Methodol Comput Appl Probab (2017) 19:665–683
DOI 10.1007/s11009-016-9514-7

On Burr XII Distribution Analysis Under Progressive Type-II Hybrid Censored Data

M. Noori Asl[1] · R. Arabi Belaghi[1] · H. Bevrani[1]

Received: 28 January 2016 / Revised: 5 May 2016
Accepted: 25 August 2016 / Published online: 6 September 2016

Abstract In the current paper, based on progressive type-II hybrid censored samples, the maximum likelihood and Bayes estimates for the two parameter Burr XII distribution are obtained. We propose the use of expectation-maximization (EM) algorithm to compute the maximum likelihood estimates (MLEs) of model parameters. Further, we derive the asymptotic variance-covariance matrix of the MLEs by applying the missing information principle and it can be utilized to construct asymptotic confidence intervals (CIs) for the parameters. The Bayes estimates of the unknown parameters are obtained under the assumption of gamma priors by using Lindley's approximation and Markov chain Monte Carlo (MCMC) technique. Also, MCMC samples are used to construct the highest posterior density (HPD) credible intervals. Simulation study is conducted to investigate the accuracy of the estimates and compare the performance of CIs obtained. Finally, one real data set is analyzed for illustrative purposes.

Keywords Bayesian estimate · EM algorithm · Missing information principle · Lindley's approximation · Importance sampling · Progressive type-II hybrid censoring

Mathematics Subject Classification (2010) 62F10 · 62N01 · 62N02

1 Introduction

In reliability and lifetime experiments, censoring is considered in order to save time and reduce the number of failed items. Two of the commonly used censoring schemes are type-I and type-II censoring schemes. The hybrid censoring scheme which is a mixture of type-I

✉ R. Arabi Belaghi
rezaarabi11@gmail.com

[1] Department of Statistics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

and type-II censoring schemes was first introduced by Epstein (1954). These three conventional censoring schemes remove the left units only at the terminal time and not at each failure time. To overcome this problem, progressive hybrid censoring (PHC) scheme which is a mixture of type-I and type-II progressive censoring schemes has been introduced. Under type-I progressive hybrid censoring (type-I PHC) scheme, Kundu and Joarder (2006) have discussed MLEs and Bayes estimates for an exponential distribution. Moreover, Childs et al. (2008) proposed the type-II progressive hybrid censoring (type-II PHC) scheme and derived the exact distribution of the MLEs for the mean of the exponential distribution. For other related works see Banerjee and Kundu (2008), Lin and Huang (2011), Lin et al. (2011) and Gurunlu Alma and Arabi Belaghi (2015).

The type-II PHC scheme overcomes the drawback of the type-I PHC scheme that the maximum likelihood may not always exist. It can be described as follows: Consider n identical items are placed simultaneously on test with the corresponding lifetimes being independent and identically distributed (i.i.d) each with probability density function (p.d.f) $f_X(x;\theta)$ and cumulative distribution function (c.d.f) $F_X(x;\theta)$, where θ denotes the vector of model parameters. $\tilde{R} = (R_1, R_2, ..., R_m)$, $1 \leq m \leq n$, is prefixed progressive type-II right censoring scheme with $R_j > 0$ and $\sum_{j=1}^{m} R_j + m = n$ is specified. Under the type-II PHC scheme, at the time of the first failure $X_{1:m:n}$, R_1 of the $n-1$ surviving units are randomly withdrawn from the experiment, then at the time of the second failure $X_{2:m:n}$, R_2 of the $n - R_1 - 2$ surviving units are withdrawn, and so on. Finally at the time of the mth failure $X_{m:m:n}$, all $R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m$ surviving units are withdrawn from the life-test. Hence, $X_{1:m:n} < \cdots < X_{m:m:n}$ denote the progressively censored failure times. The type-II PHC scheme involves the termination of the life-test at the time $T^* = \max\{X_{m:m:n}, T\}$. Let D denote the number of failures that occur up to time T. If $X_{m:m:n} \geq T$, then experiment would terminate at the mth failure with the withdrawal of units occurring after each failure according to the prefixed progressive censoring scheme $(R_1, R_2, ..., R_m)$. However, if $X_{m:m:n} < T$, then instead of terminating the experiment by removing all remaining R_m units after the mth failure, the experiment would continue to observe failures without any further withdrawals up to time T. Hence, $R_m = R_{m+1} = \cdots = R_D = 0$. In this case the failure times are represented by $X_{1:m:n} < \cdots < X_{m:m:n} < X_{m+1:n} < \cdots < X_{d:n}$ where the d is the observed value of D. We denote these two cases as case I and case II, respectively.

Case I: $\{X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}\}$, if $X_{m:m:n} \geq T$;
Case II: $\{X_{1:m:n} < \cdots < X_{m:m:n} < X_{m+1:n} < \cdots < X_{D:n}\}$, if $X_{m:m:n} < T$.

Based on the observed type-II PHC data, the likelihood function can be written as follows:

$$\text{Case I}: L(\theta) = C_1 \prod_{i=1}^{m} f_X(x_{i:m:n};\theta)[1 - F_X(x_{i:m:n};\theta)]^{R_i}, \tag{1}$$

$$\text{Case II}: L(\theta) = C_2 \prod_{i=1}^{m} f_X(x_{i:m:n};\theta)[1 - F_X(x_{i:m:n};\theta)]^{R_i} \prod_{i=m+1}^{D} f_X(x_{i:n};\theta)[1 - F_X(T;\theta)]^{R'_D}, \tag{2}$$

where $C_1 = n(n - R_1 - 1)\cdots(n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$, $C_2 = n(n - R_1 - 1)\cdots(n - R_1 - R_2 - \cdots - R_{m-1} - D + 1)$, $D = m+1, ..., n - \sum_{j=1}^{m-1} R_j$, $R_m = 0$ if $D \geq m$ and $R'_D = n - D - \sum_{j=1}^{m-1} R_j$.

In this paper, we mainly consider the analysis of progressive type-II hybrid censored data when the lifetime distribution of the individual item follows the Burr XII distribution.

The two parameter Burr XII distribution has received the most attention in the statistical literature. This distribution contains some well-known distributions, such as the Weibull, normal, log-normal, logistic, gamma and extreme value distributions, among others. Due to its flexibility and some desirable properties, applications have proved to be much wider. Applications may be found in areas of quality control, economics, duration of failure time modeling, insurance risk and reliability analysis. Some inference concerning the Burr XII distribution have been discussed by many authors. Among others, see Ali Mousa and Jaheen (2002), Soliman (2005), Abd-Elfattah et al. (2008), Ahmed et al. (2011) and Rastogi and Tripathi (2013). Reviewing the literate shows that there is no work for the parameters estimation of the Burr Model under progressive type II hybrid censored data, that's motivate us to consider this study.

The probability density function (pdf) and the cumulative distribution function (cdf) of the Burr XII distribution with two shape parameters, α and β are given, respectively, by

$$f_X(x;\alpha,\beta) = \alpha\beta x^{\beta-1}(1+x^{\beta})^{-\alpha-1}, \quad x>0, \alpha>0, \beta>0, \tag{3}$$

$$F_X(x;\alpha,\beta) = 1-(1+x^{\beta})^{-\alpha}, \quad x>0. \tag{4}$$

The objective of this paper is two-fold. In the first part, we obtain the maximum likelihood estimates (MLEs) of the unknown parameters. The Eqs. 1 and 2 are maximized numerically to obtain maximum likelihood estimates for parameters α and β. The iterative method such as the Newton-Raphson (NR) can be utilized to perform maximization. However, the MLEs via the NR method are very sensitive to their initial parameter estimation value. Moreover, in dealing with censored data, the parameter estimates of MLEs via the NR algorithm are significantly biased. In this article, we propose using the EM algorithm for computing the MLEs. The EM algorithm always converges. It has a stable global convergence because of its robustness against the initial value. The EM algorithm is also stable in numerical computation because each of its iterations increases the likelihood value; this algorithm characteristic is useful in providing valuable statistical information. This information makes it easy to monitor convergence and programming errors (Wang and Cheng 2010). From the proposed EM algorithm, the observed information matrix based on the missing value principle is computed, which can be used to construct the asymptotic confidence intervals (CIs). In the second part, we consider a Bayesian approach to estimate the parameters α and β under squared error loss (SEL) and LINEX loss functions. It is observed that the Bayes estimates cannot be obtained in nice closed form. Thus, we adopt Lindley's approximation to obtain the Bayes estimates. Since the Lindley approximation method fails to construct HPD credible intervals, we made use of the importance sampling procedure to obtain point estimates and HPD credible intervals of the parameters. We also conduct some simulation experiments to investigate the accuracy of the estimates and compare the performance of CIs obtained. The remainder of this paper is structured as follows: In Section 2 we provide the maximum likelihood estimates (MLEs) of the unknown parameters by using EM algorithm. The Fisher information matrix is also evaluated in this Section. The Bayes estimates of the unknown parameters are obtained in Section 3 using Lindely's approximation and importance sampling methods. Section 4 is devoted to the simulation study. One real data set is analyzed for illustration in Section 5. Finally, some concluding remarks are given in Section 6.

2 Maximum Likelihood Estimation

In this section, we obtain the MLEs of the parameters $\theta = (\alpha, \beta)$ using EM algorithm and derive their asymptotic variance-covariance.

