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Telegraph Process with Elastic Boundary at the Origin

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Abstract We investigate the one-dimensional telegraph random process in the presence of an elastic boundary at the origin. This process describes a finite-velocity random motion that alternates between two possible directions of motion (positive or negative). When the particle hits the origin, it is either absorbed, with probability α, or reflected upwards, with probability $1-\alpha$. In the case of exponentially distributed random times between consecutive changes of direction, we obtain the distribution of the renewal cycles and of the absorption time at the origin. This investigation is performed both in the case of motion starting from the origin and non-zero initial state. We also study the probability law of the process within a renewal cycle.

Keywords Finite velocity · Random motion · Telegraph process · Elastic boundary · Absorption time · Renewal cycle

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1 Introduction

The (integrated) telegraph process describes an alternating random motion with finite velocity. This stochastic process deserves interest in various applied fields, such as physics, finance, and mathematical biology. Among the first authors that studied the solution of the telegraph equation we recall Goldstein (1951) and Kac (1974). Several aspects and generalization of the telegraph process have been provided in a quite large literature. Orsingher (1990) studied the probability law, flow function, maximum distribution of wave-governed random motions of the telegraph type. The distributions of the first-passage time and of the maximum of the telegraph process were obtained by Foong (1992). The solutions of the one-dimensional telegraph equation on a semi-infinite line terminated by a trap, and on a finite line terminated by two traps were determined by Masoliver et al. (1992). The analysis of the telegraph process in the presence of reflecting and absorbing barriers was also investigated in Orsingher (1995) and Ratanov (1997).

Restricting the attention to some recent contributions, we also mention Beghin et al. (2001) and López and Ratanov (2014) for the asymmetric telegraph process, Bogachev and Ratanov (2011) for the distribution of the occupation time of the positive half-line for the telegraph process, Crimaldi et al. (2013) for a telegraph process driven by certain random trials, De Gregorio and Macci (2012) for the large deviation principle applied to the telegraph process, Di Crescenzo and Martinucci (2010) for a damped telegraph process, Fontbona et al. (2012) for the long-time behavior of an ergodic variant of the telegraph process, Stadje and Zacks (2004) for the telegraph process with random velocities, Pogorui et al. (2015) for estimates of the number of level-crossings for the telegraph process, Di Crescenzo and Zacks (2015) for the analysis of a generalized telegraph process perturbed by Brownian motion, De Gregorio and Orsingher (2011) and Garra and Orsingher (2014) for certain multidimensional extension of the telegraph process. Moreover, D'Ovidio et al. (2014) investigate other types of multidimensional extensions of the telegraph process, whose distribution is related to space-time fractional n-dimensional telegraph equations. A modern treatment of the one-dimensional telegraph stochastic processes, with a thorough view to their applications in financial markets, is provided in the book by Kolesnik and Ratanov (2013). See also Ratanov (2015) for a generalization of jump-telegraph process with variable velocities applied to markets modelling.

Most of the above references are concerning analytical results. However, in some instances one is forced to adopt computational methods to solve the governing equations. See, for instance, Acebrón and Ribeiro (2015), where a Monte Carlo algorithm is derived to solve the one-dimensional telegraph equations in a bounded domain subject to suitable boundary conditions.

Several applications of the telegraph process and its numerous generalizations have been stimulated by problems involving dynamical systems subject to dichotomous noise. For instance, such processes can be used for the description of stochastic dynamics of extended thermodynamic theories far from equilibrium (see Giona et al. 2016). The need to model physical systems in the presence of a variety of complex conditions encouraged several authors to analyze stochastic processes restricted by suitable boundaries, such as the elastic ones. Examples of papers dealing with elastic boundaries are provided by Veestraeten (2006) and Buonocore et al. (2003).

Analytical results on stochastic processes restricted by elastic boundaries have been obtained by various authors, such as Dominé (1995) and Dominé (1996), for the first-passage problem of the Wiener process with drift, Giorno et al. (2006) for the construction of first-passage-time densities for diffusion processes, Beghin and Orsingher (2009) for the analysis of fractional diffusion equations. Furthermore, Jacob (2012) and Jacob (2013) studied a Langevin process with partially elastic boundary at zero and related stochastic differential equations.

The analysis of finite-velocity random motions subject to elastic boundaries seems to be quite new. Along the lines of the previous papers, we investigate the distribution of a one-dimensional telegraph process $\{X(t); t \geq 0\}$ in the presence of an elastic boundary at 0. This process describes the motion of a particle over the state space $[0, +\infty)$ and starting at $x \geq 0$. The particle moves on the line up and down alternating. For simplicity, we assume that the motion has velocity 1 (upward motion) and -1 (downward motion). Initially, the motion proceeds upward for a positive random time U_1. After that, the particle moves downward for a positive random time D_1, and so on the motion alternates along the random times $U_2, D_2, U_3, D_3, \ldots$, where $\{U_i\}_{i\in\mathbb{N}}$ and $\{D_i\}_{i\in\mathbb{N}}$ are independent sequences of i.i.d. random variables. When the particle hits the origin it is either absorbed, with probability α or reflected upwards, with probability $1-\alpha$, with $0 < \alpha < 1$. Specifically, if during a downward period, say D_j, the particle reaches the origin and is not absorbed, then instantaneously the motion restarts with positive velocity, according to an independent random time U_{j+1}.

The analysis of the telegraph process and related processes is often based on the resolution of partial differential equations with proper boundary conditions. However, in this case such approach seems to be not fruitful so that we will adopt renewal theory arguments. We denote by C_x the random time till the first arrival at the origin, with starting point $x \geq 0$, and by $C_{0,i}$ the (eventual) ith interarrival time between consecutive visits at the origin following C_x, for $i \in \mathbb{N}$. Moreover, let A_x denote the time till absorption at the origin conditional on initial state $x \geq 0$. Let M be the random number of arriving at the origin, until absorption. Clearly, M has a geometric distribution, with

$$\mathbb{P}(M = m) = \alpha(1-\alpha)^{m-1}, \qquad m \in \mathbb{N}, \quad \alpha \in (0, 1). \tag{1}$$

We remark that the random variables $C_x, C_{0,1}, C_{0,2}, \ldots$ are independent. Moreover, $C_{0,1}, C_{0,2}, \ldots$ are identically distributed, and are called renewal cycles. For brevity, we denote by C_0 a random variable that is identically distributed as $C_{0,i}$, $i \in \mathbb{N}$. Clearly, the distribution of C_x is identical to that of the renewal cycles if $x = 0$. Figure 1 shows an example of sample path of $X(t)$, where D_j^* denotes the downward random period D_j truncated by the occurrence of the visit at the origin. Finally, we point out the following relation:

$$A_x = C_x + \mathbf{1}_{\{M>1\}} \sum_{i=1}^{M-1} C_{0,i}. \tag{2}$$

This is the plan of the paper. In Section 2 we provide some basic definitions and recall some useful results on the distribution of the renewal cycles when U_i are exponentially distributed and D_i have a general distribution. In Section 3 we analyze the absorption time and renewal cycles when the initial state is zero, and U_i and D_i have exponential distribution with unequal parameters. In this case we obtain the explicit expression of the probability density function (PDF), moment generating function (MGF), and moments of A_0 and C_0. In Section 4 we study the absorption time and renewal cycles for non-zero initial state. We determine the PDF, the MGF and the moments of C_x, as well as the MGF and the moments of A_x. Finally, in Section 5 we study the conditional distribution of $X(t)$ within a renewal cycle.

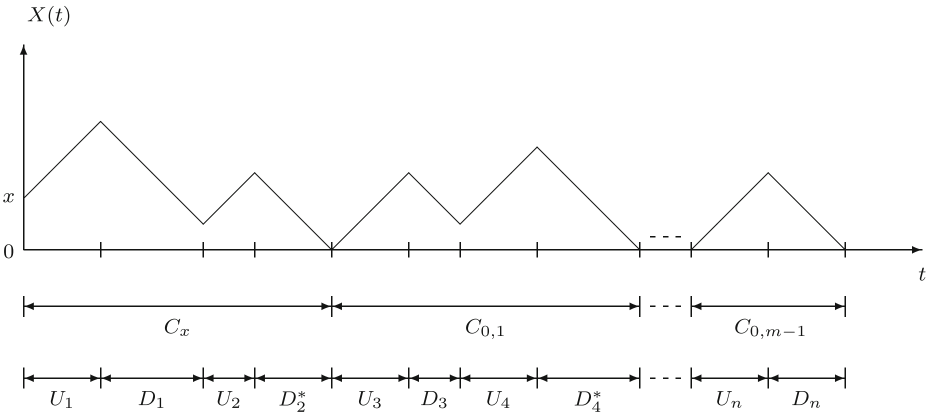

Fig. 1 A sample-path of $X(t)$

The main probabilistic characteristics of the process under investigation will be determined in an analytical form. Even if the expressions seem complicated they can be evaluated in standard computer environments, as shown in various figures throughout the paper.

