Methodology and Computing in Applied Probability (2021) 23:461–470
https://doi.org/10.1007/s11009-019-09711-x

Uniform Strong Law of Large Numbers

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Received: 2 January 2019 / Revised: 18 March 2019 / Accepted: 25 March 2019 /
Published online: 27 May 2019

Abstract
We prove the strong law of large numbers for random signed measures. The result is uniform over a family of subsets under mild assumptions.

Keywords Random signed measure · Strong law of large numbers · Uniform limit theorem over a family of subsets · Partial sum stochastic processes

Mathematics Subject Classification (2010) Primary 60F15 · Secondary 60G55, 60G57, 60G60

1 Introduction

Let $\{X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d\}$ denote a family of independent and identically distributed random variables with $\mathbf{E}\left[|X_{\boldsymbol{j}}|\right] < \infty$ for all $\boldsymbol{j} \in \mathbb{N}^d$. Throughout the paper $\mathbb{N}^d = \{1, 2, \dots\}^d$ is the set of

Based on a talk at the conference Stereology, Spatial Statistics, and Stochastic Geometry in Prague, June 25–29, 2018.

Supported by the grant 0118U003614 from Ministry of Education and Science of Ukraine (project N 2105Φ).

Supported by the grant 0118U003614 from Ministry of Education and Science of Ukraine (project N 2105Φ) and by the grant IZ7320_152292 from Swiss National Science Foundation.

Supported by the grant IZ7320_152292 from Swiss National Science Foundation.

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positive integer points in the d-dimensional Euclidean space $\mathbb{R}^d$. Denoting by $\overline{\mathbb{B}}$ the family of bounded Borel sets in $\mathbb{R}^d$ we put

$$S(B) = \sum_{\boldsymbol{j} \in B} X_{\boldsymbol{j}}$$

for all $B \in \overline{\mathbb{B}}$.

The strong law of large numbers for $\{X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d\}$ is due to Kolmogorov if $d = 1$ and to Smythe (1973) if $d > 1$. In what follows Leb (A) denotes the Lebesgue measure of a Borel set A.

Theorem 1 (Smythe (1973)) *Let $\mathbb{P}^d$ be the family of d-dimensional rectangles*

$$P(\boldsymbol{n}) = [1, n_1] \times \cdots \times [1, n_d], \qquad n_1, \ldots, n_d \in \mathbb{N}.$$

Then

$$\lim_{\mathrm{Leb}(P(\boldsymbol{n})) \to \infty} \frac{S(P(\boldsymbol{n}))}{\mathrm{Leb}\,(P(\boldsymbol{n}))} = \mu \qquad a.s. \qquad \text{for some } \mu \in \mathbb{R}$$

if and only if

$$\mathbf{E}\left[X_{\boldsymbol{j}}\right] = \mu, \qquad \mathbf{E}\left[|X_{\boldsymbol{j}}|(\log^+ |X_{\boldsymbol{j}}|)^{d-1}\right] < \infty.$$

Here "a.s." abbreviates "almost surely" and $\log^+ z = \log(e + z)$ *for* $z \geq 0$.

More strong laws of large numbers for sums of independent random variables with multiindices can be found in Klesov (2014).

A natural problem arises on obtaining similar results for sums over subsets belonging to other families of sets.

1.1 Ruled Sums

For $d = 1$, this problem goes back to the concept of ruled sums due to Baum, Katz, and Stratton (Baum et al. 1971) (further development of this idea can be found in Stratton (1972), Baum and Stratton (1973), Petrov (1974), Martikainen (1977a), Martikainen (1977b), Skovoroda (1987), and Skovoroda and Mikosch (1992)). A rule $(\cdot)$ is a function mapping $\mathbb{N}$ into $2^{\mathbb{N}}$ where (n) is some collection of n distinct positive numbers for each n. Given an appropriate sequence of independent and identically distributed random variables $\{X_j, j \in \mathbb{N}\}$, ruled sums $S_{(n)}$ are defined by

$$S_{(n)} = \sum_{j \in (n)} X_j, \qquad n \in \mathbb{N}.$$

If $(n) = \{1, 2, \ldots, n\}$, then this model corresponds to the classical model of cumulative sums of random variables. The model arising in the so called complete convergence coincides with the model of ruled sums if $(m) \cap (n) = \varnothing$ for $m \neq n$ (necessary and sufficient conditions for the strong law of large numbers are found in Hsu and Robbins (1947) and Erdös (1949) in the case of the complete convergence). It is also clear that the model of rectangular sums for $d > 1$ can be imbedded into the model of ruled sums for the corresponding rule $(\cdot)$, however not all results for rectangular sums, $d > 1$, can be derived explicitly from existing results for ruled sums.

