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Option Pricing Under Jump-Diffusion Processes with Regime Switching

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Abstract We study an incomplete market model, based on jump-diffusion processes with parameters that are switched at random times. The set of equivalent martingale measures is determined. An analogue of the fundamental equation for the option price is derived. In the case of the two-state hidden Markov process we obtain explicit formulae for the option prices. Furthermore, we numerically compare the results corresponding to different equivalent martingale measures.

Keywords Jump-telegraph process · Jump-diffusion process · Martingales · Relative entropy · Financial modelling · Option pricing · Esscher transform

Mathematics Subject Classification (2010) 91B28 · 60G44 · 60J75 · 60K99

1 Introduction

Consider the jump-diffusion process with time-dependent deterministic driving parameters that are simultaneously switched at random times,

$$X(t) = T^c(t) + J^h(t) + W^\sigma(t), \quad t \geq 0.$$

Here T^c is path-by-path integral of the alternating at random times velocity regimes $c_i(t)$, J^h we denote the associated pure jump part, i.e. the stochastic integral w.r.t. counting process $N = N(t)$ applied to the alternating functions $h_i(t)$, and W^σ denotes the Wiener part, defined by the stochastic integral (w.r.t. Brownian motion B) of the process, which is formed by deterministic functions $\sigma_i(t)$, $i \in D$, $D := \{1, \ldots, d\}$, $d \geq 2$, alternating simultaneously with T^c and J^h.

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The switchings are driven by the hidden semi-Markov random process, which is independent of the Brownian motion B. Random inter-switching time intervals are assumed to be independent and exponentially distributed with arbitrary and variable in time intensities. Generally, such a process is not a Markov process, and it's not a Lévy process.

We study the market model with a single risky asset whose price is determined by the stochastic exponential of X. In general, this model has infinitely many equivalent martingale measures, which makes the market incomplete. The model can be completed by adding other assets; for the jump-diffusion model see Runggaldier (2003), and for the telegraph-jump-diffusion model (hidden Markov model) with constant parameters see Ratanov (2010).

In the incomplete market model it is important to make a reasonable choice of a martingale measure that will be used for option pricing. One way is to choose an equivalent martingale measure that minimises the relative entropy with respect to "physical" probability measure. Another method, which is almost equivalent, is based on maximising the expected utility. Both approaches have one primary drawback, because the so-called "physical" measure might not be observable with parameters that depend on the historical data. Here we discuss another variant of choice of equivalent martingale measure based only on observable parameters $c_i, h_i,\ i \in D$.

In the case of exponentially distributed inter-switching time intervals with constant switching intensities such processes are called telegraph-jump-diffusion (or Markov modulated jump-diffusion) processes. If $d = 2$, the market model of asset pricing (with additive jumps superimposed on the diffusion) has been studied before in Ratanov (2010).

Similar models with time-dependent parameters (without a diffusion component) were considered first in Melnikov and Ratanov (2007) and more recently have been analysed in detail by Ratanov (2015). The model with missing jump component was introduced in Di Crescenzo et al. (2014).

This setting dates back to the seminal paper Runggaldier (2003), where the jump-diffusion market model has been analysed in detail. The market model without a diffusion component, so called jump-telegraph model, based on the processes with regime switching was presented first in Ratanov (2007) (see the more detailed presentation in Kolesnik and Ratanov (2013)). These models are widely used for various applications, see e.g. Weiss (1994).

In this paper we construct a viable pricing formulae by computing the expectation of a payoff function w.r.t. an equivalent martingale measure. By applying some simulating procedures we show how the reasonable equivalent martingale measure might be chosen.

The paper is organised as follows. In Section 2 we introduce the market model. Girsanov's transformation is constructed in Section 3. Then, we define the relative entropy. In Section 4 we discuss the problem of option pricing in the case of infinitely many martingale measures. The corresponding Volterra equations are deduced. Some numerical simulation results are presented in the last part of the paper, Section 5.

2 The Market Model

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. Consider a d-state semi-Markov random process $\varepsilon = \varepsilon(t) \in D,\ t \geq 0$, switching at random times $\{\tau_n\}_{n\geq 1}$, (see Jacobsen (2006)). Process ε is assumed to be right-continuous with left-hand limits.