2 Preliminaries on the Renewal Cycles

Let us denote by F and G the cumulative distribution functions of U_i and D_i, respectively. We assume that the upward periods of the motion have exponential distribution, i.e.

$$F(t) = 1 - e^{-\lambda t}, \qquad t \in [0, \infty), \quad \lambda \in (0, \infty). \tag{3}$$

Aiming to determine the distribution of the renewal cycles, we consider the auxiliary compound Poisson process

$$Y(t) = \sum_{n=1}^{N(t)} D_n, \tag{4}$$

where

$$N(t) = \max\{n \in \mathbb{N}_0 : \sum_{i=1}^{n} U_i \le t\},$$

and thus $N(t) = 0$ if $U_1 > t$. Clearly, $N(t)$ is a Poisson process with intensity λ, so that $\mathbb{P}[Y(t) = 0] = e^{-\lambda t}$, $t \in [0, \infty)$, due to Eq. 3. Moreover, if $Y(t) = s-t$, with $t \in (0, s)$, this means that the total time (from 0 to s) of moving upwards or downwards equals t or $s - t$, respectively. The PDF of the absolutely continuous component of $Y(t)$, for $t \in (0, \infty)$, is

$$h(y; t) := \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}[Y(t) \le y] = e^{-\lambda t} \sum_{n=1}^{+\infty} \frac{(\lambda t)^n}{n!} g^{(n)}(y), \qquad y \in (0, \infty), \tag{5}$$

where $g^{(n)}(y)$ is the n-fold convolution of the PDF of G.

Let us now define, for any $x \in [0, \infty)$, the following stopping time:

$$T_x = \inf\{t > 0 : Y(t) \ge x + t\}. \tag{6}$$

If the motion starts from the origin, i.e. $x = 0$, then all renewal cycles $C_{0,i}$, $i \in \mathbb{N}$, are distributed as C_0. In this case, since the first visit to the origin occurs at the first instant in which the total time downward is greater or equal to that of the time upward, we have

$$C_0 \stackrel{d}{=} 2T_0, \tag{7}$$

where $\stackrel{d}{=}$ means equality in distribution. When the initial state is away from the origin, i.e. $x \in (0, \infty)$, similarly it is

$$C_x \stackrel{d}{=} x + 2T_x. \tag{8}$$

Notice that the stopping time (6) is not necessarily a proper random variable. Indeed, $\mathbb{P}(T_x < \infty) = 1$ for $x \in [0, \infty)$ if and only if $\mathbb{E}[D_1] \geq \mathbb{E}[U_1]$, and the moments of T_x are finite only if $\mathbb{E}[D_1] > \mathbb{E}[U_1]$ (see, for instance, section 3 of Zacks et al. (1999)).

For all $x \in [0, \infty)$, let us now introduce the following subdensity,

$$g_x(y,t) := \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}[Y(t) \leq y, T_x > t], \qquad y \in (0,\infty),\ t \in (0,\infty), \tag{9}$$

and the PDF of the stopping time T_x,

$$\psi_x(t) := \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(T_x \leq t), \qquad t \in (0,\infty). \tag{10}$$

The following proposition recalls some useful results obtained by Stadje and Zacks (Stadje and Zacks 2003), concerning the functions introduced in Eqs. 9 and 10.

Proposition 1 *(i) The subdensity given in* Eq. 9 *can be expressed in terms of* $h(y,t)$*, defined in* Eq. 5*, as follows.*

– *If* $x = 0$*, then*

$$g_0(y,t) = \frac{t-y}{t} h(y,t), \qquad t \in (0,\infty),\ y \in (0,t). \tag{11}$$

– *If* $x \in (0,\infty)$*, then, for* $0 < y < x + t$ *and* $t \in (0,\infty)$*,*

$$g_x(y,t) = \mathbf{1}_{\{0<y\leq x\}} h(y;t) + \mathbf{1}_{\{x<y<x+t\}} \Big[h(y;t) - h(y; y-x) e^{-\lambda(t-y+x)} - (t-y+x) \int_{t+x-y}^{t} \frac{1}{u} h(u-t+y-x; u) h(t-u+x; t-u) du \Big]. \tag{12}$$

(ii) For all $x \in [0,\infty)$*, the PDF of the stopping time* T_x *is given by*

$$\psi_x(t) = \lambda e^{-\lambda t}\overline{G}(t+x) + \lambda \int_0^{t+x} g_x(y,t)\overline{G}(t-y+x) dy, \qquad t \in (0,\infty), \tag{13}$$

where $\overline{G}(t) = 1 - G(t) = \mathbb{P}(D_1 > t)$*.*

In the sequel we assume that the distribution of the downward random times D_n is exponential with parameter μ, i.e.

$$G(t) = 1 - e^{-\mu t}, \qquad t \in [0,\infty), \tag{14}$$

with $0 < \mu < \lambda$ in order to ensure that $\mathbb{E}[D_1] > \mathbb{E}[U_1]$.

3 Absorption Time and Renewal Cycles for Zero Initial State

In the present section we consider the special case of initial state $x = 0$. Recalling that U_1 and D_1 are exponentially distributed with parameters λ and μ, respectively, with $0 < \mu < \lambda$, from Eq. 5 we have

$$h(y,t) = \frac{\sqrt{\lambda\mu t}}{\sqrt{y}} I_1\left(2\sqrt{\lambda\mu t y}\right) e^{-\lambda t - \mu y}, \qquad y \in (0,\infty), \quad t \in (0,\infty), \tag{15}$$

where $I_1(\cdot)$ is the modified Bessel function;

$$I_n(z) := \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!(k+n)!}, \qquad n \in \mathbb{N}_0. \tag{16}$$

We are now able to obtain the PDF of Eq. 6 when $x = 0$.

Proposition 2 *Under assumptions* (3) *and* (14)*, with* $0 < \mu < \lambda$*, the PDF of* T_0 *is given by*

$$\psi_0(t) = \frac{\lambda e^{-(\lambda+\mu)t}}{t\sqrt{\lambda\mu}} I_1\left(2t\sqrt{\lambda\mu}\right), \qquad t \in (0,\infty). \tag{17}$$

Proof From Eq. 13, and recalling Eqs. 3 and 14, it follows that, for $t \in (0,\infty)$,

$$\psi_0(t) = \lambda e^{-(\lambda+\mu)t}\left[1 + \sqrt{\frac{\lambda\mu}{t}} \int_0^t \frac{(t-y)}{\sqrt{y}} I_1\left(2\sqrt{\lambda\mu t y}\right) \mathrm{d}y\right].$$

Hence, after a change of variable, and recalling Eq. (1.11.1.1) of Prudnikov et al. (1986b), we get

$$\psi_0(t) = \lambda e^{-(\lambda+\mu)t}\left\{1 + \frac{\lambda\mu t^2}{2}\left[2\,{}_1F_2(1;2,2;\lambda\mu t^2) - {}_1F_2(2;3,2;\lambda\mu t^2)\right]\right\}, \tag{18}$$

where

$${}_1F_2(a;b,c;z) = \sum_{n=0}^{+\infty} \frac{(a)_n}{(b)_n (c)_n} \frac{z^n}{n!} \tag{19}$$

is the hypergeometric function, with $(d)_0 = 1$ and $(d)_n = d(d+1)\cdots(d+n-1)$ for $n \in \mathbb{N}$ (the rising factorial). Making use of identities (1) of 'http://functions.wolfram.com/07.22.17.0005.01' and (5) of 'http://functions.wolfram.com/07.22.03.0122.01', from Eq. 18 we obtain

$$\psi_0(t) = \lambda e^{-(\lambda+\mu)t}\left[1 + \frac{\lambda\mu t^2}{2}\left\{\frac{2}{(\lambda\mu t^2)^{3/2}} I_1\left(2t\sqrt{\lambda\mu}\right) - \frac{2}{\lambda\mu t^2}\right]\right\},$$

so that Eq. 17 finally follows. □

We remark that the PDF given in Eq. 17 identifies with the busy period PDF of an M/M/1 queue with arrival rate μ and service rate λ.

