)/x(s) = x_s) = x_s e^{\hat{\alpha}(t-s)},$$

$$\widehat{\mathbb{E}}(x(t)) = x_{t_0} e^{\hat{\alpha}(t-t_0)}.$$

References

Allen LJ (2010) An introduction to stochastic processes with applications to biology. CRC Press, Boca Raton

Bezborodov V, Di Persio L, Mishura Y (2016) Option pricing with fractional stochastic volatility and discontinuous payoff function of polynomial growth. arXiv:160707392

Brennan MJ, Schwartz ES (1979) A continuous time approach to the pricing of bonds. J Bank Financ 3(2):133–155

Campillo F, Joannides M, Larramendy-Valverde I (2016) Analysis and approximation of a stochastic growth model with extinction. Methodol Comput Appl Probab 18(2):499–515

Chesney M, Elliott RJ (1995) Estimating the instantaneous volatility and covariance of risky assets. Appl Stochastic Models Data Anal 11(1):51–58

Florens-Zmirou D (1989) Approximate discrete-time schemes for statistics of diffusion processes. Statistics: A Journal of Theoretical and Applied Statistics 20(4):547–557

Giovanis A, Skiadas C (1999) A stochastic logistic innovation diffusion model studying the electricity consumption in greece and the united states. Technol Forecast Soc Chang 61(3):235–246

Gradshteyn IS, Ryzhik IM (1979) Table of integrals, series, and products . Academic Press, Cambridge

Gutiérrez R, Gutiérrez- Sánchez R, Nafidi A (2005) The stochastic Brennan Schwartz Diffusion Process: Statistical Inference and Applications. In: CIMMA: International mediterranean congress of mathematics, pp 47–57

Gutiérrez R, Gutirrez-Sánchez R, Nafidi A (2006) The stochastic Rayleigh diffusion model: Statistical inference and computational aspects. applications to modelling of real cases. Appl Math Comput 175(1):628–644

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A, Ramos E (2007) A diffusion model with cubic drift: statistical and computational aspects and application to modelling of the global CO2 emission in spain. Environmetrics 18(1):55–69

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A (2008) Emissions of greenhouse gases attributable to the activities of the land transport: modelling and analysis using I–CIR stochastic diffusion the case of spain. Environmetrics 19(2):137–161

Gutiérrez R, Gutiérrez-Sánchez R, Nafidi A (2009) Modelling and forecasting vehicle stocks using the trends of stochastic gompertz diffusion models: The case of spain. Appl Stoch Model Bus Ind 25(3):385–405

Haccou P, Jagers P, Vatutin VA (2005) Branching processes: variation, growth, and extinction of populations. 5. Cambridge University Press, Cambridge

Kloeden PE, Platen E (1992) The numerical solution of stochastic differential equations springer:berlin, Germany 1992

Kloeden PE, Platen E, Schurz H, Sørensen M (1996) On effects of discretization on estimators of drift parameters for diffusion processes. J Appl Probab 33(4):1061–1076

Kutoyants YA (2004) Statistical inference for Ergodic diffusion processes. Springer Science and Business Media, Berlin

Lewis CD (1982) Industrial and business forecasting methods: A practical guide to exponential smoothing and curve fitting. Butterworth-Heinemann, Oxford

Lin J, Ludkovski M (2014) Sequential bayesian inference in hidden markov stochastic kinetic models with application to detection and response to seasonal epidemics. Stat Comput 24(6):1047–1062

Mrkvicka T, Myllymäki M, Hahn U (2017) Multiple monte carlo testing, with applications in spatial point processes. Stat Comput 27(5):1239–1255

Nafidi A, Gutiérrez R, Gutiérrez-Sánchez R, Ramos-Ábalos E, El Hachimi S (2016) Modelling and predicting electricity consumption in spain using the stochastic gamma diffusion process with exogenous factors. Energy 113:309–318

Nicolau J (2005) Processes with volatility-induced stationarity: an application for interest rates. Statistica Neerlandica 59(4):376–396

Nobile A, Ricciardi L (1984) Growth with regulation in fluctuating environments. Biol Cybern 50(4):285–299

Phillips BL, Brown GP, Webb JK, Shine R (2006) Invasion and the evolution of speed in toads. Nature 439(7078):803–805

Skiadas CH, Giovanis AN (1997) A stochastic bass innovation diffusion model for studying the growth of electricity consumption in greece. Appl Stochastic Models Data Anal 13(2):85–101

Tintner G, Sengupta JK (1972) Stochastic economics. Academic Press, Cambridge

Vasicek O (1977) An equilibrium characterization of the term structure. J Financ Econ 5(2):177–188

Zehna PW et al (1966) Invariance of maximum likelihood estimators. Ann Math Stat 37(3):744

Yoshida N (1992) Estimation for diffusion processes from discrete observation. J Multivar Anal 41(2):220–242