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Optimal Harvesting for a Stochastic Predator-prey Model with S-type Distributed Time Delays

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Abstract In this paper, the optimization problem of harvesting for a stochastic predator-prey model with S-type distributed time delays (which contain both discrete time delays and continuously distributed time delays) is studied by using ergodic method. Sufficient and necessary conditions for the existence of optimal harvesting strategy are obtained. Moreover, the optimal harvesting effort (OHE, for short) and the maximum of expectation of sustainable yield (MESY, for short) are given. Some numerical simulations are introduced to illustrate our main results.

Keywords Optimal harvesting · Stochastic predator-prey model · Time delay · Ergodic method

Mathematics Subject Classification (2010) 60H10 · 60H30 · 92D25

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1 Introduction

Optimal harvesting problem is an important and interesting topic from both biological and mathematical point of view (Zou and Wang 2014). Since Clark's works (see Clark 1976, 1990) on the following deterministic logistic model:

$$dx(t) = x(t)\,[r - ax(t)]\,dt - hx(t)dt, \tag{1.1}$$

the problem of optimal harvesting has caused wide public concern over the recent years (see e.g. Zou and Wang 2014, Zou et al. 2013, Li et al. 2011, Liu and Bai 2015, Liu 2015, Bedington and May 1977, Li and Wang 2010, Liu and Bai 2014, a, b). Particularly, it is necessary to investigate the optimal harvesting problem of predator-prey model which is one of the most popular area in biological systems (see e.g. Thripathi et al. 2015, Liu and Bai 2014).

On the one hand, it is essential to take time delays into account since delays are ubiquitous in the real world. As is well known, systems with discrete time delays and those with continuously distributed time delays do not contain each other. However, systems with S-type distributed time delays contain both (see e.g. Wang et al. 2009, Wang and Xu 2002). Considering S-type distributed time delays, we establish the following stochastic time-delay predator-prey model with harvesting:

$$\begin{cases} \dfrac{dx(t)}{dt} =x(t)\left[r_1 - h_1 - a_{11}x(t) - \displaystyle\int_{-\tau_{12}}^{0} y(t+\theta)d\mu_{12}(\theta)\right], \\ \dfrac{dy(t)}{dt} =y(t)\left[-r_2 - h_2 + \displaystyle\int_{-\tau_{21}}^{0} x(t+\theta)d\mu_{21}(\theta) - a_{22}y(t)\right], \end{cases} \tag{1.2}$$

where $r_1 > 0$ is the intrinsic growth rate of prey and $r_2 > 0$ is the death rate of predator. $a_{11} > 0$ and $a_{22} > 0$ are intra-specific competition coefficients of prey and predator, respectively. $\int_{-\tau_{12}}^{0} y(t+\theta)d\mu_{12}(\theta)$ and $\int_{-\tau_{21}}^{0} x(t+\theta)d\mu_{21}(\theta)$ are Lebesgue-Stieltjes integrals. $\mu_{12}(\theta)$ and $\mu_{21}(\theta)$ are nondecreasing bounded variation functions defined on $[-\gamma, 0]$, $\gamma = \max\{\tau_{12}, \tau_{21}\}$. $h_1 \geq 0$ and $h_2 \geq 0$ represent, respectively, the harvesting effort of prey and predator.

On the other hand, in the real world population systems are inevitably subject to environmental noise and many scholars have studied optimal harvesting problems for many stochastic population systems (see e.g. Liu and Bai 2015, Liu 2015, Gard 1986, Mao 1994, Mao et al. 2002, Li and Mao 2009, Zhu and Yin 2009). To the best of our knowledge to date, the problem of optimal harvesting for stochastic predator-prey model with S-type distributed time delays has not been investigated in the existing literature. So, in this paper we consider the optimization problem of harvesting for the following stochastic predator-prey model with S-type distributed time delays:

$$\begin{cases} dx(t) =x(t)\left[r_1 - h_1 - a_{11}x(t) - \displaystyle\int_{-\tau_{12}}^{0} y(t+\theta)d\mu_{12}(\theta)\right]dt + \sigma_1 x(t)dB_1(t), \\ dy(t) =y(t)\left[-r_2 - h_2 + \displaystyle\int_{-\tau_{21}}^{0} x(t+\theta)d\mu_{21}(\theta) - a_{22}y(t)\right]dt + \sigma_2 y(t)dB_2(t), \end{cases} \tag{1.3}$$

where $B_1(t)$ and $B_2(t)$ are standard independent Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. σ_i^2 is the intensity of environmental noise, $i = 1, 2$.

For the optimal harvesting problem of stochastic population systems, one method is to solve the corresponding Fokker-Planck equations (see e.g. Beddington and May 1977, Li and Wang 2010, Liu and Bai 2014). However, it is difficult to obtain explicit solutions to the corresponding time-delay Fokker-Planck equations Liu and Bai 2015. Another method to investigate the optimal harvesting problem of stochastic population systems is the ergodic method proposed by Zou et al. (2013). The advantage of this method is that it is unnecessary to solve the corresponding Fokker-Planck equation (see e.g. Liu and Bai 2015, a). Hence, by combining stochastic analytical techniques with the ergodic method, we are devoted to get the optimal harvesting effort $H^* = (h_1^*, h_2^*)^{\mathrm{T}}$ such that

(i) Both the prey $x(t)$ and the predator $y(t)$ are not extinct;

(ii) The expectation of sustained yield $Y(H) = \lim_{t\to+\infty} \mathbb{E}\,[h_1 x(t) + h_2 y(t)]$ is maximal.

2 Main Results

Throughout this paper, T represents a generic positive constant whose values may vary at its different appearances. For convenience of reference, we denote $b_1 = r_1 - h_1 - \frac{\sigma_1^2}{2}$, $b_2 = r_2 + h_2 + \frac{\sigma_2^2}{2}$ and recall some inequalities stated as a lemma.

Lemma 1

$$\left(\sum_{i=1}^{n} A_i\right)^p \leq n^p \sum_{i=1}^{n} A_i^p, \ \forall p > 0, \ A_i \geq 0, \ 1 \leq i \leq n,$$
$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}, \ A \geq 0, \ B \geq 0, \ p > 0, \ q > 0, \ \frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 2 *For any initial data $(\xi(\theta), \eta(\theta))^{\mathrm{T}} \in C([-\gamma, 0], \mathbb{R}_+^2)$, system (1.3) has a unique global positive solution $z(t) = (x(t), y(t))^{\mathrm{T}}$ almost surely (a.s.). Moreover, for any $p > 0$, there exist $K_1(p) > 0$ and $K_2(p) > 0$ such that*

$$\limsup_{t\to+\infty} \mathbb{E}\left[x^p(t)\right] \leq K_1(p) \ \textit{and} \ \limsup_{t\to+\infty} \mathbb{E}\left[y^p(t)\right] \leq K_2(p). \tag{2.1}$$

The detailed proof of Lemma 2 will be given in Appendix.

Theorem 1 *For system (1.3) :*

$(\mathcal{A}_1)$ *If $b_1 < 0$, then $\lim_{t\to+\infty} x(t) = 0$ a.s. and $\lim_{t\to+\infty} y(t) = 0$ a.s.*

$(\mathcal{A}_2)$ *If $b_1 = 0$, then $\lim_{t\to+\infty} \frac{\int_0^t x(s)ds}{t} = 0$ a.s.*

($\mathcal{A}_3$) *If* $b_1 > 0$ *and* $b_1 \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} < 0$, *then* $\lim_{t\to+\infty} \frac{\int_0^t x(s)ds}{t} = \frac{b_1}{a_{11}}$ *a.s. and* $\lim_{t\to+\infty} y(t) = 0$ *a.s.*

($\mathcal{A}_4$) *If* $b_1 \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} = 0$, *then* $\lim_{t\to+\infty} \frac{\int_0^t y(s)ds}{t} = 0$ *a.s.*

($\mathcal{A}_5$) *If* $b_1 \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} > 0$, *then*

$$\begin{cases} \lim\limits_{t\to+\infty} \dfrac{\int_0^t x(s)ds}{t} = \dfrac{b_1 a_{22} + b_2 \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)}{a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta) \int_{-\tau_{21}}^{0} d\mu_{21}(\theta)} \ a.s. \\ \lim\limits_{t\to+\infty} \dfrac{\int_0^t y(s)ds}{t} = \dfrac{b_1 \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11}}{a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta) \int_{-\tau_{21}}^{0} d\mu_{21}(\theta)} \ a.s. \end{cases} \tag{2.2}$$

Proof Consider the following auxiliary system:

$$\begin{cases} dY_1(t) = Y_1(t)\left[r_1 - h_1 - a_{11}Y_1(t)\right]dt + \sigma_1 Y_1(t)dB_1(t), \\ dY_2(t) = Y_2(t)\left[-r_2 - h_2 + \int_{-\tau_{21}}^{0} Y_1(t+\theta)d\mu_{21}(\theta) - a_{22}Y_2(t)\right]dt \\ \qquad\qquad + \sigma_2 Y_2(t)dB_2(t), \\ Y_1(\theta) = \xi(\theta), \ Y_2(\theta) = \eta(\theta), \ -\gamma \le \theta \le 0. \end{cases} \tag{2.3}$$

By Itô's formula we have

$$\begin{cases} d\ln Y_1(t) = \left[b_1 - a_{11}Y_1(t)\right]dt + \sigma_1 dB_1(t), \\ d\ln Y_2(t) = \left[-b_2 + \int_{-\tau_{21}}^{0} Y_1(t+\theta)d\mu_{21}(\theta) - a_{22}Y_2(t)\right]dt + \sigma_2 dB_2(t). \end{cases} \tag{2.4}$$

In a view of system (2.4), we compute

$$\begin{cases} \ln Y_1(t) - \ln Y_1(0) \\ = b_1 t - a_{11}\int_0^t Y_1(s)ds + \sigma_1 B_1(t), \\ \ln Y_2(t) - \ln Y_2(0) \\ = -b_2 t + \int_0^t \int_{-\tau_{21}}^{0} Y_1(s+\theta)d\mu_{21}(\theta)ds - a_{22}\int_0^t Y_2(s)ds + \sigma_2 B_2(t) \\ = -b_2 t + \int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\int_0^t Y_1(s)ds - a_{22}\int_0^t Y_2(s)ds + \sigma_2 B_2(t) \\ \quad + \int_{-\tau_{21}}^{0}\int_{\theta}^{0} Y_1(s)ds d\mu_{21}(\theta) - \int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} Y_1(s)ds d\mu_{21}(\theta). \end{cases} \tag{2.5}$$

From the first equation of system (2.5) we deduce that for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for any $t > T$,

$$\begin{aligned}&(b_1-\varepsilon)t - a_{11}\int_0^t Y_1(s)ds + \sigma_1 B_1(t)\\ &< \ln Y_1(t) < (b_1+\varepsilon)t - a_{11}\int_0^t Y_1(s)ds + \sigma_1 B_1(t).\end{aligned} \tag{2.6}$$

Based on Lemma 2 in Liu and Wang (2014) and the arbitrariness of ε, we obtain

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t Y_1(s)ds = \frac{b_1}{a_{11}}\ a.s.,\ \ if\ b_1 > 0. \tag{2.7}$$

Combining (2.7) with the first equation of system (2.5) yields

$$\lim_{t\to+\infty}\tfrac{\ln Y_1(t)}{t} = 0\ a.s.,\ \ if\ b_1 > 0. \tag{2.8}$$

In view of system (2.5), we compute

$$\begin{aligned}&\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\left[\ln Y_1(t)-\ln Y_1(0)\right] + a_{11}\left[\ln Y_2(t)-\ln Y_2(0)\right]\\ &=\left(b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2 a_{11}\right)t - a_{11}a_{22}\int_0^t Y_2(s)ds\\ &\quad+\sigma_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta)B_1(t) + \sigma_2 a_{11}B_2(t)\\ &\quad+a_{11}\left(\int_{-\tau_{21}}^0\int_\theta^0 Y_1(s)dsd\mu_{21}(\theta) - \int_{-\tau_{21}}^0\int_{t+\theta}^t Y_1(s)dsd\mu_{21}(\theta)\right).\end{aligned} \tag{2.9}$$

On the basis of Eq. 2.7, for $b_1 > 0$, we get

$$\begin{aligned}&\frac{1}{t}\left|\int_{-\tau_{21}}^0\int_\theta^0 Y_1(s)dsd\mu_{21}(\theta) - \int_{-\tau_{21}}^0\int_{t+\theta}^t Y_1(s)dsd\mu_{21}(\theta)\right|\\ &\le\frac{1}{t}\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\int_{-\tau_{21}}^0 Y_1(s)ds\\ &\quad+\frac{1}{t}\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\left(\int_0^t Y_1(s)ds - \int_0^{t-\tau_{21}} Y_1(s)ds\right) \to 0,\ \ (t\to+\infty).\end{aligned} \tag{2.10}$$

