Methodol Comput Appl Probab (2017) 19:685–714
DOI 10.1007/s11009-016-9496-5

Performance Analysis of the GI/M/1 Queue with Single Working Vacation and Vacations

Qingqing Ye[1] **· Liwei Liu**[1]

Received: 2 September 2015 / Revised: 5 March 2016 /
Accepted: 29 April 2016 / Published online: 12 May 2016

Abstract In this paper, we consider a new class of the GI/M/1 queue with single working vacation and vacations. When the system become empty at the end of each regular service period, the server first enters a working vacation during which the server continues to serve the possible arriving customers with a slower rate, after that, the server may resume to the regular service rate if there are customers left in the system, or enter a vacation during which the server stops the service completely if the system is empty. Using matrix geometric solution method, we derive the stationary distribution of the system size at arrival epochs. The stochastic decompositions of system size and conditional system size given that the server is in the regular service period are also obtained. Moreover, using the method of semi-Markov process (SMP), we gain the stationary distribution of system size at arbitrary epochs. We acquire the waiting time and sojourn time of an arbitrary customer by the first-passage time analysis. Furthermore, we analyze the busy period by the theory of limiting theorem of alternative renewal process. Finally, some numerical results are presented.

Keywords Working vacation · Vacation · Stochastic decomposition · Waiting time · Sojourn time · Busy period

Mathematics Subject Classification (2010) 60K25 · 68M20

1 Introduction

Queueing systems with server vacations have attracted considerable attentions over the decades due to that they have a wide range of applications in various areas such as computer networks, communication systems and manufacturing system and so forth. For more details

✉ Qingqing Ye
yeqingzero@gmail.com

1 Nanjing University of Science and Technology, Nanjing, China

related to vacation queues, we refer the readers to the surveys of Doshi (1986), Teghem (1986), the monographs of Takagi (1991) and Tian and Zhang (2006).

The concept of working vacation was first introduced by Servi and Finn in 2002. Their work is motivated by the analysis of a reconfigurable wavelength-division multiplexing (WDM) optimal access network. Servi and Finn (2002) studied an M/M/1 queue with multiple working vacations, and obtained the transform formulae for the distributions of the system size and the sojourn time of an arbitrary customer in steady state. Based on the analysis of Servi and Finn (2002), Liu et al. (2007) re-analyzed the M/M/1 queue with multiple working vacations and obtained the concise expressions and stochastic decomposition structures for the stationary system size and sojourn time using the matrix-geometric solution method. Subsequently, utilizing the same method, Tian and Zhao (2008) studied the M/M/1 queue with single working vacation and various performance measures in steady state were derived. Extensions to M/G/1 type queue with working vacations are carried out by Wu and Takagi (2006) (using Laplace-Stieltjes transform method), Kim et al. (2003) (using the decomposition method) and Li et al. (2011) (using the matrix-analytic method). As for the GI/M/1 type queue with working vacations, Baba (2005) first presented a GI/M/1 queue with multiple working vacations using matrix analytic method and the GI/M/1 queue with single working vacation was studied by Li and Tian (2011) (using matrix-analytic method) and Chae et al. (2009) (using generating function transform method), Li et al. (2013) provided a study on a GI/M/1 queue with Bernouli-schedule-controlled vacation in which the server may choose to enter working vacation or vacation under Bernouli schedule.

In light of the classical vacation queues and working vacation queues, we consider a queue with single working vacation and vacations which is characterized by the following features: Once the system becomes empty in regular service period, a working vacation with a random period is taken, during which the possible arriving customers are served with a lower rate. After that, if there are customers left in the system, the system will be resumed to the regular service period, otherwise, the server will enter a vacation with a random period, during which the service is not rendered to any of new arrivals. Another vacation will be taken if there are still no customers in the system when the vacation ends, and so on. The server will continue the vacations until there are new arrivals at a vacation completion epoch, and then a regular service period will start again. Up to the present, no special work focused on the model seems to have appeared in open literatures.

This vacation policy has potential application in the real life situation. In order to economize the operation operation cost, the escalators in some large supermarkets and metros are always designed to operate to stay at a lower service rate period for a certain length when system just become empty, after the low service period and there is still no customers, the escalators will stop the service completely. Another example is provided in some machine systems, after the valve is closed (press control), the pump will stay at Min Speed before stopping. During the delay stop period, the pump can restart or stop immediately.

The rest of this paper is organized as follows. In Section 2, we describe the queueing model with single working vacation and vacations in detail. The steady state distribution of system size at the arrival epochs and its stochastic decomposition structure are given in Section 3. Section 4 is devoted to obtain the steady state distribution of system size at the arbitrary epochs. In Section 5, we analyze the waiting time and sojourn time of an arbitrary customer by the first-passage time analysis. Section 6 gives the analysis of the busy period. Some numerical examples are presented in Section 7. Section 8 is the conclusion. The Appendix presents the transition probability analysis for the embedded Markov chain.

2 Model Formulation and Embedded Markov Chain

We consider a GI/M/1 queue with single working vacation and vacations, this queueing model is explicitly described as follows.

- The inter-arrival times $\{T_n,\ n \geq 1\}$ are assumed to be independent and identically distributed with a general distribution function $A(t)$ with a mean $1/\lambda$ and a Laplace-Stieltjes transform (LST) $a^*(s)$.
- The vacation policy we consider is characterized by the following features: Once the system become empty during the regular service period with service rate μ, the server takes a working vacation that follows an exponential distribution with parameter θ_w, during which the server continues serving the potential arriving customers with a lower rate η $(\eta < \mu)$. At the completion epoch of the working vacation, if there are customers left in the system, the server will resume to serve the customers with the regular service rate μ, and another regular service period will start, otherwise, the system will enter into a vacation that follows an exponential distribution with parameter θ_v during which the server completely stops working. If there are customers in the system at the instant of a vacation completion, the server will resume to a regular serving level with rate μ. Otherwise, the server continues the vacations until there are arrivals in the system at the vacation completion epochs, and a regular service period will start.
- Assume that inter-arrival times, service times, working vacation time and vacation times are all mutually independent. In addition, the service discipline is First Come First Served (FCFS).

Suppose τ_n be the arrival epoch of the n-th customer with $\tau_0 = 0$, and let $L(t)$ be the number of customers in the system at time t, we choose τ_n as the imbedded points, then $L_n = L(\tau_n - 0)$ is regarded as the number of customers just before the n-th arrival epoch. Note that an arrival may occur during a regular service period, working vacation period or vacation period, so we define

$$J_n = \begin{cases} 0, & \text{the } n \text{ - th arrival occurs during the working vacation period,} \\ 1, & \text{the } n \text{ - th arrival occurs during the vacation period,} \\ 2, & \text{the } n \text{ - th arrival occurs during the regular service period.} \end{cases}$$

Since the working vacation time, vacation times, the service times during the regular service period and the working vacation period are all exponentially distributed, then process $\{(L_n,\ J_n),\ n \geq 1\}$ is a two-dimensional embedded Markov chain with the state space

$$\Omega = \{(k, 0), k \geq 0\} \cup \{(k, 1), k \geq 0\} \cup \{(k, 2), k \geq 1\}.$$

In order to express the transition matrix of (L_n, J_n), we introduce the probability measure:

$$p_{(i,j),(k,l)} = P\{L_{n+1} = k, J_{n+1} = l \mid L_n = i, J_n = j.\}$$

Using the lexicographical ordering for the states, the transition probability matrix $\tilde{P}$ of $\{(L_n,\ J_n),\ n \geq 1\}$ can be written as the Block-Jacobi matrix

$$\tilde{P} = \begin{pmatrix} B_0 & A_{01} & 0 & 0 & 0 & \cdots \\ B_1 & A_1 & A_0 & 0 & 0 & \cdots \\ B_2 & A_2 & A_1 & A_0 & 0 & \cdots \\ B_3 & A_3 & A_2 & A_1 & A_0 & \cdots \\ B_4 & A_4 & A_3 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$A_{01} = \begin{pmatrix} c_0 & 0 & d_0 \\ 0 & \varepsilon & e_0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} c_0 & 0 & d_0 \\ & \varepsilon & e_0 \\ & & b_0 \end{pmatrix}, \quad A_k = \begin{pmatrix} c_k & 0 & d_k \\ & 0 & e_k \\ & & b_k \end{pmatrix}, \quad k \geq 1,$$

$$B_0 = \begin{pmatrix} c_1' + d_1' & 1 - (c_0 + d_0) - \left(c_1' + d_1'\right) \\ e_1' & 1 - \varepsilon - e_0 - e_1' \end{pmatrix},$$

$$B_k = \begin{pmatrix} c_{k+1}' + d_{k+1}' & 1 - \sum_{i=0}^{k} (c_i + d_i) - \left(c_{k+1}' + d_{k+1}'\right) \\ e_{k+1}' & 1 - \varepsilon - e_{k+1}' - \sum_{i=0}^{k} e_i \\ b_{k+1}' & 1 - b_{k+1}' - \sum_{i=0}^{k} c_i \end{pmatrix}, \quad k \geq 1.$$

The expressions for the entries of $\tilde{P}$ are shown in detail in the Appendix. Evidently, we can find that $\tilde{P}$ is a stochastic matrix and its structure indicates that $\{(L_n, J_n), n \geq 1\}$ is irreducible and aperiodic. The limiting probabilities are defined below:

$$\pi_{n,\,j} = \lim_{k\to\infty} P\{L_k = n,\ J_k = j\},\ (n,\ j) \in \Omega;$$

$$\pi_0 = \left(\pi_{0,0},\ \pi_{0,1}\right);$$
$$\pi_n = \left(\pi_{n,0},\ \pi_{n,1},\ \pi_{n,2}\right).$$

3 Steady-State Distribution of System Size at Arrival Epochs and its Stochastic Decomposition Structure

In this section, we focus on the stationary distribution of $\{(L_n, J_n), n \geq 1\}$ by the matrix geometric solution method. Clearly, we can observe that the transition probability matrix $\tilde{P}$ is a GI/M/1-type matrix (see Neuts 1981), for such a model, it is necessary to seek the matrix R, which is the minimal nonnegative solution of

$$R = \sum_{k=0}^{\infty} R^k A_k. \tag{1}$$

In order to obtain it, we first introduce the following Lemma:

Lemma 1 *If $\rho = \lambda/\mu < 1$, the equation $z = a^*(\mu(1-z))$ has a unique root in the range $0 < z < 1$, denoted by ξ, and the equation $z = a^*(\theta_w + \eta(1-z))$ has a unique root in the range $0 < z < 1$, denoted by γ.*

Proof First, we consider the equation $z = a^*(\mu(1-z))$ and let $\psi(z) = a^*(\mu(1-z))$, evidently, $0 < \psi(0) = a^*(\mu) < \psi(1) = 1$, and for $0 < z < 1$, we have

$$\psi'(z) = \mu \int_0^\infty t e^{-\mu(1-z)t} dA(t) > 0, \;\; \psi''(z) = \mu^2 \int_0^\infty t^2 e^{-\mu(1-z)t} dA(t) > 0.$$

Meanwhile, it follows from $\rho = \lambda/\mu < 1$ that $\psi'(1) = 1/\rho > 1$. Thus the equation $z = \psi(z)$ has a unique root in the range $0 < z < 1$, denoted by ξ, similarly, we set $\Phi(z) = a^*\left(\theta_w + \eta(1-z)\right)$, then

$$0 < \Phi(0) = a^*(\theta_w + \eta) < \Phi(1) = a^*(\theta_w) < 1,$$

and, for $0 < z < 1$, we have

$$\Phi'(z) = \eta \int_0^\infty t e^{-(\theta_w + \eta(1-z))t} dA(t) > 0, \Phi''(z) = \eta^2 \int_0^\infty t^2 e^{-(\theta_w + \eta(1-z))t} dA(t) > 0.$$

Therefore, $z = \Phi(z)$ has a unique root in the range $0 < z < 1$, denoted by γ, Then the proof is completed. □

Lemma 2 *If $\rho < 1$, the matrix equation $R = \sum_{k=0}^{\infty} R^k A_k$ has the minimal non-negative solution*

$$R = \begin{pmatrix} \gamma & 0 & \alpha\,(\xi - \gamma) \\ & \varepsilon & \beta(\xi - \varepsilon) \\ & & \xi \end{pmatrix}, \tag{2}$$

where

$$\alpha = \frac{\theta_w}{\theta_w - (\mu - \eta)(1-\gamma)}, \;\; \varepsilon = a^*\left(\theta_v\right), \;\; \beta = \frac{\theta_v}{\theta_v - \mu\left(1 - a^*(\theta_v)\right)}.$$

Proof Because all A_k, for $k \geq 1$, are upper triangular, we can assume that R has the same structure as

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{pmatrix},$$

then, for $k \geq 1$, we have

$$R^k = \begin{pmatrix} r_{11}^k & r_{12} \sum_{i=0}^{k-1} r_{11}^i r_{22}^{k-1-i} & r_{13} \sum_{i=0}^{k-1} r_{11}^i r_{33}^{k-1-i} + r_{12} r_{23} \sum_{i=0}^{k-2} r_{11}^i \sum_{j=0}^{k-2-i} r_{22}^j r_{33}^{k-2-i-j} \\ & r_{22}^k & r_{23} \sum_{i=0}^{k-1} r_{22}^i r_{33}^{k-1-i} \\ & & r_{33}^k \end{pmatrix}$$

where $\sum_{i=0}^{-1} (\ldots) = 0$.