2.1 The EM Algorithm

The Expectation-Maximization (EM) algorithm (Dempster et al. 1977) is a widely applicable technique for maximum likelihood estimation with incomplete data. This algorithm enables the computationally efficient determination of the MLEs when iterative procedures are required. On each iteration of the EM algorithm, there are two steps:

E-step In E-step the missing data are replaced by their expected values when a level of parameter vector is presumed. When the likelihood function of censored data is replaced with the expected likelihood function of missing data, the pseudo-likelihood function is derived. The E-step finds the following expectation:

$$Q(\theta^{(s+1)}, \theta^{(s)}) = E[l_c(\mathbf{W}; \theta)|\mathbf{X} = \mathbf{x}, \theta^{(s)}], \tag{5}$$

where $\theta^{(s)}$ is the current parameter vector and is used to evaluate the expectation.

M-step The M-step consist in finding $\theta^{(s+1)}$, the value of θ that maximize Q:

$$\theta^{(s+1)} = \text{argmax}\, Q(\theta, \theta^{(s)}), \quad \forall \theta \in \Theta \tag{6}$$

The optimized parameter vector $\theta^{(s+1)}$ from Eq. 6 is used as the current parameter vector for Eq. 5. Each step of the EM increases the log-likelihood. In particular, if a unique finite maximum likelihood estimate of θ exists, the algorithm finds it. For a detailed discussion on the EM algorithm and its applications, the reader is referred to a book by McLachlan and Krishnan (1997).

The analysis with the data from type-II PHC scheme can be treated as an incomplete data problem and the EM algorithm can be used quite effectively to compute the MLEs by solving a two-dimensional optimization problem at each iteration. In case I, suppose that $\mathbf{X} = (X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n})$ and $\mathbf{Z} = (Z_{i1}, Z_{i2}, ..., Z_{iR_i})$ represent the observed and the censored data, respectively. Here for a given m, $\mathbf{Z}_i = (Z_{11}, ..., Z_{1R_i}, ..., Z_{m1}, ..., Z_{mR_i})$ are not observed. The censored data vector $\mathbf{Z}_i$ can be thought of as missing data. Based on $\mathbf{W} = (\mathbf{X}, \mathbf{Z}_i)^\mathsf{T}$, the complete data likelihood function will be in the form

$$L_c(\mathbf{W}; \theta) = \prod_{i=1}^{m} f_X(x_{i:m:n}; \theta) \prod_{i=1}^{m} \prod_{j=1}^{R_i} f_X(z_{ij}; \theta). \tag{7}$$

Then, the log-likelihood for the complete lifetimes of n items from the two parameter Burr XII distribution is given as follows:

$$\begin{aligned} l_c(\mathbf{W}; \alpha, \beta) \;=\; & n \log \alpha + n \log \beta + (\beta - 1) \sum_{i=1}^{m} \log x_{i:m:n} - (\alpha + 1) \sum_{i=1}^{m} \log(1 + x_{i:m:n}^{\beta}) + (\beta - 1) \sum_{i=1}^{m} \sum_{j=1}^{R_i} \log z_{ij} \\ & -(\alpha + 1) \sum_{i=1}^{m} \sum_{j=1}^{R_i} \log(1 + z_{ij}^{\beta}). \end{aligned} \tag{8}$$

The E-step of the EM algorithm involves the computation of the conditional expectation $E(l_c(\mathbf{W};\alpha,\beta)|\mathbf{X})$ which is equal to the pseudo log-likelihood function $l_c^*(\mathbf{W};\alpha,\beta)$ defined as

$$\begin{aligned} l_c^*(\mathbf{W};\alpha,\beta) \;=\; & n\log\alpha + n\log\beta + (\beta-1)\sum_{i=1}^{m}\log x_{i:m:n} - (\alpha+1)\sum_{i=1}^{m}\log(1+x_{i:m:n}^{\beta}) \\ & +(\beta-1)\sum_{i=1}^{m}\sum_{j=1}^{R_i}E[\log Z_{ij}|Z_{ij}>x_{i:m:n}] - (\alpha+1)\sum_{i=1}^{m}\sum_{j=1}^{R_i}E[\log(1+Z_{ij}^{\beta})|Z_{ij}>x_{i:m:n}].(9) \end{aligned}$$

The required expected values of a truncated Burr XII from the left at c for EM algorithm are, respectively, given by

$$A(c,\alpha,\beta) = E[\log Z_{ij}|Z_{ij}>c] \quad \text{and} \quad B(c,\alpha,\beta) = E[\log(1+Z_{ij}^{\beta})|Z_{ij}>c], \quad (10)$$

and they are presented in Rastogi and Tripathi (2013).

The M-step in a EM iteration is maximizing the log-likelihood based on complete sample over Θ; with the missing values replaced by their conditional expectations. Suppose at the s^{th} stage, the estimators of (α,β) are $(\alpha^{(s)},\beta^{(s)})$, then $(\alpha^{(s+1)},\beta^{(s+1)})$ can be obtained by maximizing

$$\begin{aligned} \lambda(\alpha,\beta) \;=\; & n\log\alpha + n\log\beta + (\beta-1)\sum_{i=1}^{m}\log x_{i:m:n} - (\alpha+1)\sum_{i=1}^{m}\log(1+x_{i:m:n}^{\beta}) \\ & +(\beta-1)\sum_{i=1}^{m}R_i A(x_{i:m:n},\alpha^{(s)},\beta^{(s)}) - (\alpha+1)\sum_{i=1}^{m}R_i B(x_{i:m:n},\alpha^{(s)},\beta^{(s)}),(11) \end{aligned}$$

with respect to α and β respectively. Note that the maximization of the Eq. 11 can be obtained quite effectively by the similar method proposed by Gupta and Kundu (2001). First, $\beta^{(s+1)}$ can be obtain by solving a fixed-point type equation

$$\varphi(\beta) = \beta$$

The function $\varphi(\beta)=\beta$ is defined as

$$\varphi(\beta) = \left[\frac{(1+\hat{\alpha}(\beta))}{n}\sum_{i=1}^{m}\left(\frac{x_{i:m:n}^{\beta}\log x_{i:m:n}}{1+x_{i:m:n}^{\beta}}\right) - \frac{\sum_{i=1}^{m}R_i A(x_{i:m:n},\alpha^{(s)},\beta^{(s)})}{n} - \frac{\sum_{i=1}^{m}\log x_{i:m:n}}{n}\right]^{-1},$$

and

$$\hat{\alpha}(\beta) = \frac{n}{\sum_{i=1}^{m}\log(1+x_{i:m:n}^{\beta}) + \sum_{i=1}^{m}R_i B(x_{i:m:n},\alpha^{(s)},\beta^{(s)})}.$$

One can follow iteration method. Once $\beta^{(s+1)}$ is obtained , $\alpha^{(s+1)}$ is obtained as $\alpha^{(s+1)} = \hat{\alpha}(\beta^{(s+1)})$.

Now, in case II, suppose that $\mathbf{X} = (X_{1:m:n}, X_{2:m:n},\cdots,X_{m:m:n},X_{m+1:n},...,X_{D:n})$ represents the observed and $\mathbf{Z}_i = (Z_{11},...,Z_{1R_i},...,Z_{m1},...,Z_{mR_i})$ and $\mathbf{Z}' = (Z_1', Z_2', ..., Z_{R_D'}')$ the censored data, respectively. The censored data $\mathbf{Z}_i$ and $\mathbf{Z}'$ can be thought of as missing data. The combination of $\mathbf{W}=(T;\mathbf{Z}_i,\mathbf{Z}')^{\mathsf{T}}$ forms the compelet data set. Based on $(T;\mathbf{Z}_i,\mathbf{Z}')^{\mathsf{T}}$ the camplete data likelihood function will be in the form

$$L_c(\mathbf{W};\theta) = \prod_{i=1}^{m}\left[f_X(x_{i:m:n};\theta)\prod_{j=1}^{R_i}f_X(z_{ij};\theta)\right]\prod_{i=m+1}^{D}\left[f_X(x_{i:n};\theta)\prod_{r=1}^{R_D'}f_X(z_r';\theta)\right].(12)$$

For complete lifetime of n items from Burr XII distribution, the Eq. 12 is obtained as

$$\begin{aligned} l_c(\mathbf{W};\alpha,\beta) &= n\log\alpha + n\log\beta + (\beta-1)\sum_{i=1}^{m}\log x_{i:m:n} - (\alpha+1)\sum_{i=1}^{m}\log(1+x_{i:m:n}^{\beta}) + (\beta-1)\sum_{i=m+1}^{D}\log x_{i:n} \\ &\quad -(\alpha+1)\sum_{i=m+1}^{D}\log(1+x_{i:n}^{\beta}) + (\beta-1)\sum_{i=1}^{m-1}\sum_{j=1}^{R_i}\log z_{ij} - (\alpha+1)\sum_{i=1}^{m-1}\sum_{j=1}^{R_i}\log(1+z_{ij}^{\beta}) \\ &\quad +(\beta-1)\sum_{r=1}^{R'_D}\log z'_r - (\alpha+1)\sum_{r=1}^{R'_D}\log(1+z_r'^{\beta}). \end{aligned} \tag{13}$$

In E-step of the EM algorithm one requires to compute the pseudo-likelihood function as follows:

$$\begin{aligned} l_c^*(\mathbf{W};\alpha,\beta) &= n\log\alpha + n\log\beta + (\beta-1)\sum_{i=1}^{m}\log x_{i:m:n} - (\alpha+1)\sum_{i=1}^{m}\log(1+x_{i:m:n}^{\beta}) + (\beta-1)\sum_{i=m+1}^{D}\log x_{i:n} \\ &\quad -(\alpha+1)\sum_{i=m+1}^{D}\log(1+x_{i:n}^{\beta}) + (\beta-1)\sum_{i=1}^{m-1}R_i E[\log Z_i|Z_i > x_{i:m:n}] + (\beta-1)\sum_{r=1}^{R'_D}E[\log Z'_r|Z'_r > T] \\ &\quad -(\alpha+1)\sum_{i=1}^{m-1}R_i E[\log(1+Z_i^{\beta})|Z_i > x_{i:m:n}] - (\alpha+1)\sum_{r=1}^{R'_D}E[\log(1+Z_r'^{\beta})|Z'_r > T]. \end{aligned} \tag{14}$$