Let us now study the renewal cycle for zero initial state. We recall that the Gauss hypergeometric function is defined as

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{20}$$

Proposition 3 *Under the assumptions of Proposition 2, the PDF of C_0 is given by*

$$ f_{C_0}(y) = \frac{1}{y}\sqrt{\frac{\lambda}{\mu}}\lambda e^{-\frac{(\lambda+\mu)}{2}y} I_1\left(y\sqrt{\lambda\mu}\right), \qquad y \in (0, \infty). \tag{21}$$

The nth moment of C_0 is

$$ \mathbb{E}(C_0^n) = \frac{\lambda\, 2^n n!}{(\lambda+\mu)^{n+1}}\, {}_2F_1\left(\frac{n+1}{2}, \frac{n+2}{2}; 2; \frac{4\lambda\mu}{(\lambda+\mu)^2}\right), \qquad n \in \mathbb{N}, \tag{22}$$

with mean and variance

$$ \mathbb{E}(C_0) = \frac{2}{\lambda-\mu}, \qquad Var(C_0) = \frac{4(\lambda+\mu)}{(\lambda-\mu)^3}. $$

Proof From assumptions (3) and (14), due to relation (7), we immediately obtain the PDF (21). Hence, the moments (22) follow recalling Eq. (3.15.1.2) of Prudnikov et al. (1992a). □

In the following proposition we obtain the expression of the MGF of the absorption time A_0.

Proposition 4 *Under the same assumptions of Proposition 2, for $s < (\sqrt{\lambda} - \sqrt{\mu})^2/2$, the MGF of A_0 is*

$$ M_{A_0}(s) := \mathbb{E}(e^{sA_0}) = \frac{2\alpha\lambda}{2\lambda(\alpha-1) + (\lambda+\mu-2s) + \sqrt{(\lambda+\mu-2s)^2 - 4\lambda\mu}}. \tag{23}$$

Proof From Eq. 2, and recalling Eq. 1, we have

$$ M_{A_0}(s) = \sum_{m=1}^{+\infty} \left[M_{C_0}(s)\right]^m \mathbb{P}(M=m) = \frac{\alpha M_{C_0}(s)}{1 + (\alpha-1) M_{C_0}(s)}, \tag{24}$$

where $M_{C_0}(s)$ is the MGF of C_0. Due to Eq. 21, and recalling Eq. (3.15.1.8) of Prudnikov et al. (1992a), for $s < (\sqrt{\lambda} - \sqrt{\mu})^2/2$ it is

$$ \begin{aligned} M_{C_0}(s) &= \sqrt{\frac{\lambda}{\mu}} \int_0^{+\infty} \frac{e^{-((\lambda+\mu)/2-s)y}}{y} I_1\left(y\sqrt{\lambda\mu}\right) dy \\ &= \frac{(\lambda+\mu-2s) - \sqrt{(\lambda+\mu-2s)^2 - 4\lambda\mu}}{2\mu}. \end{aligned} \tag{25}$$

Finally, Eq. 23 immediately follows from Eqs. 24 and 25. □

In the following theorem we obtain the PDF of the absorption time A_0.

Theorem 1 *Under the same assumptions of Proposition 2, for $y \in (0,\infty)$ we have*

$$f_{A_0}(y) = \alpha \frac{e^{-\frac{(\lambda+\mu)}{2}y}}{y}\left\{\sqrt{\frac{\lambda}{\mu}} I_1\left(y\sqrt{\lambda\mu}\right) + \sum_{m=2}^{+\infty} \frac{(\lambda y/2)^m (1-\alpha)^{m-1}}{(m-1)(m-1)!} \right.$$
$$\times \left[2m\ {}_1F_2\left(\frac{m-1}{2}; \frac{m+1}{2}, m; \frac{\lambda\mu y^2}{4}\right)\right.$$
$$\left.\left. -\ (m+1)\ {}_1F_2\left(\frac{m-1}{2}; \frac{m+1}{2}, m+1; \frac{\lambda\mu y^2}{4}\right)\right]\right\}, \qquad (26)$$

with ${}_1F_2$ defined in Eq. 19.

Proof Denoting by

$$\mathcal{L}_s[f(t)] = \int_0^{+\infty} \mathrm{e}^{-st} f(t)\mathrm{d}t, \qquad s \in [0,\infty), \qquad (27)$$

the Laplace transform of a certain integrable function $f(t)$, from Eq. 24 we have

$$\mathcal{L}_s[f_{A_0}(t)] = \sum_{m=1}^{+\infty} \left[\frac{(\lambda+\mu+2s) - \sqrt{(\lambda+\mu+2s)^2 - 4\lambda\mu}}{2\mu}\right]^m \mathbb{P}(M=m). \qquad (28)$$

We recall that, due to Eqs. (2.1.9.18) and (1.1.1.8) of Prudnikov et al. (1992b), it is, for $a = 2\sqrt{\lambda\mu}$,

$$\mathcal{L}_s\left[\int_0^t \left\{\frac{m}{x}(2\sqrt{\lambda\mu})^{m+1} I_{m-1}(2x\sqrt{\lambda\mu})\right.\right.$$
$$\left.\left. -\frac{m(m+1)}{x^2}(2\sqrt{\lambda\mu})^m I_m(2x\sqrt{\lambda\mu})\right\}\mathrm{d}x\right] = \left(s - \sqrt{s^2 - a^2}\right)^m, \qquad (29)$$

for $m \in \mathbb{N}$, $m \geq 2$, where $I_n(\cdot)$ is defined in Eq. 16. Moreover, from Eq. (1.11.1.1) of (Prudnikov et al. 1986b), we have

$$\int_0^t \left\{\frac{m}{x}(2\sqrt{\lambda\mu})^{m+1} I_{m-1}(2x\sqrt{\lambda\mu}) - \frac{m(m+1)}{x^2}(2\sqrt{\lambda\mu})^m I_m(2x\sqrt{\lambda\mu})\right\}\mathrm{d}x$$
$$= \frac{(2\lambda\mu)^m t^{m-1}}{(m-1)(m-1)!}\left\{2m\ {}_1F_2\left(\frac{m-1}{2}; \frac{m+1}{2}, m; \lambda\mu t^2\right)\right.$$
$$\left. -(m+1)\ {}_1F_2\left(\frac{m-1}{2}; \frac{m+1}{2}, m+1; \lambda\mu t^2\right)\right\}. \qquad (30)$$

Hence, from Eq. 28, taking the inverse Laplace transformation, due to Eqs. 25, 29 and 30, and recalling Eq. (1.1.1.4) of Prudnikov et al. (1992b) the proof finally follows. □

In Fig. 2 we provide some plots of the PDF $f_{A_0}(y)$ for various choices of α. Such density is decreasing in y, with $f_{A_0}(0) = \alpha\lambda/2$.

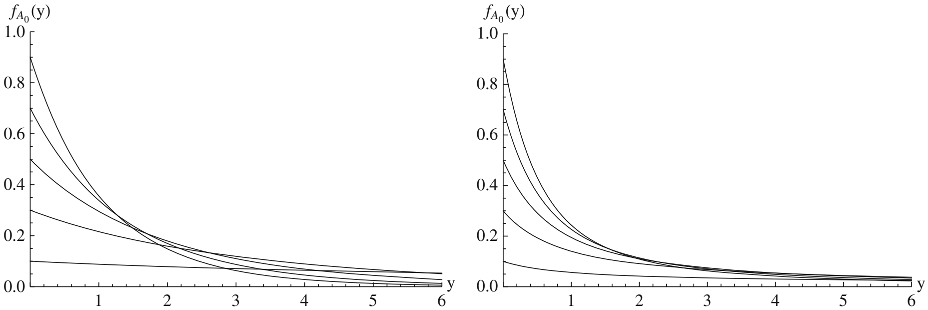

Fig. 2 Density $f_{A_0}(y)$, given in Eq. 26, for $(\lambda, \mu) = (2, 0.5)$ (*left-hand side*) and $(\lambda, \mu) = (2, 1.5)$ (*right-hand side*) with $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$ from bottom to top near the origin

We conclude this section by evaluating the moments of the absorption time A_0.