1.2 Partial Sum Process

Another concept for $d > 1$ is proposed in Bass and Pyke in Bass and Pyke (1984). For every set A, scaled versions nA, $n \in \mathbb{N}$, are considered in Bass and Pyke (1984), where

$$nA = \{\boldsymbol{y} : \boldsymbol{y} = n\boldsymbol{x} = (nx_1, \ldots, nx_d) \text{ for } \boldsymbol{x} = (x_1, \ldots, x_d) \in A\}.$$

Further we consider a family $\mathcal{A}$ of Borel subsets in $\mathbb{R}^d$, a family of independent identically distributed random variables $\{X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d\}$, and corresponding sums

$$S(nA) = \sum_{\boldsymbol{j} \in nA} X_{\boldsymbol{j}}, \qquad n \in \mathbb{N}.$$

Then the uniform one-parameter version of Theorem 1 proved in Bass and Pyke (1984) reads as follows. Put $\mathbf{0} = \underbrace{(0, 0, \ldots, 0)}_{d}$, $\mathbf{1} = \underbrace{(1, 1, \ldots, 1)}_{d}$. We say that

$$(\boldsymbol{a}, \boldsymbol{b}] = (a_1, b_1] \times \ldots (a_d, b_d] \subset \mathbb{R}^d$$

is a d-dimensional left-open and right-closed interval or a semi-interval. Let $I = (\mathbf{0}, \mathbf{1}] = (0, 1]^d$ be the d-dimensional unit semi-interval.

In what follows, $\mathcal{A}$ denotes a family of Borel subsets of the d-dimensional unit semi-interval I. For any $A \subset \mathbb{R}^d$, the symbol ∂A stands for the boundary of A with respect to the Euclidean distance ρ in $\mathbb{R}^d$. Throughout the paper $|\cdot|$ is the Euclidean norm. The δ-neighborhood of ∂A, $A \subset \mathbb{R}^d$, is denoted by $A(\delta)$,

$$A(\delta) = \{\boldsymbol{x} \in \mathbb{R}^d : \rho(\boldsymbol{x}, \partial A) < \delta\}.$$

Theorem 2 (Bass and Pyke (1984)) *If a family $\mathcal{A}$ of Borel subsets of the d-dimensional unit semi-interval I is such that*

$$r(\delta) \equiv \sup_{A \in \mathcal{A}} \text{Leb}\,(A(\delta)) \to 0, \qquad \delta \to 0, \tag{1}$$

then

$$\lim_{n \to \infty} \sup_{A \in \mathcal{A}} \left| \frac{S(nA)}{n^d} - \mu\, \text{Leb}\,(A) \right| = 0 \qquad a.s. \tag{2}$$

Other limit theorems for partial sum processes (like central limit theorem (Alexander and Pyke 1986) or law of the iterated logarithm (Bass 1985)) require an extra entropy type conditions imposed on the family $\mathcal{A}$. Further uniform limit theorems for partial sum processes can be found in Alexander (1987) and Bass and Pyke (1985) to cite a few.

This line of researches led to a notion of set-indexed processes (an important particular case is presented by the empirical processes, see Pyke (1984)). Several important applications of strong limit theorems for partial sum processes are known to various statistical problems, especially in higher dimensions, such a testing for multimodality, estimating density contour clusters, estimating nonlinear functionals of a density, density estimation, regression problems and spectral analysis (see, for example, Polonik (1995).) Nonparametric regression estimation for random fields arising in different scientific areas including econometrics, image analysis, meteorology, geostatistics is also based on dynamical properties of partial sum processes (see, for instance, El Machkouri (2007)). Other applications are in measurement error in nonlinear models (see Carroll et al. 2006) and image processing (see Müller and Song 1996).