In M-step of the s^{th} iteration of EM algorithm, by substituting (10) into $E(l_c(\mathbf{W};\alpha,\beta)|\mathbf{X})$, we obtain

$$\begin{aligned} \lambda(\alpha,\beta) &= n\log\alpha + n\log\beta + (\beta-1)\sum_{i=1}^{m}\log x_{i:m:n} - (\alpha+1)\sum_{i=1}^{m}\log(1+x_{i:m:n}^{\beta}) + (\beta-1)\sum_{i=m+1}^{D}\log x_{i:n} \\ &\quad -(\alpha+1)\sum_{i=m+1}^{D}\log(1+x_{i:n}^{\beta}) + (\beta-1)\sum_{i=1}^{m-1}R_i A(x_{i:m:n},\alpha^{(s)},\beta^{(s)}) + (\beta-1)R'_D A(T,\alpha^{(s)},\beta^{(s)}) \\ &\quad -(\alpha+1)\sum_{i=1}^{m-1}R_i B(x_{i:m:n},\alpha^{(s)},\beta^{(s)}) - (\alpha+1)R'_D B(T,\alpha^{(s)},\beta^{(s)}). \end{aligned} \tag{15}$$

For maximizing Eq. 15, first we find $\beta^{(s+1)}$ by solving the fixed-point type equation

$$\varphi(\beta) = \beta$$

The function $\varphi(\beta) = \beta$ is defined as

$$\begin{aligned} \varphi(\beta) = \Bigg[&\frac{1+\hat{\alpha}(\beta)}{n}\left\{\sum_{i=1}^{m}\frac{x_{i:m:n}^{\beta}\log x_{i:m:n}}{1+x_{i:m:n}^{\beta}} + \sum_{i=m+1}^{D}\frac{x_{i:n}^{\beta}\log x_{i:n}}{1+x_{i:n}^{\beta}}\right\} - \frac{1}{n}\sum_{i=1}^{m}\log x_{i:m:n} \\ &-\frac{1}{n}\sum_{i=1}^{m-1}R_i A(x_{i:m:n},\alpha^{(s)},\beta^{(s)}) - \frac{1}{n}\sum_{i=m+1}^{D}\log x_{i:n} - \frac{1}{n}R'_D A(T,\alpha^{(s)},\beta^{(s)})\Bigg]^{-1}, \end{aligned}$$

and

$$\hat{\alpha}(\beta) = \frac{n}{\sum_{i=1}^{m}\log(1+x_{i:m:n}^{\beta}) + \sum_{i=1}^{m-1}R_i B(x_{i:m:n},\alpha^{(s)},\beta^{(s)}) + \sum_{i=m+1}^{D}\log(1+x_{i:n}^{\beta}) + R'_D B(T,\alpha^{(s)},\beta^{(s)})}.$$

Then, $\alpha^{(s+1)}$ is obtained as $\alpha^{(s+1)} = \hat{\alpha}(\beta^{(s+1)})$.

2.2 Asymptotic Variances and Covariance of the MLEs

In this subsection, we describe the use of the missing information to compute the variance-covariance matrix of the MLEs under type-II PHCS. The idea of the missing information principle of Louis (1982) can be expressed as follows

Observed information = Complete information - Missing information

For $\theta = (\alpha, \beta)'$, we define $\mathbf{X}$, $\mathbf{Z}$ and $\mathbf{W}$ to be the observed, missing and complete data, and $I_{\mathbf{X}}(\theta)$, $I_{\mathbf{W}|\mathbf{X}}(\theta)$ and $I_{\mathbf{W}}(\theta)$ to be the corresponding Fisher information matrix, respectively. The complete information matrix $I_{\mathbf{W}}(\theta)$ is given by

$$I_{\mathbf{W}}(\theta) = -E_{\theta}\left[\frac{\partial^2 l_c(\mathbf{W};\theta)}{\partial\theta^2}\right], \tag{16}$$

Based on the conditional distribution, the Fisher information matrix in the ith observation which is censored can be computed as

$$I^{(i)}_{\mathbf{Z}|\mathbf{X}}(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log(f(z_i|x_{i:m:n},\theta))\right]. \tag{17}$$

Hence, the expected missing information can then be easily obtained as

$$I_{\mathbf{Z}|\mathbf{X}}(\theta) = \sum_{i=1}^{m} R_i I^{(i)}_{\mathbf{Z}|\mathbf{X}}(\theta). \tag{18}$$

Thus, by the missing information principle, the observed information matrix can be obtained as

$$I_{\mathbf{X}}(\theta) = I_{\mathbf{W}}(\theta) - I_{\mathbf{Z}|\mathbf{X}}(\theta). \tag{19}$$

Finally, the asymptotic variance-covariance matrix of the MLE of θ can be obtained by inverting the observed information matrix $I_{\mathbf{X}}(\hat{\theta})$. As the dimension of θ is 2, $I_{\mathbf{W}}(\theta)$ and $I_{\mathbf{W}|\mathbf{X}}(\theta)$ are both of the order 2×2. The elements of matrix $I_{\mathbf{W}}(\theta)$ for complete data set are presented in (Rastogi and Tripathi 2013). We report $I_{\mathbf{W}}(\theta)$ which have been evaluated by them here as:

$$I_{\mathbf{W}}(\theta) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where

$$\begin{aligned} a_{11}(\alpha,\beta) &= \frac{n}{\alpha^2}, \quad a_{12}(\alpha,\beta) = \frac{n}{\beta^2} + n\alpha(\alpha+1)\beta\int_0^\infty \frac{x^{2\beta-1}(\ln x)^2}{(1+x^\beta)^{\alpha+3}}dx, \\ a_{12}(\alpha,\beta) &= a_{21}(\alpha,\beta) = n\alpha\beta\int_0^\infty \frac{x^{2\beta-1}\ln x}{(1+x^\beta)^{\alpha+2}}dx. \end{aligned} \tag{20}$$

The conditional distribution required for the calculation of the missing information matrix is given by (see Ng et al. (2012))

$$f_{Z_i}(z_i|Z_i > x_{i:m:n}) = \frac{f_X(z_i)}{[1 - F_X(x_{i:m:n})]}, \quad z_i > x_{i:m:n} \tag{21}$$

Using Eq. 21, the logarithm of the pdf of the truncated Burr XII distribution for case I is

$$\log f_{Z_i}(z_i|Z_i > x_{i:m:n}) = \log\alpha + \log\beta + (\beta-1)\log z_i - (\alpha+1)\log(1+z_i^\beta) + \alpha\log(1+x_{i:m:n}^\beta). \tag{22}$$

Thus, the second partial derivatives of Eq. 22 with respect to α and β yields

$$\frac{\partial^2 \log f_{z_i}}{\partial\alpha^2} = -\frac{1}{\alpha^2}, \quad \frac{\partial^2 \log f_{z_i}}{\partial\beta^2} = -\frac{1}{\beta^2} - (\alpha+1)\frac{z_i^\beta(\log z_i)^2}{(1+z_i^\beta)^2} + \alpha\frac{x_{i:m:n}^\beta(\log x_{i:m:n})^2}{(1+x_{i:m:n}^\beta)^2},$$
$$\frac{\partial^2 \log f_{z_i}}{\partial\alpha\partial\beta} = -\frac{z_i^\beta \log z_i}{1+z_i^\beta} + \frac{x_{i:m:n}^\beta \log x_{i:m:n}}{1+x_{i:m:n}^\beta}.$$

The negative of the expected value of these three second partial derivatives are obtained as follows

$$E(\frac{\partial^2 \log f_{z_i}}{\partial\alpha^2}) = \frac{1}{\alpha^2}, \quad E(\frac{\partial^2 \log f_{z_i}}{\partial\beta^2}) = \frac{1}{\beta^2} + (\alpha+1)B^*(x_{i:m:n},\alpha,\beta) - \alpha\frac{x_{i:m:n}^\beta(\log x_{i:m:n})^2}{(1+x_{i:m:n}^\beta)^2},$$
$$E(\frac{\partial^2 \log f_{z_i}}{\partial\alpha\partial\beta}) = A^*(x_{i:m:n},\alpha,\beta) - \frac{x_{i:m:n}^\beta \log x_{i:m:n}}{1+x_{i:m:n}^\beta},$$

where

$$A^*(c,\alpha,\beta) = E(Z_i^\beta \log Z_i/(1+Z_i^\beta)|Z_i > c) \quad \text{and} \quad B^*(c,\alpha,\beta) = E(Z_i^\beta(\log Z_i)^2/(1+Z_j^\beta)^2|Z_i > c), \quad (23)$$

and they are given in Rastogi and Tripathi (2013). Using these expectations, the expected missing information matrix $I_{\mathbf{Z}|\mathbf{X}}(\theta)$ computed as in Eq. 17, and then the observed information matrix can be obtained from Eq. 19. Finally, by inverting $I_{\mathbf{X}}(\hat{\theta})$ the asymptotic variance-covariance matrix of the MLEs can be obtained.