In the light of Eqs. 2.8, 2.9 and 2.10 we observe that if $b_1 > 0$, then for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for any $t > T$,

$$\begin{cases} a_{11}\ln Y_2(t) < \left(b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2a_{11} + \varepsilon\right)t - a_{11}a_{22}\int_0^t Y_2(s)ds\\ \qquad\qquad + \sigma_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta)B_1(t) + \sigma_2a_{11}B_2(t),\\ a_{11}\ln Y_2(t) > \left(b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2a_{11} - \varepsilon\right)t - a_{11}a_{22}\int_0^t Y_2(s)ds\\ \qquad\qquad + \sigma_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta)B_1(t) + \sigma_2a_{11}B_2(t).\end{cases} \tag{2.11}$$

According to Lemma 2 in Liu and Wang (2014) and the arbitrariness of ε, we have

$$\begin{cases} \lim_{t\to+\infty} \frac{1}{t}\int_0^t Y_2(s)ds = \frac{b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2a_{11}}{a_{11}a_{22}}\ a.s., \\ if\ b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2a_{11} \geq 0; \\ \lim_{t\to+\infty} Y_2(t) = 0\ a.s.,\ if\ b_1 > 0\ and\ b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2a_{11} < 0. \end{cases} \tag{2.12}$$

From Theorem 2.1 in Bao and Yuan (2011) we obtain

$$x(t) \leq Y_1(t)\ and\ y(t) \leq Y_2(t)\ a.s.,\ \ t \in [0, +\infty). \tag{2.13}$$

Applying Itô's formula to $\ln x(t)$ and $\ln y(t)$ lead to

$$\begin{cases} d\ln x(t) = \left[b_1 - a_{11}x(t) - \int_{-\tau_{12}}^0 y(t+\theta)d\mu_{12}(\theta)\right]dt + \sigma_1 dB_1(t), \\ d\ln y(t) = \left[-b_2 + \int_{-\tau_{21}}^0 x(t+\theta)d\mu_{21}(\theta) - a_{22}y(t)\right]dt + \sigma_2 dB_2(t). \end{cases} \tag{2.14}$$

Based on system (2.14), we compute

$$\begin{cases} \ln x(t) - \ln x(0) \\ = b_1t - a_{11}\int_0^t x(s)ds - \int_0^t\int_{-\tau_{12}}^0 y(s+\theta)d\mu_{12}(\theta)ds + \sigma_1B_1(t) \\ = b_1t - a_{11}\int_0^t x(s)ds - \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_0^t y(s)ds + \sigma_1B_1(t) \\ \quad + \int_{-\tau_{12}}^0\int_{t+\theta}^t y(s)dsd\mu_{12}(\theta) - \int_{-\tau_{12}}^0\int_\theta^0 y(s)dsd\mu_{12}(\theta), \\ \ln y(t) - \ln y(0) \\ = -b_2t + \int_0^t\int_{-\tau_{21}}^0 x(s+\theta)d\mu_{21}(\theta)ds - a_{22}\int_0^t y(s)ds + \sigma_2B_2(t) \\ = -b_2t + \int_{-\tau_{21}}^0 d\mu_{21}(\theta)\int_0^t x(s)ds - a_{22}\int_0^t y(s)ds + \sigma_2B_2(t) \\ \quad + \int_{-\tau_{21}}^0\int_\theta^0 x(s)dsd\mu_{21}(\theta) - \int_{-\tau_{21}}^0\int_{t+\theta}^t x(s)dsd\mu_{21}(\theta). \end{cases} \tag{2.15}$$

From the first equation of system (2.15) we derive

$$\lim_{t\to+\infty} x(t) = 0\ a.s.,\ \ if\ b_1 < 0. \tag{2.16}$$

Hence, for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for any $t > T$,

$$\begin{aligned}
&\ln y(t) - \ln y(0) \\
\leq& - b_2 t + \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \int_0^t x(s)ds \\
&- a_{22}\int_0^t y(s)ds + \sigma_2 B_2(t) + \int_{-\tau_{21}}^{0}\int_{\theta}^{0} x(s)dsd\mu_{21}(\theta) \\
\leq& - b_2 t + \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \int_0^t x(s)ds \\
&- a_{22}\int_0^t y(s)ds + \sigma_2 B_2(t) + \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \int_{-\tau_{21}}^{0} x(s)ds \\
\leq& - b_2 t + \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \int_0^T x(s)ds + \int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\varepsilon(t-T) \\
&- a_{22}\int_0^t y(s)ds + \sigma_2 B_2(t) + \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \int_{-\tau_{21}}^{0} x(s)ds,
\end{aligned} \tag{2.17}$$

which implies that for sufficiently large t,

$$\ln y(t) \leq \left(-b_2 + \int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\varepsilon + \varepsilon\right)t - a_{22}\int_0^t y(s)ds + \sigma_2 B_2(t). \tag{2.18}$$

In view of Eq. 2.18 and the arbitrariness of ε, we obtain

$$\lim_{t\to+\infty} y(t) = 0 \ a.s., \ if \ b_1 < 0. \tag{2.19}$$

So $(\mathcal{A}_1)$ follows from combining (2.16) with Eq. 2.19.

By the first equation of system (2.15), we have

$$\ln x(t) - \ln x(0) \leq b_1 t - a_{11}\int_0^t x(s)ds + \sigma_1 B_1(t). \tag{2.20}$$

According to Lemma 2 in Liu and Wang (2014), we deduce that

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t x(s)ds \leq \frac{b_1}{a_{11}} \ a.s., \ if \ b_1 \geq 0. \tag{2.21}$$

Hence $(\mathcal{A}_2)$ follows from Eq. 2.21.

Combining the second part of Eq. 2.12 with Eq. 2.13 gives

$$\lim_{t\to+\infty} y(t) = 0 \ a.s., \ if \ b_1 > 0 \ and \ b_1 \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} < 0. \tag{2.22}$$

Accordingly, for arbitrary $\varepsilon > 0$, there exist $T > 0$ and $\Omega(\varepsilon) \subseteq \Omega$ such that $P(\Omega(\varepsilon)) > 1-\varepsilon$ and $y(t,\omega) < \varepsilon$ $(\forall\omega \in \Omega(\varepsilon))$. Therefore, based on the first equation of system (2.15), for any $t > T + \tau_{12}$, we obatin

$$\begin{cases} \ln x(t) - \ln x(0) \le b_1 t - a_{11}\int_0^t x(s)ds + \sigma_1 B_1(t) + \varepsilon\tau_{12}\int_{-\tau_{12}}^0 d\mu_{12}(\theta), \\ \ln x(t) - \ln x(0) \ge b_1 t - a_{11}\int_0^t x(s)ds - \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_0^T y(s)ds + \sigma_1 B_1(t) \\ \qquad - \varepsilon(t-T)\int_{-\tau_{12}}^0 d\mu_{12}(\theta) - \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{12}}^0 y(s)ds. \end{cases} \tag{2.23}$$

On the basis of Eq. 2.23 we observe that for sufficiently large t,

$$\begin{cases} \ln x(t) \le (b_1+\varepsilon)\,t - a_{11}\int_0^t x(s)ds + \sigma_1 B_1(t), \\ \ln x(t) \ge \left(b_1 - \varepsilon\int_{-\tau_{12}}^0 d\mu_{12}(\theta) - \varepsilon\right) t - a_{11}\int_0^t x(s)ds + \sigma_1 B_1(t). \end{cases} \tag{2.24}$$

From Lemma 2 in Liu and Wang (2014) and the arbitrariness of ε we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t x(s)ds = \frac{b_1}{a_{11}}\ a.s., \quad if\ b_1 > 0\ and\ b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2 a_{11} < 0. \tag{2.25}$$

Hence $(\mathcal{A}_3)$ follows from combining (2.22) with Eq. 2.25.

Clearly, $(\mathcal{A}_4)$ follows from combining (2.12) with Eq. 2.13.

In the light of Eqs. 2.8 and 2.13 we deduce that

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{t} \le 0\ a.s.,\ \ if\ b_1 > 0. \tag{2.26}$$

According to system (2.15), we compute

$$\begin{aligned} &\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\left(\ln x(t) - \ln x(0)\right) + a_{11}\left(\ln y(t) - \ln y(0)\right) \\ &= \left(b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2 a_{11}\right) t \\ &\quad - \left(a_{11}a_{22} + \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\right)\int_0^t y(s)ds \\ &\quad + \sigma_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta)B_1(t) + \sigma_2 a_{11}B_2(t) + t\Phi(t), \end{aligned} \tag{2.27}$$

where

$$\begin{aligned}\Phi(t) =&\frac{1}{t}\left(\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\int_{-\tau_{12}}^{0}\int_{t+\theta}^{t} y(s)dsd\mu_{12}(\theta)\right.\\&-\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\int_{-\tau_{12}}^{0}\int_{\theta}^{0} y(s)dsd\mu_{12}(\theta)\\&\left.+a_{11}\int_{-\tau_{21}}^{0}\int_{\theta}^{0} x(s)dsd\mu_{21}(\theta)-a_{11}\int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} x(s)dsd\mu_{21}(\theta)\right).\end{aligned} \tag{2.28}$$

From Eqs. 2.26 and 2.27 we derive that if $b_1 > 0$, then for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for any $t > T$,

$$\begin{aligned}a_{11}\frac{\ln y(t)}{t} >& \left(b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)-b_2a_{11}-\varepsilon\right)\\&-\left(a_{11}a_{22}+\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)\frac{\int_0^t y(s)ds}{t}\\&+\sigma_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\frac{B_1(t)}{t}+\sigma_2a_{11}\frac{B_2(t)}{t}\\&+\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\frac{\ln x(0)}{t}+a_{11}\frac{\ln y(0)}{t}+\Phi(t).\end{aligned} \tag{2.29}$$

Based on Eq. 2.13, we have

$$\begin{cases}\frac{1}{t}\int_{t-\tau_{12}}^{t} y(s)ds \le \frac{1}{t}\int_{t-\tau_{12}}^{t} Y_2(s)ds = \frac{1}{t}\int_0^t Y_2(s)ds-\frac{1}{t}\int_0^{t-\tau_{12}} Y_2(s)ds,\\ \frac{1}{t}\int_{t-\tau_{21}}^{t} x(s)ds \le \frac{1}{t}\int_{t-\tau_{21}}^{t} Y_1(s)ds = \frac{1}{t}\int_0^t Y_1(s)ds-\frac{1}{t}\int_0^{t-\tau_{21}} Y_1(s)ds.\end{cases} \tag{2.30}$$

In view of Eqs. 2.7, 2.12 and 2.30, we obtain

$$\begin{aligned}\lim_{t\to+\infty}\frac{1}{t}\int_{t-\tau_{21}}^{t} x(s)ds = \lim_{t\to+\infty}\frac{1}{t}\int_{t-\tau_{12}}^{t} y(s)ds = 0,\\ if\ b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)-b_2a_{11}\ge 0.\end{aligned} \tag{2.31}$$

On the basis of Eq. 2.31, we obtain

$$\begin{aligned}|\Phi(t)| \le&\frac{1}{t}\left(\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{t-\tau_{12}}^{t} y(s)ds\right.\\&+\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{12}}^{0} y(s)ds\\&+a_{11}\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\int_{-\tau_{21}}^{0} x(s)ds\\&\left.+a_{11}\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\int_{t-\tau_{21}}^{t} x(s)ds\right)\to 0,\ (t\to+\infty).\end{aligned} \tag{2.32}$$

In the light of Eqs. 2.29 and 2.32, we deduce that for sufficiently large t,

$$\begin{aligned} a_{11}\frac{\ln y(t)}{t} > & \left(b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} - 2\varepsilon\right) \\ & - \left(a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)\frac{\int_0^t y(s)ds}{t} \\ & + \sigma_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\frac{B_1(t)}{t} + \sigma_2 a_{11}\frac{B_2(t)}{t}. \end{aligned} \tag{2.33}$$

From Lemma 2 in Liu and Wang (2014), Eq. 2.33 and the arbitrariness of ε, we have

$$\begin{aligned} \liminf_{t\to+\infty}\frac{1}{t}\int_0^t y(s)ds \geq \frac{b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11}}{a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)}\ a.s., \\ if\ \ b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} > 0. \end{aligned} \tag{2.34}$$

Thus, for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for any $t > T$,

$$\frac{1}{t}\int_0^t y(s)ds > \frac{b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11}}{a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)} - \varepsilon. \tag{2.35}$$

Substituting (2.35) into the first equation of system (2.15) leads to

$$\begin{aligned} & \frac{\ln x(t)}{t} - \frac{\ln x(0)}{t} \\ \leq & b_1 - a_{11}\frac{\int_0^t x(s)ds}{t} + \sigma_1\frac{B_1(t)}{t} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\frac{\int_{t-\tau_{12}}^{t} y(s)ds}{t} \\ & - \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\left(\frac{b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11}}{a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)} - \varepsilon\right). \end{aligned} \tag{2.36}$$

According to Eqs. 2.31 and 2.36, we obtain that for sufficiently large t,

$$\begin{aligned} & \ln x(t) \\ \leq & \left[b_1 - \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\left(\frac{b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11}}{a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)} - \varepsilon\right) + \varepsilon\right]t \\ & - a_{11}\int_0^t x(s)ds + \sigma_1 B_1(t). \end{aligned} \tag{2.37}$$