Substituting R^k into Eq. 1 yields

$$\begin{cases} r_{11}=\sum\limits_{k=0}^{\infty}c_k r_{11}^k=a^*\left(\theta_w+\eta(1-r_{11})\right), \\ r_{12}=0, \\ r_{13}=\sum\limits_{k=0}^{\infty}d_k r_{11}^k+\sum\limits_{k=1}^{\infty}e_k r_{12}\sum\limits_{i=0}^{k-1}r_{11}^i r_{22}^{k-1-i} \\ \qquad +\sum\limits_{k=1}^{\infty}b_k\left(r_{13}\sum\limits_{i=0}^{k-1}r_{11}^i r_{33}^{k-1-i}+r_{12}r_{23}\sum\limits_{i=0}^{k-2}r_{11}^i\sum\limits_{j=0}^{k-2-i}r_{22}^j r_{33}^{k-2-i-j}\right), \\ r_{22}=\varepsilon, \\ r_{23}=\sum\limits_{k=0}^{\infty}e_k r_{22}^k+\sum\limits_{k=1}^{\infty}b_k\left(r_{23}\sum\limits_{i=0}^{k-1}r_{22}^i r_{33}^{k-1-i}\right), \\ r_{33}=\sum\limits_{k=0}^{\infty}b_k r_{33}^k=a^*\left(\mu(1-r_{33})\right), \end{cases} \tag{3}$$

where $a^*\left(\theta_v\right)$ is denoted by ε.

From Lemma 1, we observe that the first equation of Eq. 3 has the unique root $r_{11}=\gamma$ in interval (0,1) and the last equation has unique root $r_{33}=\xi$ in interval (0,1). From the third equation of Eq. 3, we can obtain

$$r_{13}=\sum_{k=0}^{\infty}d_k r_{11}^k \bigg/ \left(1-\sum_{k=1}^{\infty}b_k\sum_{i=0}^{k-1}r_{11}^i r_{33}^{k-1-i}\right). \tag{4}$$

The numerator of Eq. 4 can be computed as follows:

$$\begin{aligned} \sum_{k=0}^{\infty}d_k r_{11}^k &= \sum_{k=0}^{\infty}\gamma^k\left(\int_0^{\infty}\int_0^{t}\theta_w e^{-\theta_w x}e^{-\eta x}e^{-\mu(t-x)}\frac{1}{k!}\left(\eta x+\mu(t-x)\right)^k dx dA(t)\right) \\ &= \int_0^{\infty}\int_0^{t}\theta_w e^{-\theta_w x}e^{-\eta x}e^{-\mu(t-x)}e^{(\eta x+\mu(t-x))\gamma}dx dA(t) \\ &= \int_0^{\infty}\theta_w e^{-\mu(1-\gamma)t}\int_0^{t}e^{(\eta\gamma-\mu\gamma-\theta_w-\eta+\mu)x}dx dA\left(t\right) \\ &= \frac{\theta_w\left(a^*\left(\mu\left(1-\gamma\right)\right)-\gamma\right)}{\theta_w-(\mu-\eta)(1-\gamma)}. \end{aligned}$$

The denominator of Eq. 4 can be simplified as follows:

$$
\begin{aligned}
1-\sum_{k=1}^{\infty} b_k \sum_{i=0}^{k-1} r_{11}^{i} r_{33}^{k-1-i} &= 1-\frac{1}{\gamma-\xi}\sum_{k=0}^{\infty} b_k(\gamma^k-\xi^k) \\
&= 1-\frac{a^*(\mu(1-\gamma))-\xi}{\gamma-\xi} \\
&= \frac{\gamma-a^*(\mu-\mu\gamma)}{\gamma-\xi}.
\end{aligned}
$$

Thus $r_{13}=\alpha\,(\xi-\gamma)$, where $\alpha=\theta_w\big/(\theta_w-(\mu-\eta)(1-\gamma))$.
Inserting $r_{33}=\xi$ and $r_{22}=\varepsilon$ into the fifth equation of Eq. 3 gives

$$
r_{23}=\sum_{k=0}^{\infty} e_k r_{22}^{k} \Bigg/ \left(1-\sum_{k=1}^{\infty} b_k \sum_{i=0}^{k-1} r_{22}^{i} r_{33}^{k-1-i}\right). \tag{5}
$$

The numerator of Eq. 5 can be computed as follows:

$$
\begin{aligned}
\sum_{k=0}^{\infty} e_k r_{22}^{k} &= \sum_{k=0}^{\infty}\varepsilon^k \int_0^{\infty}\int_0^{t} \theta_v e^{-\theta_v x}\frac{(\mu(t-x))^k}{k!} e^{-\mu(t-x)} dx dA\,(t) \\
&= \int_0^{\infty}\int_0^{t} \theta_v e^{-\mu(1-\varepsilon)(t-x)-\theta_v x} dx dA\,(t) \\
&= \frac{\theta_v}{\theta_v-\mu(1-\varepsilon)}\left(a^*\,(\mu(1-\varepsilon))-\varepsilon\right).
\end{aligned}
$$

The denominator of Eq. 5 can be simplified as follows:

$$
\begin{aligned}
1-\sum_{k=1}^{\infty} b_k \sum_{j=0}^{k-1} \xi^j \varepsilon^{k-1-j} &= 1-\frac{1}{\xi-\varepsilon}\sum_{k=1}^{\infty} b_k\left(\xi^k-\varepsilon^k\right) \\
&= 1-\frac{1}{\xi-\varepsilon}\left(a^*\,(\mu(1-\xi))-a^*\,(\mu(1-\varepsilon))\right) \\
&= \frac{1}{\xi-\varepsilon}\left(a^*\,(\mu\,(1-\varepsilon))-\varepsilon\right).
\end{aligned}
$$

Thus we can get $r_{23}=\beta(\xi-\varepsilon)$, where $\beta=\theta_v\big/(\theta_v-\mu(1-\varepsilon))$. Then the proof is completed. □

Based on theorem 1.5.1 of Neuts (1981), the Markov chain is positive recurrent if and only if the spectral radius of R, denoted by $sp(R)$, is less than one, and $B[R]$ has a positive left invariant vector, where

$$
B[R]=\begin{pmatrix} B_0 & A_{01} \\ \sum\limits_{k=1}^{\infty} R^{k-1}B_k & \sum\limits_{k=1}^{\infty} R^{k-1}A_k \end{pmatrix}. \tag{6}
$$

Under the condition $\rho < 1$, we have obtained R from Lemma 2. By the structure of R, we can note that R is an upper triangular matrix and $sp(R) = \max\{\gamma,\ \varepsilon,\ \xi\} < 1$. After tedious calculation, we obtain

$$B[R] = \begin{pmatrix} c_1' + d_1' & 1-(c_0+d_0)-\left(c_1'+d_1'\right) & c_0 & 0 & d_0 \\ e_1' & 1-\varepsilon-e_0-e_1' & 0 & \varepsilon & e_0 \\ H_{31} & H_{32} & 1-\frac{c_0}{\gamma} & 0 & \frac{\alpha b_0(\xi-\gamma)}{\xi\gamma}-\frac{d_0}{\gamma} \\ H_{41} & H_{42} & 0 & 0 & \frac{\beta b_0(\xi-\varepsilon)}{\xi\varepsilon}-\frac{e_0}{\varepsilon} \\ H_{51} & H_{52} & 0 & 0 & 1-\frac{b_0}{\xi} \end{pmatrix}, \tag{7}$$

where

$$H_{31} = \frac{a^*(\theta_w)-\gamma}{\gamma(1-\gamma)}\left(1-\frac{\mu}{\eta}\alpha\right)+\frac{\alpha}{\xi}\frac{\mu(\xi-a^*(\theta_w))}{\theta_w-\mu(1-\xi)}-\frac{c_1'+d_1'}{\gamma}+\alpha b_1'\frac{(\gamma-\xi)}{\xi\gamma},$$

$$H_{32} = -H_{31}+\frac{c_0+d_0}{\gamma}-\alpha b_0\frac{\xi-\gamma}{\gamma\xi},$$

$$H_{41} = \frac{\mu\beta(a^*(\theta_w)-\varepsilon)}{\varepsilon(\theta_w-\theta_v)}+\frac{\beta\mu}{\xi}\frac{\xi-a^*(\theta_w)}{\theta_w-\mu(1-\xi)}-\frac{e_1'}{\varepsilon}-\frac{\beta b_1'}{\xi}+\frac{\beta b_1'}{\varepsilon},$$

$$H_{42} = 1-H_{41}-\frac{\beta b_0(\xi-\varepsilon)}{\xi\varepsilon}+\frac{e_0}{\varepsilon},$$

$$H_{51} = \frac{\mu(\xi-a^*(\theta_w))}{\xi(\theta_w-\mu(1-\xi))}-\frac{b_1'}{\xi},$$

$$H_{52} = \frac{b_0}{\xi}-H_{51}.$$

We can verify that $B[R]$ is a stochastic matrix and has the left invariant vector

$$K'(1,\ p,\ \gamma,\ p\varepsilon,\ \alpha(\xi-\gamma)+p\beta(\xi-\varepsilon)), \tag{8}$$

where K' is a constant and

$$p = \frac{1-\frac{a^*(\theta_w)-\gamma}{1-\gamma}\left(1-\frac{\mu}{\eta}\alpha\right)-\frac{\alpha\mu(\xi-a^*(\theta_w))}{\theta_w-\mu(1-\xi)}}{\frac{\mu\beta(a^*(\theta_w)-\varepsilon)}{\theta_w-\theta_v}+\beta\frac{\mu(\xi-a^*(\theta_w))}{\theta_w-\mu(1-\xi)}}. \tag{9}$$

Then the Markov chain is positive recurrent if and only if $sp(R) < 1$. Based on it, the joint stationary probability distribution of system size at arrival epochs is obtained in the following Theorem.