Let $N = N(t) = \max\{n \mid \tau_n \leq t\},\ t \geq 0$, be the counting process and $B = B(t),\ t \geq 0$, be a standard Brownian motion.

Let $c_i = c_i(t),\ h_i = h_i(t)$ and $\sigma_i = \sigma_i(t),\ t \ge 0,\ i \in D$, be deterministic measurable functions. We assume, that functions c_i are locally integrable, h_i are bounded and σ_i are locally square integrable.

Define a jump-diffusion process with the parameters $\langle c_i, h_i, \sigma_i\rangle,\ i \in D$, simultaneously switching at times $\tau_n,\ \tau_0 = 0$:

$$\begin{aligned} X = X(t) &= T^c(t) + J^h(t) + W^\sigma(t) \\ &= \sum_{n=1}^{N(t)} \left[\int_0^{T_n} c_{\varepsilon(\tau_{n-1})}(u)\mathrm{d}u + h_{\varepsilon(\tau_{n-1})}(T_n) + \int_0^{T_n} \sigma_{\varepsilon(\tau_{n-1})}(u)\mathrm{d}B(u) \right] \\ &+ \int_0^{t-\tau_{N(t)}} c_{\varepsilon(\tau_{N(t)})}(u)\mathrm{d}u + \int_0^{t-\tau_{N(t)}} \sigma_{\varepsilon(\tau_{N(t)})}(u)\mathrm{d}B(u), \qquad t > 0. \end{aligned} \tag{2.1}$$

Let the inter-switching times $T_n = \tau_n - \tau_{n-1},\ n \ge 1$, be exponentially distributed with switching rates $\lambda_{ij} = \lambda_{ij}(t) > 0,\ t \ge 0,\ i, j \in D$. Here $\lambda_{ij}(t),\ t \ge 0$, are positive locally integrable functions.

Denote $\lambda_i = \sum\limits_{j \in D\setminus\{i\}} \lambda_{ij}$. We assume the non-exploding condition to be hold:

$$\int_0^\infty \lambda_i(u)\mathrm{d}u = +\infty, \qquad i \in D. \tag{2.2}$$

Let $T = \tau_1$ be the first switching time. Thus, the survival functions are well-defined:

$$\overline{F}_i^{\mathbb{P}}(t) = \overline{F}_i(t) := \mathbb{P}\{T \ge t \mid \varepsilon(0) = i\} = \exp\left(-\int_0^t \lambda_i(u)\mathrm{d}u\right) \mathbb{1}_{\{t>0\}}, \quad i \in D. \tag{2.3}$$

Therefore, the transition density functions from state $i \in D$ to state $j \in D,\ j \ne i$, are given by

$$f_{ij}(t) := \mathbb{P}\{T_n \in \mathrm{d}t,\ \varepsilon(\tau_n) = j \mid \varepsilon(\tau_{n-1}) = i\}/\mathrm{d}t = \lambda_{ij}(t)\exp\left(-\int_0^t \lambda_i(u)\mathrm{d}u\right)\mathbb{1}_{\{t>0\}}. \tag{2.4}$$

The following result is known (see e.g. (Ratanov 2015, Theorem 1) or (Di Crescenzo and Ratanov 2015, Theorem 3.1)).

Theorem 1 *Let $X = X(t),\ t \ge 0$, be the process with the regimes $\langle c_i, h_i, \sigma_i\rangle,\ i \in D$, switching at random instants $\tau_n,\ n \ge 0$.*

Process X is a $(\mathscr{F}_t, \mathbb{P})$-martingale, if and only if

$$\lambda_i(t)h_i(t) + c_i(t) \equiv 0, \qquad t \ge 0,\ i \in D. \tag{2.5}$$

Remark 1 These type of conditions for the jump-telegraph processes with constant deterministic parameters c, h, λ appears first in (Ratanov 2007, Theorem 1) (see also Kolesnik and Ratanov (2013)). In this case condition (2.5) is intuitively obvious. It means that the mean displacement $c_i\lambda_i^{-1}$, which is performed by the telegraph process during a time-period τ, is identical to the jump's size performed in the opposite direction.

This intuitively explains why (2.5) is a martingale condition.

Let $U,\ U > 0$, be the fixed time horizon.