Proposition 5 *Under the same assumptions of Proposition 2, for $n \in \mathbb{N}$ the nth moment of A_0 is given by*

$$\begin{aligned}\mathbb{E}(A_0^n) = & \frac{2\alpha\lambda\, n!}{[4\lambda\alpha(\mu+\lambda(\alpha-1))]^{n+1}}\Bigg\{[2\mu+2\lambda(\alpha-1)][8\lambda(\alpha-1)]^n \\ & + \sum_{h=1}^{n}[4\lambda\alpha(\mu+\lambda(\alpha-1))]^h\,[8\lambda(\alpha-1)]^{n-h}\frac{\lambda\mu\, 2^{h+1}}{(\lambda+\mu)^{h+1}} \\ & \times {}_2F_1\left(\frac{h+1}{2}, \frac{h+2}{2}; 2; \frac{4\lambda\mu}{(\lambda+\mu)^2}\right)\Bigg\}. \end{aligned} \tag{31}$$

The mean and the variance of A_0 are given by

$$\mathbb{E}(A_0) = \frac{2}{\alpha(\lambda-\mu)}, \qquad Var(A_0) = \frac{4[\lambda+\mu(2\alpha-1)]}{\alpha^2(\lambda-\mu)^3}.$$

Proof From Eqs. 23, 25 and 22 we have that the MGF of A_0, for $s < (\sqrt{\lambda}-\sqrt{\mu})^2/2$ can be rewritten as

$$\begin{aligned} M_{A_0}(s) &= \frac{2\alpha\lambda\left[2\lambda(\alpha-1)+(\lambda+\mu-2s)-\sqrt{(\lambda+\mu-2s)^2-4\lambda\mu}\right]}{4\lambda\alpha(\mu+\lambda(\alpha-1))-8\lambda(\alpha-1)s} \\ &= \frac{2\alpha\lambda\left[2\lambda(\alpha-1)+2\mu+\sum_{r=1}^{+\infty}\frac{2^{r+1}\lambda\mu}{(\lambda+\mu)^{r+1}}\,{}_2F_1\left(\frac{r+1}{2}, \frac{r+2}{2}; 2; \frac{4\lambda\mu}{(\lambda+\mu)^2}\right)s^r\right]}{4\lambda\alpha(\mu+\lambda(\alpha-1))-8\lambda(\alpha-1)s}. \end{aligned}$$

Hence, the proof follows after straightforward calculations. □

4 Absorption Time and Renewal Cycles for Non-Zero Initial State

In this section we obtain the distribution of the renewal cycles in the case $x \in (0, \infty)$. We first determine the PDF of the first-passage-time (6) for non-zero initial state.

Proposition 6 *Under assumptions* (3) *and* (14), *for* $0 < \mu < \lambda$, *the PDF of* T_x, $x > 0$, *for* $t \in (0, \infty)$ *is given by*

$$\psi_x(t) = \lambda e^{-(\lambda+\mu)t} e^{-\mu x} \Bigg\{ I_0(2\sqrt{\lambda\mu t(t+x)}) + \frac{1}{2}\sum_{r=0}^{+\infty} \frac{(\lambda\mu t x)^r}{r!(r+1)!} \sum_{j=0}^{r} \binom{r}{j} \times (j+r+1)\left(\frac{t}{x}\right)^j \left[-1 + {}_1F_2\left(-\frac{1}{2}; \frac{(j+r+1)}{2}, 1+\frac{(j+r)}{2}; \lambda\mu t^2\right)\right]\Bigg\}, \quad (32)$$

where $I_0(\cdot)$ *and* ${}_1F_2(a; b, c; \cdot)$ *are defined in* Eqs. 16 *and* 19, *respectively.*

Proof Substituting Eq. 15 in Eq. 12, considering the series form of I_1, and making use of Eq. (2.2.6.1) of (Prudnikov et al. 1986a), we have

$$\begin{aligned} g_x(y,t) &= \mathbf{1}_{\{0<y\le x\}} \mathrm{e}^{-\lambda t-\mu y} \frac{\sqrt{\lambda\mu t}}{\sqrt{y}} I_1(2\sqrt{\lambda\mu t y}) + \mathbf{1}_{\{x<y<x+t\}} \mathrm{e}^{-\lambda t-\mu y} \\ &\times \left\{ \frac{\sqrt{\lambda\mu t}}{\sqrt{y}} I_1(2\sqrt{\lambda\mu t y}) - \frac{\sqrt{\lambda\mu(y-x)}}{\sqrt{y}} I_1(2\sqrt{\lambda\mu(y-x)y}) \right. \\ &- \sum_{k=0}^{+\infty}\sum_{r=0}^{+\infty}\sum_{j=0}^{k} \frac{y^r}{r!}(j+k)! \frac{(\lambda\mu)^{k+r+2}(t+x-y)^{k+1-j}(y-x)^{k+r+j+2}}{(k+r+j+2)!(k+1)!j!(k-j)!} \\ &\left. \times\ {}_2F_1\left(j+k+1, -r; j+k+r+3; \frac{y-x}{y}\right) \right\}, \end{aligned}$$

with ${}_2F_1(a, b; c; \cdot)$ defined in Eq. 20. Hence, using the above expression of $g_x(y,t)$ in Eq. 13, and recalling that $\overline{G}(t) = \mathrm{e}^{-\mu t}$, $t \in [0, \infty)$, for $0 < \mu < \lambda$ (due to Eq. 14), we obtain

$$\begin{aligned} \psi_x(t) &= \lambda \mathrm{e}^{-(\lambda+\mu)t-\mu x} \left\{ 1 + \sqrt{\lambda\mu t} \int_0^{t+x} \frac{I_1(2\sqrt{\lambda\mu t y})}{\sqrt{y}} \mathrm{d}y \right. \\ &- \int_x^{t+x} \frac{\sqrt{\lambda\mu(y-x)} I_1(2\sqrt{\lambda\mu(y-x)y})}{\sqrt{y}} \mathrm{d}y \\ &- \sum_{k=0}^{+\infty}\sum_{r=0}^{+\infty}\sum_{j=0}^{k} \frac{(\lambda\mu)^{k+r+2}(j+k)!}{(k+r+j+2)!(k+1)!j!r!(k-j)!} \\ &\times \int_x^{t+x} y^r (t+x-y)^{k+1-j}(y-x)^{k+r+j+2} \\ &\left. \times\ {}_2F_1\left(j+k+1, -r; j+k+r+3; \frac{y-x}{y}\right) \mathrm{d}y \right\}, \end{aligned} \quad (33)$$

for $t \in (0, \infty)$. Due to Eq. (1.11.1.1) of Prudnikov et al. (1986b) and Eq. (7.14.2.84) of Prudnikov et al. (1990), we have

$$\int_0^{t+x} \frac{I_1(2\sqrt{\lambda\mu t y})}{\sqrt{y}} \mathrm{d}y = \frac{1}{\sqrt{\lambda\mu t}} \left[I_0(2\sqrt{\lambda\mu t(t+x)}) - 1 \right], \quad (34)$$

whereas, from Eq. (2.2.6.1) of (Prudnikov et al. 1986a), it is

$$\int_x^{t+x} \frac{\sqrt{\lambda\mu y(y-x)}I_1(2\sqrt{\lambda\mu(y-x)y})}{\sqrt{y}}\mathrm{d}y = \sum_{k=0}^{+\infty}\frac{(\lambda\mu)^{k+1}}{k!(k+1)!}\int_0^t (x+z)^k z^{k+1}\mathrm{d}z$$
$$= \lambda\mu t^2 \sum_{k=0}^{+\infty}\frac{(\lambda\mu t x)^k}{k!(k+2)!}\,{}_2F_1\left(-k, k+2; k+3; -\frac{t}{x}\right). \tag{35}$$

Moreover, recalling Eq. (2.21.1.4) of Prudnikov et al. (1990), we get

$$\int_x^{t+x} y^r(t+x-y)^{k+1-j}(y-x)^{k+r+j+2}$$
$$\times\,{}_2F_1\left(j+k+1, -r; j+k+r+3; \frac{y-x}{y}\right)\mathrm{d}y$$
$$= \frac{t^{2k+r+4}x^r(k+1-j)!(k+r+j+2)!}{(2k+r+4)!}\,{}_2F_1\left(-r, r+2; 2k+r+5; -\frac{t}{x}\right). \tag{36}$$