Theorem 1 *If $\rho < 1$, the joint stationary probability distribution of the Markov process $\{(L_n, J_n), n \geq 1\}$ is*

$$\begin{cases} \pi_{k,0} = K\gamma^k, & k \geq 0, \\ \pi_{k,1} = Kp\varepsilon^k, & k \geq 0, \\ \pi_{k,2} = K\left(\alpha\left(\xi^k-\gamma^k\right)+p\beta\left(\xi^k-\varepsilon^k\right)\right), & k \geq 1, \end{cases} \tag{10}$$

where

$$K = \frac{(1-\gamma)(1-\varepsilon)(1-\xi)}{(1-\xi+\alpha(\xi-\gamma))(1-\varepsilon)+p(1-\xi+\beta(\xi-\varepsilon))(1-\gamma)}.$$

Proof Using Theorem 1.5.1 of Neuts (1981), the invariant probability vector π of the Markov process is given by

$$\pi_k = \pi_1 R^{k-1}, k \geq 1, \tag{11}$$

and $(\pi_{00},\ \pi_{01},\ \pi_{10},\ \pi_{11},\ \pi_{12})$ is the positive left invariant vector of $B[R]$. It means that it has the type as $K\left(1,\ p,\ \gamma,\ \varepsilon,\ \alpha\left(\xi-\gamma\right)+p\beta\left(\xi-\varepsilon\right)\right)$, in which the constant K is determined by the normalization condition

$$\pi_{0,0} + \pi_{0,1} + \left(\pi_{1,0},\ \pi_{1,1},\ \pi_{1,2}\right)(I-R)^{-1} e = 1, \tag{12}$$

where e denotes 3 dimensional column vector with all elements being equal to 1.

Noting

$$(I-R)^{-1} = \begin{pmatrix} \frac{1}{1-\gamma} & 0 & \frac{\alpha(\xi-\gamma)}{(1-\gamma)(1-\xi)} \\ & \frac{1}{1-\varepsilon} & \frac{\beta(\xi-\varepsilon)}{(1-\varepsilon)(1-\xi)} \\ & & \frac{1}{1-\xi} \end{pmatrix}, \tag{13}$$

and substituting $(I-R)^{-1}$ into Eq. 12, we obtain

$$K = \frac{(1-\gamma)(1-\varepsilon)(1-\xi)}{(1-\xi+\alpha(\xi-\gamma))(1-\varepsilon)+p(1-\xi+\beta(\xi-\varepsilon))(1-\gamma)}. \tag{14}$$

Note that

$$R^k = \begin{pmatrix} \gamma^k & 0 & \alpha\left(\xi^k-\gamma^k\right) \\ & \varepsilon^k & \beta\left(\xi^k-\varepsilon^k\right) \\ & & \xi^k \end{pmatrix}, \tag{15}$$

substituting R^{k-1} into Eq. 11, we can get Eq. 10, then the proof is completed. □

Special case 1 If $\theta_w \to \theta_v$ and $\eta \to 0$, it means there is no service in the working vacation and the working vacation reduces to the vacation, then our model becomes the standard GI/M/1 queue with multiple vacations. To verify that, let $\theta_w \to \theta_v$ and $\eta \to 0$, then $\gamma \to \varepsilon$, $\alpha \to \beta$, the distribution of stationary system size becomes

$$\begin{cases} \pi_{k,0} = K\varepsilon^k, & k \geq 0, \\ \pi_{k,1} = Kp\varepsilon^k, & k \geq 0, \\ \pi_{k,2} = K(\beta+p\beta)\left(\xi^k-\varepsilon^k\right), & k \geq 1, \end{cases} \tag{16}$$

where

$$p = \left(1 - \frac{\beta\mu\left(\xi - a^*(\theta_v)\right)}{\theta_v - \mu(1-\xi)}\right) \Bigg/ \left(\beta\mu\delta + \frac{\beta\mu\left(\xi - a^*(\theta_v)\right)}{\theta_v - \mu(1-\xi)}\right), \tag{17}$$

$$\delta = \frac{d}{ds} a^*(s)\bigg|_{s=\theta_v}, \quad K = \frac{(1-\varepsilon)(1-\xi)}{(1-\xi+\beta(\xi-\varepsilon))(1+p)}. \tag{18}$$

Note here that states $(k, 0)$ and $(k, 1)$, for $k \geq 0$, both represent the system in vacation period. In fact, $(k, 0)$ means that the system stays in the first vacation after the regular service period. So the probability that there are k customers in the system and the server stays in vacation period is $\pi_{k,0} + \pi_{k,1}$. Therefore, we can find that the distribution of stationary system size coincides with that of Theorem 2 in Tian et al. (1989).

Special case 2 If $\theta_v \to \infty$, it means there is no vacation period, then the vacation period reduces to the idle period, so, our model becomes the standard GI/M/1 queue with single working vacation. To verify that, let $\theta_v \to \infty$, then $\varepsilon \to 0$ and $\beta \to 1$, the distribution of stationary system size becomes

$$\begin{cases} \pi_{k,0} = K\gamma^k, & k \geq 0, \\ \pi_{0,1} = Kp, & k \geq 0, \\ \pi_{k,2} = K\left(\alpha\left(\xi^k - \gamma^k\right) + p\xi^k\right), & k \geq 1, \end{cases} \tag{19}$$

where π_{01} represents that the server is in the idle period and

$$p = \left(1 - \frac{a^*(\theta_w) - \gamma}{1-\gamma}\left(1 - \frac{\mu}{\eta}\alpha\right) - \frac{\alpha\mu\left(\xi - a^*(\theta_w)\right)}{\theta_w - \mu(1-\xi)}\right) \Big/ \frac{\mu\left(\xi - a^*(\theta_w)\right)}{\theta_w - \mu(1-\xi)}, \tag{20}$$

$$K = \frac{(1-\gamma)(1-\xi)}{(1-\xi+\alpha(\xi-\gamma)) + p(1-\gamma)}. \tag{21}$$

We can find that the stationary distribution of system size coincides with that of Theorem 2 in Li and Tian (2011) and Lemma 2 in Chae et al. (2009).

The state probabilities of the server in steady state are shown as

$$P_0 = P\{J = 0\} = K\frac{1}{1-\gamma}, \tag{22}$$

$$P_1 = P\{J = 1\} = Kp\frac{1}{1-\varepsilon}, \tag{23}$$

$$P_2 = P\{J = 2\} = K\left(\alpha\frac{\xi-\gamma}{(1-\xi)(1-\gamma)} + p\beta\frac{\xi-\varepsilon}{(1-\xi)(1-\varepsilon)}\right). \tag{24}$$

Theorem 2 *If $\rho < 1$ and $\mu > \eta$, the stationary system size L can be decomposed into the sum of two independent random variables: $L = L_0 + L_d$, in which L_0 is the stationary system size of a classical GI/M/1 queue without vacation, and follows a geometric distribution with parameter $1 - \xi$, and the additional queue length L_d has the distribution*

$$P\{L_d = 0\} = K^*(1-\gamma)(1-\varepsilon)(1+p), \tag{25}$$

$$\begin{aligned} P\{L_d = k\} =& K^*(1-\varepsilon)(\alpha-1)(\xi-\gamma)(1-\gamma)\gamma^{k-1} \\ &+ K^*p(1-\gamma)(\beta-1)(\xi-\varepsilon)(1-\varepsilon)\varepsilon^{k-1}, \quad k \geq 1, \end{aligned} \tag{26}$$

where $K^* = ((1-\xi+\alpha(\xi-\gamma))(1-\varepsilon) + p(1-\xi+\beta(\xi-\varepsilon))(1-\gamma))^{-1}$.

Proof From Eq. 10, the PGF of L can be shown as follows:

$$\begin{aligned} L(z) &= \sum_{k=0}^{\infty} \pi_{k0} z^k + \sum_{k=0}^{\infty} \pi_{k1} z^k + \sum_{k=1}^{\infty} \pi_{k1} z^k \\ &= \frac{1-\xi}{1-\xi z} K^* (1-\gamma)(1-\varepsilon) \left(\frac{1-\xi z}{1-\gamma z} + \frac{p(1-\xi z)}{1-\varepsilon z} + \alpha \frac{z(\xi-\gamma)}{1-\gamma z} + p\beta \frac{z(\xi-\varepsilon)}{1-\varepsilon z} \right) \\ &= \frac{1-\xi}{1-\xi z} K^* (1-\gamma)(1-\varepsilon) \left(\sum_{k=0}^{\infty} (\gamma z)^k + (\alpha(\xi-\gamma)-\xi) \sum_{k=1}^{\infty} z^k \gamma^{k-1} + p \sum_{k=0}^{\infty} (\varepsilon z)^k \right. \\ &\quad \left. + p(\alpha(\xi-\varepsilon)-\xi) \sum_{k=1}^{\infty} z^k \varepsilon^{k-1} \right) \\ &= \frac{1-\xi}{1-\xi z} K^* (1-\gamma)(1-\varepsilon) \left(1 + p + (\alpha-1)(\xi-\gamma) \sum_{k=1}^{\infty} z^k \gamma^{k-1} \right. \\ &\quad \left. + p(\beta-1)(\xi-\varepsilon) \sum_{k=1}^{\infty} z^k \varepsilon^{k-1} \right). \end{aligned} \tag{27}$$

From the above expression, we obtain the Theorem 2 directly. □

Remark 1 From Theorem 2, the stationary additional queue length L_d has the special probability explanation that it is the mixture of three random variables. That is

$$\begin{aligned} L_d = & K^* (1-\gamma)(1-\varepsilon)(1+p) X_0 + K^* (1-\varepsilon)(\alpha-1)(\xi-\gamma) X_1 \\ & + K^* p (1-\gamma)(\beta-1)(\xi-\varepsilon) X_2, \end{aligned} \tag{28}$$

where X_0 equals to 0, X_1 follows a geometric distribution with parameter $1-\gamma$ and X_2 follows a geometric distribution with parameter $1-\varepsilon$.

Remark 2 From the distributions of L_0 and L_d, we can know that the system size at the arrival epochs follows a PH-distribution with representation (α_1, T_1), where

$$\begin{aligned} \alpha_1 &= \left(\xi, \ (1-\xi) K^* (1-\varepsilon)(\alpha-1)(\xi-\gamma), \ (1-\xi) K^* p (1-\gamma)(\beta-1)(\xi-\varepsilon) \right), \\ T_1 &= \begin{pmatrix} \xi & (1-\xi) K^* (1-\varepsilon)(\alpha-1)(\xi-\gamma) & (1-\xi) K^* p (1-\gamma)(\beta-1)(\xi-\varepsilon) \\ 0 & \gamma & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}. \end{aligned}$$

Let $Q^{(0)}$, $Q^{(1)}$ and $Q^{(2)}$ be the conditional system size just before a new customer arrives, given that the arrival occurs during working vacation, vacation and regular service period, respectively. We can get the distributions of $Q^{(0)}$, $Q^{(1)}$ and $Q^{(2)}$ in the following Corollary.