By Lemma 2 in Liu and Wang (2014), Eq. 2.37 and the arbitrariness of ε, we get

$$\begin{aligned} \limsup_{t\to+\infty}\frac{1}{t}\int_0^t x(s)ds \leq \frac{b_1 a_{22} + b_2\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)}{a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)}\ a.s., \\ if\ \ b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} > 0. \end{aligned} \tag{2.38}$$

Hence, for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for any $t > T$,

$$\frac{1}{t}\int_0^t x(s)ds \le \frac{b_1a_{22} + b_2\int_{-\tau_{12}}^0 d\mu_{12}(\theta)}{a_{11}a_{22} + \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{21}}^0 d\mu_{21}(\theta)} + \varepsilon. \tag{2.39}$$

Substituting (2.39) into the second equation of system (2.15) gives

$$\begin{aligned}&\frac{\ln y(t)}{t} - \frac{\ln y(0)}{t}\\ \le& - b_2 + \int_{-\tau_{21}}^0 d\mu_{21}(\theta)\left(\frac{b_1a_{22} + b_2\int_{-\tau_{12}}^0 d\mu_{12}(\theta)}{a_{11}a_{22} + \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{21}}^0 d\mu_{21}(\theta)} + \varepsilon\right)\\ &- a_{22}\frac{\int_0^t y(s)ds}{t} + \sigma_2\frac{B_2(t)}{t} + \frac{1}{t}\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\int_{-\tau_{21}}^0 x(s)ds.\end{aligned} \tag{2.40}$$

From Eq. 2.40 we obtain that for sufficiently large t,

$$\begin{aligned}&\ln y(t)\\ \le&\left[\varepsilon - b_2 + \int_{-\tau_{21}}^0 d\mu_{21}(\theta)\left(\frac{b_1a_{22} + b_2\int_{-\tau_{12}}^0 d\mu_{12}(\theta)}{a_{11}a_{22} + \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{21}}^0 d\mu_{21}(\theta)} + \varepsilon\right)\right]t\\ &- a_{22}\int_0^t y(s)ds + \sigma_2 B_2(t).\end{aligned} \tag{2.41}$$

Based on Lemma 2 in Liu and Wang (2014), Eq. 2.41 and the arbitrariness of ε, we obtain

$$\begin{aligned}\limsup_{t\to+\infty}\frac{1}{t}\int_0^t y(s)ds \le \frac{b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2a_{11}}{a_{11}a_{22} + \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{21}}^0 d\mu_{21}(\theta)}\ a.s.,\\ if\ b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2a_{11} > 0.\end{aligned} \tag{2.42}$$

Accordingly, for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for any $t > T$,

$$\frac{1}{t}\int_0^t y(s)ds < \frac{b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2a_{11}}{a_{11}a_{22} + \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{21}}^0 d\mu_{21}(\theta)} + \varepsilon. \tag{2.43}$$

Substituting (2.43) into the second equation of system (2.15) leads to

$$\begin{aligned}&\frac{\ln x(t)}{t} - \frac{\ln x(0)}{t}\\ \ge& b_1 - \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\left(\frac{b_1\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2a_{11}}{a_{11}a_{22} + \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{21}}^0 d\mu_{21}(\theta)} + \varepsilon\right)\\ &- a_{11}\frac{\int_0^t x(s)ds}{t} + \sigma_1\frac{B_1(t)}{t} - \frac{1}{t}\int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{12}}^0 y(s)ds.\end{aligned} \tag{2.44}$$

On the basis of Eq. 2.44 we obtain that for sufficiently large t,

$$\begin{aligned}
&\ln x(t)\\
&\geq \left[b_1 - \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\left(\frac{b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11}}{a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)} + \varepsilon\right) - \varepsilon\right] t \\
&\quad - a_{11}\int_0^t x(s)ds + \sigma_1 B_1(t).
\end{aligned} \tag{2.45}$$

In view of Lemma 2 in Liu and Wang (2014), Eq. 2.45 and the arbitrariness of ε, we have

$$\begin{aligned}
&\liminf_{t\to+\infty}\frac{1}{t}\int_0^t x(s)ds \geq \frac{b_1 a_{22} + b_2\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)}{a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)}\ a.s.,\\
&\qquad\qquad\qquad\qquad if\ \ b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} > 0.
\end{aligned} \tag{2.46}$$

Consequently, $(\mathcal{A}_5)$ follows from Eqs. 2.34, 2.38, 2.42 and 2.46. The proof is complete. □

Lemma 3 *Let* $(x(t;\phi), y(t;\phi))^{\mathrm{T}}$ *and* $(x(t;\phi^*), y(t;\phi^*))^{\mathrm{T}}$ *be, respectively, the solution to system* (1.3) *with initial data* ϕ *and* $\phi^* \in C([-\gamma, 0], \mathbb{R}_+^2)$. *If* $a_{11} - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) > 0,\ a_{22} - \int_{-\tau_{12}}^{0} d\mu_{12}(\theta) > 0$, *then*

$$\lim_{t\to+\infty}\mathbb{E}\sqrt{|x(t;\phi) - x(t;\phi^*)|^2 + |y(t;\phi) - y(t;\phi^*)|^2} = 0. \tag{2.47}$$

Proof For this purpose, we only need to show

$$\lim_{t\to+\infty}\mathbb{E}\left|x(t;\phi) - x(t;\phi^*)\right| = 0, \tag{2.48}$$

and

$$\lim_{t\to+\infty}\mathbb{E}\left|y(t;\phi) - y(t;\phi^*)\right| = 0. \tag{2.49}$$

Define

$$\begin{cases} W_1(t;\phi,\phi^*) = \left|\ln x(t;\phi^*) - \ln x(t;\phi)\right|, \\ W_2(t;\phi,\phi^*) = \left|\ln y(t;\phi^*) - \ln y(t;\phi)\right|. \end{cases} \tag{2.50}$$

By Itô's formula, we have

$$\begin{aligned}
\mathcal{L}[W_1(t;\phi,\phi^*)] =& sign\left(x(t;\phi^*) - x(t;\phi)\right)\Big\{-a_{11}\left[x(t;\phi^*) - x(t;\phi)\right]\\
&- \int_{-\tau_{12}}^{0}\left[y(t+\theta;\phi^*) - y(t+\theta;\phi)\right]d\mu_{12}(\theta)\Big\}\\
\leq& -a_{11}\left|x(t;\phi^*) - x(t;\phi)\right|\\
&+ \int_{-\tau_{12}}^{0}\left|y(t+\theta;\phi^*) - y(t+\theta;\phi)\right|d\mu_{12}(\theta).
\end{aligned} \tag{2.51}$$

In the same way, we get

$$\begin{aligned}\mathcal{L}[W_2(t;\phi,\phi^*)] =& sign\left(y(t;\phi^*)-y(t;\phi)\right)\Big\{-a_{22}\left[y(t;\phi^*)-y(t;\phi)\right]\\ &+\int_{-\tau_{21}}^{0}\left[x(t+\theta;\phi^*)-x(t+\theta;\phi)\right]d\mu_{21}(\theta)\Big\}\\ \le& -a_{22}\left|y(t;\phi^*)-y(t;\phi)\right|\\ &+\int_{-\tau_{21}}^{0}\left|x(t+\theta;\phi^*)-x(t+\theta;\phi)\right|d\mu_{21}(\theta).\end{aligned}\tag{2.52}$$

Define

$$W(t;\phi,\phi^*)=W_1(t;\phi,\phi^*)+W_2(t;\phi,\phi^*)+W_3(t;\phi,\phi^*),\tag{2.53}$$

where

$$\begin{aligned}W_3(t;\phi,\phi^*)=&\int_{-\tau_{12}}^{0}\int_{t+\theta}^{t}\left|y(s;\phi^*)-y(s;\phi)\right|dsd\mu_{12}(\theta)\\ &+\int_{-\tau_{21}}^{0}\int_{t+\theta}^{t}\left|x(s;\phi^*)-x(s;\phi)\right|dsd\mu_{21}(\theta).\end{aligned}\tag{2.54}$$

From Itô's formula, Eqs. 2.51, 2.52, 2.53, and 2.54 we obtain

$$\begin{aligned}\mathcal{L}[W(t;\phi,\phi^*)] =& \mathcal{L}[W_1(t;\phi,\phi^*)]+\mathcal{L}[W_2(t;\phi,\phi^*)]+\tfrac{dW_3(t;\phi,\phi^*)}{dt}\\ \le& -\left(a_{11}-\int_{-\tau_{21}}^{0}d\mu_{21}(\theta)\right)\left|x(t;\phi^*)-x(t;\phi)\right|\\ &-\left(a_{22}-\int_{-\tau_{12}}^{0}d\mu_{12}(\theta)\right)\left|y(t;\phi^*)-y(t;\phi)\right|.\end{aligned}\tag{2.55}$$

According to Eq. 2.55, we have

$$\begin{aligned}&\mathbb{E}[W(t;\phi,\phi^*)]\\ \le& W(0;\phi,\phi^*)-\left(a_{11}-\int_{-\tau_{21}}^{0}d\mu_{21}(\theta)\right)\int_0^t\mathbb{E}\left[\left|x(s;\phi^*)-x(s;\phi)\right|\right]ds\\ &-\left(a_{22}-\int_{-\tau_{12}}^{0}d\mu_{12}(\theta)\right)\int_0^t\mathbb{E}\left[\left|y(s;\phi^*)-y(s;\phi)\right|\right]ds,\end{aligned}\tag{2.56}$$

which implies

$$\begin{cases}0\le\int_0^t\mathbb{E}\left[\left|x(s;\phi^*)-x(s;\phi)\right|\right]ds\le\dfrac{W(0;\phi,\phi^*)}{a_{11}-\int_{-\tau_{21}}^{0}d\mu_{21}(\theta)},\\ 0\le\int_0^t\mathbb{E}\left[\left|y(s;\phi^*)-y(s;\phi)\right|\right]ds\le\dfrac{W(0;\phi,\phi^*)}{a_{22}-\int_{-\tau_{12}}^{0}d\mu_{12}(\theta)}.\end{cases}\tag{2.57}$$

From Eq. 2.57 we obtain

$$\begin{cases}\int_0^{+\infty}\mathbb{E}\left[\left|x(t;\phi^*)-x(t;\phi)\right|\right]dt<+\infty,\\ \int_0^{+\infty}\mathbb{E}\left[\left|y(t;\phi^*)-y(t;\phi)\right|\right]dt<+\infty.\end{cases}\tag{2.58}$$

Define $F(t)$ and $G(t)$ as follows:

$$\begin{cases} F(t) = F(t;\phi,\phi^*) = \mathbb{E}\left[\left|x(t;\phi^*) - x(t;\phi)\right|\right], \\ G(t) = G(t;\phi,\phi^*) = \mathbb{E}\left[\left|y(t;\phi^*) - y(t;\phi)\right|\right]. \end{cases} \tag{2.59}$$

Then for any $t_1, t_2 \in [0, +\infty)$, we compute

$$\begin{aligned} |F(t_2) - F(t_1)| =& \left|\mathbb{E}\left[\left|x(t_2;\phi^*) - x(t_2;\phi)\right| - \left|x(t_1;\phi^*) - x(t_1;\phi)\right|\right]\right| \\ \leq& \mathbb{E}\left[\left|\left|x(t_2;\phi^*) - x(t_2;\phi)\right| - \left|x(t_1;\phi^*) - x(t_1;\phi)\right|\right|\right] \\ \leq& \mathbb{E}\left[\left|\left(x(t_2;\phi^*) - x(t_2;\phi)\right) - \left(x(t_1;\phi^*) - x(t_1;\phi)\right)\right|\right] \\ \leq& \mathbb{E}\left[\left|x(t_2;\phi^*) - x(t_1;\phi^*)\right| + |x(t_2;\phi) - x(t_1;\phi)|\right] \\ =& \mathbb{E}\left[\left|x(t_2;\phi^*) - x(t_1;\phi^*)\right|\right] + \mathbb{E}\left[|x(t_2;\phi) - x(t_1;\phi)|\right]. \end{aligned} \tag{2.60}$$

In the same way, we obtain

$$|G(t_2) - G(t_1)| \leq \mathbb{E}\left[\left|y(t_2;\phi^*) - y(t_1;\phi^*)\right|\right] + \mathbb{E}\left[|y(t_2;\phi) - y(t_1;\phi)|\right]. \tag{2.61}$$

Clearly, system (1.3) is equivalent to the following stochastic integral system:

$$\begin{cases} x(t) =x(0) + \displaystyle\int_0^t x(s)\left[r_1 - h_1 - a_{11}x(s) - \int_{-\tau_{12}}^0 y(s+\theta)d\mu_{12}(\theta)\right]ds \\ \qquad + \displaystyle\int_0^t \sigma_1 x(s)dB_1(s), \\ y(t) =y(0) + \displaystyle\int_0^t y(s)\left[-r_2 - h_2 + \int_{-\tau_{21}}^0 x(s+\theta)d\mu_{21}(\theta) - a_{22}y(s)\right]ds \\ \qquad + \displaystyle\int_0^t \sigma_2 y(s)dB_2(s). \end{cases} \tag{2.62}$$