Theorem 2.1 *(For case II). Given $X_{1:m:n} = x_{1:m:n}, X_{2:m:n} = x_{2:m:n}, \cdots, X_{i:m:n} = x_{i:m:n}$, and $X_{m+1:n} = x_{m+1:n}, X_{m+2:n} = x_{m+2:n} \cdots, X_{D:n} = x_{D:n}$, the conditional distribution of Z_{ij} and Z'_r is*

$$f_{\mathbf{Z}_i,\mathbf{Z}'|\mathbf{X},T}(z_{jk}, z'_r|x_{1:m:n}, x_{2:m:n}, \cdots, x_{i:m:n}, x_{m+1:n}, \cdots, x_{D:n}, T) = \frac{f_X(z_{ij})f_X(z'_r)}{[1-F_X(x_{i:m:n})][1-F_X(T)]}.$$

Proof See Gurunlu Alma and Arabi Belaghi (Gurunlu Alma and Arabi Belaghi 2015). □

By using theorem (2.1), the logarithm of the left truncated Burr XII pdf for case II is

$$\log f_{z_i,z'_r}(z_i, z'_p|\mathbf{X}, T) = 2\log\alpha + 2\log\beta + (\beta-1)[\log z_i + \log z'_r] - (\alpha+1)[\log(1+z_i^\beta) + \log(1+{z'_r}^\beta)]$$
$$+\alpha[\log(1+x_{i:m:n}^\beta) + \log(1+T^\beta)].$$

The second partial derivatives with respect to α and β are obtained by

$$\frac{\partial^2 \log f_{z_i,z'_r}}{\partial\beta^2} = -\frac{2}{\beta^2} - (\alpha+1)\frac{z_i^\beta(\log z_i)^2}{(1+z_i^\beta)^2} + \alpha\frac{x_{i:m:n}^\beta(\log x_{i:m:n})^2}{(1+x_{i:m:n}^\beta)^2} - (\alpha+1)\frac{{z'_r}^\beta(\log z'_r)^2}{(1+{z'_r}^\beta)^2} + \alpha\frac{T^\beta(\log T)^2}{(1+T^\beta)^2},$$
$$\frac{\partial^2 \log f_{z_j,z'_r}}{\partial\alpha\partial\beta} = \frac{x_{i:m:n}^\beta \log x_{i:m:n}}{1+x_{i:m:n}^\beta} - \frac{z_i^\beta \log z_i}{1+z_i^\beta} - \frac{{z'_r}^\beta \log z'_r}{1+{z'_r}^\beta} + \frac{T^\beta \log T}{1+T^\beta},$$
$$\frac{\partial^2 \log f_{z_i,z'_r}}{\partial\alpha^2} = -\frac{2}{\alpha^2}$$

Using the expectations in Eq. 23, the Fisher information matrix based observations which are truncated at the time $x_{i:m:n}$ and T can be computed by straightforward replacing and the following expectations can be obtained

$$\begin{aligned}
E(\frac{\partial^2 \log f_{z_i,z'_r}}{\partial \beta^2}) &= \frac{2}{\beta^2} + (\alpha+1)B^*(x_{i:m:n},\alpha,\beta) - \alpha\frac{x^{\beta}_{i:m:n}(\log x_{i:m:n})^2}{(1+x^{\beta}_{i:m:n})^2} + (\alpha+1)B^*(T,\alpha,\beta) - \alpha\frac{T^{\beta}(\log T)^2}{(1+T^{\beta})^2},\\
E(\frac{\partial^2 \log f_{z_i,z'_r}}{\partial \alpha \partial \beta}) &= -\frac{x^{\beta}_{i:m:n}\log x_{i:m:n}}{1+x^{\beta}_{i:m:n}} - \frac{T^{\beta}\log T}{1+T^{\beta}} + A^*(x_{j:m:n},\alpha,\beta) + A^*(T,\alpha,\beta),\\
E(\frac{\partial^2 \log f_{z_i,z'_r}}{\partial \alpha^2}) &= \frac{2}{\alpha^2}.
\end{aligned}$$

Thus, the expected information for conditional distribution of $\mathbf{Z}$ given $\mathbf{X}$ can be obtained using $I_{\mathbf{Z}|\mathbf{X}}(\theta)$ and hence $I_{\mathbf{X}}(\theta)$. Inverting $I_{\mathbf{X}}(\theta)$ yields the variance-covariance matrix of $\hat{\theta} = (\alpha, \beta)'$.

The asymptotic normality of the MLE can be utilized to construct the approximate confidence intervals for the parameters α and β become, respectively,

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{\sigma}^2_{\alpha}} \quad \text{and} \quad \hat{\beta} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{\sigma}^2_{\beta}},$$

where $Z_{\gamma/2}$ is the upper $\gamma/2$ quantile of the standard normal distribution.

3 Bayesian Estimation

In this section, we deal with the problem of estimating the parameters α and β under square error loss (SEL) and LINEX loss functions. These loss functions for a parameter δ are as follows, respectively,

$$L_1(\delta, \hat{\delta}) = (\hat{\delta} - \delta)^2, \tag{24}$$

$$L_2(\Delta) \propto e^{w\Delta} - w\Delta - 1, \quad w \neq 0. \tag{25}$$

where $\Delta = (\delta - \hat{\delta})$ denote the scalar estimation error in using $\hat{\delta}$ to estimate δ. The sign of w represents the direction and its magnitude represents the degree of symmetry. This has been proposed by Varian (1975) and its proporties have been studied by Zellner (1986). For $w = 1$, the LINEX loss function is quite asymmetric about zero with overestimation being more costly than underestimation. If $w < 0$, $L_2(\Delta)$ rises exponentially when $\Delta < 0$ (underestimation) and almost linearly when $\Delta > w$ (overestimation). For w closed to zero, the LINEX is approximately SEL and therefore almost symmetric (see Zellner (1986)).

For a Bayes estimation, we need to assume some prior distributions for the unknown parameters. We assume that α has a gamma prior, $GA(a_1, b_1)$ with pdf given by

$$\pi_1(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha}, \tag{26}$$

where $a_1 > 0$ and $b_1 > 0$; and the prior of β, $\pi_2(\beta)$, has the support on $(0, \infty)$, and it is independent of the prior of α. By applying then the joint prior distribution α and β, we obtain the joint density function of α, β and $\underline{x}$ for the two cases as follows:

Case I:

$$\ell(\alpha, \beta, \underline{x}) \propto \alpha^{m+a_1-1}\beta^m\pi_2(\beta)e^{-b_1\alpha}\prod_{i=1}^{m} x^{\beta-1}_{i:m:n}(1+x^{\beta}_{i:m:n})^{-(\alpha(R_i+1)+1)}, \tag{27}$$

where $\underline{x} = (x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:m:n})$.

Case II:

$$\ell(\alpha,\beta,\underline{x}) \propto \alpha^{a_1+D-1}\beta^{D}\pi_2(\beta)e^{-b_1\alpha}\prod_{i=1}^{m}x_{i:m:n}^{\beta-1}(1+x_{i:m:n}^{\beta})^{-(\alpha(R_i+1)+1)}$$
$$\times\prod_{i=m+1}^{D}x_{i:n}^{\beta-1}(1+x_{i:n}^{\beta})^{-\alpha-1}(1+T^{\beta})^{-\alpha R'_D}.$$

where $\underline{x}=(x_{1:m:n},x_{2:m:n},\cdots,x_{m:m:n},x_{m+1:n},\cdots,x_{D:n})$.

Based on $\ell(\alpha,\beta,\underline{x})$, the joint posterior density function of α and β, is given by

$$\pi(\alpha,\beta|\underline{x})=\frac{\ell(\alpha,\beta,\underline{x})}{\int_0^\infty\int_0^\infty\ell(\alpha,\beta,\underline{x})\mathrm{d}\alpha\mathrm{d}\beta}. \tag{28}$$

It is clear that Eq. 28 cannot be obtained analytically even when $\pi_2(\beta)$ is known. Therefore, we adopt Lindley's approximation (Lindley 1980) and the MCMC technique to compute Bayes estimates.

3.1 Lindley's Approximation

In this section, we compute the Bayes estimates of α and β by using Lindley's approximation method. Based on Lindley's approximation, the expectation of any function of α and β in the form

$$\begin{aligned}\hat{\phi}(\alpha,\beta) &= E_{\alpha,\beta|\underline{x}}[\phi(\alpha,\beta)]\\ &= \frac{\int_0^\infty\int_0^\infty\phi(\alpha,\beta)\ell(\alpha,\beta,\underline{x})\mathrm{d}\alpha\mathrm{d}\beta}{\int_0^\infty\int_0^\infty\ell(\alpha,\beta,\underline{x})\mathrm{d}\alpha\mathrm{d}\beta},\end{aligned} \tag{29}$$

can be evaluated as

$$\hat{\phi}=\phi(\hat{\alpha},\hat{\beta})+\frac{1}{2}[A+L_{30}B_{12}+L_{03}B_{21}+L_{21}C_{12}+L_{12}C_{21}]+\rho_1A_{12}+\rho_2A_{21}, \tag{30}$$