Corollary 1 *If $\rho < 1$, the random variables $Q^{(0)}$ and $Q^{(1)}$ follow geometric distributions with parameter γ, ε, respectively. The random variable $Q^{(2)}$ can be decomposed into the sum of two independent random variables: $Q^{(2)} = L_0 + L_d^{(1)}$, in which L_0 follows a* *geometric distribution with parameter* $1-\xi$, *and the additional queue length* $L_d^{(1)}$ *has the distribution*

$$P\left\{L_d^{(1)}=k\right\}=\delta\left(1-\gamma\right)\gamma^{k-1}+\left(1-\delta\right)\left(1-\varepsilon\right)\varepsilon^{k-1},\ k\geq 1, \tag{29}$$

where

$$\delta=\frac{\alpha\left(\xi-\gamma\right)\left(1-\varepsilon\right)}{\alpha\left(\xi-\gamma\right)\left(1-\varepsilon\right)+p\beta\left(\xi-\varepsilon\right)\left(1-\gamma\right)}. \tag{30}$$

Proof The distributions of $Q^{(0)}$ and $Q^{(1)}$ can be obtained by direct computation as follows:

$$P\left\{Q^{(0)}=k\right\}=\pi_{k0}\big/P_0=\left(1-\gamma\right)\gamma^k,\ k\geq 0, \tag{31}$$

$$P\left\{Q^{(1)}=k\right\}=\pi_{k1}\big/P_1=\left(1-\varepsilon\right)\varepsilon^k,\ k\geq 0. \tag{32}$$

For $k\geq 1$,

$$\begin{aligned}&P\left\{Q^{(2)}=k\right\}=\pi_{k2}\big/P_2\\&=\frac{(1-\xi)(1-\gamma)(1-\varepsilon)}{\alpha(\xi-\gamma)(1-\varepsilon)+p\beta(\xi-\varepsilon)(1-\gamma)}\left(\alpha\left(\xi-\gamma\right)\sum_{i=0}^{k-1}\xi^i\gamma^{k-i}+p\beta\left(\xi-\varepsilon\right)\sum_{j=0}^{k-1}\xi^j\varepsilon^{k-j}\right).\end{aligned} \tag{33}$$

Multiplying both sides of the above expression by z^k and summing over $k=1,2,\ldots,$ we get

$$\begin{aligned}Q^{(2)}\left(z\right)=\frac{1-\xi}{1-\xi z}&\left(\frac{\alpha\left(\xi-\gamma\right)\left(1-\varepsilon\right)}{\alpha\left(\xi-\gamma\right)\left(1-\varepsilon\right)+p\beta\left(\xi-\varepsilon\right)\left(1-\gamma\right)}\frac{\left(1-\gamma\right)z}{1-\gamma z}\right.\\&\left.+\frac{p\beta\left(\xi-\varepsilon\right)\left(1-\gamma\right)}{\alpha\left(\xi-\gamma\right)\left(1-\varepsilon\right)+p\beta\left(\xi-\varepsilon\right)\left(1-\gamma\right)}\frac{\left(1-\varepsilon\right)z}{1-\varepsilon z}\right).\end{aligned} \tag{34}$$

Then we can obtain Eq. 29 from the above expression, then the proof is completed. □

Remark 3 From Corollary 1, we know that random variable $Q^{(2)}$ follows a PH-distribution with representation $(\alpha_2,\ T_2)$, where

$$\alpha_2=\left(\xi,\ \left(1-\xi\right)\sigma,\ \left(1-\xi\right)\left(1-\sigma\right)\right),$$

$$T_1=\begin{pmatrix}\xi & (1-\xi)\sigma & (1-\xi)(1-\sigma)\\ 0 & \gamma & 0\\ 0 & 0 & \varepsilon\end{pmatrix}.$$

From Theorem 2 and Corollary 1, we can get

$$E\left(L_d\right)=K^*\left(1-\varepsilon\right)\frac{\mu-\eta}{\theta_w}\alpha\left(\xi-\gamma\right)+K^*\left(1-\gamma\right)p\frac{\mu}{\theta_v}\beta\left(\xi-\varepsilon\right), \tag{35}$$

$$E\left(L\right)=\frac{\xi}{1-\xi}+K^*\left(1-\varepsilon\right)\frac{\mu-\eta}{\theta_w}\alpha\left(\xi-\gamma\right)+K^*\left(1-\gamma\right)p\frac{\mu}{\theta_v}\beta\left(\xi-\varepsilon\right), \tag{36}$$

$$E\left(Q^{(2)}\right)=\frac{\xi}{1-\xi}+\delta\frac{\gamma}{1-\gamma}+\left(1-\delta\right)\frac{\varepsilon}{1-\varepsilon}. \tag{37}$$

We can verify that

$$P\{L=0\}=P\{L_0=0\}P\{L_d=0\}, \tag{38}$$

$$P\{L=k\}=\sum_{i=0}^{k}P\{L_0=i\}P\{L_d=k-i\},\ k\geq 1. \tag{39}$$

4 The Steady-State Distribution of System Size at Arbitrary Epochs

In this section, we are concerned on deriving the limiting distribution of continuous-parameter process $\{L(t),\ t\geq 0\}$ by the method of semi-Markov process (SMP). To this end, we let $N_\tau(t)$ be the number of customers in the system at the most recent arrival and $J_\tau(t)$ be the state of server at time when the most recent arrival occurs, if this new customer arrives at the system in working vacation, vacation or regular service period, then $J_\tau(t)$ is corresponding to 0, 1 or 2, respectively. Clearly, $\{(N_\tau(t),\ J_\tau(t)),\ t\geq 0\}$ is a SMP that has $\{(L_n,\ J_n),n\geq 1\}$ for its basic embedded Markov chain.

Theorem 3 *If $\rho<1$ and $\theta_w,\ \theta_v>0$, the limiting distribution of continuous parameter process exists. If we denote $p_k=\lim_{t\to\infty}P\{L(t)=k\}$, for $k\geq 0$, then we have*

$$\begin{cases} p_0=1-K\left((\alpha+p\beta)\dfrac{\lambda}{\mu(1-\xi)}+\dfrac{\lambda(1-\alpha)}{\theta_w+\eta(1-\gamma)}+p\dfrac{\lambda}{\theta_v}(1-\beta)\right), \\ p_k=K(\alpha+p\beta)\dfrac{\lambda}{\mu}\xi^{k-1}+K\dfrac{\lambda(1-\alpha)(1-\gamma)}{\theta_w+\eta(1-\gamma)}\gamma^{k-1}+Kp\dfrac{\lambda}{\theta_v}(1-\beta)(1-\varepsilon)\varepsilon^{k-1},\ k\geq 1. \end{cases} \tag{40}$$

Proof Based on theory of the SMP (see Gross and Harris 1985), if the Markov chain $\{(L_n,\ J_n),n\geq 0\}$ is irreducible, aperiodic and positive recurrent, then the limiting distribution of the $\{(N_\tau(t),\ J_\tau(t)),\ t\geq 0\}$ exists. Let $m_{k,j}$ be the time the SMP is in state (k,j), for $(k,j)\in\Omega$. By definition, we know that $P\left\{m_{k,j}\leqslant t\right\}=A(t)$, $E\left(m_{k,j}\right)=\lambda^{-1}$, for all $(k,j)\in\Omega$. Let

$$v_{k,j}=\lim_{t\to\infty}P\{N_\tau(t)=k,\ J_\tau(t)=j\}, \tag{41}$$

then

$$v_{k,j}=\frac{\pi_{k,j}E\left(m_{k,j}\right)}{\sum_{(h,i)\in\Omega}\pi_{h,i}E\left(m_{h,i}\right)} \tag{42}$$

where $\pi_{k,j}$ is the limiting distribution of basic MC $\{(L_n,\ J_n),n\geq 0\}$ shown in (10). But, because all $E\left(m_{k,j}\right)=\lambda^{-1}$, we have $v_{k,j}=\pi_{k,j}$, for $(k,j)\in\Omega$. Thus, the SMP and its basic MC have an identical limiting distribution. Then we have

$$p_k=\sum_{(j,i)\in\Omega}v_{j,i}\sum_{l=0}^{2}\int_0^\infty P\left\{\begin{matrix}\text{requried changes in } t \text{ to bring}\\ \text{state from } (j,i) \text{ to } (k,l)\end{matrix}\right\}\lambda(1-A(t))\,dt. \tag{43}$$

Now, for $k \geq 1$,

$$
\begin{aligned}
p_k = & \sum_{i=k-1}^{\infty} \pi_{i,2} \int_0^{\infty} \frac{(\mu t)^{i+1-k}}{(i+1-k)!} e^{-\mu t} \lambda (1 - A(t)) dt \\
& + \sum_{i=k-1}^{\infty} \pi_{i,0} \int_0^{\infty} e^{-\theta_w t} \frac{(\eta t)^{i+1-k}}{(i+1-k)!} e^{-\eta t} \lambda (1 - A(t)) dt \\
& + \sum_{i=k-1}^{\infty} \pi_{i,0} \sum_{j=0}^{i+1-k} \int_0^{\infty} \int_0^{t} \theta_w e^{-\theta_w x} \frac{(\eta x)^j}{j!} e^{-\eta x} \frac{(\mu(t-x))^{i+1-k-j}}{(i+1-k-j)!} e^{-\mu(t-x)} \lambda (1 - A(t))\, dx dt \\
& + \sum_{i=k-1}^{\infty} \pi_{i,1} \int_0^{\infty} \int_0^{t} \theta_v e^{-\theta_v x} \frac{(\mu(t-x))^{i-k+1}}{(i-k+1)!} e^{-\mu(t-x)} \lambda (1 - A(t))\, dx dt \\
& + \pi_{k-1,1} \int_0^{\infty} e^{\theta_v t} \lambda (1 - A(t)) dx dt,
\end{aligned}
$$

substituting Eq. 10 in above equation, we can obtain that

$$
\begin{aligned}
p_k = & K \sum_{i=k-1}^{\infty} \left(\alpha \left(\xi^i - \gamma^i \right) + p\beta \left(\xi^i - \varepsilon^i \right) \right) \int_0^{\infty} \frac{(\mu t)^{i+1-k}}{(i+1-k)!} e^{-\mu t} \lambda (1 - A(t)) dt \\
& + K \sum_{i=k-1}^{\infty} \gamma^i \int_0^{\infty} e^{-\theta_w t} \frac{(\eta t)^{i+1-k}}{(i+1-k)!} e^{-\eta t} \lambda (1 - A(t)) dt \\
& + K \sum_{i=k-1}^{\infty} \gamma^i \sum_{j=0}^{i+1-k} \int_0^{\infty} \int_0^{t} \theta_w e^{-\theta_w x} \frac{(\eta x)^j}{j!} e^{-\eta x} \frac{(\mu(t-x))^{i+1-k-j}}{(i+1-!k-j)!} e^{-\mu(t-x)} \lambda (1 - A(t))\, dx dt \\
& + Kp \sum_{i=k-1}^{\infty} \varepsilon^i \int_0^{\infty} \int_0^{t} \theta_v e^{-\theta_v x} \frac{(\mu(t-x))^{i-k+1}}{(i-k+1)!} e^{-\mu(t-x)} \lambda (1 - A(t))\, dx dt \\
& + Kp\varepsilon^{k-1} \int_0^{\infty} e^{\theta_v t} \lambda (1 - A(t)) dx dt
\end{aligned} \tag{44}
$$

We now compute each part of the above equation.
Firstly,

$$
\begin{aligned}
& \sum_{i=k-1}^{\infty} \xi^i \int_0^{\infty} \frac{(\mu t)^{i-k+1}}{(i-k+1)!} e^{-\mu t} \lambda (1 - A(t))\, dt = \xi^{k-1} \int_0^{\infty} e^{-\mu(1-\xi)t} \lambda (1 - A(t))\, dt \\
& = \lambda \xi^{k-1} \frac{1}{\mu(1-\xi)} \left(1 - a^* (\mu(1-\xi)) \right) = \frac{\lambda \xi^{k-1}}{\mu}.
\end{aligned} , \tag{45}
$$

Secondly,

$$\sum_{i=k-1}^{\infty}\gamma^{i}\int_{0}^{\infty}\frac{(\mu t)^{i-k+1}}{(i-k+1)!}e^{-\mu t}\lambda\left(1-A\left(t\right)\right)dt=\gamma^{k-1}\int_{0}^{\infty}e^{-\mu(1-\gamma)t}\lambda\left(1-A\left(t\right)\right)dt \\ =\lambda\gamma^{k-1}\frac{1}{\mu\left(1-\gamma\right)}\left(1-a^{*}\left(\mu\left(1-\gamma\right)\right)\right), \tag{46}$$

$$\sum_{i=k-1}^{\infty}\varepsilon^{i}\int_{0}^{\infty}\frac{(\mu t)^{i-k+1}}{(i-k+1)!}e^{-\mu t}\lambda\left(1-A\left(t\right)\right)dt=\varepsilon^{k-1}\int_{0}^{\infty}e^{-\mu(1-\varepsilon)t}\lambda\left(1-A\left(t\right)\right)dt \\ =\lambda\varepsilon^{k-1}\frac{1}{\mu\left(1-\varepsilon\right)}\left(1-a^{*}\left(\mu\left(1-\varepsilon\right)\right)\right). \tag{47}$$