Assume functions h_i to be bounded away from -1 uniformly in $t \in [0, U]$:

$$-1 + \delta \le h_i(t) \le M, \qquad i \in D,\ \delta > 0. \tag{2.6}$$

Let the parameters of X satisfy (2.6) and (2.5).

The stochastic exponential of X is defined by

$$\begin{aligned}\mathscr{E}_t(X) &= \exp\left(T^{c-\sigma^2/2}(t) + J^{\ln(1+h)}(t) + W^{\sigma}(t)\right)\\ &= \exp\left(T^{c-\sigma^2/2}(t) + W^{\sigma}(t)\right)\prod_{n=1}^{N(t)}\left(1 + h_{\varepsilon(\tau_n)}(T_n)\right), \quad t \geq 0, \qquad (2.7)\end{aligned}$$

see Jeanblanc et al. (2009). Process X is the martingale, then the stochastic exponential $\mathscr{E}_t(X),\ t \geq 0$, is the strictly positive martingale, $\mathbb{E}[\mathscr{E}_t(X)] \equiv 1$.

Consider the market of one risky asset with the price given by the geometric jump-diffusion process under regime switching,

$$S(t) = S(x;t) = x\mathscr{E}_t(X), \qquad t \in [0, U], \qquad (2.8)$$

where $x = S(0)$ is the initial asset price.

It is assumed that the market interest rate depends on the current state of the market. Let $r_i = r_i(t),\ t \geq 0,\ i \in D$, be deterministic non-negative locally integrable functions that are switched at times τ_n, when the market switched the state.

Consider a bond (numeraire) with the price

$$R(t) = \exp\left(T^r(t)\right), \qquad t \in [0, U]. \qquad (2.9)$$

Note that the discounted process $R(t)^{-1}S(t)$ is again the jump-diffusion process with regime switching based on the triplet $\langle c_i - r_i,\ h_i,\ \sigma_i\rangle,\ i \in D$. Without loss of generality hereafter we assume that the interest rate is null, so $R(t) \equiv 1$.

Let $\mathscr{F}_t \subset \mathscr{F},\ t \geq 0$, be the natural (right-continuous) filtration generated by ε and B. Recall that measure $\mathbb{Q}$ is the equivalent martingale measure if it is equivalent to $\mathbb{P}$ and such that the discounted price process $R(t)^{-1}S(t)$ is a $(\mathscr{F}_t, \mathbb{Q})$-martingale.

Following Dybvig Ph and Ross (2008) and Delbaen and Schachermayer (1994), we say, that

- the model is arbitrage-free, if there is at least one equivalent martingale measure;
- the arbitrage-free model is complete, if the equivalent martingale measure is unique.

To define the measure transform we will restrict the consideration to measures $\mathbb{Q}$ with Radon-Nikodym density, which is $\mathbb{P}$-square integrable martingale over the finite time interval $[0, U]$. Under fixed time horizon $t,\ t \in [0, U]$, the Radon-Nikodym density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t) = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathscr{F}_t},\ t \in [0, U]$, is defined by the stochastic exponential, Eq. 2.7,

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t) = \mathscr{E}_t(X^*) = \exp\left(T^{c^*-\sigma^{*2}/2}(t) + J^{\ln(1+h^*)}(t) + W^{\sigma^*}(t)\right), \qquad (2.10)$$

see (Bellamy and Jeanblanc 2000, Proposition 3.1). Here $\sigma_i^*(t)$ are some square-integrable, $c_i^*(t)$ are locally integrable and $h_i^*(t),\ t \in [0, U],\ i \in D$, are measurable bounded functions satisfying (2.5) and (2.6):

$$\lambda_i(t)h_i^*(t) + c_i^*(t) \equiv 0, \quad -1 + \delta \leq h_i^* \leq M, \quad t \in [0, U],\ i \in D. \qquad (2.11)$$

3 Girsanov's Theorem. Relative Entropy

Recall the following well-known result, which generalises the classical Girsanov theorem.

Theorem 2 *Let the jump-diffusion process X with the triplet $\langle c_i, h_i, \sigma_i\rangle$, $i \in D$, be defined by* Eq. 2.1 *on the filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$.*

Consider measure $\mathbb{Q}$ defined by the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t)$, $t \in [0, U]$, with parameters $\langle c^*, h^*, \sigma^*\rangle$, see Eqs. 2.10-2.11.