1.3 Random Signed Measures

Another look at set-indexed processes in presented in Klesov and Molchanov (2019). There $S(nA)$ is treated as a random signed measure. In particular, if all random variables $X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d$, are nonnegative, then $S(nA)$ is a random measure over an appropriate family of subsets. Let $\mathbb{R}_+^d$ be the subset of $\mathbb{R}^d$ of points with positive coordinates. Let $\boldsymbol{t} = (t_1, \ldots, t_d) \in \mathbb{R}_+^d$ and

$$\boldsymbol{t}A = \{\boldsymbol{y} : \boldsymbol{y} = \boldsymbol{t}\boldsymbol{x} = (t_1x_1, \ldots, t_dx_d) \text{ for } \boldsymbol{x} = (x_1, \ldots, x_d) \in A\}, \qquad A \subset \mathbb{R}^d.$$

Sometimes we write $\boldsymbol{t} \cdot A$ instead of $\boldsymbol{t}A$. We say that a random signed measure ξ satisfies the multiparameter strong law of large numbers if

$$\lim_{|\boldsymbol{t}|\to\infty} \frac{\xi(\boldsymbol{t} \cdot I) - \mathbf{E}\,[\xi(\boldsymbol{t} \cdot I)]}{|\boldsymbol{t}|} = 0 \qquad \text{a.s.}$$

Further put $C_m(\boldsymbol{k}) = \frac{1}{m}(\boldsymbol{k} - \boldsymbol{1}, \boldsymbol{k}]$, where $\boldsymbol{k} \in \mathbb{N}^d$ and $m \in \mathbb{N}$. Also let

$$B'_m = \bigcup_{\boldsymbol{k} : C_m(\boldsymbol{k}) \subseteq B} C_m(\boldsymbol{k}),$$

$$B''_m = \bigcup_{\boldsymbol{k} : C_m(\boldsymbol{k}) \cap B \neq \varnothing} C_m(\boldsymbol{k})$$

for all $B \subset \mathbb{R}_+^d$ and $m \in \mathbb{N}$.

Theorem 3 (Klesov and Molchanov (2019)) *Let $\mathcal{A}$ be a family of Borel sets of the d-dimensional unit semi-interval and let ξ be a random signed measure that satisfies the multiparameter strong law of large numbers. Assume that*

$$\lim_{m\to\infty} \limsup_{|\boldsymbol{t}|\to\infty} \sup_{A\in\mathcal{A}} \left| \frac{\mathbf{E}\left[\xi(\boldsymbol{t} \cdot (A \setminus A'_m))\right]}{|\boldsymbol{t}|} \right| = 0$$

and $|\xi(A)| \leq \eta(A)$ for all Borel sets A and a random measure η that satisfies the multiparameter strong law of large numbers and such that

$$\lim_{m\to\infty} \limsup_{|\boldsymbol{t}|\to\infty} \sup_{A\in\mathcal{A}} \frac{\mathbf{E}\left[\eta(\boldsymbol{t} \cdot (A''_m \setminus A'_m))\right]}{|\boldsymbol{t}|} = 0.$$

Then ξ satisfies the uniform strong law of large numbers, that is,

$$\lim_{|\boldsymbol{t}|\to\infty} \sup_{A\in\mathcal{A}} \left| \frac{\xi(\boldsymbol{t} \cdot A) - \mathbf{E}\,[\xi(\boldsymbol{t} \cdot A)]}{|\boldsymbol{t}|} \right| = 0 \qquad a.s.$$

Remark 1 The difference between settings in Bass and Pyke (1984) and in Klesov and Molchanov (2019) is twofold. First, the scaling parameter $\boldsymbol{t}$ in Klesov and Molchanov (2019) is multiparameter while that in Bass and Pyke (1984) is one-parameter. Thus the result in Klesov and Molchanov (2019) is related to Theorem 1 rather than to Kolmogorov's strong law of large numbers as in (Bass and Pyke 1984). Second, $\boldsymbol{t}$ in Klesov and Molchanov (2019) is continuous, while n in Bass and Pyke (1984) is discrete. As a result, the condition in Klesov and Molchanov (2019) is stronger than in Bass and Pyke (1984) (however it leads to a stronger conclusion, as well).