Hence, making use of Eqs. 34, 35 and 36 in Eq. 33, for $t \in (0, \infty)$ we obtain

$$\psi_x(t) = \lambda \mathrm{e}^{-(\lambda+\mu)t-\mu x}\Bigg\{ I_0(2\sqrt{\lambda\mu t(t+x)})$$
$$-\lambda\mu t^2\sum_{k=0}^{+\infty}\frac{(\lambda\mu t x)^k}{k!(k+2)!}\,{}_2F_1\left(-k, k+2; k+3; -\frac{t}{x}\right)$$
$$-2\sum_{k=0}^{+\infty}\sum_{r=0}^{+\infty}\frac{(\lambda\mu)^{k+r+2}t^{2k+r+4}x^r(2k+1)!}{(k+2)!k!r!(2k+r+4)!}\,{}_2F_1\left(-r, r+2; 2k+r+5; -\frac{t}{x}\right)\Bigg\}. \tag{37}$$

Finally, recalling the integral form of the Gauss Hypergeometric function (see, for instance, Eq. 15.3.1 of Abramowitz and Stegun (1994)), and making use of Eq. (7.2.1.2) of Prudnikov et al. (1990) and Eq. (2.15.1.1) of Prudnikov et al. (1986b), the proof follows from Eq. 37 after some calculations. □

We are now able to obtain the PDF of the first renewal cycle when $x \in (0, \infty)$.

Proposition 7 *Under the same assumptions of Proposition 6, the PDF of C_x for $y > x$ is given by*

$$f_{C_x}(y) = \frac{1}{2}\lambda e^{-\lambda\frac{y-x}{2}-\mu\frac{y+x}{2}}\Bigg\{ I_0\left(\sqrt{\lambda\mu(y^2-x^2)}\right)$$
$$+\frac{1}{2}\sum_{r=0}^{+\infty}\frac{(\lambda\mu x(y-x)/2)^r}{r!(r+1)!}\sum_{j=0}^{r}\binom{r}{j}(j+r+1)\left(\frac{y-x}{2x}\right)^j$$
$$\times\left[-1+{}_1F_2\left(-\frac{1}{2}; \frac{(j+r+1)}{2}, 1+\frac{(j+r)}{2}; \lambda\mu\left(\frac{y-x}{2}\right)^2\right)\right]\Bigg\}. \tag{38}$$

Proof The proof immediately follows from Proposition 6, and recalling Eq. 8. □

Some plots of the PDF $f_{C_x}(y)$ are provided in Fig. 3. We note that $f_{C_x}(x) = \lambda e^{-\mu x}/2$, $x \in (0, \infty)$.

Remark 1 It is not hard to show that if $x \to 0^+$, then the PDF of T_x, given in Eq. 32, tends to the PDF of T_0, shown in Eq. 17. Indeed, by virtue of Eq. (2.15.1.1) of Prudnikov et al. (1986b), for any fixed $t \in (0, \infty)$, we have

$$\lim_{x\to 0^+} \psi_x(t) = \lambda e^{-(\lambda+\mu)t}\Big\{ I_0\left(2t\sqrt{\lambda\mu}\right) - \int_0^1 \frac{1}{z} I_1\left(2tz\sqrt{\lambda\mu}\right) I_1\left(2t(1-z)\sqrt{\lambda\mu}\right) dz \Big\}.$$

The latter expression is identical to Eq. 17, due to Eq. (2.15.19.9) of Prudnikov et al. (1986b) and the following well-known recurrence relation for the Bessel function (see, for instance, (9.6.26) of Abramowitz and Stegun (1994)): $I_{n-1}(z) - I_{n+1}(z) = (2n/z)I_n(z)$.

Similarly, one can show that if $x \to 0^+$, then the PDF given in Eq. 38 tends to the PDF (21).

In the following proposition we obtain the MGF of the first-passage-time defined in Eq. 6.

Proposition 8 *Under the same assumptions of Proposition 6, for $s < (\sqrt{\lambda} - \sqrt{\mu})^2$ we have*

$$M_{T_x}(s) := \mathbb{E}(e^{sT_x}) = \frac{\lambda+\mu-s-\sqrt{(\lambda+\mu-s)^2-4\lambda\mu}}{2\mu} \times e^{\frac{x}{2}\left[\lambda-\mu-s-\sqrt{(\lambda+\mu-s)^2-4\lambda\mu}\right]}. \tag{39}$$

Proof Recalling Eq. 27, from Eq. 32 we have

$$M_{T_x}(s) = \lambda e^{-\mu x}\mathcal{L}_{\lambda+\mu-s}\left[I_0\left(2\sqrt{\lambda\mu}\sqrt{t^2+tx}\right)\right] + \frac{1}{2}\lambda e^{-\mu x}\sum_{r=0}^{+\infty}\frac{(\lambda\mu)^r}{r!(r+1)!}\sum_{j=0}^{r}\binom{r}{j}(j+r+1)x^{r-j} \times \mathcal{L}_{\lambda+\mu-s}\left[t^{r+j}\left(-1+{}_1F_2\left(-\frac{1}{2};\frac{(j+r+1)}{2},1+\frac{(j+r)}{2};\lambda\mu t^2\right)\right)\right].$$

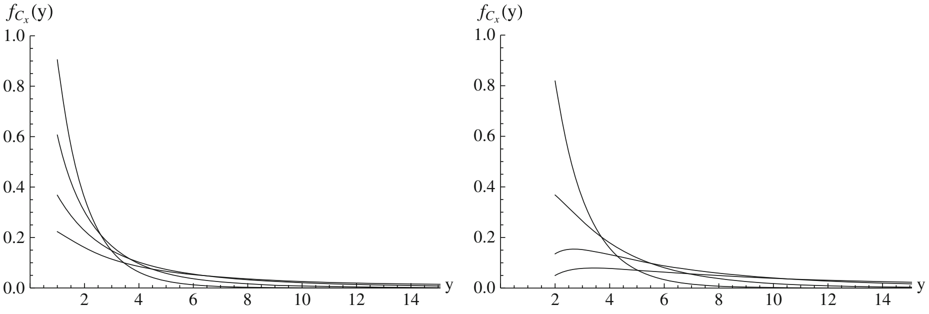

Fig. 3 Density $f_{C_x}(y)$, given in Eq. 38, for $\lambda = 2$, $x = 1$ (*left-hand side*) and $\lambda = 2$, $x = 2$ (*right-hand side*) with $\mu = 0.1, 0.5, 1, 1.5$ from top to bottom near the origin

Hence, due to Eqs. (3.15.3.1) of Prudnikov et al. (1992a) and (4.23.17) of Erdelyi (1954), we have for $s < (\sqrt{\lambda} - \sqrt{\mu})^2$

$$M_{T_x}(s) = \lambda e^{-\mu x} \frac{e^{\frac{x}{2}\left[\lambda+\mu-s-\sqrt{(\lambda+\mu-s)^2-4\lambda\mu}\right]}}{\sqrt{(\lambda+\mu-s)^2-4\lambda\mu}} + \frac{1}{2}\lambda e^{-\mu x}\left[-1+\sqrt{1-\frac{4\lambda\mu}{(\lambda+\mu-s)^2}}\right]$$
$$\times \frac{1}{\lambda+\mu-s}\sum_{r=0}^{+\infty}\left(\frac{\lambda\mu x}{\lambda+\mu-s}\right)^r \sum_{j=0}^{r}\left(\frac{1}{x(\lambda+\mu-s)}\right)^j \binom{j+r+1}{j}\frac{1}{(r-j)!}.$$

Consequently, due to Eqs. (7.2.2.1), (7.2.2.8) and (7.11.1.15) of (Prudnikov et al. 1990), we obtain

$$M_{T_x}(s) = \lambda e^{-\mu x} \frac{e^{\frac{x}{2}\left[\lambda+\mu-s-\sqrt{(\lambda+\mu-s)^2-4\lambda\mu}\right]}}{\sqrt{(\lambda+\mu-s)^2-4\lambda\mu}} + \frac{1}{2}\lambda e^{-\mu x}\left[-1+\sqrt{1-\frac{4\lambda\mu}{(\lambda+\mu-s)^2}}\right]$$
$$\times \frac{1}{\lambda+\mu-s} e^{\lambda\mu x/(\lambda+\mu-s)} \sum_{j=0}^{+\infty}\left[\frac{\lambda\mu}{(\lambda+\mu-s)^2}\right]^j \mathbf{L}_j^{j+1}\left(-\frac{\lambda\mu x}{\lambda+\mu-s}\right),$$

where $\mathbf{L}_n^{\beta}$, $n \in \mathbb{N}$, denotes the generalized Laguerre polynomials. Finally, recalling Eq. (5.11.4.7) of Prudnikov et al. (1986b), it is

$$M_{T_x}(s) = \frac{4\lambda\mu - \left[\lambda+\mu-s-\sqrt{(\lambda+\mu-s)^2-4\lambda\mu}\right]^2}{4\mu\left[\sqrt{(\lambda+\mu-s)^2}-4\lambda\mu\right]} e^{\frac{x}{2}\left[\lambda-\mu-s-\sqrt{(\lambda+\mu-s)^2-4\lambda\mu}\right]},$$

so that Eq. 39 immediately follows. □

Hereafter we obtain the MGF of the absorption time at the origin.