Under measure $\mathbb{Q}$

(a) *the process $\widetilde{B}(t) = B(t) - L^*(t)$, $t \geq 0$, is the standard $\mathbb{Q}$-Brownian motion.*

Here $L^*(t)$, $t \geq 0$, is the piecewise deterministic process with switching velocities σ_i^*, $i \in D$, i.e.

$$L^*(t) := T^{\sigma^*}(t) = \sum_{n=1}^{N(t)} \int_0^{T_n} \sigma^*_{\varepsilon(\tau_{n-1})}(u)\mathrm{d}u + \int_0^{t-\tau_{N(t)}} \sigma^*_{\varepsilon(\tau_{N(t)})}(u)\mathrm{d}u;$$

(b) *process X becomes*

$$X(t) = T^{c+\sigma\sigma^*}(t) + J^h(t) + \widetilde{W}^\sigma(t), \qquad t \in [0, U]. \tag{3.1}$$

It is characterised by the triplet $\langle c_i + \sigma_i\sigma_i^*, h_i, \sigma_i\rangle$, $i \in D$, with the switching intensities of interarrival times $\lambda_i^* = \lambda_i^*(t)$ given by

$$\lambda_i^*(t) = (1 + h_i^*(t))\lambda_i(t) = \lambda_i(t) - c_i^*(t), \quad t \in [0, U],\ i \in D. \tag{3.2}$$

The non-exploding condition

$$\int_0^\infty \lambda_i^*(u)\mathrm{d}u = +\infty, \qquad i \in D, \tag{3.3}$$

holds.

Proof Part (a) follows from the classical theorem, see e.g. (Jeanblanc et al. 2009, Proposition 1.7.3.1). Part (b) consists in the special version of Girsanov's Theorem, see Di Crescenzo and Ratanov (2015). Condition (3.3) follows from Eq. 2.2, Eq. 3.2 and $h_i^* > -1 + \delta$, see Eq. 2.11. □

Under measure $\mathbb{Q}$ the asset price becomes

$$S(x; t) = x \cdot \exp\left(T^{c+\sigma\sigma^* - \sigma^2/2}(t) + \widetilde{W}^\sigma(t)\right) \prod_{n=1}^{N(t)} \left(1 + h_{\varepsilon(\tau_n)}(T_n)\right), \tag{3.4}$$

with new switching intensities λ_i^*, $i \in D$, see Eqs. 3.1–3.2.

One can easily derive the following description of equivalent martingale measures of the model (2.8).

Proposition 1 *Let measure $\mathbb{Q}$ be defined by the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t)$, $t \in [0, U]$, with parameters $\langle c^*, h^*, \sigma^*\rangle$, see* Eq. 2.10–2.11.

Processes $X = X(t)$ and $S(x; t) = x\mathscr{E}_t(X)$ are $\mathbb{Q}$-martingales if and only if

$$c_i(t) + \sigma_i(t)\sigma_i^*(t) + \lambda_i^*(t)h_i(t) \equiv 0, \quad t \in [0, U],\ i \in D, \tag{3.5}$$

where $\lambda_i^*(t)$ is defined by Eq. 3.2.

Proof By Theorem 2 process X is splitted into the $\mathbb{Q}$-martingale $\widetilde{W}^\sigma$ and $T^{c+\sigma\sigma^*} + J^h$. The latter is the $\mathbb{Q}$-martingale if and only if Eq. 3.5 holds, see Eq. 2.5. □

We use Proposition 1 to classify models (2.8)-(2.9).

Proposition 2 *Consider the market model with the risky asset price* $S = S(t),\ t \in [0, U]$, *given by* Eq. 2.8, *and with zero interest rate*, $B(t) \equiv 1$.