where

$$\begin{aligned}
A &= \sum_{i=1}^{2}\sum_{j=1}^{2}\phi_{ij}\sigma_{ij}, \quad L_{ij}=\frac{\partial^{i+j}L(\alpha,\beta)}{\partial\alpha^i\partial\beta^j}, \quad i,j=0,1,2,3,\ i+j=3,\\
\rho_1 &= \frac{\partial\log\varrho}{\partial\alpha}, \quad \rho_2=\frac{\partial\log\varrho}{\partial\beta}, \quad \varrho=\log\pi(\alpha,\beta),\\
\phi_1 &= \frac{\partial\phi(\alpha,\beta)}{\partial\alpha}, \quad \phi_2=\frac{\partial\phi(\alpha,\beta)}{\partial\beta},\\
\phi_{11} &= \frac{\partial^2\phi(\alpha,\beta)}{\partial\alpha^2}, \quad \phi_{12}=\frac{\partial^2\phi(\alpha,\beta)}{\partial\alpha\partial\beta}, \quad \phi_{22}=\frac{\partial^2\phi(\alpha,\beta)}{\partial\beta^2},\\
A_{ij} &= \phi_i\sigma_{ii}+\phi_j\sigma_{ji}, \quad B_{ij}=(\phi_i\sigma_{ii}+\phi_j\sigma_{ij})\sigma_{ii}, \quad C_{ij}=3\phi_i\sigma_{ii}\sigma_{ij}+\phi_j(\sigma_{ii}\sigma_{jj}+2\sigma_{ij}^2),
\end{aligned}$$

where $L(.)$ is the log-likelihood function of the observed data, σ_{ij} is the (i,j)-th elements of the inverse of the Fisher information matrix, $\pi(\alpha,\beta)$ is the joint prior density function of (α,β), $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of α and β, respectively and all the quantities are evaluated at $(\hat{\alpha}, \hat{\beta})$. In case I, with the specification $\beta \sim GA(a_2, b_2)$, the log-likelihood function is given by

$$\log L(\alpha,\beta) = \text{const.} + m\log\alpha + m\log\beta + (\beta-1)\sum_{i=1}^{m}\log x_{i:m:n} - \sum_{i=1}^{m}[\alpha(R_i+1)+1]\log(1+x_{i:m:n}^{\beta}).$$

To apply Lindley's approximation in Eq. 30, we obtain

$$L_{11} = -\sum_{i=1}^{m}(1+R_i)\frac{x_{i:m:n}^{\hat{\beta}}\log x_{i:m:n}}{(1+x_{i:m:n}^{\hat{\beta}})}, \quad L_{12} = -\sum_{i=1}^{m}(R_i+1)\frac{x_{i:m:n}^{\hat{\beta}}(\log x_{i:m:n})^2}{(1+x_{i:m:n}^{\hat{\beta}})^2}, \quad L_{21}=0,$$

$$L_{20} = -\frac{m}{\hat{\alpha}^2}, \quad L_{02} = -\frac{m}{\hat{\beta}^2} - \sum_{i=1}^{m}[\alpha(1+R_i)+1]\frac{x_{i:m:n}^{\hat{\beta}}(\log x_{i:m:n})^2}{(1+x_{i:m:n}^{\hat{\beta}})^2}, \quad L_{30} = \frac{2m}{\hat{\alpha}^3},$$

$$L_{03} = \frac{2m}{\hat{\beta}^3} - \sum_{i=1}^{m}(\alpha(1+R_i)+1)\frac{x_{i:m:n}^{\hat{\beta}}(\log x_{i:m:n})^3(1-x_{i:m:n}^{\hat{\beta}})}{(1+x_{i:m:n}^{\hat{\beta}})^3}.$$

Furthermore, we have

$$\rho_1 = \frac{\partial \log\pi(\alpha,\beta)}{\partial\alpha} = \frac{a_1-1}{\hat{\alpha}} - b_1, \quad \rho_2 = \frac{\partial \log\pi(\alpha,\beta)}{\partial\beta} = \frac{a_2-1}{\hat{\beta}} - b_2.$$

The log-likelihood function in case II is given by

$$\begin{aligned}\log L(\alpha,\beta) &= \text{const.} + D\log\alpha + D\log\beta - \sum_{i=1}^{m}[\alpha(R_i+1)+1]\log(1+x_{i:m:n}^{\beta}) \\ &+(\beta-1)\left[\sum_{i=1}^{m}\log x_{i:m:n} + \sum_{i=m+1}^{D}\log x_{i:n}\right](\alpha+1)\sum_{i=m+1}^{D}\log(1+x_{i:n}^{\beta}) - \alpha R_D'\log(1+T^{\beta}). \quad (31)\end{aligned}$$

From Eq. 31, we obtain

$$\begin{aligned}L_{11} &= -\sum_{i=1}^{m}(R_i+1)\frac{x_{i:m:n}^{\hat{\beta}}\log x_{i:m:n}}{1+x_{i:m:n}^{\hat{\beta}}} - \sum_{i=m+1}^{D}\frac{x_{i:n}^{\hat{\beta}}\log x_{i:n}}{1+x_{i:n}^{\hat{\beta}}} - R_D'\frac{T^{\hat{\beta}}\log T}{1+T^{\hat{\beta}}}, \\ L_{12} &= -\sum_{i=1}^{m}(R_i+1)\frac{x_{i:m:n}^{\hat{\beta}}(\log x_{i:m:n})^2}{(1+x_{i:m:n}^{\hat{\beta}})^2} - \sum_{i=m+1}^{D}\frac{x_{i:n}^{\hat{\beta}}(\log x_{i:n})^2}{(1+x_{i:n}^{\hat{\beta}})^2} - R_D'\frac{T^{\hat{\beta}}(\log T)^2}{(1+T^{\hat{\beta}})^2}, \\ L_{02} &= -\frac{D}{\hat{\beta}^2} - \sum_{i=1}^{m}[\alpha(1+R_i)+1]\frac{x_{i:m:n}^{\hat{\beta}}(\log x_{i:m:n})^2}{(1+x_{i:m:n}^{\hat{\beta}})^2} - (\alpha+1)\sum_{i=m+1}^{D}\frac{x_{i:n}^{\hat{\beta}}(\log x_{i:n})^2}{(1+x_{i:n}^{\hat{\beta}})^2} - \alpha R_D'\frac{T^{\hat{\beta}}(\log T)^2}{(1+T^{\hat{\beta}})^2}, \\ L_{03} &= \frac{2D}{\hat{\beta}^3} - \sum_{i=1}^{m}[\alpha(1+R_i)+1]\frac{x_{i:m:n}^{\hat{\beta}}(\log x_{i:m:n})^3(1-x_{i:m:n}^{\hat{\beta}})}{(1+x_{i:m:n}^{\hat{\beta}})^3} - (\alpha+1)\sum_{i=m+1}^{D}\frac{x_{i:n}^{\hat{\beta}}(\log x_{i:n})^3(1-x_{i:n}^{\hat{\beta}})}{(1+x_{i:n}^{\hat{\beta}})^3} \\ &-\alpha R_D'\frac{T^{\hat{\beta}}(\log T)^3(1-T^{\hat{\beta}})}{(1+T^{\hat{\beta}})^3}, \\ L_{21} &= 0, \quad L_{20} = -\frac{D}{\hat{\alpha}^2}, \quad L_{30} = \frac{2D}{\hat{\alpha}^3}.\end{aligned}$$

The approximate Bayes estimates of α and β under SEL function are given by

$$\begin{aligned}\hat{\alpha}_{SEL} &= \hat{\alpha} + (\rho_1\hat{\sigma}_{11} + \rho_2\hat{\sigma}_{12}) + \frac{1}{2}[\hat{\sigma}_{11}^2 L_{30} + \hat{\sigma}_{21}\hat{\sigma}_{22}L_{03} + \hat{\sigma}_{11}\hat{\sigma}_{22}L_{12} + 2\hat{\sigma}_{21}^2 L_{12}], \\ \hat{\beta}_{SEL} &= \hat{\beta} + (\rho_1\hat{\sigma}_{21} + \rho_2\hat{\sigma}_{22}) + \frac{1}{2}[\hat{\sigma}_{11}\hat{\sigma}_{12}L_{30} + \hat{\sigma}_{22}^2 L_{03} + 3\hat{\sigma}_{21}\hat{\sigma}_{22}L_{12}]. \quad (32)\end{aligned}$$

Also, the Bayes estimates of α and β under the LINEX loss function are, respectively, given by

$$\begin{aligned}\hat{\alpha}_{LL} &= -\frac{1}{w}\log[E(e^{-w\hat{\alpha}}|\underline{x})] \\ &= -\frac{1}{w}\log\left\{e^{-w\hat{\alpha}} - we^{-w\hat{\alpha}}(\hat{\rho}_1\hat{\sigma}_{11} + \hat{\rho}_2\hat{\sigma}_{12}) + \frac{1}{2}\left[w^2e^{-w\hat{\alpha}}\hat{\sigma}_{11} - we^{-w\hat{\alpha}}\left(\hat{\sigma}_{11}^2L_{30} + \hat{\sigma}_{21}\hat{\sigma}_{22}L_{03}\right.\right.\right. \\ &\quad \left.\left.\left.+\hat{\sigma}_{11}\hat{\sigma}_{22}L_{12} + 2\hat{\sigma}_{21}^2L_{12}\right)\right]\right\}, \end{aligned} \tag{33}$$

and

$$\begin{aligned}\hat{\beta}_{LL} &= -\frac{1}{w}\log[E(e^{-w\hat{\beta}}|\underline{x})] \\ &= -\frac{1}{w}\log\left\{e^{-w\hat{\beta}} - we^{-w\hat{\beta}}(\hat{\rho}_1\hat{\sigma}_{12} + \hat{\rho}_2\hat{\sigma}_{22}) + \frac{1}{2}\left[w^2e^{-w\hat{\beta}}\hat{\sigma}_{22} - we^{-w\hat{\beta}}\left(\hat{\sigma}_{11}\hat{\sigma}_{12}L_{30}\right.\right.\right. \\ &\quad \left.\left.\left.+\hat{\sigma}_{22}^2L_{03} + 3\hat{\sigma}_{21}\hat{\sigma}_{22}L_{12}\right)\right]\right\}. \end{aligned} \tag{34}$$