Proposition 9 *Under the same assumptions of Proposition 6, for $s < (\sqrt{\lambda} - \sqrt{\mu})^2/2$, the MGF of A_x is*

$$M_{A_x}(s) := \mathbb{E}(e^{sA_x}) = \frac{2\alpha\lambda e^{\frac{x}{2}\left[\lambda-\mu-\sqrt{(\lambda+\mu-2s)^2-4\lambda\mu}\right]}}{2\lambda(\alpha-1)+(\lambda+\mu-2s)+\sqrt{(\lambda+\mu-2s)^2-4\lambda\mu}}. \quad (40)$$

Proof From Eqs. 1 and 2, we have the following relation:

$$M_{A_x}(s) = \frac{\alpha M_{C_x}(s)}{1+(\alpha-1)M_{C_0}(s)}.$$

Hence, recalling Eq. 25 and noting that

$$M_{C_x}(s) := \mathbb{E}(e^{sC_x}) = \frac{\lambda+\mu-2s-\sqrt{(\lambda+\mu-2s)^2-4\lambda\mu}}{2\mu}$$
$$\times\ e^{\frac{x}{2}\left[\lambda-\mu-\sqrt{(\lambda+\mu-2s)^2-4\lambda\mu}\right]} \quad (41)$$

for $s < (\sqrt{\lambda} - \sqrt{\mu})^2/2$, Eq. 40 follows after some calculations. □

Let us now determine the moments of the renewal cycle when the initial state is non-zero.

Proposition 10 *Under the same assumptions of Proposition 6, for $n \in \mathbb{N}$ the nth moment of C_x is given by*

$$\mathbb{E}(C_x^n) = \frac{\lambda}{\lambda+\mu} e^{\frac{x}{2}(\lambda-\mu)} \frac{2^n}{(\lambda+\mu)^n} \sum_{h=0}^{n} \left(-\frac{\lambda+\mu}{(\sqrt{\lambda}-\sqrt{\mu})^2} \right)^h \times {}_2F_1\left(\frac{1+n-h}{2}, \frac{2+n-h}{2}; 2; \frac{4\lambda\mu}{(\lambda+\mu)^2} \right) \times \sum_{j=0}^{+\infty} \left[-\frac{(\lambda-\mu)x}{2} \right]^j \frac{1}{j!} \binom{j/2}{h} {}_2F_1\left(-h, -\frac{j}{2}; \frac{j}{2}+1-h; \left(\frac{\sqrt{\lambda}-\sqrt{\mu}}{\sqrt{\lambda}+\sqrt{\mu}} \right)^2 \right), \tag{42}$$

with ${}_2F_1$ given in Eq. 20, *and* $\binom{x}{h} := x(x-1)(x-2)\ldots(x-h+1)/h!$ *for $x \in \mathbb{R}$ and $h \in \mathbb{N}$.*

Proof Comparing the MGFs (25) and (41), for $s < (\sqrt{\lambda}-\sqrt{\mu})^2/2$ we have

$$M_{C_x}(s) = M_{C_0}(s) \cdot e^{\frac{x}{2}\left[\lambda-\mu-\sqrt{(\lambda+\mu-2s)^2-4\lambda\mu}\right]}. \tag{43}$$

We note that

$$e^{\frac{x}{2}\left[\lambda-\mu-\sqrt{(\lambda+\mu-2s)^2-4\lambda\mu}\right]} = \sum_{n=0}^{+\infty} \frac{x^n}{2^n n!} \left[\lambda-\mu-\sqrt{(\lambda+\mu-2s)^2-4\lambda\mu} \right]^n$$
$$= \sum_{n=0}^{+\infty} \frac{x^n}{2^n n!} \sum_{j=0}^{n} \binom{n}{j} (\lambda-\mu)^{n-j} (-1)^j [(\lambda+\mu-2s)^2-4\lambda\mu]^{j/2},$$

where

$$[(\lambda+\mu-2s)^2-4\lambda\mu]^{j/2}$$
$$= (\lambda-\mu)^j \sum_{k=0}^{+\infty} \binom{j/2}{k} \left(-\frac{2s}{(\sqrt{\lambda}-\sqrt{\mu})^2} \right)^k \sum_{l=0}^{+\infty} \binom{j/2}{l} \left(-\frac{2s}{(\sqrt{\lambda}+\sqrt{\mu})^2} \right)^l$$
$$= (\lambda-\mu)^j \sum_{r=0}^{+\infty} s^r \sum_{h=0}^{r} \binom{j/2}{h} \binom{j/2}{r-h} \left(-\frac{2}{(\sqrt{\lambda}-\sqrt{\mu})^2} \right)^{r-h} \left(-\frac{2}{(\sqrt{\lambda}+\sqrt{\mu})^2} \right)^h,$$

so that

$$e^{\frac{x}{2}\left[\lambda-\mu-\sqrt{(\lambda+\mu-2s)^2-4\lambda\mu}\right]}$$
$$= e^{\frac{x}{2}(\lambda-\mu)} \sum_{r=0}^{+\infty} s^r \left(-\frac{2}{(\sqrt{\lambda}-\sqrt{\mu})^2} \right)^r \sum_{j=0}^{+\infty} \left[-\frac{(\lambda-\mu)x}{2} \right]^j \frac{1}{j!}$$
$$\times \sum_{h=0}^{r} \binom{j/2}{h} \binom{j/2}{r-h} \left(\frac{\sqrt{\lambda}-\sqrt{\mu}}{\sqrt{\lambda}+\sqrt{\mu}} \right)^{2h} = e^{\frac{x}{2}(\lambda-\mu)} \sum_{r=0}^{+\infty} s^r \left(-\frac{2}{(\sqrt{\lambda}-\sqrt{\mu})^2} \right)^r$$
$$\times \sum_{j=0}^{+\infty} \left[-\frac{(\lambda-\mu)x}{2} \right]^j \frac{1}{j!} \binom{j/2}{r} {}_2F_1\left(-r, -j/2; j/2+1-r; \left(\frac{\sqrt{\lambda}-\sqrt{\mu}}{\sqrt{\lambda}+\sqrt{\mu}} \right)^2 \right). \tag{44}$$

Hence, the moments of C_x can be obtained from Eq. 43 and taking into account Eqs. 22 and 44, after some calculations. □

We can now provide the moments of A_x.

Proposition 11 *Under the same assumptions of Proposition 6, the nth moment of A_x, for $n \in \mathbb{N}$, is given by*

$$\mathbb{E}(A_x^n) = 2\alpha\lambda e^{\frac{x}{2}(\lambda-\mu)} \sum_{h=0}^{n} \left(-\frac{2}{(\sqrt{\lambda}-\sqrt{\mu})^2}\right)^h \frac{(8\lambda(\alpha-1))^{n-h}}{(4\lambda\alpha(\mu+\lambda(\alpha-1)))^{n-h+1}}$$
$$\times \left[2\mu + 2\lambda(\alpha-1) + \frac{2\lambda\mu}{\lambda+\mu} \sum_{m=1}^{n-h} \left(\frac{\alpha\mu+\alpha\lambda(\alpha-1)}{(\alpha-1)(\lambda+\mu)}\right)^m \right.$$
$$\left. \times\ {}_2F_1\left(\frac{m+1}{2}, \frac{m+2}{2}; 2; \frac{4\lambda\mu}{(\lambda+\mu)^2}\right)\right]$$
$$\times \sum_{j=0}^{+\infty} \left[-\frac{(\lambda-\mu)x}{2}\right]^j \frac{1}{j!} \binom{j/2}{h}\, {}_2F_1\left(-h, -\frac{j}{2}; \frac{j}{2}+1-h; \left(\frac{\sqrt{\lambda}-\sqrt{\mu}}{\sqrt{\lambda}+\sqrt{\mu}}\right)^2\right). \tag{45}$$

Proof Due to Eqs. 23 and 40, the following relation holds:

$$M_{A_x}(s) = M_{A_0}(s) \cdot e^{\frac{x}{2}\left[\lambda-\mu-\sqrt{(\lambda+\mu-2s)^2-4\lambda\mu}\right]}.$$

The moments (45) then follow from Eqs. (31) and (44), similarly as in the proof of Proposition 10. □

From Propositions 10 and 11 the following results immediately follow.