Our aim in this paper is to prove an analogue of Theorem 2 for signed measures rather than for sums of random variables (see Section 2). In doing so we use some other conditions as compared to Bass and Pyke (1984) and Klesov and Molchanov (2019). In Section 3 we show that these conditions are weaker than in Bass and Pyke (1984). The method of the proof of Theorem 4 below is close to that in Klesov and Molchanov (2019).

2 Main Result

A random measure is a mapping defined on $\overline{\mathbb{B}} \times \Omega$ and such that, for every fixed $\omega \in \Omega$, it is a measure, and, for every fixed $B \in \overline{\mathbb{B}}$, it is a random variable. As usual, $[r]$, $r \in \mathbb{R}$, denotes the integer part of a real number; similarly, $[\boldsymbol{x}] = [(x_1, x_2, \dots, x_d)] = ([x_1], [x_2], \dots, [x_d])$ is the integer part of a d-dimensional vector.

Theorem 4 *Let ξ be a random signed measure. Assume that*

$$\lim_{n\to\infty} \frac{\xi(nC) - \mathbf{E}\,[\xi(nC)]}{n^d} = 0 \qquad a.s. \tag{3}$$

for all $C = (\boldsymbol{0}, \boldsymbol{x}] \subset I = (0, 1]^d$.

Let $\mathcal{A}$ be a certain family of Borel subsets of I. Suppose that there exists a random measure η such that

1. $|\xi(A)| \le \eta(A)$ *for all $A \in \overline{\mathbb{B}}$;*
2. *For all $C = (\boldsymbol{0}, \boldsymbol{x}] \subset I$,*

$$\lim_{n\to\infty} \frac{\eta(nC) - \mathbf{E}\,[\eta(nC)]}{n^d} = 0 \qquad a.s.; \tag{4}$$

3.

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \frac{\mathbf{E}\left[\eta(n(A''_m \backslash A'_m))\right]}{n^d} = 0. \tag{5}$$

Then

$$\lim_{n\to\infty} \sup_{A\in\mathcal{A}} \left| \frac{\xi(nA) - \mathbf{E}\,[\xi(nA)]}{n^d} \right| = 0 \qquad a.s. \tag{6}$$

Proof First we show that equality (3) (as well as (4)) holds for all sets of the form $C = (\mathbf{x}, \mathbf{y}] \subset I$. Note that if (3) holds for two disjoint sets C_1 and C_2, then (3) holds for $C_1 \cup C_2$, as well. Indeed,

$$\begin{aligned} &\lim_{n\to\infty} \frac{\xi(n(C_1\cup C_2)) - \mathbf{E}\,[\xi(n(C_1\cup C_2))]}{n^d} \\ &= \lim_{n\to\infty} \frac{\xi(nC_1) - \mathbf{E}\,[\xi(nC_1)]}{n^d} + \lim_{n\to\infty} \frac{\xi(nC_2) - \mathbf{E}\,[\xi(nC_2)]}{n^d} = 0 \qquad a.s. \end{aligned}$$

Similarly, if (3) holds for $C_1 \subset C_2$, then it holds for $C_2 \backslash C_1$:

$$\begin{aligned} &\lim_{n\to\infty} \frac{\xi(n(C_2\backslash C_1)) - \mathbf{E}\,[\xi(n(C_2\backslash C_1))]}{n^d} \\ &= \lim_{n\to\infty} \frac{\xi(nC_1) - \mathbf{E}\,[\xi(nC_1)]}{n^d} - \lim_{n\to\infty} \frac{\xi(nC_2) - \mathbf{E}\,[\xi(nC_2)]}{n^d} = 0 \qquad a.s. \end{aligned}$$

Sets of the form $(\boldsymbol{x}, \boldsymbol{y}] \subset I$ can easily be constructed with the help of the second operation above from sets of the form $(\boldsymbol{0}, \boldsymbol{x}] \subset I$. We prove this by induction over k, where $k \in \{0, 1, 2, \dots, d\}$ is the minimal number such that $x_l = 0$ for all l, $k < l \le d$. This result is obvious for $k = 0$. Assume it holds for $k = s < d$ and let us prove it for $s + 1$. Indeed, if