Thirdly,

$$\sum_{i=k-1}^{\infty}\gamma^{i}\int_{0}^{\infty}e^{-\theta_{w}t}\frac{(\eta t)^{i+1-k}}{(i+1-k)!}e^{-\eta t}\lambda\left(1-A\left(t\right)\right)dt=\gamma^{k-1}\int_{0}^{\infty}e^{-(\theta_{w}+\eta(1-\gamma))t}\lambda\left(1-A\left(t\right)\right)dt \\ =\lambda\gamma^{k-1}\frac{1-a^{*}\left(\theta_{w}+\eta\left(1-\gamma\right)\right)}{\theta_{w}+\eta\left(1-\gamma\right)}=\frac{\lambda\gamma^{k-1}\left(1-\gamma\right)}{\theta_{w}+\eta\left(1-\gamma\right)}. \tag{48}$$

Fourthly, we get

$$\sum_{i=k-1}^{\infty}\gamma^{i}\sum_{j=0}^{i+1-k}\int_{0}^{\infty}\int_{0}^{t}\theta_{w}e^{-\theta_{w}x}\frac{(\eta x)^{j}}{j!}e^{-\eta x}\frac{\left(\mu\left(t-x\right)\right)^{i+1-k-j}}{(i+1-k-j)!}e^{-\mu(t-x)}\lambda\left(1-A\left(t\right)\right)dxdt \\ =\gamma^{k-1}\sum_{i=k-1}^{\infty}\gamma^{i-k+1}\sum_{j=0}^{i+1-k}\int_{0}^{\infty}\int_{0}^{t}\theta_{w}e^{-\theta_{w}x}\frac{(\eta x)^{j}}{j!}e^{-\eta x}\frac{\left(\mu\left(t-x\right)\right)^{i+1-k-j}}{(i+1-k-j)!} \\ \times e^{-\mu(t-x)}\lambda\left(1-A\left(t\right)\right)dxdt \\ =\gamma^{k-1}\int_{0}^{\infty}\int_{0}^{t}\theta_{w}e^{-\theta_{w}x}e^{\eta\gamma x}e^{-\eta x}e^{\mu(t-x)\gamma}e^{-\mu(t-x)}\lambda\left(1-A\left(t\right)\right)dxdt \\ =\lambda\alpha\gamma^{k-1}\int_{0}^{\infty}\left(e^{-\mu(1-\gamma)t}-e^{-(\theta_{w}+\eta(1-\gamma))t}\right)\left(1-A\left(t\right)\right)dt \\ =\lambda\alpha\gamma^{k-1}\left(\frac{1-a^{*}\left(\mu\left(1-\gamma\right)\right)}{\mu(1-\gamma)}-\frac{1-\gamma}{\theta_{w}+\eta\left(1-\gamma\right)}\right). \tag{49}$$

Fifthly,

$$\begin{aligned}
&\sum_{i=k-1}^{\infty}\varepsilon^{i}\int_{0}^{\infty}\int_{0}^{t}\theta_v e^{-\theta_v x}\frac{(\mu\,(t-x))^{i-k+1}}{(i-k+1)!}e^{-\mu(t-x)}\lambda\,(1-A\,(t))\,dxdt\\
&\quad=\varepsilon^{k-1}\int_{0}^{\infty}\int_{0}^{t}\theta_v e^{-\theta_v x}e^{-\mu(1-\gamma)(t-x)}\lambda\,(1-A\,(t))\,dxdt\\
&\quad=\lambda\beta\varepsilon^{k-1}\int_{0}^{\infty}\left(e^{-\mu(1-\varepsilon)t}-e^{-\theta_v t}\right)(1-A\,(t))\,dt\\
&\quad=\lambda\beta\varepsilon^{k-1}\left(\frac{1-a^{*}\,(\mu\,(1-\varepsilon))}{\mu\,(1-\varepsilon)}-\frac{1}{\theta_v}\,(1-\varepsilon)\right).
\end{aligned}\tag{50}$$

Finally,

$$\int_{0}^{\infty}e^{-\theta_v t}\lambda\,(1-A\,(t))dt=\frac{\lambda}{\theta_v}\,(1-\varepsilon)\,.\tag{51}$$

Substituting Eqs. 45–51 into Eq. 44, we can get

$$p_k=K\,(\alpha+p\beta)\,\frac{\lambda}{\mu}\xi^{k-1}+K\frac{\lambda\,(1-\alpha)\,(1-\gamma)}{\theta_w+\eta\,(1-\gamma)}\gamma^{k-1}+Kp\frac{\lambda}{\theta_v}\,(1-\beta)\,(1-\varepsilon)\,\varepsilon^{k-1},\,k\geq 1,$$

and the p_0 can be determined by the normalization condition. Then the proof is completed. □

Let L_a denote the steady state system size at arbitrary epochs. From Theorem 3, we can obtain

$$E\,(L_a)=\sum_{k=0}^{\infty}kp_k=K\,(\alpha+p\beta)\,\frac{\lambda}{\mu}\frac{1}{(1-\xi)^2}+K\frac{\lambda\,(1-\alpha)}{\theta_w+\eta\,(1-\gamma)}\frac{1}{(1-\gamma)}+Kp\frac{\lambda}{\theta_v}\frac{1-\beta}{1-\varepsilon}.\tag{52}$$

5 Waiting Time and Sojourn Time

In this section, we analyze the waiting time(the time a customer spends in the queue) and sojourn time (the time a customer spends in the system) by the first passage time analysis.

Let H_0, H_1 and H_2 be the probabilities that the new arrival need to wait for its service, given that server is in working vacation, vacation and regular service period, respectively, when the new arrival comes. We can easily get

$$H_0=\sum_{k=1}^{\infty}\pi_{k,0}=K\sum_{k=1}^{\infty}\gamma^{k}=K\frac{\gamma}{1-\gamma},\tag{53}$$

$$H_1=\sum_{k=0}^{\infty}\pi_{k,1}=K\sum_{k=0}^{\infty}\varepsilon^{k}=Kp\frac{1}{1-\varepsilon},\tag{54}$$

$$H_2=\sum_{k=1}^{\infty}\pi_{k,2}=K\left(\alpha\frac{\xi-\gamma}{(1-\xi)\,(1-\gamma)}+p\beta\frac{\xi-\varepsilon}{(1-\xi)\,(1-\varepsilon)}\right),\tag{55}$$

Let W and $W^*(s)$ be the waiting time of a tagged customer and its corresponding LST, S and $S^*(s)$ be the sojourn time of a tagged customer and its corresponding LST. To find the distributions of W and S, an absorbing Markov process is introduced to describe the change of position of a tagged customer in the queue. The transition rate matrix Q_1 for the absorbing Markov chain is given by

$$Q_1 = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ D_1' & D_0 & & & & \cdots \\ & D_1 & D_0 & & & \cdots \\ & & D_1 & D_0 & & \cdots \\ & & & D_1 & D_0 & \cdots \\ & & & & \ddots & \ddots \end{bmatrix}, \tag{56}$$

where {0, 1, 2, 3, ...} represents position of a tagged customer in queue.

$$D_1' = \begin{pmatrix} \eta \\ 0 \\ \mu \end{pmatrix}, D_0 = \begin{pmatrix} -(\theta_w + \eta) & 0 & \theta_w \\ & -\theta_v & \theta_v \\ & & -\mu \end{pmatrix}, D_1 = \begin{pmatrix} \eta & & \\ & 0 & \\ & & \mu \end{pmatrix}. \tag{57}$$

Note that the stationary distribution of system size just before the new customer arrivals is given by Eq. 10 which is also the stationary distribution of system size as seen by the tagged customer at its arrival epoch. Then we can obtain the following two theorems.

Theorem 4 *The LST of the waiting time of an arbitrary customer is given by*

$$\begin{aligned} W^*(s) = \bar{H} + H_0 \left\{ q_0 \frac{\theta_w + \eta(1-\gamma)}{s + \theta_w + \eta(1-\gamma)} + (1-q_0) \frac{\theta_w + \eta(1-\gamma)}{s + \theta_w + \eta(1-\gamma)} \frac{\mu(1-\gamma)}{s + \mu(1-\gamma)} \right\} \\ + H_1 \frac{(s+\mu)(1-\varepsilon)}{s + \mu(1-\varepsilon)} \frac{\theta_v}{s + \theta_v} + H_2 \left\{ q_1 \frac{(s+\mu)(1-\xi)}{s + \mu(1-\xi)} \frac{\mu(1-\gamma)}{s + \mu(1-\gamma)} \right. \\ \left. + (1-q_1) \frac{(s+\mu)(1-\xi)}{s + \mu(1-\xi)} \frac{\mu(1-\varepsilon)}{s + \mu(1-\varepsilon)} \right\}, \end{aligned} \tag{58}$$

where H_0, H_1 *and* H_2 *are given by* Eqs. 53–55, $\bar{H} = 1 - H_0 - H_1 - H_2$ *and*

$$q_0 = \frac{\eta(1-\gamma)}{\theta_w + \eta(1-\gamma)}, \quad q_1 = \frac{\alpha(\xi-\gamma)(1-\varepsilon)}{\alpha(\xi-\gamma)(1-\varepsilon) + p\beta(\xi-\varepsilon)(1-\gamma)}. \tag{59}$$

Proof According to the equation (3.9.5) in chapter 3.9 of Neuts (1981), we can get the following LST of the waiting time

$$W^*(s) = \pi_{00} + \pi_{01} \frac{\theta_v}{s + \theta_v} + \sum_{k=1}^{\infty} (\pi_{k0}, \pi_{k1}, \pi_{k2}) \left((sI - D_0)^{-1} D_1 \right)^k e, \tag{60}$$

where e denotes a column vector with appropriate dimension, whose elements are all equal to one.

By direct computation, we can obtain

$$(sI-D_0)^{-1}D_1=\begin{pmatrix}\frac{1}{s+\theta_w+\eta} & 0 & \frac{\theta_w}{(s+\mu)(s+\theta_w+\eta)}\\ & \frac{1}{s+\theta_v} & \frac{\theta_v}{(s+\mu)(s+\theta_v)}\\ & & \frac{1}{s+\mu}\end{pmatrix}\begin{pmatrix}\eta & & \\ & 0 & \\ & & \mu\end{pmatrix}$$
$$=\begin{pmatrix}\frac{\eta}{s+\theta_w+\eta} & 0 & \frac{\theta_w\mu}{(s+\mu)(s+\theta_w+\eta)}\\ & 0 & \frac{\theta_v\mu}{(s+\mu)(s+\theta_v)}\\ & & \frac{\mu}{s+\mu}\end{pmatrix}.$$

For convenience of the following calculations, we denote $\frac{\eta}{s+\theta_w+\eta}$, $\frac{\theta_w\mu}{(s+\mu)(s+\theta_w+\eta)}$, $\frac{\theta_v\mu}{(s+\mu)(s+\theta_v)}$, $\frac{\mu}{s+\mu}$ by $r'_{11}, r'_{13}, r'_{23}, r'_{33}$, respectively, that is

$$(sI-D_0)^{-1}D_1=\begin{pmatrix}r'_{11} & 0 & r'_{13}\\ & 0 & r'_{23}\\ & & r'_{33}\end{pmatrix}. \tag{61}$$