1. *Let the diffusion component is missed,* $\sigma_i = 0,\ i \in D$, *the jump component be defined by* $h_i(t) \neq 0,\ \forall t \in [0, U],\ i \in D$, *and* Eq. 2.6 *holds.*
   - (a) *If* $c_i(t)/h_i(t) > 0$ *in some interval* $t \in [\alpha, \beta],\ 0 \le \alpha < \beta \le U$, *then the model possesses arbitrage opportunities*;
   - (b) *The model is arbitrage-free, if* $c_i(t)/h_i(t) < 0,\ \forall t \in [0, U],\ i \in D$. *In this case the model is complete.*
2. *Let* $\sigma_i(t) \neq 0,\ \forall t \in [0, U],\ i \in D$. *Hence the market model is arbitrage-free and incomplete.*

Proof If the diffusion component is missed, model (2.8)-(2.9) is similar to well-studied jump-telegraph model, see (Kolesnik and Ratanov 2013, Chapter 5) and references therein.

If $\sigma_i(t) \neq 0,\ \forall t \in [0, U],\ i \in D$, then by applying the Radon-Nikodym derivative (2.10) one can arbitraraly change the tendency and then, construct an arbitrary distribution of inter-switching time intervals. More precisely, Eq. 3.5 has infinitely many solutions $\sigma_i^*, h_i^*;\ h_i^* > -1$, if $\sigma_i(t) \neq 0,\ \forall t \in [0, U],\ i \in D$. □

Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent measures defined by the Radon-Nikodym derivative $\dfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$, see Eq. 2.10. Under the time horizon $U,\ U > 0$, the relative entropy of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ is defined by the set of functions $H_i(U),\ i \in D$:

$$H_i(U) = H_i^{\mathbb{Q},\mathbb{P}}(U) := \mathbb{E}^{\mathbb{Q}}\left[\ln \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(U) \middle| \varepsilon(0) = i\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(U) \ln \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(U) \;\middle|\; \varepsilon(0) = i\right], \tag{3.6}$$

see Frittelli (2000). Here the Radon-Nikodym derivative $\dfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(U) = \mathcal{E}_U\,(X^*)$ is introduced by Eq. 2.10 with restrictions (2.11).

Let $\mathfrak{M}$ be the set of martingale measures $\mathbb{Q}$ equivalent to $\mathbb{P}$. For option pricing we are interested in maximisation of the relative entropy, $\min\limits_{\mathfrak{M}} H_i^{\mathbb{Q},\mathbb{P}}(U)$.

In the two-state case with constant parameters and the intensities of switchings the relative entropy functions of the equivalent martingale measure, $H_1(U),\ H_2(U)$, are given by

$$\begin{aligned} H_1(U) &= H_1(U; \lambda_1^*, \lambda_2^*) = BU + A_1\left[1 - \mathrm{e}^{-(\lambda_1^*+\lambda_2^*)U}\right], \\ H_2(U) &= H_2(U; \lambda_1^*, \lambda_2^*) = BU + A_2\left[1 - \mathrm{e}^{-(\lambda_1^*+\lambda_2^*)U}\right], \end{aligned} \tag{3.7}$$

where

$$A_1 = \frac{\lambda_1^*(b_1 - b_2)}{(\lambda_1^* + \lambda_2^*)^2}, \quad A_2 = \frac{\lambda_2^*(b_2 - b_1)}{(\lambda_1^* + \lambda_2^*)^2}, \quad B = \frac{\lambda_2^* b_1 + \lambda_1^* b_2}{\lambda_1^* + \lambda_2^*}. \tag{3.8}$$

$$b_i = \lambda_i - \lambda_i^* + \lambda_i^* \ln\left[\frac{\lambda_i^*}{\lambda_i}\right] + \frac{1}{2}(\sigma_i^*)^2, \qquad i \in \{0, 1\}. \tag{3.9}$$

See the detailed proof in Di Crescenzo and Ratanov (2015).

Remark 2 Note that in model (2.8) the minimal entropy martingale measure depends on the time horizon U. This is in contrast with the Lévy model (Fujiwara and Miyahara 2003) and with the regime switching diffusion model (Elliott et al. 2005).

By using formulae (3.7)-(3.9) one can see that the minimal entropy corresponds to the switching intensities $\lambda_i^* = \lambda_i^*(U)$ that are moving in the interval between λ_i and $-c_i/h_i$, $i \in \{0, 1\}$, see Di Crescenzo and Ratanov (2015).

4 The Option Pricing

Let model (2.8)-(2.9) be arbitrage-free, see parts 1b and 2 of Proposition 2. Consider an option with the nonnegative payoff function $\mathscr{H} = \mathscr{H}(\cdot)$, executed at the maturity time $U,\ U > 0$. Let $\mathbb{Q} \in \mathfrak{M}$ be an equivalent martingale measure.