Based on system (2.62), we compute

$$\begin{cases} x(t_2;\phi) - x(t_1;\phi) \\ = \displaystyle\int_{t_1}^{t_2} x(s;\phi)\left[r_1 - h_1 - a_{11}x(s;\phi) - \int_{-\tau_{12}}^0 y(s+\theta;\phi)d\mu_{12}(\theta)\right]ds \\ \quad + \displaystyle\int_{t_1}^{t_2} \sigma_1 x(s;\phi)dB_1(s), \\ y(t_2;\phi) - y(t_1;\phi) \\ = \displaystyle\int_{t_1}^{t_2} y(s;\phi)\left[-r_2 - h_2 + \int_{-\tau_{21}}^0 x(s+\theta;\phi)d\mu_{21}(\theta) - a_{22}y(s;\phi)\right]ds \\ \quad + \displaystyle\int_{t_1}^{t_2} \sigma_2 y(s;\phi)dB_2(s). \end{cases} \tag{2.63}$$

For $t_2 > t_1$ and $p > 1$, on the basis of Hölder's inequality, the first equation of system (2.63) and Lemma 1, we have

$$\begin{aligned}
&(\mathbb{E}\left[|x(t_2;\phi)-x(t_1;\phi)|\right])^p \leq \mathbb{E}\left[|x(t_2;\phi)-x(t_1;\phi)|^p\right]\\
\leq&\mathbb{E}\left[\left(\int_{t_1}^{t_2} x(s;\phi)\left|r_1-h_1-a_{11}x(s;\phi)-\int_{-\tau_{12}}^{0} y(s+\theta;\phi)d\mu_{12}(\theta)\right|ds\right.\right.\\
&\left.\left.+\left|\int_{t_1}^{t_2}\sigma_1 x(s;\phi)dB_1(s)\right|\right)^p\right]\\
\leq&2^p\mathbb{E}\left[\left(\int_{t_1}^{t_2} x(s;\phi)\left|r_1-h_1-a_{11}x(s;\phi)-\int_{-\tau_{12}}^{0} y(s+\theta;\phi)d\mu_{12}(\theta)\right|ds\right)^p\right]\\
&+2^p\mathbb{E}\left[\left|\int_{t_1}^{t_2}\sigma_1 x(s;\phi)dB_1(s)\right|^p\right].
\end{aligned} \tag{2.64}$$

In the same manner, we deduce

$$\begin{aligned}
&(\mathbb{E}\left[|y(t_2;\phi)-y(t_1;\phi)|\right])^p \leq \mathbb{E}\left[|y(t_2;\phi)-y(t_1;\phi)|^p\right]\\
\leq&\mathbb{E}\left[\left(\int_{t_1}^{t_2} y(s;\phi)\left|-r_2-h_2-a_{22}y(s;\phi)+\int_{-\tau_{21}}^{0} x(s+\theta;\phi)d\mu_{21}(\theta)\right|ds\right.\right.\\
&\left.\left.+\left|\int_{t_1}^{t_2}\sigma_2 y(s;\phi)dB_2(s)\right|\right)^p\right]\\
\leq&2^p\mathbb{E}\left[\left(\int_{t_1}^{t_2} y(s;\phi)\left|-r_2-h_2-a_{22}y(s;\phi)+\int_{-\tau_{21}}^{0} x(s+\theta;\phi)d\mu_{21}(\theta)\right|ds\right)^p\right]\\
&+2^p\mathbb{E}\left[\left|\int_{t_1}^{t_2}\sigma_2 y(s;\phi)dB_2(s)\right|^p\right].
\end{aligned} \tag{2.65}$$

From Hölder's inequality and Lemma 1 again we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{t_1}^{t_2} x(s;\phi)\left|r_1-h_1-a_{11}x(s;\phi)-\int_{-\tau_{12}}^{0} y(s+\theta;\phi)d\mu_{12}(\theta)\right|ds\right)^p\right]\\
\leq&\mathbb{E}\left[\left(\int_{t_1}^{t_2}\left(|r_1-h_1|\,x(s;\phi)+a_{11}x^2(s;\phi)\right.\right.\right.\\
&\left.\left.\left.+\int_{-\tau_{12}}^{0} x(s;\phi)y(s+\theta;\phi)d\mu_{12}(\theta)\right)ds\right)^p\right]\\
\leq&(t_2-t_1)^{p-1}\,\mathbb{E}\left[\int_{t_1}^{t_2}\left(|r_1-h_1|\,x(s;\phi)+a_{11}x^2(s;\phi)\right.\right.\\
&\left.\left.+\int_{-\tau_{12}}^{0} x(s;\phi)y(s+\theta;\phi)d\mu_{12}(\theta)\right)^p ds\right]\\
\leq&(t_2-t_1)^{p-1}\,\mathbb{E}\left[\int_{t_1}^{t_2} 3^p\left(|r_1-h_1|^p\,x^p(s;\phi)+a_{11}^p x^{2p}(s;\phi)\right.\right.\\
&\left.\left.+\left(\int_{-\tau_{12}}^{0} x(s;\phi)y(s+\theta;\phi)d\mu_{12}(\theta)\right)^p\right)ds\right]
\end{aligned}$$

$$
\begin{aligned}
=&3^p|r_1-h_1|^p\,(t_2-t_1)^{p-1}\int_{t_1}^{t_2}\mathbb{E}\left[x^p(s;\phi)\right]ds\\
&+3^p a_{11}^p\,(t_2-t_1)^{p-1}\int_{t_1}^{t_2}\mathbb{E}\left[x^{2p}(s;\phi)\right]ds\\
&+3^p\,(t_2-t_1)^{p-1}\,\mathbb{E}\left[\int_{t_1}^{t_2}\left(\int_{-\tau_{12}}^0 x(s;\phi)y(s+\theta;\phi)d\mu_{12}(\theta)\right)^p ds\right].
\end{aligned}
\tag{2.66}
$$

Similarly, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_{t_1}^{t_2} y(s;\phi)\left|-r_2-h_2-a_{22}y(s;\phi)+\int_{-\tau_{21}}^0 x(s+\theta;\phi)d\mu_{21}(\theta)\right|ds\right)^p\right]\\
\le&\mathbb{E}\left[\left(\int_{t_1}^{t_2}\left(|r_2+h_2|\,y(s;\phi)+a_{22}y^2(s;\phi)\right.\right.\right.\\
&\left.\left.\left.+\int_{-\tau_{21}}^0 y(s;\phi)x(s+\theta;\phi)d\mu_{21}(\theta)\right)ds\right)^p\right]\\
\le&(t_2-t_1)^{p-1}\,\mathbb{E}\left[\int_{t_1}^{t_2}\left(|r_2+h_2|\,y(s;\phi)+a_{22}y^2(s;\phi)\right.\right.\\
&\left.\left.+\int_{-\tau_{21}}^0 y(s;\phi)x(s+\theta;\phi)d\mu_{21}(\theta)\right)^p ds\right]\\
\le&3^p|r_2+h_2|^p\,(t_2-t_1)^{p-1}\int_{t_1}^{t_2}\mathbb{E}\left[y^p(s;\phi)\right]ds\\
&+3^p a_{22}^p\,(t_2-t_1)^{p-1}\int_{t_1}^{t_2}\mathbb{E}\left[y^{2p}(s;\phi)\right]ds\\
&+3^p\,(t_2-t_1)^{p-1}\,\mathbb{E}\left[\int_{t_1}^{t_2}\left(\int_{-\tau_{21}}^0 y(s;\phi)x(s+\theta;\phi)d\mu_{21}(\theta)\right)^p ds\right].
\end{aligned}
\tag{2.67}
$$

By Hölder's inequality and Lemma 1 once again, we get

$$
\begin{aligned}
&\mathbb{E}\left[\int_{t_1}^{t_2}\left(\int_{-\tau_{12}}^0 x(s;\phi)y(s+\theta;\phi)d\mu_{12}(\theta)\right)^p ds\right]\\
\le&\mathbb{E}\left[\int_{t_1}^{t_2}\left(\frac{1}{2}\int_{-\tau_{12}}^0 d\mu_{12}(\theta)x^2(s;\phi)+\frac{1}{2}\int_{-\tau_{12}}^0 y^2(s+\theta;\phi)d\mu_{12}(\theta)\right)^p ds\right]\\
\le&\mathbb{E}\left[\int_{t_1}^{t_2}\left(\left(\int_{-\tau_{12}}^0 d\mu_{12}(\theta)\right)^p x^{2p}(s;\phi)+\left(\int_{-\tau_{12}}^0 y^2(s+\theta;\phi)d\mu_{12}(\theta)\right)^p\right)ds\right]\\
\le&\mathbb{E}\left[\int_{t_1}^{t_2}\left(\left(\int_{-\tau_{12}}^0 d\mu_{12}(\theta)\right)^p x^{2p}(s;\phi)\right.\right.\\
&\left.\left.+\left(\int_{-\tau_{12}}^0 d\mu_{12}(\theta)\right)^{p-1}\int_{-\tau_{12}}^0 y^{2p}(s+\theta;\phi)d\mu_{12}(\theta)\right)ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta) \right)^{p} \int_{t_1}^{t_2} \mathbb{E}\left[x^{2p}(s;\phi) \right] ds \\
&\quad + \left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta) \right)^{p-1} \int_{t_1}^{t_2} \int_{-\tau_{12}}^{0} \mathbb{E}\left[y^{2p}(s+\theta;\phi) \right] d\mu_{12}(\theta) ds.
\end{aligned}
\tag{2.68}
$$

Analogously, we derive

$$
\begin{aligned}
&\mathbb{E}\left[\int_{t_1}^{t_2} \left(\int_{-\tau_{21}}^{0} y(s;\phi)x(s+\theta;\phi)d\mu_{21}(\theta) \right)^{p} ds \right] \\
\le& \mathbb{E}\left[\int_{t_1}^{t_2} \left(\frac{1}{2} \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) y^2(s;\phi) + \frac{1}{2} \int_{-\tau_{21}}^{0} x^2(s+\theta;\phi) d\mu_{21}(\theta) \right)^{p} ds \right] \\
\le& \mathbb{E}\left[\int_{t_1}^{t_2} \left(\left(\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \right)^{p} y^{2p}(s;\phi) + \left(\int_{-\tau_{21}}^{0} x^2(s+\theta;\phi) d\mu_{21}(\theta) \right)^{p} \right) ds \right] \\
\le& \mathbb{E}\left[\int_{t_1}^{t_2} \left(\left(\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \right)^{p} y^{2p}(s;\phi) \right.\right. \\
&\left.\left. + \left(\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \right)^{p-1} \int_{-\tau_{21}}^{0} x^{2p}(s+\theta;\phi) d\mu_{21}(\theta) \right) ds \right] \\
=& \left(\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \right)^{p} \int_{t_1}^{t_2} \mathbb{E}\left[y^{2p}(s;\phi) \right] ds \\
&+ \left(\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \right)^{p-1} \int_{t_1}^{t_2} \int_{-\tau_{21}}^{0} \mathbb{E}\left[x^{2p}(s+\theta;\phi) \right] d\mu_{21}(\theta) ds.
\end{aligned}
\tag{2.69}
$$

In view of Theorem 7.1 in Mao (2007), for $t_2 > t_1$ and $p \ge 2$, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left| \int_{t_1}^{t_2} \sigma_1 x(s;\phi) dB_1(s) \right|^{p} \right] \\
\le& \left(\tfrac{p(p-1)}{2} \right)^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \mathbb{E}\left[\int_{t_1}^{t_2} |\sigma_1 x(s;\phi)|^{p} ds \right] \\
=& |\sigma_1|^{p} \left(\tfrac{p(p-1)}{2} \right)^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} \mathbb{E}\left[x^{p}(s;\phi) \right] ds.
\end{aligned}
\tag{2.70}
$$

And

$$
\begin{aligned}
&\mathbb{E}\left[\left| \int_{t_1}^{t_2} \sigma_2 y(s;\phi) dB_2(s) \right|^{p} \right] \\
\le& \left(\tfrac{p(p-1)}{2} \right)^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \mathbb{E}\left[\int_{t_1}^{t_2} |\sigma_2 y(s;\phi)|^{p} ds \right] \\
=& |\sigma_2|^{p} \left(\tfrac{p(p-1)}{2} \right)^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} \mathbb{E}\left[y^{p}(s;\phi) \right] ds.
\end{aligned}
\tag{2.71}
$$