Then we have

$$\begin{aligned}W^*(s)&=\pi_{00}+\pi_{01}\frac{\theta_v}{s+\theta_v}+\sum_{k=1}^{\infty}(\pi_{k0},\ \pi_{k1},\ \pi_{k2})\left((sI-D_1)^{-1}D_2\right)^k e\\
&=K+Kp\frac{\theta_v}{s+\theta_v}\frac{1}{1-r'_{33}\varepsilon}+K\frac{r'_{11}\gamma}{1-r'_{11}\gamma}+Kr'_{13}\gamma\frac{1}{1-r'_{11}\gamma}\frac{1}{1-r'_{33}\gamma}\\
&\quad+K\alpha\left(\frac{1}{1-r'_{33}\xi}-\frac{1}{1-r'_{33}\gamma}\right)+Kp\beta\left(\frac{1}{1-r'_{33}\xi}-\frac{1}{1-r'_{33}\varepsilon}\right)\\
&=K+Kp\frac{\theta_v}{s+\theta_v}\frac{s+\mu}{s+\mu(1-\varepsilon)}+K\frac{\gamma\eta}{s+\theta_w+\eta(1-\gamma)}\\
&\quad+K\frac{\theta_w}{s+\theta_w+\eta(1-\gamma)}\frac{\mu\gamma}{s+\mu(1-\gamma)}+K\alpha(s+\mu)\frac{\mu(\xi-\gamma)}{(s+\mu(1-\xi))(s+\mu(1-\gamma))}\\
&\quad+Kp\beta(s+\mu)\frac{\mu(\xi-\varepsilon)}{(s+\mu(1-\xi))(s+\mu(1-\varepsilon))}.\end{aligned} \tag{62}$$

By routine calculation, Eq. 62 can be transformed to

$$\begin{aligned}W^*(s)&=\bar{H}+H_0\left\{q_0\frac{\theta_w+\eta(1-\gamma)}{s+\theta_w+\eta(1-\gamma)}+(1-q_0)\frac{\theta_w+\eta(1-\gamma)}{s+\theta_w+\eta(1-\gamma)}\frac{\mu(1-\gamma)}{s+\mu(1-\gamma)}\right\}\\
&\quad+H_1\frac{(s+\mu)(1-\varepsilon)}{s+\mu(1-\varepsilon)}\frac{\theta_v}{s+\theta_v}+H_2\left\{q_1\frac{(s+\mu)(1-\xi)}{s+\mu(1-\xi)}\frac{\mu(1-\gamma)}{s+\mu(1-\gamma)}\right.\\
&\quad\left.+(1-q_1)\frac{(s+\mu)(1-\xi)}{s+\mu(1-\xi)}\frac{\mu(1-\varepsilon)}{s+\mu(1-\varepsilon)}\right\}\end{aligned} \tag{63}$$

where H_0, H_1 and H_2 are given by Eqs. 53–55, $\bar{H}=1-H_0-H_1-H_2$, q_0 and q_1 are given by Eq. 59. □

Theorem 5 *If $\rho < 1$, and $\theta_w,\ \theta_v > 0$, the LST of the steady-state sojourn time is given by*

$$S^*(s) = P_0\left(q_0\frac{\theta_w+\eta(1-\gamma)}{s+\theta_w+\eta(1-\gamma)} + (1-q_0)\frac{\theta_w+\eta(1-\gamma)}{s+\theta_w+\eta(1-\gamma)}\frac{\mu(1-\gamma)}{s+\mu(1-\gamma)}\right)$$
$$+P_1\left(\frac{\theta_v}{s+\theta_v}\frac{\mu(1-\varepsilon)}{s+\mu(1-\varepsilon)}\right) + P_2\left(q_1\frac{\mu(1-\xi)}{s+\mu(1-\xi)}\frac{\mu(1-\gamma)}{s+\mu(1-\gamma)}\right.$$
$$\left.+(1-q_1)\frac{\mu(1-\xi)}{s+\mu(1-\xi)}\frac{\mu(1-\varepsilon)}{s+\mu(1-\varepsilon)}\right), \tag{64}$$

where P_0, P_1 and are given by Eqs. 22–24, *q_0 and q_1 are given by* Eq. 58.

Proof On the basis of Q_1, we can obtain LST of sojourn time of an arbitrary customer

$$S^*(s) = (\pi_{00},\ \pi_{01},\ 0)(sI-D_1)^{-1}D_2e + \sum_{k=1}^{\infty}(\pi_{k0},\ \pi_{k1},\ \pi_{k2})\left((sI-D_1)^{-1}D_2\right)^{k+1}e \tag{65}$$

The following procedure is similar to that of Theorem 5, we omit it. □

Remark 4 From Theorem 4, we know that the steady-state waiting time of an arbitrary customer has the following probability explanation, that is, with probability $\bar{H}$, the waiting time equals to zero; with probability H_0q_0, it equals to an exponential random variable with parameter $\theta_w + \eta(1-\gamma)$; with probability $H_0(1-q_0)$, it equals to sum of one exponential random variable with parameter $\theta_w + \eta(1-\gamma)$ plus one exponential random variable with parameter$\mu(1-\gamma)$; with probability H_1, it equals to the sum of one modified exponential variable with parameter $\mu(1-\varepsilon)$ plus an exponential random variable with the parameter θ_v; with probability H_2q_1, it equals sum of one modified exponential variable with parameter $\mu(1-\xi)$ plus an exponential random variable with parameter $\mu(1-\gamma)$; with probability $H_2(1-q_1)$, it equals to sum of one modified exponential variable with parameter $\mu(1-\xi)$ plus an exponential random variable with the rate $\mu(1-\varepsilon)$.

Remark 5 From Theorem 5, we know that the steady-state sojourn time of an arbitrary customer has the following probability explanation, that is, with probability P_0q_0, it equals to an exponential random variable with parameter $\theta_w+\eta(1-\gamma)$; with probability $P_0(1-q_0)$, it equals to sum of one exponential random variable with parameter $\theta_w + \eta(1-\gamma)$ plus one exponential random variable with parameter $\mu(1-\gamma)$; with probability P_1, it equals to the sum of one modified exponential variable with parameter $\mu(1-\varepsilon)$ plus an exponential random variable with parameter θ_v; with probability P_2q_1, it equals sum of one modified exponential variable with parameter $\mu(1-\xi)$ plus an exponential random variable with parameter $\mu(1-\gamma)$; with probability $P_2(1-q_1)$, it equals to sum of one modified exponential variable with parameter $\mu(1-\xi)$ plus an exponential random variable with parameter $\mu(1-\varepsilon)$.

From Theorem 4 ,Theorem 5 and the Remark 4 and Remark 5, we can easily get the means of waiting time and sojourn time of an arbitrary customer.

$$E(W) = H_0\frac{\theta_w+\mu(1-\gamma)}{(\theta_w+\eta(1-\gamma))(\mu(1-\gamma))} + H_1\left(\frac{\varepsilon}{\mu(1-\varepsilon)}+\frac{1}{\theta_v}\right)$$
$$+H_2\left(\frac{\xi}{\mu(1-\xi)}+\frac{q_1}{\mu(1-\gamma)}+\frac{1-q_1}{\mu(1-\varepsilon)}\right), \tag{66}$$

$$E(S) = P_0\left(\frac{1}{\theta_w+\eta(1-\gamma)} + (1-q_0)\frac{1}{\mu(1-\gamma)}\right) + P_1\left(\frac{1}{\theta_v} + \frac{1}{\mu(1-\varepsilon)}\right) + P_2\left(\frac{1}{\mu(1-\xi)} + q_1\frac{1}{\mu(1-\gamma)} + (1-q_1)\frac{1}{\mu(1-\varepsilon)}\right). \tag{67}$$

We can verify that $E(S)$ is equivalent to the following equation

$$E(S) = K(\alpha+p\beta)\frac{1}{\mu(1-\xi)^2} + K\frac{1-\alpha}{(\theta_w+\eta(1-\gamma))(1-\gamma)} + Kp\frac{1-\beta}{\theta_v(1-\varepsilon)}. \tag{68}$$

Comparing the Eqs. 52 and 68, we can verify the Little's formula:

$$E(L_a) = \lambda E(S). \tag{69}$$

6 Busy Period Analysis

In this section, we consider the busy period by the theory of limiting theorem of alternative renewal process.

The duration in which a server works at a rate of μ continuously is called regular busy period, denoted by B, Also, the continuous durations that the server is in working vacation and vacation are denoted by V_w, V_v, respectively. A regular busy period can start at the instant when a working vacation or a vacation finishes, then a busy cycle C is composed of a working vacation V_w, several vacations V_v(if exists) and subsequent regular busy period B. Note that there is just a working vacation in a busy cycle C, and $E(V_w) = \theta_w^{-1}$. Using the theory of limiting theorem of alternative renewal process and Eq. 22, we have

$$K\frac{1}{1-\gamma} = P\{J=0\} = \frac{E(V_w)}{E(C)}. \tag{70}$$

Thus the expected busy cycle is

$$E(C) = \frac{1-\gamma}{K\theta_w}. \tag{71}$$

Table 1 Steady-state probabilities in the D/M/1 queue with single working vacation and vacations

k	π_{k0}	π_{k1}	π_{k2}	π_k	p_k
0	0.1647	0.0905	0	0.2552	0.1148
1	0.0294	0.0356	0.2062	0.2711	0.2816
2	0.0052	0.0140	0.1715	0.1907	0.2321
3	9.3429e-004	0.0055	0.1128	0.1193	0.1532
4	1.6663e-004	0.0022	0.0684	0.0708	0.0932
5	2.9720e-005	8.5106e-004	0.0399	0.0408	0.0545
⋮	⋮	⋮	⋮	⋮	⋮
14	5.4275e-012	1.9138e-007	1.9722e-004	1.9741e-004	2.7101e-004
15	9.6802e-013	7.5257e-008	1.0782e-004	1.0790e-004	1.4818e-004
16	1.7265e-013	2.9594e-008	5.8914e-005	5.8943e-005	8.0971e-005
Sum	0.2004	0.1492	0.6503	0.9999	0.9999

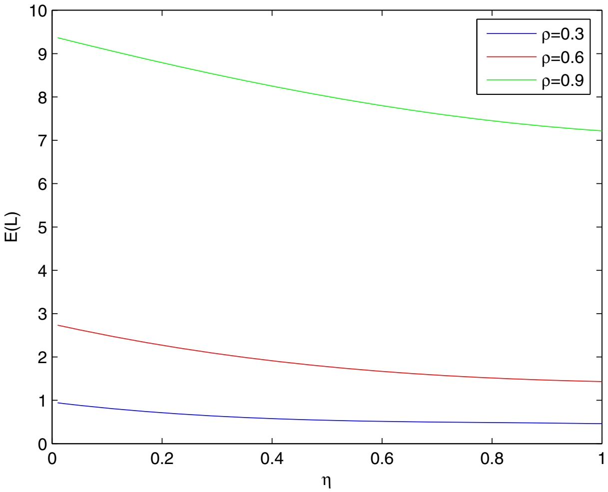

Fig. 1 The mean system size at the arrival epochs for changing η

Similarly, using the alternative renewal theorem and Eqs. 23 and 24, we have

$$E\left(V_v\right) = P\left\{J=1\right\} E\left(C\right) = \frac{p\left(1-\gamma\right)}{\theta_w\left(1-\varepsilon\right)}, \tag{72}$$

$$E\left(B\right) = P\left\{J=2\right\} E\left(C\right) = \frac{1-\gamma}{\theta_w}\left(\alpha\frac{\xi-\gamma}{\left(1-\xi\right)\left(1-\gamma\right)} + p\beta\frac{\xi-\varepsilon}{\left(1-\xi\right)\left(1-\varepsilon\right)}\right). \tag{73}$$

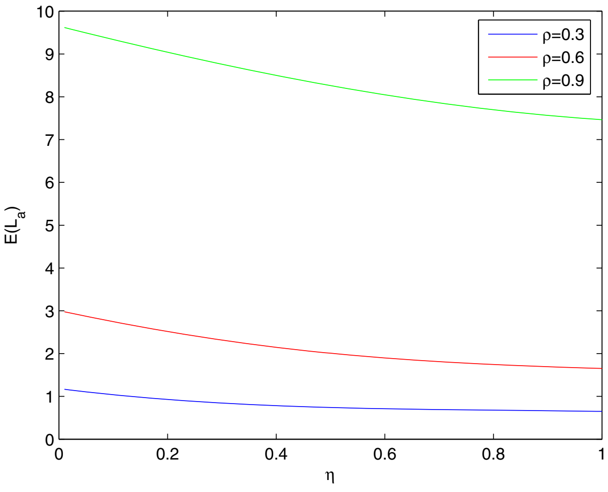

Fig. 2 The mean system size at the arbitrary epochs for changing η

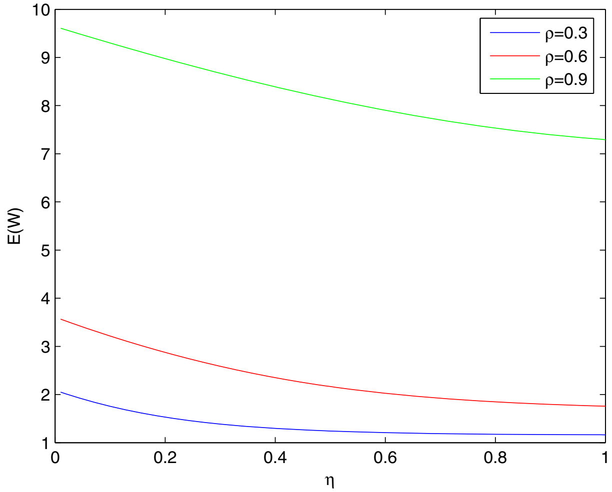

Fig. 3 mean waiting time for changing η

7 Numerical Examples

The results of some numerical examples are depicted in this section. We consider two types of inter-arrival time distributions: deterministic and Erlang distributions.