In the case without a diffusion component and without arbitrage opportunities (part 1b of Proposition 2), the equivalent martingale measure $\mathbb{Q}$ exists and it is unique. In this case the option price is unique and it is determined by the expectation

$$\Phi_i(x; U) = \mathbb{E}_i^{\mathbb{Q}}\left[\mathscr{H}(S(x; U))\right], \qquad x > 0,\ i \in D, \tag{4.1}$$

where $S(x; \cdot)$ is given by Eq. 3.4 and i refers to the initial market state, $\varepsilon(0) = i$. For the constant parameters, the exponentially distributed inter-switching times and $d = 2$ the option pricing in this arbitrage-free and complete case is well-studied, see e.g. Kolesnik and Ratanov (2013).

Consider the incomplete model with the diffusion component (part 2 of Proposition 2). Now, the equivalent martingale measure is not unique. Let $\mathbb{Q}$ be the martingale measure defined by the density $\dfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t),\ \ t \in [0, U]$, see Eq. 2.10, with parameters satisfying conditions (2.11) and (3.5). Then, the viable option price is defined by the expectation (4.1).

Proposition 3 *Functions $\Phi_i(\cdot;\cdot),\ i \in D$, satisfy the following Volterra equations:*

$$\begin{aligned}
\Phi_i(x; U) \;=\; & \overline{F}_i^{\mathbb{Q}}(U)\int_{-\infty}^{\infty}\mathscr{H}\left(x\mathrm{e}^{l_i(U)+y}\right)\psi_i(y, U)\mathrm{d}y \\
& + \sum_{j \in D\setminus\{i\}}\int_0^U \lambda_{ij}^*(u)\exp\left(-\int_0^u \lambda_i^*(u')\mathrm{d}u'\right) \\
& \times\left[\int_{-\infty}^{\infty}\Phi_j\left(x(1+h_i(u))\mathrm{e}^{l_i(u)+y}, U-u\right)\psi_i(y, u)\mathrm{d}y\right]\mathrm{d}u, \qquad i \in D.
\end{aligned} \tag{4.2}$$

Here λ_{ij}^* and λ_i^* are the new switching intensities under measure $\mathbb{Q}$ defined by Theorem 2; $l_i(u)$, see Eq. 3.4*, is defined by*

$$l_i(u) = \int_0^u\left(c_i(u') + \sigma_i(u')\sigma_i^*(u') - \sigma_i(u')^2/2\right)\mathrm{d}u', \qquad u \in [0, U],$$

and $\psi_i = \psi_i(y, u)$ is the $\mathbb{Q}$-density function of the Gaussian random variable $w_i(u) = \int_0^u \sigma_i(u')\mathrm{d}\widetilde{B}(u')$, $u \in [0, U],\ i \in D$,

$$\psi_i(y, u) = \frac{1}{\Sigma_i(u)\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\Sigma_i(u)^2}\right), \qquad y \in (-\infty, \infty),$$

where $\Sigma_i(u)^2 := \int_0^u \sigma_i(u')^2 \mathrm{d}u' = \mathrm{Var}[w_i(u)],\ u \in [0, U],\ i \in D$. *Survival functions* $\overline{F}_i^{\mathbb{Q}}$ *are defined by*

$$\overline{F}_i^{\mathbb{Q}}(u) = \exp\left(-\int_0^u \lambda_i^*(u')\mathrm{d}u'\right).$$

Proof Conditioning on the first switching one can easily derive (4.2) from Eqs. 4.1 and 3.4. □

To describe of the option price, if the elapsed time after the last switching is not 0, consider the conditional expectations

$$\Phi_i(x; U \mid t, s) = \mathbb{E}^{\mathbb{Q}}\left[\mathscr{H}\left(S(x; t, U)\right) \mid \mathscr{F}_t,\ s = t - \tau_{N(t)}\right], \qquad 0 \le s < t \le U,\ i \in D. \tag{4.3}$$

Here $S(x; t, U) = xS(x; U)/S(x; t)$ describes the evolution of asset price beginning from time t, and $s = t - \tau_{N(t)}$, corresponds to the elapsed time after the last regime switching. Subscript i refers to the market state at time $t,\ t \in [0, U], \varepsilon(t) = i$.