From Lemma 2 we observe that there exist $K_1^{**}(p) > 0$ and $K_2^{**}(p) > 0$ such that $\sup_{t\geq-\gamma}\mathbb{E}\left[x^p(t)\right] \leq K_1^{**}(p)$ and $\sup_{t\geq-\gamma}\mathbb{E}\left[y^p(t)\right] \leq K_2^{**}(p)$. Hence, based on Eqs. 2.64, 2.66, 2.68 and 2.70, we deduce that for $p \geq 2$ and $|t_2 - t_1| \leq \delta$,

$$\begin{aligned}
&(\mathbb{E}\left[|x(t_2;\phi) - x(t_1;\phi)|\right])^p \\
\leq &2^p\left[|\sigma_1|^p\left(\tfrac{p(p-1)}{2}\right)^{\frac{p}{2}} K_1^{**}(p)\,(t_2-t_1)^{\frac{p}{2}}\right] \\
&+ 2^p\left[3^p|r_1-h_1|^pK_1^{**}(p)\,(t_2-t_1)^p + 3^pa_{11}^pK_1^{**}(2p)\,(t_2-t_1)^p\right] \\
&+ 2^p3^p\left(\int_{-\tau_{12}}^0 d\mu_{12}(\theta)\right)^p\left[K_1^{**}(2p)+K_2^{**}(2p)\right](t_2-t_1)^p \\
\leq &M_1^{**}\,|t_2-t_1|^{\frac{p}{2}},
\end{aligned} \tag{2.72}$$

where

$$\begin{aligned}
M_1^{**} = &|\sigma_1|^p\,[2p(p-1)]^{\frac{p}{2}}\,K_1^{**}(p) + [36\delta]^{\frac{p}{2}}\left[|r_1-h_1|^pK_1^{**}(p) + a_{11}^pK_1^{**}(2p)\right] \\
&+ [36\delta]^{\frac{p}{2}}\left(\int_{-\tau_{12}}^0 d\mu_{12}(\theta)\right)^p\left[K_1^{**}(2p)+K_2^{**}(2p)\right].
\end{aligned} \tag{2.73}$$

Thus, by combining (2.60) with Eq. 2.72 we obtain

$$\begin{aligned}
|F(t_2)-F(t_1)| \leq &\mathbb{E}\left[\left|x(t_2;\phi^*)-x(t_1;\phi^*)\right|\right] + \mathbb{E}\left[|x(t_2;\phi)-x(t_1;\phi)|\right] \\
\leq &2\sqrt[p]{M_1^{**}}\sqrt{|t_2-t_1|}.
\end{aligned} \tag{2.74}$$

Similarly, on the basis of Eqs. 2.65, 2.67, 2.69 and 2.71, we derive that for $p \geq 2$ and $|t_2 - t_1| \leq \delta$,

$$\begin{aligned}
&(\mathbb{E}\left[|y(t_2;\phi) - y(t_1;\phi)|\right])^p \\
\leq &2^p\left[|\sigma_2|^p\left(\tfrac{p(p-1)}{2}\right)^{\frac{p}{2}} K_2^{**}(p)\,(t_2-t_1)^{\frac{p}{2}}\right] \\
&+ 2^p\left[3^p|r_2+h_2|^pK_2^{**}(p)\,(t_2-t_1)^p + 3^pa_{22}^pK_2^{**}(2p)\,(t_2-t_1)^p\right] \\
&+ 2^p3^p\left(\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\right)^p\left[K_1^{**}(2p)+K_2^{**}(2p)\right](t_2-t_1)^p \\
\leq &M_2^{**}\,|t_2-t_1|^{\frac{p}{2}},
\end{aligned} \tag{2.75}$$

where

$$\begin{aligned}
M_2^{**} = &|\sigma_2|^p\,[2p(p-1)]^{\frac{p}{2}}\,K_2^{**}(p) + [36\delta]^{\frac{p}{2}}\left[|r_2+h_2|^pK_2^{**}(p) + a_{22}^pK_2^{**}(2p)\right] \\
&+ [36\delta]^{\frac{p}{2}}\left(\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\right)^p\left[K_1^{**}(2p)+K_2^{**}(2p)\right].
\end{aligned} \tag{2.76}$$

Therefore, it follows from Eqs. 2.61 and 2.75 that

$$\begin{aligned}
|G(t_2)-G(t_1)| \leq &\mathbb{E}\left[|y(t_2;\phi)-y(t_1;\phi)|\right] + \mathbb{E}\left[\left|y(t_2;\phi^*)-y(t_1;\phi^*)\right|\right] \\
\leq &2\sqrt[p]{M_2^{**}}\sqrt{|t_2-t_1|}.
\end{aligned} \tag{2.77}$$

So Eq. 2.48 follows from the first inequality of Eqs. 2.58, 2.74 and Barbalat's conclusion in Barbalat (1959); Eq. 2.49 follows from the second inequality of Eq. 2.58, 2.77 and Barbalat's conclusion in Barbalat (1959). The proof is complete. □

Now, let us denote by $\mathcal{P}([-\gamma, 0], \mathbb{R}_+^2)$ the space of all probability measures on $\mathcal{C}([-\gamma, 0], \mathbb{R}_+^2)$. For $P_1, P_2 \in \mathcal{P}([-\gamma, 0], \mathbb{R}_+^2)$, define

$$d_{BL}(P_1, P_2) = \sup_{f \in BL} \left| \int_{\mathbb{R}_+^2} f(z) P_1(dz) - \int_{\mathbb{R}_+^2} f(z) P_2(dz) \right|, \tag{2.78}$$

where

$$BL = \left\{ f : \mathcal{C}([-\gamma, 0], \mathbb{R}_+^2) \to \mathbb{R} : |f(z_1) - f(z_2)| \le \|z_1 - z_2\|, |f(\cdot)| \le 1 \right\}. \tag{2.79}$$

Denote by $p(t, \phi, dz)$ the transition probability of the process $z(t)$. Denote by $P(t, \phi, A)$ the probability of event $z(t; \phi) = (x(t; \phi), y(t; \phi))^{\mathrm{T}} \in A$ with initial data $\phi \in \mathcal{C}([-\gamma, 0], \mathbb{R}_+^2)$.

Lemma 4 *If* $a_{11} - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) > 0$, $a_{22} - \int_{-\tau_{12}}^{0} d\mu_{12}(\theta) > 0$, *then for any initial data* $\phi \in \mathcal{C}([-\gamma, 0], \mathbb{R}_+^2)$, $\{p(t, \phi, \cdot) : t \ge 0\}$ *is cauchy in the metric space* $\mathcal{P}([-\gamma, 0], \mathbb{R}_+^2)$.

Proof By Eq. 2.78 and the Markov property of $z(t)$, for any $t > 0$ and $s > 0$, we compute

$$\begin{aligned}
& d_{BL}(p(t+s, \phi, \cdot), p(t, \phi, \cdot)) \\
&= \sup_{f \in BL} \left| \int_{\mathbb{R}_+^2} f(z(t+s; \phi)) p(t+s, \phi, dz) - \int_{\mathbb{R}_+^2} f(z(t; \phi)) p(t, \phi, dz) \right| \\
&= \sup_{f \in BL} |\mathbb{E}[f(z(t+s; \phi))] - \mathbb{E}[f(z(t; \phi))]| \\
&= \sup_{f \in BL} \left|\mathbb{E}\left[\mathbb{E}\left[f(z(t+s; \phi))|\mathcal{F}_s\right]\right] - \mathbb{E}[f(z(t; \phi))]\right| \\
&= \sup_{f \in BL} \left| \int_{\mathbb{R}_+^2} \mathbb{E}[f(z(t; \psi))]\, p(s, \phi, d\psi) - \mathbb{E}[f(z(t; \phi))] \right| \\
&= \sup_{f \in BL} \left| \int_{\mathbb{R}_+^2} \mathbb{E}[f(z(t; \psi))]\, p(s, \phi, d\psi) - \int_{\mathbb{R}_+^2} \mathbb{E}[f(z(t; \phi))]\, p(s, \phi, d\psi) \right| \\
&= \sup_{f \in BL} \left| \int_{\mathbb{R}_+^2} \mathbb{E}[f(z(t; \psi)) - f(z(t; \phi))]\, p(s, \phi, d\psi) \right| \\
&\le \sup_{f \in BL} \int_{\mathbb{R}_+^2} \mathbb{E}[|f(z(t; \psi)) - f(z(t; \phi))|]\, p(s, \phi, d\psi) \\
&\le \sup_{f \in BL} \int_{\mathbb{B}_N} \mathbb{E}[|f(z(t; \psi)) - f(z(t; \phi))|]\, p(s, \phi, d\psi) \\
&\quad + \sup_{f \in BL} \int_{\mathbb{R}_+^2 \backslash \mathbb{B}_N} \mathbb{E}[|f(z(t; \psi)) - f(z(t; \phi))|]\, p(s, \phi, d\psi),
\end{aligned} \tag{2.80}$$

where $\mathbb{B}_N = \left\{(x, y)^{\mathrm{T}} \in \mathbb{R}_+^2 : 0 \le \sqrt{x^2 + y^2} \le N\right\}$. According to Lemma 2 and Chebyshsev's inequality we observe that for any initial data $\phi \in \mathcal{C}([-\gamma, 0], \mathbb{R}_+^2)$, the family of transition probability $p(t, \phi, \cdot)$ is tight. That is, for any $\epsilon > 0$, there exists a compact subset D of $\mathbb{R}^2_+$ such that for any $t \geq 0$,

$$P(t, \phi, D) \geq 1 - \epsilon. \tag{2.81}$$

So, for sufficiently large $N > 0$, we get

$$\sup_{f \in BL} \int_{\mathbb{R}^2_+ \setminus \mathbb{B}_N} \mathbb{E}\left[|f(z(t;\psi)) - f(z(t;\phi))|\right] p(s, \phi, d\psi) \leq 2P(s, \phi, \mathbb{R}^2_+ \setminus \mathbb{B}_N) \leq 2\epsilon. \tag{2.82}$$

On the other hand, based on Eq. 2.79 and Lemma 3, we observe that there exists $T > 0$ such that for any $t > T$,

$$\begin{aligned} &\sup_{f \in BL} \int_{\mathbb{B}_N} \mathbb{E}\left[|f(z(t;\psi)) - f(z(t;\phi))|\right] p(s, \phi, d\psi) \\ \leq &\sup_{f \in BL} \int_{\mathbb{B}_N} \mathbb{E}\left[|z(t;\psi) - z(t;\phi)|\right] p(s, \phi, d\psi) \\ \leq &\sup_{f \in BL} \int_{\mathbb{B}_N} \epsilon p(s, \phi, d\psi) = \epsilon P(s, \phi, \mathbb{B}_N) \leq \epsilon. \end{aligned} \tag{2.83}$$

Hence the desired result follows from Eqs. 2.80, 2.82 and 2.83. The proof is complete. □

Lemma 5 *If* $a_{11} - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) > 0$, $a_{22} - \int_{-\tau_{12}}^{0} d\mu_{12}(\theta) > 0$, *then system* (1.3) *is asymptotically stable in distribution, i.e., there exists a unique probability measure* $\nu(\cdot)$ *such that for any initial data* $\phi \in \mathcal{C}([-\gamma, 0], \mathbb{R}^2_+)$, *the transition probability* $p(t, \phi, \cdot)$ *of* $z(t)$ *converges weakly to* $\nu(\cdot)$ *when* $t \to +\infty$.

Proof We only need to show that for any initial data $\phi \in \mathcal{C}([-\gamma, 0], \mathbb{R}^2_+)$,

$$\lim_{t \to +\infty} d_{BL}(p(t, \phi, \cdot), \nu(\cdot)) = 0. \tag{2.84}$$

From Lemma 4 we observe that for $\phi_0 = (c_{10}, c_{20})^{\mathrm{T}} \in \mathcal{C}([-\gamma, 0], \mathbb{R}^2_+)$, where c_{10} and c_{20} are positive constants, $\{p(t, \phi_0, \cdot) : t \geq 0\}$ is cauchy in the metric space $\mathcal{P}([-\gamma, 0], \mathbb{R}^2_+)$. Therefore, there exists a probability measure $\nu(\cdot)$ such that

$$\lim_{t \to +\infty} d_{BL}(p(t, \phi_0, \cdot), \nu(\cdot)) = 0. \tag{2.85}$$

By the triangle inequality, we have

$$d_{BL}(p(t, \phi, \cdot), \nu(\cdot)) \leq d_{BL}(p(t, \phi, \cdot), p(t, \phi_0, \cdot)) + d_{BL}(p(t, \phi_0, \cdot), \nu(\cdot)). \tag{2.86}$$

Based on Eqs. 2.78 and 2.79 we compute

$$\begin{aligned} &d_{BL}(p(t, \phi, \cdot), p(t, \phi_0, \cdot)) \\ = &\sup_{f \in BL} \left| \int_{\mathbb{R}^2_+} f(z(t;\phi)) p(t, \phi, dz) - \int_{\mathbb{R}^2_+} f(z(t;\phi_0)) p(t, \phi_0, dz) \right| \\ = &\sup_{f \in BL} \left| \mathbb{E}\left[f(z(t;\phi))\right] - \mathbb{E}\left[f(z(t;\phi_0))\right] \right| \leq \sup_{f \in BL} \mathbb{E}\left[|f(z(t;\phi)) - f(z(t;\phi_0))|\right] \\ \leq &\mathbb{E}\left[|z(t;\phi) - z(t;\phi_0)|\right]. \end{aligned} \tag{2.87}$$