For the D/M/1 queue with single working vacation and vacations, we assume that $\lambda = 0.75$, $\mu = 1$, $\theta_w = 0.8$, $\theta_v = 0.7$ and $\eta = 0.6$, and use Table 1 to present the steady-state probabilities π_{k0}, π_{k1} ,π_{k2} and p_k under the condition that the parameters are fixed. From Table 1, we can observe that when k =16, π_k and p_k become so smaller that the probabilities even can be omitted. The sums of π_k and p_k from 0 to 16 also verify this result.

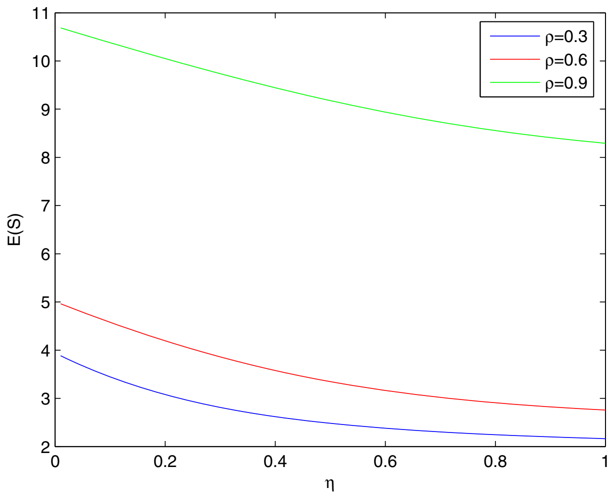

Fig. 4 mean sojourn time for changing η

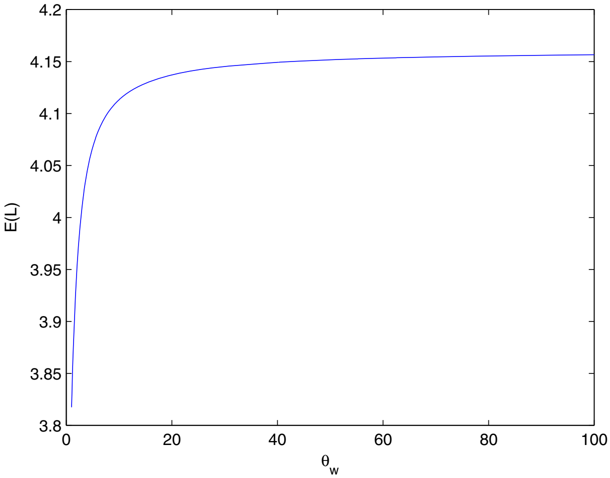

Fig. 5 The effects of θ_w on mean system size at arrival epochs

For the E_2/M/1 queue with single working vacation and vacations, we focus on the effects of service rate in the working vacation η on the performance measures of interest such as mean system size at arrival epochs and arbitrary epochs, the mean waiting time and mean sojourn time. To this end we assume that regular service rate $\mu = 1, \theta_w = 0.3$ and $\theta_v = 0.6$ are fixed, and choose λ as 0.3, 0.5 and 0.8 to represent light, medium and high traffic intensities, that is, $\rho = \lambda/\mu = 0.3$, 0.5 and 0.8, respectively. In Figs. 1, 2, 3, and 4, we plot the change trends of various performance measures as η increase from 0 to 1 under the corresponding traffic intensities. It is obvious that, if η is fixed, the higher ρ is, the larger the mean system size at the arrival epochs, mean system size at the arbitrary epochs, mean waiting time and mean sojourn time become. We also observe that increased η leads to the smaller of these performance measures.

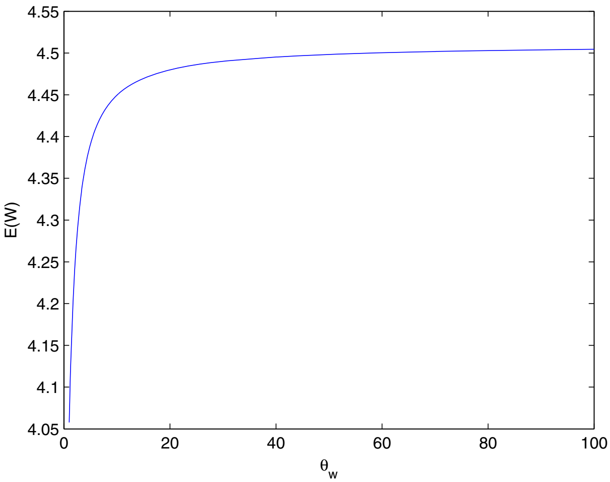

Fig. 6 The effects of θ_v on mean sojourn time

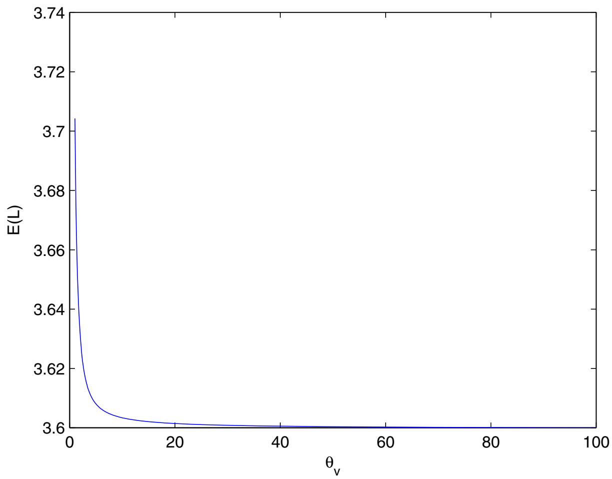

Fig. 7 The effects of θ_w on mean sojourn time

Assuming that $\lambda = 0.8$, $\mu = 1$, $\eta = 0.6$, $\theta_v = 0.6$, Figs. 5 and 6 display the expected system size at arrival epochs and expected sojourn time as θ_w varies. We can find that the expected system size at the arrival epochs and expected waiting time increase and both converge to fixed constants as θ_w increases, as we expect.

In Figs. 7 and 8, we assume that$\lambda = 0.8$, $\mu = 1$, $\eta = 0.6$, $\theta_w = 0.6$, and plot the mean system size at arrival epochs and the mean sojourn time of an arbitrary, with θ_v varying. A similar property as the Figs. 5 and 6 can be found that the mean system size at arrival epochs and mean waiting time both decrease and converge to fixed constants as θ_v increases, as we expect.

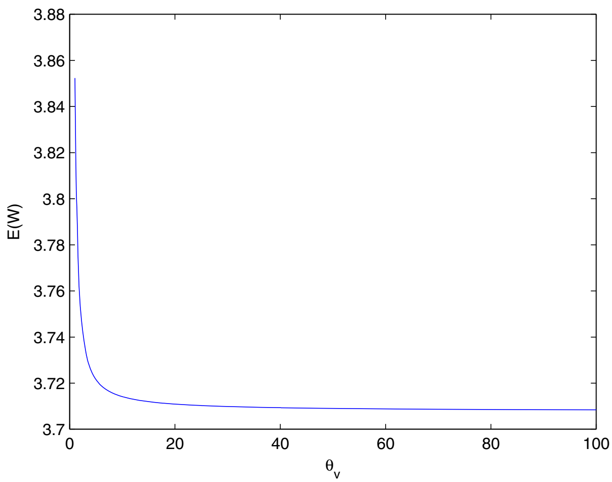

Fig. 8 The effects of θ_v on mean sojourn time

8 Conclusion

In this paper, we analyze GI/M/1 queue with single working vacation and vacations and have done works in several aspects:

1. Using matrix geometric solution method, we derive the stationary distribution of the system size of our model at arrival epochs and its corresponding stochastic decomposition.
2. Using the method of semi-Markov process (SMP), we gain the stationary distribution of system size of our model at arbitrary epochs.
3. By the first-passage time analysis, we acquire the waiting time and sojourn time of an arbitrary customer.
4. Some numerical examples are performed to study the effect of some parameters on performance measure of interest

Appendix

In this appendix, we construct the transition matrix $\tilde{P}$ explicitly and the expressions of the entries in $\tilde{P}$ are also shown here.

Case 1 Consider the transition from state $(i,\ 2)$ to state $(j,\ 2)$, for $1 \leqslant j \leqslant i+1$. Define

$$b_k = \int_0^\infty \frac{(\mu t)^k}{k!} e^{-\mu t} dA\left(t\right), \quad k \geq 0, \tag{74}$$

then b_k $(k \geq 0)$ represent the probabilities that there are k customers served completely during an inter-arrival time in the regular service period. Therefore,

$$p_{(i,\ 2),(j,\ 2)} = b_{i-j+1}. \tag{75}$$

Case 2 Consider the transition from state $(i,\ 2)$ to state $(0,\ 0)$. Define

$$b'_k = \int_0^\infty \int_0^t \frac{\mu\left(\mu x\right)^{k-1}}{i!} e^{-\mu x} e^{-\theta_w (t-x)} dx dA\left(t\right), \text{ for } k \geq 1, \tag{76}$$

then b'_k represents the probability that k customers are served in regular period, then the server enters the working vacation period and the working vacation has not ended before the next arrival. So

$$P_{(i,2),(0,0)} = b'_{i+1}. \tag{77}$$

Case 3 Consider transition from state from $(i,\ 2)$ to $(0,\ 1)$, that may occur when i+1 customers are served in the regular period, then the server enters into the working vacation and subsequently into the vacation period before the next arrival. Hence

$$\begin{aligned} p_{(i,2)(0,1)} &= \int_0^\infty \int_0^t \frac{\mu\,(\mu x)^i}{i!}\left(1 - e^{-\theta_w (t-x)}\right) dx dA\,(t) \\ &= \int_0^\infty \int_0^t \frac{\mu\,(\mu x)^i}{i!} dx dA\,(t) - \int_0^\infty \int_0^t \frac{\mu\,(\mu x)^i}{i!} e^{-\theta_w (t-x)} dx dA\,(t) \\ &= \int_0^\infty \left(1 - \sum_{k=0}^{i} \frac{(\mu t)^i}{i!} e^{-\mu t}\right) dA\,(t) - \int_0^\infty \int_0^t \frac{\mu\,(\mu x)^i}{i!} e^{-\theta_w (t-x)} dx dA\,(t) \\ &= 1 - \sum_{k=0}^{i} b_k - b'_{i+1}. \end{aligned} \tag{78}$$