3.2 MCMC method

In this subsection, we apply the importance sampling method to compute the Bayes estimates and the HPD credible intervals of the unknown parameters. Based on the independent gamma priors $\alpha \sim GA(a_1, b_1)$ and $\beta \sim GA(a_2, b_2)$, the posterior pdfs of α and β are given by

Case I:

$$\pi_1(\alpha, \beta|\underline{x}) \propto f_{GA}(\alpha|\beta; m + a_1, \xi_1)f_{GA}(\beta; m + a_2, \xi_2)h_1(\alpha, \beta), \tag{35}$$

where $\xi_1 = b_1 + \sum_{i=1}^{m}(R_i + 1)\log(1 + x_{i:m:n}^{\beta})$, $\xi_2 = b_2 - \sum_{i=1}^{m}\log x_{i:m:n}$ and

$$h_1(\alpha, \beta) = \frac{\exp(-\sum_{i=1}^{m}\log(1 + x_{i:m:n}^{\beta}))}{\left[b_1 + \sum_{i=1}^{m}(R_i + 1)\log(1 + x_{i:m:n}^{\beta})\right]^{m+a_1}}. \tag{36}$$

Case II:

$$\pi_2(\alpha, \beta|\underline{x}) \propto f_{GA}(\alpha|\beta; D + a_1, \xi_1')f_{GA}(\beta; D + a_2, \xi_2')h_2(\alpha, \beta), \tag{37}$$

where $\xi_1' = \xi_1 + \sum_{i=m+1}^{D}\log(1 + x_{i:n}^{\beta}) + R_D'\log(1 + T^{\beta})$, $\xi_2' = \xi_2 - \sum_{i=m+1}^{D}\log x_{i:n}$ and

$$h_2(\alpha, \beta) = \frac{\exp\left\{-\left(\sum_{i=1}^{m}\log(1 + x_{i:m:n}^{\beta}) + \sum_{i=m+1}^{D}\log(1 + x_{i:n}^{\beta})\right)\right\}}{\left[\xi_1 + \sum_{i=m+1}^{D}\log(1 + x_{i:n}^{\beta}) + R_D'\log(1 + T^{\beta})\right]^{D+a_1}}, \tag{38}$$

and $f_{GA}(.; a, b)$ is a gamma density with shape and scale parameters a and b, respectively.

Analoguesly as in Kundu and Pradhan (2009), we use the following algorithm to compute Bayes estimates of $\phi(\alpha, \beta)$, say $\hat{\phi}(\alpha, \beta)$, and to construct its HPD credible intervals.

Step 1: Generate $\beta_1 \sim GA(\beta; m + a_2, \xi_2)$.
Step 2: Given β_1 generated in step 1, generate α_1 from $GA(\alpha|\beta; m + a_1, \xi_1)$.
Step 3: Repeat Steps 1-2 M times to obtain the importance sample $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_M, \beta_M)$.

The approximate Bayes estimates of $\phi(\alpha, \beta)$ under squared error as well as LINEX loss functions can be obtained as

$$\hat{\phi}_{SEL}(\alpha, \beta) = \frac{\sum_{i=1}^{M} \phi(\alpha_i, \beta_i) h_1(\alpha_i, \beta_i)}{\sum_{i=1}^{M} h_1(\alpha_i, \beta_i)},$$
$$\hat{\phi}_{LL}(\alpha, \beta) = -\frac{1}{w} \log\left[\frac{\sum_{i=1}^{M} e^{-w\phi(\alpha_i, \beta_i)} h_1(\alpha_i, \beta_i)}{\sum_{i=1}^{M} h_1(\alpha_i, \beta_i)}\right]. \tag{39}$$

respectively. Similarly, the above algorithm can be written for case II.

Now, we obtain HPD credible intervals of α and β using the generated importance sample. In this work we mainly adopted the method proposed by Chen and Shao (1999).

Suppose that $\pi(\theta|\underline{x})$ and $\Pi(\theta|\underline{x})$ are the posterior density and posterior distribution functions of θ, respectively. Further, let θ_p be the pth quantile of θ as

$$\theta_p = \inf\{\theta : \Pi(\theta|\underline{x}) \geq p\}; \quad 0 < p < 1.$$

For a given θ^*, we have

$$\Pi(\theta^*|\underline{x}) = E(1_{\theta \leq \theta^*}|\underline{x}),$$

where $1_{\theta \leq \theta^*}$ is the indicator function defined as

$$1_{\theta \leq \theta^*} = \begin{cases} 1 & \theta \leq \theta^* \\ 0 & \theta > \theta^* \end{cases}$$

Then a simulation consist estimator of $\Pi(\theta^*|\underline{x})$ can be obtained as

$$\Pi(\theta^*|\underline{x}) = \frac{\sum_{i=1}^{M} 1_{\theta \leq \theta^*} h_j(\alpha_i, \beta_i)}{\sum_{i=1}^{M} h_j(\alpha_i, \beta_i)}, \quad j = 1, 2.$$

Let $\theta_{(i)}; i = 1, \cdots, M$ denote the ordered values of θ_i, and

$$w_i = \frac{h_j(\alpha_i, \beta_i)}{\sum_{i=1}^{M} h_j(\alpha_i, \beta_i)}$$

be the associated weight then

$$\Pi(\theta^*|\underline{x}) = \begin{cases} 0 & \theta^* < \theta_{(1)} \\ \sum_{k=1}^{i} w_k & \theta_{(i)} \leq \theta^* < \theta_{(i+1)} \\ 1 & \theta^{(M)} \leq \theta_*. \end{cases}$$

Thus, the approximate of θ_p can be obtained as

$$\theta_p = \begin{cases} \theta_{(1)} & p = 0 \\ \theta_i & \sum_{k=1}^{i-1} w_k < p \leq \sum_{k=1}^{i} w_k. \end{cases}$$

Now, let $R_k(M) = (\hat{\theta}^{(k/M)}, \hat{\theta}^{((k+[(1-p)M])/M)}$ and $[x]$ denote the greatest integer less than or equal to x. Therefore, the HPD credible interval for θ can be obtained by choosing $R_k(M)$ among all intervals such that it has the smallest width.

4 Simulation Study

In this section, we present some simulation results to compare the performance of the various estimates and confidence intervals of the unknown parameters of the Burr XII distribution. We mainly compare the performances of the MLEs obtained by EM algorithm and Bayes estimates obtained by Lindley's approximation as well as the MCMC technique in terms of root mean square error (RMSE) values. We also construct the %95 confidence intervals for the parameters using the estimated asymptotic variances of the MLEs obtained

Table 1 Average values of the different estimates and the corresponding RMSEs (in parentheses) when $\alpha = 1.5$, $\beta = 0.5$, $w = 2$

						Lindely		MCMC	
T	n	m		Schems	EM	SEL	LINEX	SEL	LINEX
5	25	15	α	$(10, 0^{*14})$	1.60926 (0.45716)	1.59356 (0.44633)	1.42188 (0.54859)	1.52671 (0.44510)	1.38346 (0.48054)
				$(0, 0, 0, 10, 0^{*11})$	1.69690 (0.33554)	1.67349 (0.31806)	1.51382 (0.31071)	1.53258 (0.40376)	1.38595 (0.45316)
			β	$(10, 0^{*14})$	0.81591 (0.39766)	0.83677 (0.42404)	0.79819 (0.61718)	0.53565 (0.12525)	0.52711 (0.17229)
				$(0, 0, 0, 10, 0^{*11})$	0.70017 (0.23928)	0.70319 (0.24369)	0.68680 (0.33575)	0.53168 (0.11073)	0.52262 (0.15014)
		20	α	$(5, 0^{*19})$	1.27708 (0.34066)	1.27117 (0.34112)	1.81991 (0.42525)	1.51304 (0.46234)	1.40448 (0.53471)
				$(0, 0, 0, 5, 0^{*16})$	1.31533 (0.35588)	1.30752 (0.35528)	1.22294 (0.42765)	1.51094 (0.43034)	1.40289 (0.52609)
			β	$(5, 0^{*19})$	0.60055 (0.15252)	0.60460 (0.15821)	0.59138 (0.19917)	0.54933 (0.13274)	0.54223 (0.18662)
				$(0, 0, 0, 5, 0^{*16})$	0.59038 (0.14541)	0.59265 (0.14957)	0.58074 (0.18893)	0.54411 (0.12543)	0.53719 (0.17794)
	30	20	α	$(10, 0^{*19})$	1.46187 (0.21223)	1.45321 (0.21084)	1.35003 (0.30484)	1.52452 (0.44737)	1.41563 (0.52892)
				$(0, 0, 0, 10, 0^{*16})$	1.64404 (0.42022)	1.62944 (0.41048)	1.50772 (0.58187)	1.54892 (0.42631)	1.43506 (0.51369)
			β	$(10, 0^{*19})$	0.70011 (0.21579)	0.70587 (0.22187)	0.68963 (0.30652)	0.54822 (0.13223)	0.54145 (0.19146)
				$(0, 0, 0, 10, 0^{*16})$	0.66744 (0.19500)	0.66980 (0.19705)	0.65764 (0.27829)	0.53123 (0.10783)	0.52527 (0.15606)
	50	25	α	$(25, 0^{*24})$	1.82162 (0.61448)	1.81093 (0.60681)	1.68501 (0.66052)	1.51553 (0.38255)	1.42921 (0.45417)
				$(0, 0, 0, 25, 0^{*21})$	1.73435 (0.45735)	1.71662 (0.44610)	1.60161 (0.73188)	1.52860 (0.35951)	1.43862 (0.44532)
			β	$(25, 0^{*24})$	0.75972 (0.36211)	0.77317 (0.37463)	0.75320 (0.52919)	0.54219 (0.11863)	0.53727 (0.17371)
				$(0, 0, 0, 25, 0^{*21})$	0.73187 (0.25300)	0.74113 (0.26204)	0.72931 (0.39051)	0.52421 (0.09006)	0.51968 (0.13095)
		30	α	$(20, 0^{*29})$	1.54585 (0.31686)	1.15398 (0.31330)	1.45710 (0.45855)	1.48050 (0.40969)	1.41148 (0.48641)
				$(0, 0, 0, 20, 0^{*26})$	1.64681 (0.40042)	1.63632 (0.39375)	1.54892 (0.57744)	1.51907 (0.34213)	1.44561 (0.43567)
			β	$(20, 0^{*29})$	0.70839 (0.22972)	0.71377 (0.23532)	0.70192 (0.34825)	0.54583 (0.11864)	0.54190 (0.17295)
				$(0, 0, 0, 20, 0^{*26})$	0.68151 (0.19962)	0.68566 (0.20385)	0.67691 (0.29871)	0.53167 (0.09989)	0.52767 (0.14820)