Hence the desired result follows from Lemma 3, Eqs. 2.85, 2.86 and 2.87 . The proof is complete. □

Theorem 2 *Let* $a_{11} - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) > 0,\ a_{22} - \int_{-\tau_{12}}^{0} d\mu_{12}(\theta) > 0$. *Denote*

$$\begin{aligned} h_1^* =& \frac{2a_{11}a_{22}+\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)-\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)}{4a_{11}a_{22}-\left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)-\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)^2}\left(r_1-\frac{\sigma_1^2}{2}\right) \\ &+ \frac{a_{11}\left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)+\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)}{4a_{11}a_{22}-\left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)-\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)^2}\left(r_2+\frac{\sigma_2^2}{2}\right), \\ h_2^* =& \frac{a_{22}\left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)+\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)}{4a_{11}a_{22}-\left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)-\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)^2}\left(r_1-\frac{\sigma_1^2}{2}\right) \\ &+ \frac{\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)-\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)-2a_{11}a_{22}}{4a_{11}a_{22}-\left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)-\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)^2}\left(r_2+\frac{\sigma_2^2}{2}\right), \\ Y^*(H) =& -a_{22}h_1^2 + \left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta) - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)h_1h_2 - a_{11}h_2^2 \\ &+ \left[\left(r_1-\frac{\sigma_1^2}{2}\right)a_{22} + \left(r_2+\frac{\sigma_2^2}{2}\right)\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\right]h_1 \\ &+ \left[\left(r_1-\frac{\sigma_1^2}{2}\right)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - \left(r_2+\frac{\sigma_2^2}{2}\right)a_{11}\right]h_2. \end{aligned} \tag{2.88}$$

$(\mathcal{B}_1)$ *If*

$$\begin{cases} \left(b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2a_{11}\right)\big|_{h_1=h_1^*\geq 0,\ h_2=h_2^*\geq 0} > 0, \\ 4a_{11}a_{22} - \left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta) - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)^2 > 0, \end{cases} \tag{2.89}$$

then the optimal harvesting strategy exists. Moreover, $H^* = (h_1^*, h_2^*)^{\mathrm{T}}$ *and*

$$MESY = \frac{Y^*(H^*)}{a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)}. \tag{2.90}$$

$(\mathcal{B}_2)$ *If one of the following conditions holds, then the optimal harvesting strategy does not exist:*
$(\mathcal{C}_1)$ $b_1|_{h_1=h_1^*} \leq 0$;
$(\mathcal{C}_2)$ $\left(b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2a_{11}\right)|_{h_1=h_1^*,\ h_2=h_2^*} \leq 0$;
$(\mathcal{C}_3)$ $h_1^* < 0$ *or* $h_2^* < 0$;
$(\mathcal{C}_4)$ $4a_{11}a_{22} - \left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta) - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)^2 < 0$.

Proof Let

$$\mathcal{U} = \left\{H = (h_1, h_2)^{\mathrm{T}} \in \mathbb{R}^2 \mid b_1\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2a_{11} > 0,\ h_1 \geq 0,\ h_2 \geq 0\right\}. \tag{2.91}$$

On the one hand, from the definition of $\mathcal{U}$ (i.e. Eq. 2.91) and Theorem 1 ($\mathcal{A}_5$) we observe that for every $H \in \mathcal{U}$, equality system (2.2) holds. On the other hand, if the OHE H^* exists, then $H^* \in \mathcal{U}$.

Proof of ($\mathcal{B}_1$). Based on the first condition of Eq. 2.89 we obtain that $\mathcal{U}$ is not empty. According to Lemma 5 we observe that there exists a unique invariant measure $\nu(\cdot)$ for system (1.3). It then follows from Corollary 3.4.3 in Prato and Zabczyk (1996) that $\nu(\cdot)$ is strong mixing. By Theorem 3.2.6 in Prato and Zabczyk (1996), we obtain that the measure $\nu(\cdot)$ is ergodic. On the basis of Theorem 3.3.1 in Prato and Zabczyk (1996), for $H = (h_1, h_2)^{\mathrm{T}} \in \mathcal{U}$, we have

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t H^{\mathrm{T}} z(s)ds = \int_{\mathbb{R}^2_+} H^{\mathrm{T}} z\nu(dz). \tag{2.92}$$

Let $\varrho(z)$ be the stationary probability density of system (1.3), then we get

$$Y(H) = \lim_{t\to+\infty} \mathbb{E}\left[h_1 x(t) + h_2 y(t)\right] = \lim_{t\to+\infty} \mathbb{E}\left[H^{\mathrm{T}} z(t)\right] = \int_{\mathbb{R}^2_+} H^{\mathrm{T}} z\varrho(z)dz. \tag{2.93}$$

Note that the invariant measure of system (1.3) is unique and that there exists a one-to-one correspondence between $\varrho(z)$ and its corresponding invariant measure, we deduce

$$\int_{\mathbb{R}^2_+} H^{\mathrm{T}} z\varrho(z)dz = \int_{\mathbb{R}^2_+} H^{\mathrm{T}} z\nu(dz). \tag{2.94}$$

In the light of Eqs. 2.92, 2.93, 2.94 and Theorem 1 ($\mathcal{A}_5$), we have

$$\begin{aligned}
Y(H) &= \lim_{t\to+\infty} \frac{1}{t}\int_0^t H^{\mathrm{T}} z(s)ds \\
&= h_1 \lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)ds + h_2 \lim_{t\to+\infty} \frac{1}{t}\int_0^t y(s)ds \\
&= h_1 \frac{b_1 a_{22} + b_2 \int_{-\tau_{12}}^0 d\mu_{12}(\theta)}{a_{11}a_{22} + \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{21}}^0 d\mu_{21}(\theta)} \\
&\quad + h_2 \frac{b_1 \int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2 a_{11}}{a_{11}a_{22} + \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{21}}^0 d\mu_{21}(\theta)} \\
&= \frac{Y^*(H)}{a_{11}a_{22} + \int_{-\tau_{12}}^0 d\mu_{12}(\theta)\int_{-\tau_{21}}^0 d\mu_{21}(\theta)}.
\end{aligned} \tag{2.95}$$

Based on the third equation of Eq. 2.88, we compute

$$\begin{cases} \dfrac{\partial Y^*(H)}{\partial h_1} = -2a_{22}h_1 + \left(\displaystyle\int_{-\tau_{12}}^0 d\mu_{12}(\theta) - \int_{-\tau_{21}}^0 d\mu_{21}(\theta)\right) h_2 \\ \qquad + \left[\left(r_1 - \dfrac{\sigma_1^2}{2}\right) a_{22} + \left(r_2 + \dfrac{\sigma_2^2}{2}\right) \displaystyle\int_{-\tau_{12}}^0 d\mu_{12}(\theta)\right], \\ \dfrac{\partial Y^*(H)}{\partial h_2} = \left(\displaystyle\int_{-\tau_{12}}^0 d\mu_{12}(\theta) - \int_{-\tau_{21}}^0 d\mu_{21}(\theta)\right) h_1 - 2a_{11}h_2 \\ \qquad + \left[\left(r_1 - \dfrac{\sigma_1^2}{2}\right) \displaystyle\int_{-\tau_{21}}^0 d\mu_{21}(\theta) - \left(r_2 + \dfrac{\sigma_2^2}{2}\right) a_{11}\right], \\ \dfrac{\partial^2 Y^*(H)}{\partial h_1^2} = -2a_{22}, \ \dfrac{\partial^2 Y^*(H)}{\partial h_1 \partial h_2} = \displaystyle\int_{-\tau_{12}}^0 d\mu_{12}(\theta) - \int_{-\tau_{21}}^0 d\mu_{21}(\theta), \\ \dfrac{\partial^2 Y^*(H)}{\partial h_2 \partial h_1} = \displaystyle\int_{-\tau_{12}}^0 d\mu_{12}(\theta) - \int_{-\tau_{21}}^0 d\mu_{21}(\theta), \ \dfrac{\partial^2 Y^*(H)}{\partial h_2^2} = -2a_{11}. \end{cases} \tag{2.96}$$

Note that

$$4a_{11}a_{22} > \left(\int_{-\tau_{12}}^0 d\mu_{12}(\theta) - \int_{-\tau_{21}}^0 d\mu_{21}(\theta)\right)^2, \tag{2.97}$$

we deduce that $Y^*(H)$ has a unique maximum, and the unique maximum value point of $Y^*(H)$ is $H^* = (h_1^*, h_2^*)^{\mathrm{T}}$. Hence (2.90) follows from Eq. 2.95.

Proof of $(\mathcal{B}_2)$. To begin with, from Theorem 1 $(\mathcal{A}_1)$ and Theorem 1 $(\mathcal{A}_2)$ one can obtain that under condition $(\mathcal{C}_1)$, the optimal harvesting strategy does not exist. Then, let us show that the optimal harvesting strategy does not exist, provided that either $(\mathcal{C}_2)$ or $(\mathcal{C}_3)$ holds. The proof is by contradiction. Suppose that the OHE is $\widetilde{H}^* = \left(\widetilde{h}_1^*, \widetilde{h}_2^*\right)^{\mathrm{T}}$. Then $\widetilde{H}^* \in \mathcal{U}$. In other words, we have

$$\left(b_1 \int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2 a_{11}\right)|_{h_1=\widetilde{h}_1^*,\ h_2=\widetilde{h}_2^*} > 0, \ \widetilde{h}_1^* \geq 0, \ \widetilde{h}_2^* \geq 0. \tag{2.98}$$

On the other hand, since $\widetilde{H}^* = \left(\widetilde{h}_1^*, \widetilde{h}_2^*\right)^{\mathrm{T}} \in \mathcal{U}$ is the OHE, then $\left(\widetilde{h}_1^*, \widetilde{h}_2^*\right)^{\mathrm{T}}$ must be the unique solution to the following system:

$$\frac{\partial Y^*(H)}{\partial h_1} = 0, \ \frac{\partial Y^*(H)}{\partial h_2} = 0. \tag{2.99}$$

Note that $H^* = (h_1^*, h_2^*)^{\mathrm{T}}$ is the unique solution to system (2.99), we deduce that $(h_1^*, h_2^*)^{\mathrm{T}} = \left(\widetilde{h}_1^*, \widetilde{h}_2^*\right)^{\mathrm{T}}$. Hence, Eq. 2.98 becomes

$$\left(b_1 \int_{-\tau_{21}}^0 d\mu_{21}(\theta) - b_2 a_{11}\right)|_{h_1=h_1^*,\ h_2=h_2^*} > 0, \ h_1^* \geq 0, \ h_2^* \geq 0, \tag{2.100}$$

which contradicts with both $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$.

Now we are in the position to prove that if the following condition holds, then the optimal harvesting strategy does not exist:

$$\begin{cases} \left(b_1 \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11}\right) |_{h_1=h_1^*\geq 0,\ h_2=h_2^*\geq 0} > 0, \\ 4a_{11}a_{22} - \left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta) - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)^2 < 0. \end{cases} \tag{2.101}$$

From the first inequality of Eq. 2.101 we observe that $\mathcal{U}$ is not empty. Hence both (2.95) and Eq. 2.96 are true. $-2a_{22} < 0$ implies that the Hessian matrix is not positive semidefinite. The second inequality of Eq. 2.101 indicates that the Hessian matrix is not negative semidefinite. Namely, the Hessian matrix is indefinite. Thus the third equation of Eq. 2.88 does not exist extreme point. So the OHE does not exist. The proof is complete. □

If $\mu_{12}(\theta)$ and $\mu_{21}(\theta)$ are defined as follows:

$$\mu_{12}(\theta) = \begin{cases} -a_{12}, & -\tau_{12} \leq \theta \leq -\tau_1, \\ 0, & -\tau_1 < \theta \leq 0, \end{cases} \quad \mu_{21}(\theta) = \begin{cases} -a_{21}, & -\tau_{21} \leq \theta \leq -\tau_2, \\ 0, & -\tau_2 < \theta \leq 0, \end{cases} \tag{2.102}$$

then system (1.3) becomes the following stochastic predator-prey system with discrete time delays:

$$\begin{cases} dx(t) = x(t)\left[r_1 - h_1 - a_{11}x(t) - a_{12}y(t-\tau_1)\right]dt + \sigma_1 x(t)dB_1(t), \\ dy(t) = y(t)\left[-r_2 - h_2 + a_{21}x(t-\tau_2) - a_{22}y(t)\right]dt + \sigma_2 y(t)dB_2(t). \end{cases} \tag{2.103}$$

Remark 1 Theorem 1 contains Theorem 1 in Liu et al. (2013) as a special case.

Remark 2 Theorem 2 contains Theorem 1 in Liu (2015) as a special case.