The viable option price $V(t)$ at time t is defined by

$$\begin{aligned} V(t) &= \Phi_i(S(t); U \mid t, t - \tau_{N(t)}), \qquad t > 0,\\ V(0) &= \Phi_i(x; U). \end{aligned}$$

The following equation is derived similarly to Eq. 4.2. For $i \in D$

$$\begin{aligned} \Phi_i(x; U \mid t, s) &= \overline{F}_i^{\mathbb{Q}}(t, U) \int_{-\infty}^{\infty} \mathscr{H}\left(x\mathrm{e}^{l_i(U) - l_i(u) + y}\right) \psi_i(y, t, U)\mathrm{d}y\\ &\quad + \sum_{j \in D\setminus\{i\}} \int_t^U \lambda_{ij}(u) \exp\left(-\int_0^u \lambda_i^*(u')\mathrm{d}u'\right)\\ &\quad \times \left[\int_{-\infty}^{\infty} \Phi_j\left(x(1 + h_i(u - t + s))\mathrm{e}^{l_i(u) - l_i(t) + y}, U - u\right) \psi_i(y, t, u)\mathrm{d}y\right] \mathrm{d}u. \end{aligned} \tag{4.4}$$

Here $\Phi_j\left(x(1 + h_i(u - t + s))\mathrm{e}^{l_i(u) - l_i(t) + y}, U - u\right)$ is given as the solution of Eq. 4.2,

$$\overline{F}_i^{\mathbb{Q}}(t, U) = \exp\left(-\int_t^U \lambda_i^*(u')\mathrm{d}u'\right), \qquad t \in [0, U],$$

and $\psi_i(\cdot, t, u)$ is the density function of normally distributed $w_i(t, u) = \int_t^u \sigma_i(u')\mathrm{d}\widetilde{B}(u')$, $t < u$.

Equations 4.2 and 4.4 serve as an analogue of the fundamental equation of the Black-Scholes option pricing.

From a practical viewpoint one can choose the martingale measure $\mathbb{Q}$ with a minimal relative entropy (3.6). This measure is situated between the following two extremal points.

The first one is the equivalent martingale measure constructed by the so-called Esscher transform. In the case of the model under regime switching the Esscher transform is defined by

$$\frac{\mathrm{d}\mathbb{Q}^\theta}{\mathrm{d}\mathbb{P}} \mid_{\mathscr{F}_t} = \exp\left(W^\theta(t) + T^{-\theta^2/2}(t)\right), \qquad t \in [0, U], \tag{4.5}$$

see Elliott et al. (2005). Here $\theta_i = \theta_i(t),\ t \in [0, U],\ i \in D$, are square integrable functions.

Note that Eq. 4.5 corresponds to Radon-Nikodym derivative (2.10) with

$$\sigma_i^* = \theta_i, \quad c_i^* = 0, \quad h_i^* = 0, \qquad i \in D.$$

Under measure $\mathbb{Q}^\theta$ the distributions of inter-switching times have not changed, $\lambda_i^* = \lambda_i,\ i \in D$, see Theorem 2, Eq. 3.2. Due to Eq. 3.5 measure $\mathbb{Q}^\theta$ is the martingale measure if $\theta_i = \theta_i(t) = -\dfrac{c_i(t) + \lambda_i(t)h_i(t)}{\sigma_i(t)}$.

Another extremal point might be more suitable for option pricing. This is provided by the Radon-Nikodym derivative (2.10) of the form

$$\frac{\mathrm{d}\mathbb{Q}_\theta}{\mathrm{d}\mathbb{P}} \mid_{\mathscr{F}_t} = \exp\left(T^\theta(t) + J^{\ln(1-\theta/\lambda)}(t)\right), \quad t \in [0, U], \tag{4.6}$$

which corresponds to the equivalent martingale measure $\mathbb{Q}_\theta$ defined by Eq. 2.10 with $\sigma_i^* = 0,\ c_i^* = \theta_i$ and $h_i^* = -\theta_i/\lambda_i,\ i \in D$. This transformation does not affect the Brownian motion B, see Theorem 2. It changes only the distributions of inter-switching times. The new switching intensities are defined by means of observable parameters: $\lambda_i^*(t) = -c_i(t)/h_i(t),\ t \in [0, U]$, see Eq. 3.5.