Table 2 Average values of the different estimates and the corresponding RMSEs (in parentheses) when $\alpha = 1.5$, $\beta = 0.5$, $w = 2$

						Lindely		MCMC	
T	n	m		Schems	EM	SEL	LINEX	SEL	LINEX
10	25	15	α	$(10, 0^{*14})$	1.51194 (0.40466)	1.49688 (0.39595)	1.33816 (0.52373)	1.47433 (0.43757)	1.34079 (0.44156)
				$(0, 0, 0, 10, 0^{*11})$	1.60449 (0.26787)	1.58274 (0.25559)	1.43637 (0.27225)	1.48336 (0.35845)	1.34739 (0.39146)
			β	$(10, 0^{*14})$	0.75396 (0.32508)	0.77751 (0.36040)	0.74109 (0.48811)	0.53820 (0.12685)	0.52932 (0.16492)
				$(0, 0, 0, 10, 0^{*11})$	0.64687 (0.17793)	0.65000 (0.18284)	0.63600 (0.23648)	0.52329 (0.10305)	0.51434 (0.13123)
		20	α	$(5, 0^{*19})$	1.27708 (0.34066)	1.27117 (0.34112)	1.18991 (0.42525)	1.45701 (0.44277)	1.35643 (0.47877)
				$(0, 0, 0, 5, 0^{*16})$	1.31533 (0.35588)	1.30752 (0.35528)	1.22294 (0.42765)	1.46927 (0.38176)	1.36808 (0.43308)
			β	$(5, 0^{*19})$	0.60055 (0.15252)	0.60460 (0.15821)	0.59138 (0.19917)	0.55272 (0.14296)	0.54593 (0.19797)
				$(0, 0, 0, 5, 0^{*16})$	0.59038 (0.14541)	0.59265 (0.14957)	0.58074 (0.18893)	0.53926 (0.12059)	0.53230 (0.16492)
	30	20	α	$(10, 0^{*19})$	1.46187 (0.21223)	1.45321 (0.21084)	1.35003 (0.30484)	1.47989 (0.43900)	1.37758 (0.47081)
				$(0, 0, 0, 10, 0^{*16})$	1.65373 (0.42115)	1.63913 (0.41132)	1.51625 (0.59148)	1.51440 (0.35044)	1.40716 (0.39842)
			β	$(10, 0^{*19})$	0.70011(0.21579)	0.70587 (0.22187)	0.68963 (0.30652)	0.54787 (0.13176)	0.54123 (0.17831)
				$(0, 0, 0, 10, 0^{*16})$	0.65526 (0.18131)	0.65760 (0.18355)	0.64579 (0.25486)	0.52753 (0.10473)	0.52090 (0.14268)
	50	25	α	$(25, 0^{*24})$	1.39177 (0.15506)	1.38549 (0.15161)	1.26524 (0.26559)	1.45993 (0.41968)	1.37963 (0.50026)
				$(0, 0, 0, 25, 0^{*21})$	1.60961 (0.31508)	1.59247 (0.30893)	1.49023 (0.37195)	1.50603 (0.32457)	1.41943 (0.40235)
			β	$(25, 0^{*24})$	0.85076 (0.35818)	0.87868 (0.38648)	0.84695 (0.46962)	0.54501 (0.12029)	0.54027 (0.17092)
				$(0, 0, 0, 25, 0^{*21})$	0.71352 (0.23791)	0.72406 (0.24875)	0.71212 (0.35681)	0.52699 (0.09076)	0.52229 (0.12917)
		30	α	$(20, 0^{*29})$	1.72830 (0.40252)	1.72061 (0.39671)	1.62203 (0.43909)	1.47389 (0.39983)	1.40595 (0.47010)
				$(0, 0, 0, 20, 0^{*26})$	1.51724 (0.23904)	1.50791 (0.23681)	1.43376 (0.24993)	1.51344 (0.32893)	1.44105 (0.41633)
			β	$(20, 0^{*29})$	0.70816 (0.22605)	0.71330 (0.23360)	0.70165 (0.29753)	0.55008 (0.12743)	0.54612 (0.18684)
				$(0, 0, 0, 20, 0^{*26})$	0.64626 (0.15904)	0.64947 (0.16252)	0.64199 (0.19100)	0.53046 (0.09799)	0.52657 (0.14382)

Table 3 Approximate CIs and HPD credible intervals when $\alpha = 1.5, \beta = 0.5$

n	m	Schems	%95 Approximate CIs α	%95 Approximate CIs β	%95 HPD credible intervals α	%95 HPD credible intervals β
$T = 5$						
25	15	$(10, 0^{*14})$	[0.66948,2.54903]	[0.30462,1.32721]	[0.50075,2.41446]	[0.35614,0.86146]
		$(0, 0, 0, 10, 0^{*11})$	[1.12173,2.27208]	[0.42264,0.97770]	[0.65474,2.41154]	[0.36644,0.79128]
	20	$(5, 0^{*19})$	[0.71257,1.84159]	[0.34927,0.85184]	[0.40412,2.43356]	[0.36469,0.86929]
		$(0, 0, 0, 5, 0^{*16})$	[0.64868,1.98198]	[0.34076,0.84000]	[0.50737,2.38473]	[0.36920,0.84505]
30	20	$(10, 0^{*19})$	[1.04008,1.88366]	[0.53695,0.86327]	[0.49009,2.37607]	[0.36911,0.89228]
		$(0, 0, 0, 10, 0^{*16})$	[0.84650,2.44158]	[0.46542,0.86941]	[0.66436,2.51651]	[0.38183,0.79391]
50	25	$(25, 0^{*24})$	[0.78380,2.85944]	[0.25956,1.25987]	[0.53296,2.23561]	[0.37834,0.83598]
		$(0, 0, 0, 25, 0^{*21})$	[0.95587,2.51282]	[0.53123,0.93251]	[0.78007,2.25830]	[0.39137,0.73459]
	30	$(20, 0^{*29})$	[0.93080,2.16090]	[0.51876,0.89801]	[0.43619,2.21129]	[0.38493,0.80538]
		$(0, 0, 0, 20, 0^{*26})$	[0.91129,2.38233]	[0.51747,0.84555]	[0.71324,2.21000]	[0.39848,0.79045]
$T = 10$						
25	15	$(10, 0^{*14})$	[0.64348,2.38040]	[0.31824,1.18967]	[0.36186,2.27693]	[0.36416,0.87782]
		$(0, 0, 0, 10, 0^{*11})$	[1.06399,2.14498]	[0.42677,0.86696]	[0.75777,2.20681]	[0.37024,0.77147]
	20	$(5, 0^{*19})$	[0.71257,1.84159]	[0.34927,0.85184]	[0.28425,2.25624]	[0.37455,0.90772]
		$(0, 0, 0, 5, 0^{*16})$	[0.64868,1.98198]	[0.34076,0.84000]	[0.58714,2.17685]	[0.37479,0.83369]
30	20	$(10, 0^{*19})$	[1.04008,1.88366]	[0.53695,0.86327]	[0.38900,2.23940]	[0.36969,0.85005]
		$(0, 0, 0, 10, 0^{*16})$	[0.85623,2.45124]	[0.46483,0.84569]	[0.68836,2.22666]	[0.37892,0.77564]
50	25	$(25, 0^{*24})$	[1.08395,1.69960]	[0.64976,1.05176]	[0.43519,2.20755]	[0.38353,0.84397]
		$(0, 0, 0, 25, 0^{*21})$	[1.00237,2.21684]	[0.49780,0.92923]	[0.85552,2.19568]	[0.39484,0.74872]
	30	$(20, 0^{*29})$	[0.93250,2.52409]	[0.49114,0.92518]	[0.55701,2.17919]	[0.38817,0.89778]
		$(0, 0, 0, 20, 0^{*26})$	[0.85638,2.17810]	[0.47310,0.81942]	[0.82404,2.18204]	[0.39277,0.78099]

by the missing information principle. For comparison purpose we also consider the %95 HPD credible intervals based on 1000 MCMC samples. We consider two different sampling scheme as follows:

- Scheme 1 : $R_1 = n - m$ and $R_2 = R_3 = \cdots = R_m = 0$;
- Scheme 2 : $R_1 = R_2 = R_3 = 0,\ R_4 = n - m$ and $R_5 = ... = R_m = 0$.