3 An Example

By the method in Glasserman (2003), for $a_{11} = 1.0$, $a_{22} = 0.5$, $\tau_{12} = \ln 2$, $\tau_{21} = \ln 2$, $\mu_{12}(\theta) = 0.8e^{\theta}$, $\mu_{21}(\theta) = 1.5e^{\theta}$, $\xi(\theta) = 1.4e^{\theta}$, $\eta(\theta) = 0.8e^{\theta}$, $\theta \in [-\ln 2, 0]$ and step size $\Delta t = \frac{\ln 2}{15}$, we numerically simulate the solutions of the following system to support our results:

$$\begin{cases} dx(t) = x(t)\left[r_1 - h_1 - x(t) - 0.8\int_{-\ln 2}^{0} y(t+\theta)e^{\theta}d\theta\right]dt + \sigma_1 x(t)dB_1(t), \\ dy(t) = y(t)\left[-r_2 - h_2 + 1.5\int_{-\ln 2}^{0} x(t+\theta)e^{\theta}d\theta - 0.5y(t)\right]dt + \sigma_2 y(t)dB_2(t), \\ x(\theta) = \xi(\theta) = 1.4e^{\theta}, \quad y(\theta) = \eta(\theta) = 0.8e^{\theta}, \quad \theta \in [-\ln 2, 0]. \end{cases} \tag{3.1}$$

(I) For $r_1 = 2.0$, $r_2 = 1.0$, $h_1 = 0$, $h_2 = 0$, $\sigma_1 = 2.5$ and $\sigma_2 = 0.1$, we have (Fig. 1a):

$$b_1 = -1.125 < 0. \tag{3.2}$$

By Theorem 1 ($\mathcal{A}_1$), both $x(t)$ and $y(t)$ are extinctive a.s. From Theorem 2 ($\mathcal{C}_1$), the optimal harvesting strategy does not exist.

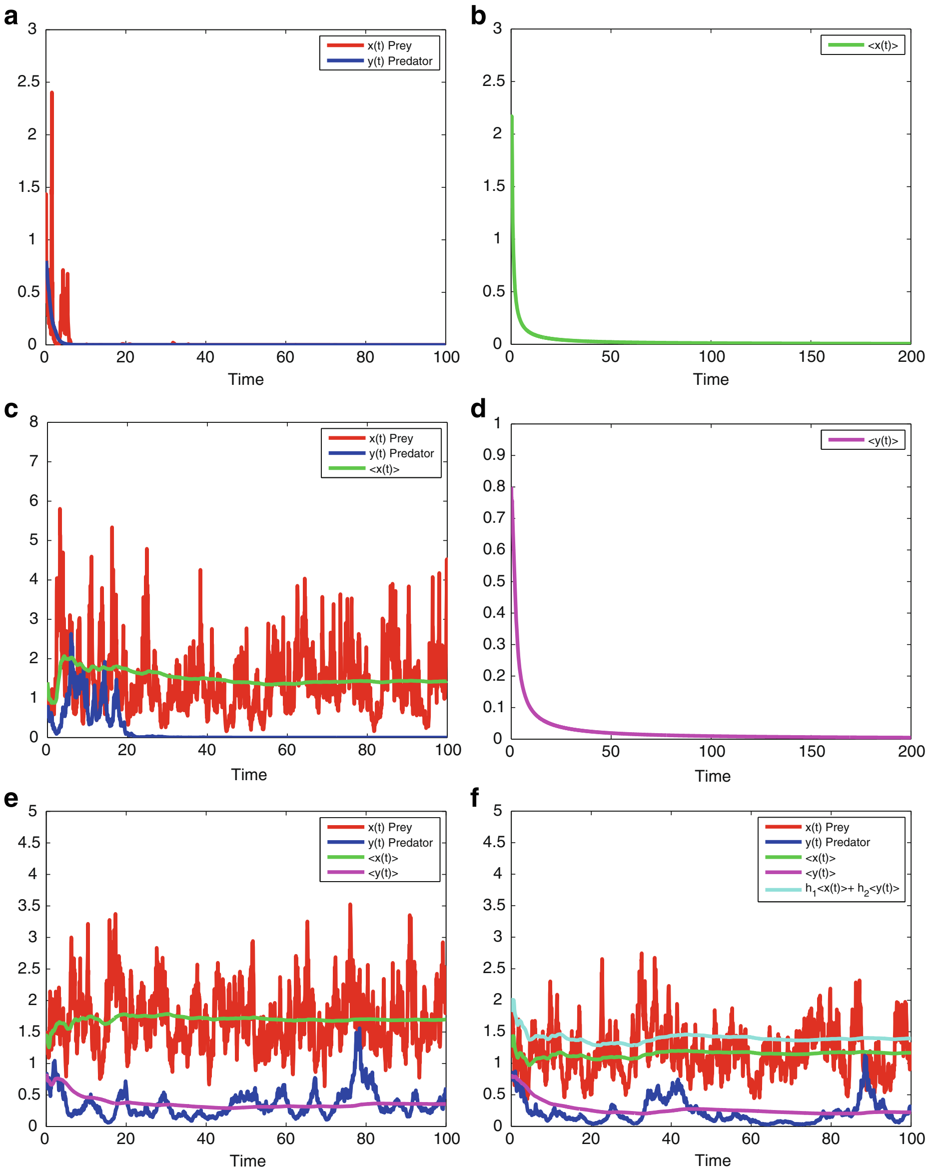

Fig. 1 Solutions of system (3.1)

(II) For $r_1 = 2.0$, $r_2 = 1.0$, $h_1 = 0$, $h_2 = 0$, $\sigma_1 = 2.0$ and $\sigma_2 = 0.1$, we have (Fig. 1b):

$$b_1 = 0. \tag{3.3}$$

In view of Theorem 1 ($\mathcal{A}_2$), $x(t)$ is nonpersistent in mean a.s. Based on Theorem 2 ($\mathcal{C}_1$), the optimal harvesting strategy does not exist.

(III) For $r_1 = 2.0$, $r_2 = 1.0$, $h_1 = 0$, $h_2 = 0$, $\sigma_1 = 1.0$ and $\sigma_2 = 0.6$, we have (Fig. 1c):

$$b_1 = 1.5 > 0,\ \ b_1 \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} = -0.055 < 0. \tag{3.4}$$

According to Theorem 1 ($\mathcal{A}_3$), $x(t)$ is persistent in mean while $y(t)$ is extinctive a.s. In line with Theorem 2 ($\mathcal{C}_2$), the optimal harvesting strategy does not exist.

(IV) For $r_1 = 2.0$, $r_2 = 1.0$, $h_1 = 0$, $h_2 = 0$, $\sigma_1 = 1.0$ and $\sigma_2 = 0.5$, we have (Fig. 1d):

$$b_1 \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} = 0. \tag{3.5}$$

In the light of Theorem 1 ($\mathcal{A}_4$), $y(t)$ is nonpersistent in mean a.s. On the basis of Theorem 2 ($\mathcal{C}_2$), the optimal harvesting strategy does not exist.

(V) For $r_1 = 2.0$, $r_2 = 1.0$, $h_1 = 0$, $h_2 = 0$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$, we have (Fig. 1e):

$$b_1 \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} = 0.36125 > 0. \tag{3.6}$$

Based on Theorem 1 ($\mathcal{A}_5$), both $x(t)$ and $y(t)$ are persistent in mean a.s.

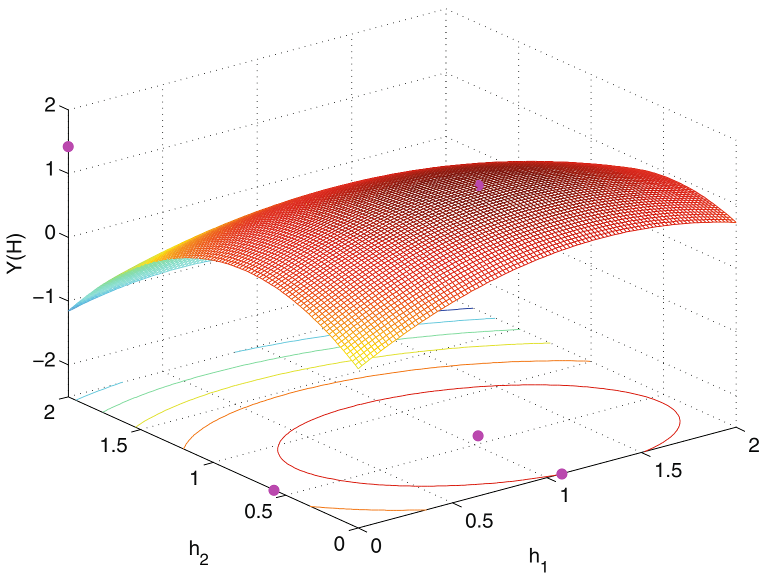

Fig. 2 $Y(H)$ of system (3.1) for $r_1 = 2.5$, $r_2 = 0.2$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$. Then the maxima is $H^* = (1.0829893475, 0.5786018642)^{\mathrm{T}}$ and the maximum is $Y(H^*) = 1.4256624501$

(VI) For $r_1 = 2.5$, $r_2 = 0.2$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$, we have (Figs. 1f and 2):

$$\begin{aligned}
&a_{11} - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) = 0.25 > 0,\ a_{22} - \int_{-\tau_{12}}^{0} d\mu_{12}(\theta) = 0.1 > 0,\\
&h_1^* = 1.0829893475 > 0,\ h_2^* = 0.5786018642 > 0,\\
&4a_{11}a_{22} - \left(\int_{-\tau_{12}}^{0} d\mu_{12}(\theta) - \int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)^2 = 1.8775 > 0,\\
&b_1 \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) - b_2 a_{11} = 0.1454061252 > 0,\\
&Y^*(H) = -0.5h_1^2 - 0.35h_1h_2 - h_2^2 + 1.2855h_1 + 1.53625h_2,\\
&a_{11}a_{22} + \int_{-\tau_{12}}^{0} d\mu_{12}(\theta) \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) = 0.8.
\end{aligned} \tag{3.7}$$

From Theorem 2 ($\mathcal{B}_1$), the optimal harvesting strategy exists. Moreover, the OHE $H^* = (1.0829893475, 0.5786018642)^{\mathrm{T}}$ and $MESY = 1.4256624501$.

4 Conclusions and Outlook

This paper is devoted to studying the optimal harvesting problem of a stochastic predator-prey model with S-type distributed time delays (which contain both discrete time delays and continuously distributed time delays). By combining stochastic analytical techniques with the ergodic method proposed in Zou et al. (2013), sufficient and necessary conditions for the existence of optimal harvesting strategy are obtained. Moreover, both the OHE and MESY are given. Some existing results are generalized. Our analytical results reveal that the existence of optimal harvesting strategy has close relationships with both time delays and stochastic noise.

Some interesting topics deserve further investigations. First, it is interesting to investigate more realistic and complex systems in lieu of the considered system, for example, stochastic time-delay predator-prey model with Markovian switching (see e.g. Zou and Wang 2014, Bao et al. 2009) and Lévy noise (see e.g. Zou and Wang 2014, Liu and Bai 2016b, Liu and Wang 2014). Next, it is of interest to investigate the optimal harvesting problem of other stochastic population systems with S-type distributed time delays, for instance, competitive systems and cooperative systems. Finally, motivated by the works in Liu and Bai (2014) and Liu and Bai (2016a), we may also study the optimal harvesting of stochastic food-chain model with S-type distributed time delays and Lévy noise. We leave these investigations for future work.

Appendix

Proof of Lemma 2 Since the coefficients of system (1.3) are locally Lipschitz continuous, from Mao (1994) (Theorem 3.2.2) and Wei and Wang (2007) we observe that for any initial data $(\xi(\theta), \eta(\theta))^{\mathrm{T}} \in C([-\gamma, 0], \mathbb{R}_+^2)$, system (1.3) has a unique local solution $z(t) = (x(t), y(t))^{\mathrm{T}}$ on $t \in [-\gamma, \tau_e)$, where τ_e is the explosion time. We need to prove $\tau_e = +\infty$ a.s. For this purpose, let $k_0 > 0$ be sufficiently large that both $\xi(0)$ and $\eta(0)$ lie in $[\frac{1}{k_0}, k_0]$. For every integer $k > k_0$, define stopping times as follows:

$$\tau_k = \inf\left\{t \in [0, \tau_e) : \ x(t) \notin \left(\tfrac{1}{k}, k\right) \ or \ y(t) \notin \left(\tfrac{1}{k}, k\right)\right\}. \tag{A.1}$$

Set $\tau_\infty = \lim_{k\to+\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$ a.s. Hence, we only need to show $\tau_\infty = +\infty$ a.s. If this deduction is false, then there exist $T > 0$ and $\epsilon \in (0, 1)$ such that $P(\tau_\infty \leq T) > \epsilon$. Thus, there exists an integer $k_1 \geq k_0$ such that

$$P(\tau_k \leq T) > \epsilon, \ \ k > k_1. \tag{A.2}$$

For $(x, y)^{\mathrm{T}} \in \mathbb{R}^2_+$, define

$$V_1(x) = x - 1 - \ln x, \ \ V_2(y) = y - 1 - \ln y. \tag{A.3}$$

By Itô's formula, we have

$$\begin{cases} dV_1(x) =\mathcal{L}[V_1(x)]dt + \sigma_1(x-1)dB_1(t), \\ dV_2(y) =\mathcal{L}[V_2(y)]dt + \sigma_2(y-1)dB_2(t), \end{cases} \tag{A.4}$$

where

$$\begin{cases} \mathcal{L}[V_1(x)] =\dfrac{\sigma_1^2}{2} + (x-1)\left[r_1 - h_1 - a_{11}x - \displaystyle\int_{-\tau_{12}}^{0} y(t+\theta)d\mu_{12}(\theta)\right], \\ \mathcal{L}[V_2(y)] =\dfrac{\sigma_2^2}{2} + (y-1)\left[-r_2 - h_2 + \displaystyle\int_{-\tau_{21}}^{0} x(t+\theta)d\mu_{21}(\theta) - a_{22}y\right]. \end{cases} \tag{A.5}$$