$\boldsymbol{x} = (x_1, x_2, \ldots, x_{s+1}, 0, \ldots, 0)$, $\boldsymbol{x}' = (x_1, x_2, \ldots, x_s, 0, \ldots, 0)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_d)$, then

$$(\boldsymbol{x}, \boldsymbol{y}] = (\boldsymbol{x}', \boldsymbol{y}] \backslash (\boldsymbol{x}', (y_1, y_2, \ldots, y_s, x_{s+1}, y_{s+2}, \ldots, y_d)].$$

It is clear that, for all $m \in \mathbb{N}$,

$$\limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \left| \frac{\xi(nA) - \mathbf{E}\,[\xi(nA)]}{n^d} \right| \leq X_m + Y_m + Z_m,$$

where

$$X_m = \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \left| \frac{\xi(nA) - \xi(nA'_m)}{n^d} \right|,$$

$$Y_m = \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \left| \frac{\xi(nA'_m) - \mathbf{E}\left[\xi(nA'_m)\right]}{n^d} \right|,$$

$$Z_m = \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \left| \frac{\mathbf{E}\left[\xi(nA'_m)\right] - \mathbf{E}\,[\xi(nA)]}{n^d} \right|.$$

Thus our result is proved if $X_m \to 0$ and $Y_m \to 0$ a.s. and $Z_m \to 0$ as $m \to \infty$.

It is clear that

$$\begin{aligned} Z_m &= \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \left| \frac{\mathbf{E}\left[\xi(n(A_m \backslash A'_m))\right]}{n^d} \right| \\ &\leq \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \frac{\mathbf{E}\left[\left|\xi(n(A_m \backslash A'_m))\right|\right]}{n^d} \leq \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \frac{\mathbf{E}\left[\eta(n(A''_m \backslash A'_m))\right]}{n^d}, \end{aligned}$$

for all $m \geq 1$. Also, $Z_m \to 0$ as $m \to \infty$ by (5).

The equality $Y_m = 0$ a.s. follows, since, for all m, the cardinality of the set $\{A'_m \mid A \in \mathcal{A}\}$ is finite and since,

$$\lim_{n\to\infty} \frac{\xi(nA'_m) - \mathbf{E}\left[\xi(nA'_m)\right]}{n^d} = 0 \qquad \text{a.s.}$$

for all A'_m. The latter equality holds, since $A'_m = \bigcup_{\boldsymbol{k}\,:\,C_m(\boldsymbol{k})\subseteq A} C_m(\boldsymbol{k})$ is the union of disjoint sets of the form $(\boldsymbol{x}, \boldsymbol{y}] \subset I$ and equality (3) holds for each of these sets.

Further,

$$\begin{aligned} X_m &\leq \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \frac{\eta(n(A''_m \backslash A'_m))}{n^d} \\ &\leq \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \frac{\mathbf{E}\left[\eta(n(A''_m \backslash A'_m))\right]}{n^d} \\ &\quad + \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \left| \frac{\eta(n(A''_m \backslash A'_m)) - \mathbf{E}\left[\eta(n(A''_m \backslash A'_m))\right]}{n^d} \right|. \end{aligned}$$

The first term tends to 0 as $m \to \infty$ by condition (5); the second term equals 0 a.s. (this is proved by analogy with the proof of the equality $Y_m = 0$ a.s.) □

3 Comparison of Theorems 2 and 4

Now we show that Theorem 4 is an extension of Theorem 2 in a certain sense.

Proposition 1 *If* (1) *holds with* $\mathcal{A} \subset 2^I$, $\xi(B) \equiv S(B)$, *and* $\eta(B) \equiv T(B) \equiv \sum_{j\in B} |X_j|$, *then all assumptions of Theorem 4 are satisfied.*

Proposition 2 *Let condition* (1) *hold for a family* $\mathcal{A} \subset 2^I$. *Denote by* $W(A)$ *the number of positive integer points in* A *for* $A \in \mathbb{R}^d$. *If*

$$\lim_{n\to\infty} \sup_{A\in\mathcal{A}} \left| \frac{S(nA) - \mu W(nA)}{n^d} \right| = 0 \;\; a.s.,$$

then (2) *also holds.*

Clearly Proposition 2 is weaker than Theorem 2. However the proof of (2) below is different from the proof in Bass and Pyke (1984).