Table 4 Different progressive type-II hybrid censored data sets

(T, m, n)	Censoring scheme	type-II PHCS data
(0.1, 20, 12)	$\tilde{R_1} = (8, 0^{*11})$	0.529,0.665,0.683,0.698,0.788,0.866,0.879,0.881, 0.917,1.050,1.110,1.138
	$\tilde{R_2} = (0, 2, 0, 3, 3, 0^{*7})$	0.529,0.554,0.653,0.665,0.683,0.698,0.828,0.829, 0.866,0.881,0.917,1.050

Using the algorithm presented in (Gurunlu Alma and Arabi Belaghi 2015), we generate the type-II progressively hybrid censored samples from the Burr XII distribution for a given set $n, m, R_1, ..., R_m$ and T. We notice that risk expression of none of these estimates can be evaluated in closed form. So, we performed simulation to evaluate RMSE values of all estimates. Without loss of generality, we take $\alpha = 1.5$ and $\beta = 0.5$ and we simulate the whole process $N = 1000$ times in each case. For computing the Bayes estimates under squared error and LINEX loss functions, it has been assumed that α and β have priors GA(a_1, b_1) and GA(a_2, b_2), respectively. Moreover we use the non-informative priors of both α and β. This corresponds to the case when hyperparameters take values of $a_1 = a_2 = b_1 = b_2 = 0$. The average estimates, the RMSEs of the MLEs and Bayes estimates under different loss functions are presented in Tables 1 and 2. The %95 approximate confidence intervals of the parameters based on MLEs are included in Table 3. Apart from these, the corresponding HPD credible intervals are also presented in the table. The results of the Monte Carlo simulation study are presented in Tables 1–3. From these tables the following conclusions are made:

1. For fixed m and T as sample size (n) increases the average estimates and the RMSEs decreases.
2. For fixed n and m as T increases, the average estimates and the RMSEs decreases. Similar trend is observed when n and T are kept fixed and m is allowed to increase.
3. Average length of approximate confidence/HPD credible intervals decrease when n or T increases. Also, as the m increases, the average length of approximate CIs increases while the average length of HPD credible intervals are narrow down. Overall, it is clear from the tabulated interval estimates that HPD intervals are superior to the corresponding approximate confidence intervals.
4. Bayes estimates are very good in respect of RMSE. The MCMC technique is better than the Lindley approximation procedure in respect of RMSE. Its is observed that MCMC estimates derived under the squared error loss function shows steady behavior for all tabulated combinations of n, m and T. Moreover, Bayes estimates under LINEX loss function based on MCMC method are better choice among all its rivals and for all values of n, m and T.

Table 5 The MLE and Bayes estimates of the parameters for the real data set

Censoring scheme	$\tilde{R}_1$	$\tilde{R}_2$
α_{ML}	3.0693	4.1640
β_{ML}	5.1195	5.9963
α_{SEL}	2.9200	4.1444
α_{MCSEL}	2.7869	2.7997
α_{LL}	2.7577	3.9355
α_{MCLL}	2.7166	2.7281
β_{SEL}	5.8925	6.0070
β_{MCSEL}	6.3660	6.0354
β_{LL}	5.9120	5.8694
β_{MCLL}	6.2184	5.9189

Table 6 Different CIs of α and β parameters for the real data set

(T, m, n)	Censoring schems	%95 Approximate CIs		%95 HPD credible intervals	
		α	β	α	β
(0.1,20,12)	$\tilde{R}_1 = (8, 0^{*11})$	[0.6917,6.8301]	[1.5612,11.8004]	[2.0225,4.1789]	[5.4047,7.7698]
	$\tilde{R}_2 = (0, 2, 0, 3, 3, 0^{*7})$	[0.3227,8.0052]	[0.5836,11.4095]	[2.0727,4.2516]	[5.3832,7.2688]

5. In general, as the sample proportion m/n increases performance of all estimates improves in terms of RMSEs. In studying the effect of different censoring schemes, we observed that the RMSE in Scheme 2 is smaller than Scheme 1.

5 Real Data Analysis

For illustrative purpose, in this section we have analyzed one data set from Wingo (1993). The data were originally collected from a clinical trial designed to access the effectiveness of an antibiotic ointment in relieving pain. The data set are listed as follows:
0.828,0.881,1.138,0.879,0.554,0.653,0.698,0.566,0.665,0.917,
0.529,0.786,1.110,0.866,1.037,0.788,1.050,0.899,0.683,0.829.
Here, we have $n = 20$. We generate two progressive type-II hybrid censored samples from the above data set using two different censoring schemes from the above progressive type-II hybrid censored sample with $m = 12$ and $T = 0.1$. The different censoring schemes and the corresponding progressive type-II hybrid censored samples are presented in the second and third columns of Table 4, respectively. In all the two cases we calculate the ML and Bayes estimates of the parameters. In Bayes estimation we use non-informative priors as we have no prior information about the parameters. For importance sampling procedure, we take $M = 1000$. The estimates are listed in Table 5. Further, the %95 approximate confidence intervals and HPD intervals are provided in Table 6.

6 Concluding Remarks

In this paper, the Bayes and classical estimates have been obtained for a two parameter Burr XII distribution when samples are available from type-II progressive hybrid censoring scheme. By EM algorithm iteration and asymptotic normality theory, we have derived the MLEs and approximate confidence intervals of the unknown parameters. We have also proposed a Bayesian approach to estimate the model parameters. Bayes estimates are obtained using the Lindley approximation method. Since the Lindley approximation method fails to construct HPD credible intervals, we made use of the importance sampling procedure to obtain point estimates and HPD credible intervals of the parameters. Further, we carried out a simulation study to evaluate the performance of all the methods of estimation and it was observed that the Bayes estimates overall perform better than MLEs in the sense of RMSE.

Acknowledgments The authors are grateful to the Editor in chief and an anonymous referee for making many helpful comments and suggestions on an earlier version of this paper. We also appreciate Dr. William Volterman for his expert suggestion in the earlier version of this paper.

References

Abd-Elfattah AM, Hassan AS, Nassr SG (2008) Estimation in step-stress partially accelerated life tests for the Burr type XII distribution using type I censoring. Stat Methodol 5:502–514

Ahmed A, Soliman AH, Abd Ellah NA, Abou-Elheggag A, Modhesh A (2011) Bayesian inference and prediction of Burr type XII distribution for progressive first failure censored sampling. Intell Inform Manage 3:175–185

Ali Mousa MAM, Jaheen ZF (2002) Statistical inference for the Burr model based on progressively censored data. Computers & Mathematics with Applications 43:1441–1449

Banerjee A, Kundu D (2008) Inference based on type-II hybrid censored data from a Weibull distribution. IEEE Trans Reliab 57:369–378

Chen MH, Shao QM (1999) Monte Carlo estimation of Bayesian credible and HPD intervals. J Comput Graph Stat 8:69–92

Childs A, Chandrasekar B, Balakrishnan N (2008) Exact likelihood inference for an exponential parameter under progressive hybrid censoring schemes. In: Vonta F, Nikulin M, Limnios N, Huber-Carol C (eds) Statistical models and methods for biomedical and technical systems. Birkhäuser, pp 323-334

Dempster AP, Laird NM, Rubin DB (1977) Maximum likelihood from incomplete data via the EM algorithm (with discussion). J R Stat Soc Ser B 39:1–38

Epstein B (1954) Truncated life tests in the exponential case. Ann Math Stat 25:555–564

Gurunlu Alma O, Arabi Belaghi R (2015) On the estimation of the extreme value and normal distribution parameters based on progressive type-II hybrid-censored data. J Stat Comput Simul 86(3):569–596. http://dx.doi.org/10.1080/00949655.2015.1025785

Gupta RD, Kundu D (2001) Exponentiated exponential family an alternative to Gamma and Weibull. Biometr J 33:117–130

Kundu D, Joarder A (2006) Analysis of type-II progressively hybrid censored data. Computational Statistics & Data Analysis 50:2509–2528

Kundu D, Pradhan B (2009) Estimating the parameters of the generalized exponential distribution in prescence of hybrid censoring. Communications in Statistics–Theory and Methods 38:2030–2041

Lindley DV (1980) Approximate bayesian method. Trabajos de Estadistica 31:223–237

Lin CT, Huang YL (2011) On progressive hybrid censored exponential distribution. J Stat Comput Simul. 1st published on: 21 June 2011 (iFirst)

Lin CT, Huang YL, Balakrishnan N (2011) Exact Bayesian variable sampling plans for the exponential distribution with progressive hybrid censoring. J Stat Comput Simul 81:873–882

Louis TA (1982) Finding the observed information matrix using the EM algorithm. Journal of Royal Statistical Society: Series B 44:226–233

McLachlan GJ, Krishnan T (1997) The EM algorithm and extensions. Wiley, New York

Ng HKT, Chan PS, Balakrihnan N (2012) Estimation of parameters from progressively censored data using EM algorithm. Computational Statistics & Data Analysis 39:371–386

Rastogi MK, Tripathi YM (2013) Inference on unknown parameters of a Burr distribution under hybrid censoring. Stat Pap 54:619–643

Soliman AA (2005) Estimation of parameters of life from progressively censored data using Burr XII model. IEEE Trans Reliab 54:34–42

Varian HR (1975) A Bayesian approach to real estate assessment. In: Stephen EF, Zellner A (eds) Studies in bayesian econometrics and statistics in honor of Leonard J. Savage, North-Holland, pp 195–208

Wang F, Cheng YF (2010) EM algorithm for estimating the Burr XII parameters with multiple censored data. Qual Reliab Eng Int 26:615–630

Wingo DR (1993) Maximum likelihood methods for fitting the Burr type XII distribution to multiply (progressively) censored life test data. Metrika 40:203–210

Zellner A (1986) Bayesian estimation and prediction using asymptotic loss function. J Am Stat Assoc 81:446–451