For any positive integer n, applying Lemma 1 to Eq. A.5 gives

$$\begin{cases} \mathcal{L}[V_1(x)] \leq\dfrac{\sigma_1^2}{2} - (r_1 - h_1) + \dfrac{n^2}{2}\displaystyle\int_{-\tau_{12}}^{0} d\mu_{12}(\theta) \\ \qquad\qquad + (r_1 - h_1)x + a_{11}x - a_{11}x^2 + \dfrac{1}{2n^2}\displaystyle\int_{-\tau_{12}}^{0} y^2(t+\theta)d\mu_{12}(\theta), \\ \mathcal{L}[V_2(y)] \leq\dfrac{\sigma_2^2}{2} + (r_2 + h_2) - (r_2 + h_2)y + a_{22}y - a_{22}y^2 \\ \qquad\qquad + \dfrac{y^2}{2n}\displaystyle\int_{-\tau_{21}}^{0} d\mu_{21}(\theta) + \dfrac{n}{2}\displaystyle\int_{-\tau_{21}}^{0} x^2(t+\theta)d\mu_{21}(\theta). \end{cases} \tag{A.6}$$

Define

$$\begin{aligned} V_0(x, y, t) =&\alpha\left(V_1(x) + \frac{1}{2n^2}\int_{-\tau_{12}}^{0}\int_{t+\theta}^{t} y^2(s)dsd\mu_{12}(\theta)\right) \\ &+ \beta\left(V_2(y) + \frac{n}{2}\int_{-\tau_{21}}^{0}\int_{t+\theta}^{t} x^2(s)dsd\mu_{21}(\theta)\right), \end{aligned} \tag{A.7}$$

where α and β are positive constants satisfying:

$$\begin{cases} - a_{11}\alpha + \dfrac{n}{2}\displaystyle\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\beta < 0, \\ \dfrac{1}{2n^2}\displaystyle\int_{-\tau_{12}}^{0} d\mu_{12}(\theta)\alpha + \left(-a_{22} + \dfrac{1}{2n}\displaystyle\int_{-\tau_{21}}^{0} d\mu_{21}(\theta)\right)\beta < 0. \end{cases} \tag{A.8}$$

Then from Itô's formula, Eqs. A.6, A.7 and A.8, we obtain that for sufficiently large n, there exists $\mathcal{K} > 0$ such that

$$d[V_0(x, y, t)] \leq \mathcal{K}dt + \sigma_1\alpha(x-1)dB_1(t) + \sigma_2\beta(y-1)dB_2(t). \tag{A.9}$$

Integrating both sides of Eq. A.9 from 0 to $\tau_k \wedge T$ and then taking the expectations yield

$$\begin{aligned} &\mathbb{E}[\alpha V_1(x(\tau_k \wedge T)) + \beta V_2(y(\tau_k \wedge T))] \\ \leq &\mathbb{E}[V_0(x(\tau_k \wedge T), y(\tau_k \wedge T), \tau_k \wedge T)] \\ \leq &\mathbb{E}[V_0(x(0), y(0), 0)] + \mathbb{E}\int_0^{\tau_k \wedge T} \mathcal{K}ds \\ \leq &V_0(x(0), y(0), 0) + \mathcal{K}T. \end{aligned} \tag{A.10}$$

Let $\Omega_k = \{\tau_k \leq T\}$. In view of Eq. A.2, we have $P(\Omega_k) > \epsilon$. Note that for every $\omega \in \Omega_k$, one of the following equalities holds:

$$x(\tau_k, \omega) = \tfrac{1}{k},\ \ x(\tau_k, \omega) = k,\ \ y(\tau_k, \omega) = \tfrac{1}{k},\ \ y(\tau_k, \omega) = k. \tag{A.11}$$

Hence, based on Eqs. A.3 and A.11 we get

$$\begin{aligned} &\alpha V_1\left(x(\tau_k, \omega)\right) + \beta V_2\left(y(\tau_k, \omega)\right) \\ \geq &\min\{\alpha, \beta\} \min\left\{\sqrt{k} - 1 - \ln\sqrt{k},\ \tfrac{1}{\sqrt{k}} - 1 + \ln\sqrt{k}\right\}. \end{aligned} \tag{A.12}$$

Thus it follows from Eqs. A.10 and A.12 that

$$\begin{aligned} &V_0(x(0), y(0), 0) + \mathcal{K}T \\ \geq &\mathbb{E}\left[I_{\Omega_k}(\omega)\left[\alpha V_1\left(x(\tau_k, \omega)\right) + \beta V_2\left(y(\tau_k, \omega)\right)\right]\right] \\ > &\epsilon \min\{\alpha, \beta\} \min\left\{\sqrt{k} - 1 - \ln\sqrt{k},\ \tfrac{1}{\sqrt{k}} - 1 + \ln\sqrt{k}\right\}, \end{aligned} \tag{A.13}$$

where I_{Ω_k} is the indicator function of Ω_k. Letting $k \to +\infty$ leads to the contradiction $+\infty > V_0(x(0), y(0), 0) + \mathcal{K}T \geq +\infty$.

Now, we are in the position to prove (2.1). On the one hand, for any $p > 0$, by Itô's formula we obtain

$$d[\mathrm{e}^t x^p] = \mathcal{L}[\mathrm{e}^t x^p]dt + p\sigma_1 \mathrm{e}^t x^p dB_1(t), \tag{A.14}$$

where

$$\begin{aligned} \mathcal{L}[\mathrm{e}^t x^p] = &\mathrm{e}^t x^p \left\{1 + \tfrac{p(p-1)\sigma_1^2}{2} + p\left[r_1 - h_1 - a_{11}x - \int_{-\tau_{12}}^0 y(t+\theta)d\mu_{12}(\theta)\right]\right\} \\ \leq &\mathrm{e}^t\left\{\left[1 + p(r_1 - h_1) + \tfrac{p(p-1)\sigma_1^2}{2}\right]x^p - pa_{11}x^{p+1}\right\} \leq K_1(p)\mathrm{e}^t. \end{aligned} \tag{A.15}$$

In other words, we have

$$d[\mathrm{e}^t x^p] \leq K_1(p)\mathrm{e}^t dt + p\sigma_1 \mathrm{e}^t x^p dB_1(t). \tag{A.16}$$

Integrating both sides of Eq. A.16 from 0 to t and then taking the expectations lead to

$$\mathbb{E}[\mathrm{e}^t x^p(t)] - \xi^p(0) \leq K_1(p)(\mathrm{e}^t - 1). \tag{A.17}$$

According to Eq. A.17 we derive

$$\mathbb{E}[x^p(t)] \leq K_1(p) + \mathrm{e}^{-t}\left[\xi^p(0) - K_1(p)\right]. \tag{A.18}$$

On the other hand, applying Itô's formula to $\mathrm{e}^t y^p(t)$ yields

$$d[\mathrm{e}^t y^p] = \mathcal{L}[\mathrm{e}^t y^p]dt + p\sigma_2 \mathrm{e}^t y^p dB_2(t). \tag{A.19}$$

For sufficiently large positive integer n, on the basis of Lemma 1 we obtain

$$\begin{aligned}
\mathcal{L}[\mathrm{e}^t y^p] =&\mathrm{e}^t \left\{\left[1 - p(r_2 + h_2) + \tfrac{p(p-1)\sigma_2^2}{2}\right] y^p - pa_{22}y^{p+1}\right.\\
&\left.+p\int_{-\tau_{21}}^0 x(t+\theta)y^p(t)d\mu_{21}(\theta)\right\}\\
\leq&\mathrm{e}^t \left\{\left[1 - p(r_2 + h_2) + \tfrac{p(p-1)\sigma_2^2}{2}\right] y^p - pa_{22}y^{p+1}\right.\\
&\left.+p\int_{-\tau_{21}}^0 \left[\frac{p}{p+1}\frac{y^{p+1}}{n^{\frac{p+1}{p}}} + \frac{n^{p+1}}{p+1}x^{p+1}(t+\theta)\right]d\mu_{21}(\theta)\right\}\\
=&\mathrm{e}^t \left\{\left[1 - p(r_2 + h_2) + \frac{p(p-1)\sigma_2^2}{2}\right] y^p\right.\\
&-p\left[a_{22} - \frac{p}{p+1}n^{-\frac{p+1}{p}}\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\right]y^{p+1}\\
&\left.+\frac{p}{p+1}n^{p+1}\int_{-\tau_{21}}^0 x^{p+1}(t+\theta)d\mu_{21}(\theta)\right\}.
\end{aligned} \tag{A.20}$$

Define $\widetilde{Q}(t)$ as follows:

$$\widetilde{Q}(t) = \mathrm{e}^{\tau_{21}}\frac{pn^{p+1}}{p+1}Q(t),\ \ Q(t) = \int_{-\tau_{21}}^0\int_{t+\theta}^t \mathrm{e}^s x^{p+1}(s)dsd\mu_{21}(\theta). \tag{A.21}$$

Compute

$$\begin{aligned}
\frac{dQ(t)}{dt} =&\int_{-\tau_{21}}^0 \mathrm{e}^t x^{p+1}(t)d\mu_{21}(\theta) - \int_{-\tau_{21}}^0 \mathrm{e}^{t+\theta}x^{p+1}(t+\theta)d\mu_{21}(\theta)\\
\leq&\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\mathrm{e}^t x^{p+1}(t) - \mathrm{e}^{-\tau_{21}}\int_{-\tau_{21}}^0 \mathrm{e}^t x^{p+1}(t+\theta)d\mu_{21}(\theta).
\end{aligned} \tag{A.22}$$

In the light of Itô's formula, Eqs. A.20, A.21 and A.22 we deduce

$$\begin{aligned}
\mathcal{L}\left[\mathrm{e}^t y^p + \widetilde{Q}(t)\right] \leq&\mathrm{e}^t\left\{\left[1 - p(r_2 + h_2) + \frac{p(p-1)\sigma_2^2}{2}\right]y^p\right.\\
&-p\left[a_{22} - \frac{p}{p+1}n^{-\frac{p+1}{p}}\int_{-\tau_{21}}^0 d\mu_{21}(\theta)\right]y^{p+1}\\
&\left.+\mathrm{e}^{\tau_{21}}\frac{pn^{p+1}}{p+1}\int_{-\tau_{21}}^0 d\mu_{21}(\theta)x^{p+1}(t)\right\}.
\end{aligned} \tag{A.23}$$

Define $Q^*(x, y, t)$ as follows:

$$Q^*(x, y, t) = C^*\mathrm{e}^t x^p + \left(\mathrm{e}^t y^p + \widetilde{Q}(t)\right),\ C^* = a_{11}^{-1}\mathrm{e}^{\tau_{21}}n^{p+1}\int_{-\tau_{21}}^0 d\mu_{21}(\theta). \tag{A.24}$$

In view of Itô's formula, Eqs. A.15, A.23 and A.24, we observe that there exists $K_2(p) > 0$ such that

$$\begin{aligned}\mathcal{L}\left[Q^*(x,y,t)\right] \leq & \mathrm{e}^t \left\{ \left[1 - p(r_2+h_2) + \frac{p(p-1)\sigma_2^2}{2} \right] y^p \right. \\ & - p\left[a_{22} - \frac{p}{p+1} n^{-\frac{p+1}{p}} \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) \right] y^{p+1} \\ & + C^* \left[1 + p(r_1-h_1) + \frac{p(p-1)\sigma_1^2}{2} \right] x^p \\ & \left. - \mathrm{e}^{\tau_{21}} \frac{p^2}{p+1} n^{p+1} \int_{-\tau_{21}}^{0} d\mu_{21}(\theta) x^{p+1} \right\} \leq K_2(p)\mathrm{e}^t. \end{aligned} \tag{A.25}$$

According to Eqs. A.24 and A.25, we get

$$\mathbb{E}\left[\mathrm{e}^t y^p(t)\right] \leq \mathbb{E}\left[Q^*(x,y,t)\right] \leq \mathbb{E}\left[Q^*(\xi(0),\eta(0),0)\right] + K_2(p)\left(\mathrm{e}^t - 1\right). \tag{A.26}$$

In other words, we have

$$\mathbb{E}\left[y^p(t)\right] \leq K_2(p) + \mathrm{e}^{-t}\left[Q^*(\xi(0),\eta(0),0) - K_2(p)\right]. \tag{A.27}$$

Consequently, Eq. 2.1 follows from taking superior limits on both sides of Eqs. A.18 and A.27. The proof is complete.

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