Proposition 12 *For $0 < \mu < \lambda$ and $\alpha \in (0, 1)$, the means of C_x and A_x are*

$$\mathbb{E}(C_x) = \frac{2+(\lambda+\mu)x}{\lambda-\mu}, \qquad \mathbb{E}(A_x) = \frac{2+\alpha(\lambda+\mu)x}{\alpha(\lambda-\mu)},$$

whereas their variances are given by

$$Var(C_x) = \frac{4\mu}{(\lambda-\mu)^3} - \frac{2(\lambda^2-\mu^2-2\lambda\mu)x}{(\lambda-\mu)^3} - \frac{(\lambda+\mu)^2x^2}{2(\lambda-\mu)^2},$$
$$Var(A_x) = \frac{4\mu}{\alpha(\lambda-\mu)^3} - \frac{2(\lambda^2-\mu^2-2\alpha\lambda\mu)x}{\alpha(\lambda-\mu)^3} - \frac{(\lambda+\mu)^2x^2}{2(\lambda-\mu)^2}.$$

It is easy to see that $\mathbb{E}(A_x)$ is decreasing in α, and clearly tends to $\mathbb{E}(C_x)$ as $\alpha \to 1^-$. Indeed, A_x identifies with C_x when $\alpha = 1$.

5 Conditional Distribution of the Process within a Renewal Cycle C_0

In this section we derive the conditional distribution of $X(t)$ within a renewal cycle in the case of zero initial state. Specifically, let us consider renewal cycles that start with $X(0) = 0$ and ends at C_0. We recall that $Y(t)$ is the compound Poisson process defined in Eq. 4

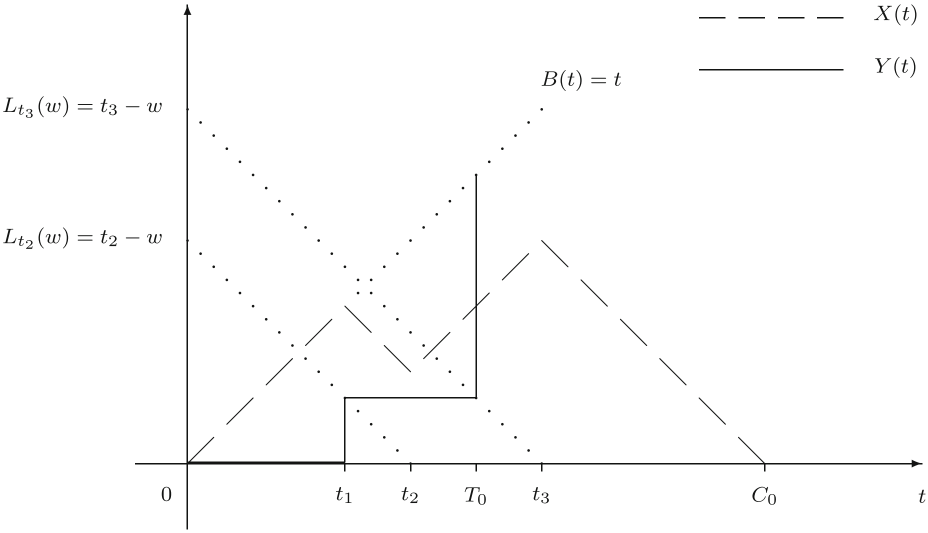

Fig. 4 Sample paths of the processes $X(t)$ and $Y(t)$, with $x = 0$

and T_0 is the stopping time introduced in Eq. 6 for $x = 0$. Given T_0, we consider any sample path of $Y(t)$ which crosses the boundary $\{B(t) = t,\ t > 0\}$ at T_0. For any given $t \in (0, C_0)$, let $W(t)$ be the time coordinate at which the sample path of $Y(t)$ crosses the line $\{L_t(w) = t - w,\ \ w \in (0, t)\}$. The value of $X(t)$ within a renewal cycle is then $X(t) = 2W(t) - t$, in the case of zero initial state. Notice that $W(t)$ is the total time in $(0, t]$ at which the telegraph process is moving upwards. As example, Figs. 4 and 5 provide sample paths of such processes. We observe that, for every $t \in (0, C_0)$ and given $T_0 = C_0/2$, it results $t/2 < W(t) \leq T_0$.

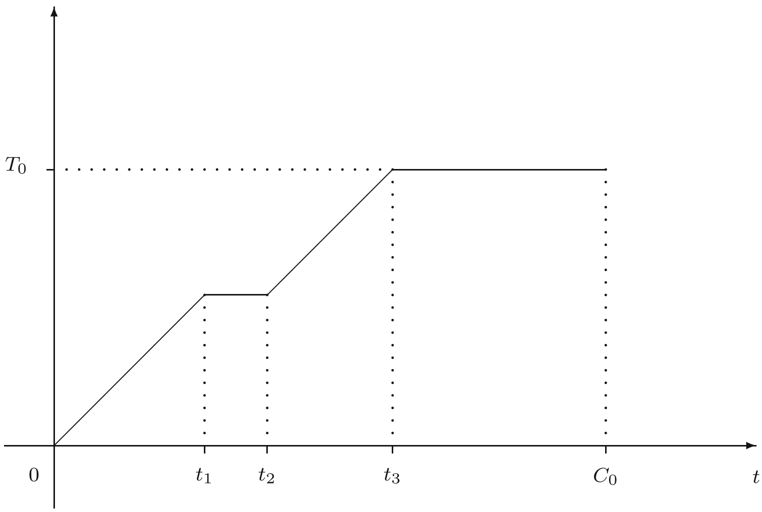

Fig. 5 Sample path of $W(t)$ corresponding to the case of Fig. 4

Let us now determine the subdistribution function of $Y(t)$ and T_0, in the case of zero initial state. For any $t \in (0, \infty)$ it is defined as

$$F_{Y(t),T_0}(y, \tau)\mathrm{d}\tau := \mathbb{P}[Y(t) \le y, T_0 \in \mathrm{d}\tau], \qquad y \in (0, t), \quad \tau \in (t, \infty). \tag{46}$$

Proposition 13 *Under assumptions* (3) *and* (14)*, for* $0 < \mu < \lambda$*, the subdistribution function defined in* Eq. 46 *is given by*

$$\begin{aligned}
F_{Y(t),T_0}(y, \tau) &= \lambda \mathrm{e}^{-(\lambda+\mu)\tau} \Bigg\{ I_0\left(2\sqrt{\lambda\mu\tau(t-\tau)}\right) \\
&+\frac{1}{2}\sum_{r=0}^{+\infty} \frac{[\lambda\mu t(\tau-t)]^r}{r!(r+1)!} \sum_{j=0}^{r} \binom{r}{j} (j+r+1) \left(\frac{\tau-t}{t}\right)^j \\
&\times \left[-1 + {}_1F_2\left(-\frac{1}{2}; \frac{j+r+1}{2}, 1+\frac{j+r}{2}; \lambda\mu(\tau-t)^2\right)\right] \\
&+\lambda\,\mu\, y \sum_{j=0}^{+\infty} \frac{[\lambda\mu y(\tau-t)]^j}{j!}\, {}_0F_1\,(;\, j+1;\, \lambda\mu(t-\tau)(y-\tau)) \\
&\times \left[\frac{t}{(j+1)!}\, {}_1F_2\,(1;\, j+2, 2;\, \lambda\mu t y) - \frac{y}{(j+2)!}\, {}_1F_2\,(2;\, j+3, 2;\, \lambda\mu t y)\right] \\
&+\frac{\lambda\,\mu}{2} \sum_{r=0}^{+\infty} \frac{[\lambda\mu(\tau-t)]^r}{r!(r+1)!} \sum_{s=0}^{r} \binom{r}{r-s} (2r+1-s)(\tau-t)^{r-s} \\
&\times \left[-1 + {}_1F_2\left(-\frac{1}{2}; r-\frac{s-1}{2}, r+1-\frac{s}{2}; \lambda\mu(\tau-t)^2\right)\right] \\
&\times \sum_{k=0}^{s+1} \binom{s+1}{k} (t-y)^{s+1-k} \frac{y^{k+1}}{k+1}\, {}_1F_2\,(1;\, k+2, 2;\, \lambda\mu t y) \Bigg\},
\end{aligned} \tag{47}$$

where

$${}_0F_1\,(;\, b;\, z) = \sum_{n=0}^{+\infty} \frac{z^n}{(b)_n n!} \tag{48}$$

and ${}_1F_2(a; b, c; z)$ *is defined in* Eq. 19.