Proof of Proposition 1 Clearly $S(B)$ is a random signed measure. Our aim is to prove that

$$\lim_{n\to\infty} \frac{S(nC) - \mathbf{E}\,[S(nC)]}{n^d} = 0 \qquad \text{a.s. for all } C = (0, \mathbf{x}] \subset I; \tag{7}$$

$$|S(B)| \le T(B) \qquad \text{for all } B \in \overline{\mathbb{B}}; \tag{8}$$

$$\lim_{n\to\infty} \frac{T(nC) - \mathbf{E}\,[T(nC)]}{n^d} = 0 \qquad \text{a.s. for all } C = (\mathbf{x}, \mathbf{y}] \subset I; \tag{9}$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \frac{\mathbf{E}\left[T(n(A''_m \backslash A'_m))\right]}{n^d} = 0. \tag{10}$$

It is obvious that $\mathbf{E}\,[S(nC)] = \mu W(nC)$ and thus (7) follows from

$$\lim_{n\to\infty} \frac{S(nC) - \mu W(nC)}{n^d} = 0 \qquad \text{a.s.}$$

Note that $\frac{S(nC)-\mu W(nC)}{n^d} = \frac{W(nC)}{n^d}\left(\frac{S(nC)}{W(nC)} - \mu\right)$. The second factor tends to 0 a.s. by the strong law of large numbers, while the first one does not exceed 1.

Bound (8) is obvious.

Equality (9) is proved similarly to the analogous result for S (since $|X_j|$ also are independent and identically distributed random variables).

Since $v = \mathbf{E}\left[|X_j|\right] < +\infty$, for the proof equality (10), we need to show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sup_{A\in\mathcal{A}} \frac{W(n(A''_m \backslash A'_m))}{n^d} = 0. \tag{11}$$

First we prove that

$$\mathrm{Leb}\left(A''_m \backslash A'_m\right) \le \mathrm{Leb}\left(A\left(\frac{\sqrt{d}}{m}\right)\right) \qquad \text{for all} \quad m \in \mathbb{N}. \tag{12}$$

Note that (12) follows from $A''_m \backslash A'_m \subset A(\frac{\sqrt{d}}{m})$. Note that $A''_m \backslash A'_m$ is constituted by those d-dimensional semi-intervals $C_m(\boldsymbol{k})$ that contain both a point of A and a point that does not belong to A.

We prove that each semi-interval $C_m(\boldsymbol{k})$ with this property belongs to $A(\frac{\sqrt{d}}{m})$. Choose points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $C_m(\boldsymbol{k})$ such that $\boldsymbol{x} \in A$ and $\boldsymbol{y} \notin A$. The segment of the line connecting these points (this interval is denoted by L) contains a boundary point of A. Indeed, let $v = \sup\{r \in \mathbb{R} | (\bar{U}(\boldsymbol{x}, r) \cap L) \subset A\}$, where $\bar{U}(\boldsymbol{x}, r)$ is the closed ball in $\mathbb{R}^d$ with radius r centered at the point $\boldsymbol{x}$, and let $\boldsymbol{z} \in L$ be a point such that $\rho(\boldsymbol{x}, \boldsymbol{z}) = v$. Then $\boldsymbol{z}$ is a boundary point of A. Further, a $\frac{\sqrt{d}}{m}$-neighborhood of this point contains a semi-interval $C_m(\boldsymbol{k})$, since

$\frac{\sqrt{d}}{m}$ is its diameter (this value is not attained, that is $\frac{\sqrt{d}}{m}$ is larger than the distance between any two points belonging to the semi-interval). Thus a $\frac{\sqrt{d}}{m}$-neighborhood of ∂A contains $A''_m \backslash A'_m$.