Case 4 Consider the transition from state $(i,\ 0)$ to $(j,\ 0)$, for $i \geqslant 1,\ 1 \leq j \leqslant i+1$. Define

$$c_k = \int_0^\infty e^{-\theta_w t} \frac{(\eta t)^k}{k!} e^{-\eta t} dA(t)), \tag{79}$$

then $c_k\ (k \geqslant 0)$ represents the probability that the residual working vacation is longer than an inter-arrival time and k services are completed during the inter-arrival time. Hence

$$p_{(i,\,0),(j,\,0)} = c_{i-j+1}. \tag{80}$$

Case 5 Consider the transition from state $(i,\ 0)$ to $(j,\ 2)$, for $i \geqslant 1,\ 1 \leqslant j \leq i+1$. Define

$$d_k = \int_0^\infty \int_0^t \theta_w e^{-\theta_w x} \sum_{j=0}^{k} \frac{(\eta x)^j}{j!} e^{-\eta x} \times \frac{(\mu (t-x))^{k-j}}{(k-j)!} e^{-\mu (t-x)} dx dA(t), \tag{81}$$

then $d_k\ (k \geqslant 0)$ represents the probability that an inter-arrival time is longer than the residual working vacation, and the server resumes to the regular service period and k customers are served in the inter-arrival time. Hence

$$p_{(i,\,0),(j,\,2)} = d_{i-j+1}. \tag{82}$$

Case 6 Consider the transition from state $(i,\ 0)$ to $(0,\ 0)$, and we would like to point out that there are two possible ways to cause the transition from $(i,\ 0)$ to $(0,\ 0)$. One is that if the residual working vacation is longer than an inter-arrival time, and more than $i+1$ customers can be served during the inter-arrival time, the other is that if the ongoing working vacation ends during an inter-arrival time but before there are $i+1$ service completions, and the $(i+1)$st service completion occurs during a regular service period, and a newly started working vacation does not complete during the remaining inter-arrival time. Therefore,

$$\begin{aligned} p_{(i,0),(0,0)} &= \int_0^\infty e^{-\theta_w t} \int_0^t \frac{\eta\,(\eta x)^i}{i!} e^{-\eta x}\, dx dA\,(t) \\ &+ \sum_{k=0}^{i} \int_0^\infty \int_0^t \theta_w e^{-\theta_w x} \frac{(\eta x)^k}{k!} e^{-\eta x} \int_x^t \frac{\mu(\mu\,(y-x))^{i-k}}{(i-k)!} e^{-\mu(y-x)} e^{-\theta_w (t-y)} dy dx dA(t) \\ &= c'_{i+1} + d'_{i+1}, \end{aligned} \tag{83}$$

where

$$c'_i = \int_0^\infty e^{-\theta_w t} \int_0^t \frac{\eta\,(\eta x)^{i-1}}{(i-1)!} e^{-\eta x}\, dx dA\,(t)\,, \tag{84}$$

$$d'_i = \sum_{k=0}^{i-1} \int_0^\infty \int_0^t \theta_w e^{-\theta_w x} \frac{(\eta x)^k}{k!} e^{-\eta x} \int_x^t \frac{\mu(\mu\,(y-x))^{i-1-k}}{(i-1-k)!} e^{-\mu(y-x)} e^{-\theta_w (t-y)} dy dx dA(t). \tag{85}$$

From the definition, we can find that c_k $(k = 0, 1, 2...)$ and c'_i $(i = 0, 1, 2...)$ satisfy the following relationship:

$$\sum_{k=0}^{i} c_k + c'_{i+1} = P\left\{V'_w \geqslant A\right\} = a^*\,(\theta_w)\,, \tag{86}$$

where V'_w is the residual working vacation and A stands for the limit of T_n as $n \to \infty$.

Case 7 Consider the transition from state $(i,\ 0)$ to $(0,\ 1)$. Note that there are also two possible situations that may cause $(i,\ 0)$ to $(0,\ 1)$. One is that if the residual working vacation is longer than an inter-arrival time, more than $i+1$ customers can be served during the working vacation, the other is that there are k $(k < i+1)$ customers served during the working vacation and the residual customers are served in the regular busy period, the system enters another working vacation during which no one comes, and the next arrival occur in the vacation period.

$$p_{(i,0),(0,1)} = \int_0^\infty \int_0^t \theta_w e^{-\theta_w x} \int_0^x \frac{\eta(\eta x)^i}{i!} e^{-\eta x} dx dA(t)$$
$$+ \int_0^\infty \int_0^t \theta_w e^{-\theta_w x} \sum_{k=0}^{i} \frac{(\eta x)^k}{k!} e^{-\eta x} \int_x^t \frac{\mu(\mu(y-x))^{i-k}}{(i-k)!} e^{-\mu(y-x)} \left(1 - e^{-\theta_w(t-y)}\right) dy dx dA(t)$$
$$= \int_0^\infty \int_0^t \theta_w e^{-\theta_w x} \left(1 - \sum_{k=0}^{i} \frac{(\eta x)^k}{k!} e^{-\eta x}\right) dx dA(t)$$
$$+ \int_0^\infty \int_0^t \theta_w e^{-\theta_w x} \sum_{k=0}^{i} \frac{(\eta x)^k}{k!} e^{-\eta x} \left(1 - \sum_{j=0}^{i-k} \frac{(\mu(t-x))^j}{j!} e^{-\mu(t-x)}\right) dx dA(t)$$
$$- \int_0^\infty \int_0^t \theta_w e^{-\theta_w x} \sum_{k=0}^{i} \frac{(\eta x)^k}{k!} e^{-\eta x} \int_x^t \frac{\mu(\mu(y-x))^{i-k}}{(i-k)!} e^{-\mu(y-x)} e^{-\theta_w(t-y)} dy dx dA(t)$$
$$= 1 - a^*(\theta_w) - \sum_{k=0}^{i} d_i - d'_{i+1}$$
$$= 1 - \sum_{k=0}^{i} c_i - c'_{i+1} - \sum_{k=0}^{i} d_i - d'_{i+1}. \tag{87}$$

In the above equation, we use the following equalities:

$$\sum_{k=0}^{i} d_i = \int_0^\infty \int_0^t \theta_w e^{-\theta_w x} \sum_{k=0}^{i} \frac{(\eta x)^k}{k!} e^{-\eta x} \sum_{j=0}^{i-k} \frac{(\mu(t-x))^j}{j!} e^{-\mu(t-x)} dx dA(t),$$

$$a^*(\theta_w) = \sum_{k=0}^{i} c_k + c'_{i+1}.$$

Case 8 Consider transition from state $(i,\ 1)$ to state $(i+1,\ 1)$, for $i \geq 0$, which can occur if and only if the residual vacation time is longer than an arrival time. Therefore, we have

$$P\{A < V_v\} = \int_0^\infty e^{-\theta_v t} dA(t) = a^*(\theta_v). \tag{88}$$

where V_v stands for the residual vacation time. For convenience, we denote $a^*(\theta_v)$ by ε.Thus,

$$p_{(i,1),(i+1,1)} = \varepsilon. \tag{89}$$

Case 9 Consider transition from state $(i,\ 1)$ to state $(j,\ 2)$, for $i \geq 1,\ 1 \leq j \leq i+1$. It can occur if that the inter-arrival time is longer than the residual vacation time, then the server enters into regular service period, and there are *i*-*j*+1 customers served with the rate μ and then the next customer arrives. This gives

$$p_{(i,1),(j,2)} = \int_0^\infty \int_0^t \theta_v e^{-\theta_v x} \frac{(\mu(t-x))^{i-j+1}}{(i-j+1)!} e^{-\mu(t-x)} dx dA(t) = e_{i-j+1}. \tag{90}$$

Case 10 Consider transition from state $(i,\ 1)$ to state $(0,\ 0)$ which can occur if the residual vacation time is shorter than an inter-arrival time, after the vacation, the server will enter the regular service during which *i*+1 customers are served with rate μ , and the next arrival occurs in the subsequent partial close-down time. Therefore

$$p_{(i,1),(0,0)} = \int_0^\infty \int_0^t \theta_v e^{-\theta_v x} \int_x^t \frac{\mu\left(\mu\left(y-x\right)\right)^i}{i!} e^{-\mu(y-x)} e^{-\theta_w(t-y)} dy dx dA(t) = e'_{i+1} \tag{91}$$

Case 11 Consider the situation from state (i, 1) to state (0, 1) which may occur if an inter-arrival time is greater than the residual vacation, $i+1$ service times, and the subsequent working vacation, and the next arrival will occur in the vacation period. Hence

$$\begin{aligned} P_{(i,1)(0,1)} &= \int_0^\infty \theta_v e^{-\theta_v x} \int_x^t \frac{\mu\left(\mu\left(y-x\right)\right)^i}{i!} e^{-\mu(y-x)} \left(1-e^{-\theta_w(t-y)}\right) dy dx dA\left(t\right) \\ &= \int_0^\infty \theta_v e^{-\theta_v x} \left(1-\sum_{k=0}^{i} \frac{\left(\mu\left(t-x\right)\right)^k}{k!} e^{-\mu(t-x)}\right) dx dA\left(t\right) \\ &- \int_0^\infty \theta_v e^{-\theta_v x} \int_x^t \frac{\mu\left(\mu\left(y-x\right)\right)^i}{i!} e^{-\mu(y-x)} e^{-\theta_w(t-y)} dy dx dA\left(t\right) \\ &= 1-\varepsilon-\sum_{k=0}^{i} e_i - e'_{i+1}. \end{aligned} \tag{92}$$

References

Baba Y (2005) Analysis of a GI/m/1 queue with multiple working vacations. Oper Res Lett 33:654–681

Chae K, Lim D, Yang W (2009) The GI/m/1 queue and the GI/geo/1 queue both with single working vacation. Perform Eval 66:356–367

Doshi BT (1986) Queueing systems with vacations—a survey. Queueing Syst 1:29–66

Gross D, Harris C (1985) Fundamentals of queueing theory, 2nd edn. Wiley, New York

Kim JD, Choi DW, Chae KC (2003) Analysis of queue-length distribution of the M/G/1 queue with working vacation. In: Hawaii International Conference on Statistics and Related Fields

Li J, Tian N (2011) Performance analysis of an G/M/1 Queue with single working vacation. Appl Math Comput 217:4960–4971

Li J, Tian N, Zhang ZG, Luh HP (2011) Analysis of the M/G/1 Queue with exponentially working vacations-a matrix analytic approach. Queueing Sys 61:139–166

Li T, Wang Z, Liu Z (2013) The GI/m/1 queue with bernouli-schedule-controlled vacation and vacation interruption. Appl Math Modell 37:3724–3735

Liu W, Xu X, Tian N (2007) Stochastic decompositions in the M/M/1 queue with working vacations. Oper Res Lett 35:595–600
Neuts M (1981) Matrix-Geometric Solutions in stochastic models. Hopkins University Press, Baltimore
Servi LD, Finn SG (2002) M/m/1 queues with working vacation (m/m/1/WV). Perform Eval 50:41–52
Takagi H (1991) Queueing analysis: a foundation of performance evaluation, vol. 1, vacation and priority systems, part 1. North-Holland Elsevier, New York
Teghem J (1986) Control of the service process in a queueing system. Eur J Oper Res 23:141–158
Tian N, Zhang ZG (2006) Vacation queueing models-theory and application. Springer, New York
Tian N, Zhao X (2008) The M/M/1 queue with single working vacation. Int J Inf Manag Sci 19(4):621–634
Tian N, Zhang D, Cao C (1989) The GI/M/1 Queue with exponential vacations. Queueing Syst 5:331–334
Wu D, Takagi H (2006) M/G/1 Queue with multiple working vacations. Perform Eval 63:654–681