Proof Due to Eq. 46, we note that, for $t > 0$,

$$F_{Y(t),T_0}(y, \tau) = \mathbb{P}[Y(t) = 0, T_0 \in \mathrm{d}\tau]/\mathrm{d}\tau + \int_0^y p_{Y(t),T_0}(x, \tau)\mathrm{d}x, \tag{49}$$

where p is the subdensity $p_{Y(t),T_0}(x, \tau) := \frac{\partial}{\partial x} F_{Y(t),T_0}(x, \tau)$. We point out that for $\tau \in (t, \infty)$ and $y \in (0, t)$, it is

$$\mathbb{P}[Y(t) = 0, T_0 \in \mathrm{d}\tau]/\mathrm{d}\tau = \mathrm{e}^{-\lambda t} \psi_t(\tau - t), \tag{50}$$

since $\mathbb{P}[Y(t) = 0] = \mathbb{P}[U_1 > t] = \mathrm{e}^{-\lambda t}$, $t > 0$, and $\mathbb{P}[T_0 \in \mathrm{d}\tau \mid Y(t) = 0] = \mathbb{P}(t + T_t \in \mathrm{d}\tau)$, $\tau > t$. By a similar reasoning, one also has

$$p_{Y(t),T_0}(x, \tau) = g_0(x, t)\psi_{t-x}(\tau - t), \tag{51}$$

where g_0 and ψ_{t-x} are defined in Eqs. 12 and 13, respectively. Making use of Eqs. 50 and 51 in 49, the function $F_{Y(t),T_0}(y, \tau)$ can be obtained by recalling the expressions of $g_0(y, t)$

and $\psi_x(t)$ provided by Eqs. 11, 15 and 32. The resulting expression of $F_{Y(t),T_0}(y,\tau)$ involves the following identities:

$$\int_0^y \frac{(t-x)^{r-j+1}}{\sqrt{x}} I_1\left(2\sqrt{\lambda\mu t x}\right)\mathrm{d}x$$
$$= \sqrt{\lambda\mu t}\sum_{k=0}^{r-j+1}\binom{r-j+1}{k}(t-y)^{r-j+1-k}\frac{y^{k+1}}{k+1}\,{}_1F_2\left(1;k+2,2;\lambda\mu t y\right)$$

and

$$\frac{\lambda\sqrt{\lambda\mu}}{\sqrt{t}}\mathrm{e}^{-(\lambda+\mu)\tau}\int_0^y \frac{t-x}{\sqrt{x}} I_1\left(2\sqrt{\lambda\mu t x}\right) I_0\left(2\sqrt{\lambda\mu(\tau-t)(\tau-x)}\right)\mathrm{d}x$$
$$= \lambda^2\mu\mathrm{e}^{-(\lambda+\mu)\tau}\times\sum_{r=0}^{+\infty}\frac{[\lambda\mu(\tau-t)]^r}{r!^2}\sum_{j=0}^{r}\binom{r}{j}(\tau-y)^{r-j}\frac{y^{j+1}}{j+1}$$
$$\times\left\{t\,{}_1F_2\left(1;j+2,2;\lambda\mu t y\right)-\frac{y}{j+2}\,{}_1F_2\left(2;j+3,2;\lambda\mu t y\right)\right\}$$
$$= \lambda^2\,\mu\,y\,\mathrm{e}^{-(\lambda+\mu)\tau}\sum_{j=0}^{+\infty}\frac{[\lambda\mu y(\tau-t)]^j}{j!}\,{}_0F_1\left(;j+1;\lambda\mu(t-\tau)(y-\tau)\right)$$
$$\times\left[\frac{t}{(j+1)!}\,{}_1F_2\left(1;j+2,2;\lambda\mu t y\right)-\frac{y}{(j+2)!}\,{}_1F_2\left(2;j+3,2;\lambda\mu t y\right)\right],$$

the latter being due to Eq. (2.15.2.5) of Prudnikov et al. (1986b) and identity

$$I_0\left(2\sqrt{\lambda\mu(\tau-t)(\tau-x)}\right)=\sum_{r=0}^{+\infty}\frac{[\lambda\mu(\tau-t)]^r}{r!^2}\sum_{j=0}^{r}\binom{r}{j}(\tau-y)^{r-j}(y-x)^j.$$

The proof thus follows after some calculations. □

We conclude this paper by giving the expression of the conditional distribution of $X(t)$ given T_0, within C_0. The proof is omitted since it immediately follows from the definition of $W(t)$.

Proposition 14 *The conditional distribution of $X(t)$ given T_0, during a renewal cycle C_0, is expressed as*

$$\mathbb{P}[X(t)\le x\,|\,T_0=\tau]=\mathbb{P}[W(t)>t/2\,|\,T_0=\tau]-\mathbb{P}[W(t)>(t+x)/2\,|\,T_0=\tau],\quad(52)$$

for $t\in(0,\tau)$ and $x\in[0,t]$, where

$$\mathbb{P}[W(t)>w\,|\,T_0=\tau]=\frac{F_{Y(w),T_0}(t-w,\tau)}{\psi_0(\tau)},\qquad w\in\left(\frac{t}{2},t\right),$$

with $\psi_0(x)$ and $F_{Y(w),T_0}(y,\tau)$ given in Eqs. 17 *and* 47*, respectively.*

We omit the explicit expression of the distribution (52), being too cumbersome. Some plots of the corresponding PDF are given in Fig. 6 for some choices of μ. We remark that the corresponding discrete component of such distribution is

$$\mathbb{P}[X(t)=t\,|\,T_0=\tau]=\frac{\mathrm{e}^{-\lambda t}\psi_t(\tau-t)}{\psi_0(\tau)},\qquad t\in(0,\tau).$$

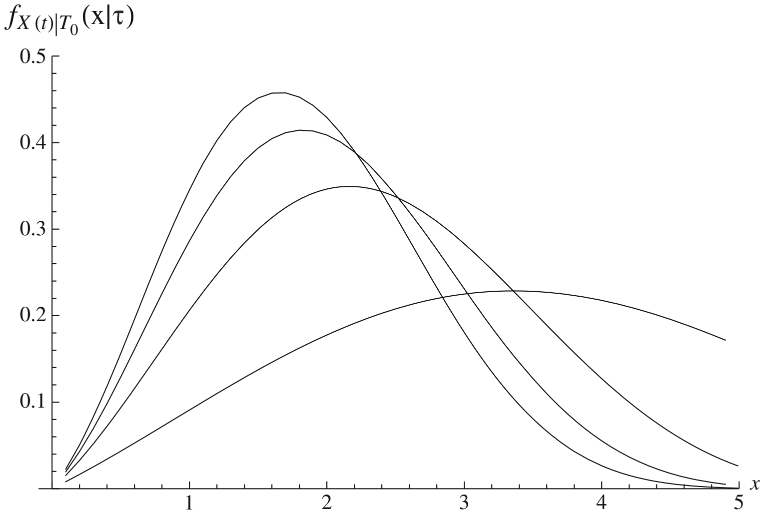

Fig. 6 Conditional density of $X(t)$ given $T_0 = \tau$, for $\tau = 6$, $t = 5$ and $\lambda = 2$ with $\mu = 0.1, 0.5, 1, 1.5$ from bottom to top near the origin

Finally, we omit the determination of the conditional distribution of $X(t)$ within a renewal cycle in the case of non-zero initial state, since the involved calculations are very cumbersome.

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