Now we prove that $W(C) \leq 3^d \operatorname{Leb}(C)$ if C is of the form $(\boldsymbol{x}, \boldsymbol{y}]$, where $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_d)$ and $y_k \geq x_k + 1$ for all k. This follows from

$$\begin{aligned} W(C) &\leq W(([\boldsymbol{x}], [\boldsymbol{y}] + \boldsymbol{1}]) = \prod_{k=1}^{d}([y_k] + 1 - [x_k]) \leq \prod_{k=1}^{d}(y_k + 2 - x_k) \\ &\leq \prod_{k=1}^{d} 3(y_k - x_k) = 3^d \operatorname{Leb}(C). \end{aligned}$$

Now we turn back to the proof of (11). If $n > m$, then $n(A''_m \backslash A'_m)$ is a union of disjoint sets $nC_m(\boldsymbol{k})$ of the form $(\boldsymbol{x}, \boldsymbol{y}]$, where $y_k \geq x_k + 1$ for all k. Thus

$$\frac{W(n(A''_m \backslash A'_m))}{n^d} \leq \frac{3^d \operatorname{Leb}\left(n(A''_m \backslash A'_m)\right)}{n^d} = 3^d \operatorname{Leb}\left(A''_m \backslash A'_m\right) \leq 3^d \operatorname{Leb}\left(A\left(\frac{\sqrt{d}}{m}\right)\right)$$

for $n > m$, whence

$$\sup_{A \in \mathcal{A}} \frac{W(n(A''_m \backslash A'_m))}{n^d} \leq 3^d \sup_{A \in \mathcal{A}} \operatorname{Leb}\left(A\left(\frac{\sqrt{d}}{m}\right)\right) = 3^d r\left(\frac{\sqrt{d}}{m}\right)$$

for all $n > m$. Hence

$$\limsup_{n \to \infty} \sup_{A \in \mathcal{A}} \frac{W(n(A''_m \backslash A'_m))}{n^d} \leq 3^d r\left(\frac{\sqrt{d}}{m}\right).$$

It is obvious that $3^d r\left(\frac{\sqrt{d}}{m}\right) \to 0$ as $m \to \infty$. □

Proof of Proposition 2 It is sufficient to prove that

$$\lim_{n \to \infty} \sup_{A \in \mathcal{A}} \left| \frac{W(nA)}{n^d} - \operatorname{Leb}(A) \right| = 0$$

or, in other words, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sup_{A \in \mathcal{A}} \left| \frac{W(nA)}{n^d} - \operatorname{Leb}(A) \right| \leq \varepsilon$$

for all $n > N$.

Choose $\delta > 0$ such that $r(\delta) < \varepsilon$ and let N be such that $\delta > \frac{\sqrt{d}}{N}$. Then $\delta > \frac{\sqrt{d}}{n}$ for all $n > N$. Now we prove that

$$\sup_{A \in \mathcal{A}} \left| \frac{W(nA)}{n^d} - \operatorname{Leb}(A) \right| \leq \varepsilon \quad \text{for all} \quad n > N.$$

This follows from

$$\left| \frac{W(nA)}{n^d} - \operatorname{Leb}(A) \right| \leq \operatorname{Leb}(A(\delta)) \quad \text{for all} \quad A \in \mathcal{A} \text{ and } n > N. \tag{13}$$

To prove (13) note that the numbers $\operatorname{Leb}(A)$ and $\frac{W(nA)}{n^d}$ lie between the numbers $\operatorname{Leb}\left(A'_n\right)$ and $\operatorname{Leb}\left(A''_n\right)$. This is obvious for $\operatorname{Leb}(A)$. Let us show that

$$n^d \cdot \operatorname{Leb}\left(A'_n\right) \leq W(nA) \leq n^d \cdot \operatorname{Leb}\left(A''_n\right). \tag{14}$$

Consider the following three subsets of $\mathbb{N}^d$:

$$\mathbb{W}_1 = \left\{ \boldsymbol{k} \in \mathbb{N} : \frac{1}{n}(\boldsymbol{k}-\boldsymbol{1}, \boldsymbol{k}] \subset A \right\};$$
$$\mathbb{W}_2 = \left\{ \boldsymbol{k} \in \mathbb{N} : \frac{\boldsymbol{k}}{n} \subset A \right\};$$
$$\mathbb{W}_3 = \left\{ \boldsymbol{k} \in \mathbb{N} : \frac{1}{n}(\boldsymbol{k}-\boldsymbol{1}, \boldsymbol{k}] \cap A \neq \varnothing \right\}.$$

It is clear that $\mathbb{W}_1 \subseteq \mathbb{W}_2 \subseteq \mathbb{W}_3$ and that the numbers of points in these sets are equal to $n^d \cdot \mathrm{Leb}\left(A'_n\right)$, $W(nA)$, and $n^d \cdot \mathrm{Leb}\left(A''_n\right)$, respectively. This implies (14).

Thus

$$\left| \frac{W(nA)}{n^d} - \mathrm{Leb}\,(A) \right| \leq \mathrm{Leb}\left(A''_n\right) - \mathrm{Leb}\left(A'_n\right) = \mathrm{Leb}\left(A''_n \backslash A'_n\right),$$

and it is sufficient to show that $\mathrm{Leb}\left(A''_n \backslash A'_n\right) \leq \mathrm{Leb}\,(A(\delta))$. This follows from (12), since $\delta > \frac{\sqrt{d}}{n}$. □

4 A Remark on the Set *I*

In fact, Theorem 2 in Bass and Pyke (1984) is proved for $I = [\mathbf{0}, \mathbf{1}]$ rather than for $I = (\mathbf{0}, \mathbf{1}]$. In Proposition 3 below we prove that the statement in Theorem 2 with $I = (\mathbf{0}, \mathbf{1}] = (0, 1]^d$ is equivalent to that with $I' = [\mathbf{0}, \mathbf{1}]$ instead of $I = (\mathbf{0}, \mathbf{1}]$.

Remark 2 On the other hand, the authors are not aware whether or not Theorem 4 holds with $I = [\mathbf{0}, \mathbf{1}]$.

Proposition 3 *Theorem 2 with $I = (\boldsymbol{0}, \boldsymbol{1}]$ is equivalent to that with $I' = [\boldsymbol{0}, \boldsymbol{1}]$.*

Proof It is clear that the statement with I' implies that with I. Assume that this statement holds for I and let us prove it for I'. Let $\mathcal{A}$ be the family of Borel subsets of I'. For every $A \in \mathcal{A}$, let $F(A) = A \cap I$ and $F(\mathcal{A}) = \{F(A) \mid A \in \mathcal{A}\}$. Then condition (1) with I holds for $F(\mathcal{A})$. Indeed, if $A \in \mathcal{A}$ is an arbitrary subset, then $\mathrm{Leb}\,(F(A)(\delta)) \leq \mathrm{Leb}\,(A(\delta)) + \mathrm{Leb}\,(I(\delta))$, since $\partial F(A) \subset (\partial A \cup \partial I)$. Thus

$$\sup_{A \in \mathcal{A}} \mathrm{Leb}\,(F(A)(\delta)) \leq \sup_{A \in \mathcal{A}} \mathrm{Leb}\,(A(\delta)) + \mathrm{Leb}\,(I(\delta)) \to 0, \qquad \delta \to 0.$$

Therefore (2) holds for I:

$$\lim_{n\to\infty} \sup_{A \in F(\mathcal{A})} \left| \frac{S(nA)}{n^d} - \mu\,\mathrm{Leb}\,(A) \right| = 0 \qquad a.s.$$

Since $S(nF(A)) = S(nA)$ and $\mathrm{Leb}\,(F(A)) = \mathrm{Leb}\,(A)$, we get

$$\lim_{n\to\infty} \sup_{A \in \mathcal{A}} \left| \frac{S(nA)}{n^d} - \mu\,\mathrm{Leb}\,(A) \right| = \lim_{n\to\infty} \sup_{A \in \mathcal{A}} \left| \frac{S(nF(A))}{n^d} - \mu\,\mathrm{Leb}\,(F(A)) \right|$$
$$= \lim_{n\to\infty} \sup_{A \in F(\mathcal{A})} \left| \frac{S(nA)}{n^d} - \mu\,\mathrm{Leb}\,(A) \right| = 0 \qquad a.s.,$$

which is what had to be proved. □

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