Notice that if the diffusion component is missed, $\sigma_i \equiv 0,\ i \in D$, (Proposition 2, 1b) the market model is complete and the unique equivalent martingale measure is determined by Eq. 4.6. Thus, the choice of the equivalent martingale measure as in Eq. 4.6 corresponds to the proximity to the complete case. See also the results of numerical simulations in the next section.

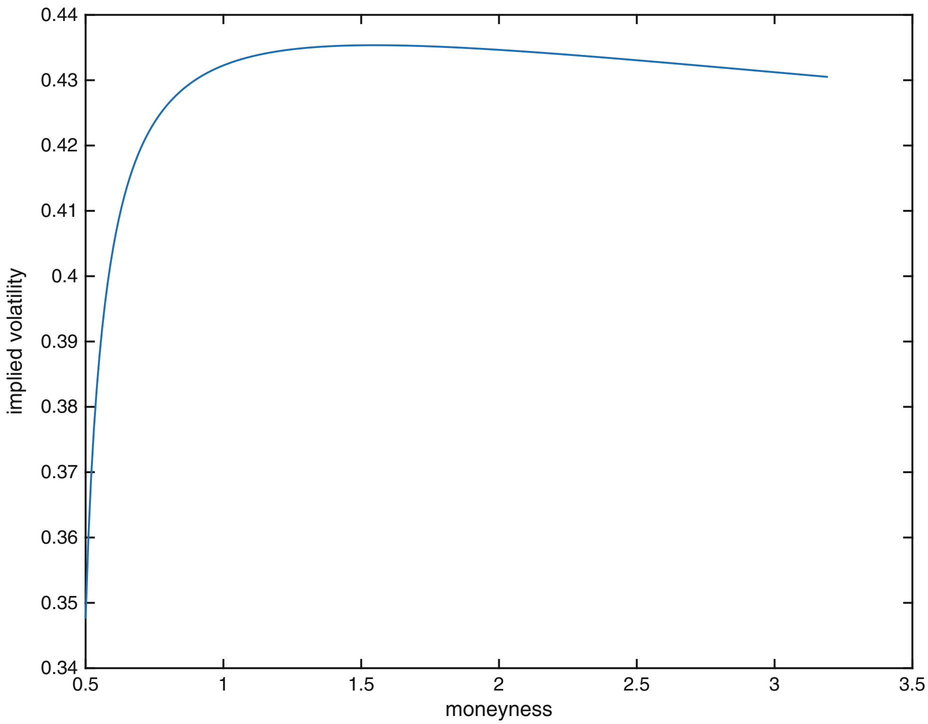

Fig. 1 $\lambda_1^* = \lambda_2^* = 10,\ c_1 = -c_2 = -1,\ h_1 = -h_2 = 0.1,\ \sigma_1 = \sigma_2 = 0.32;\ \delta = 0.2237$

5 The Two-States Markov Model and Numerical Study

To simplify numerical computing we consider the case of two-state semi-Markov process ε, with constant alternating parameters c_i, h_i, σ_i, $i \in \{0, 1\}$. Assume the inter-switching times T_n to be exponentially distributed with the constant intensities $\lambda_i > 0$, $i \in \{0, 1\}$.

Let $\mathbb{Q}$ be the equivalent martingale measure defined by the density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(t)$, $t \in (0, U)$, Eq. 2.10 with constant parameters $\langle c^*, h^*, \sigma^* \rangle$, $h_i^* > -1, i \in \{0, 1\}$. Under measure $\mathbb{Q}$ the inter-switching times are again exponentially distributed with the alternating intensities $\lambda_i^* = (1 + h_i^*)\lambda_i$, $i \in \{0, 1\}$, see Theorem 2.

The option price $\Phi_i(x; U)$ satisfies the Volterra Eq. 4.2, which is difficult to solve. For the case of two-state process with constant parameters one can use explicit formulae which are obtained below as a mix of the Black-Scholes.

Fix the initial state i, $i \in \{0, 1\}$, $\varepsilon_i(0) = i$. Denote by $\mathscr{T}_i(t) := \int_0^t \mathbb{1}_{\{\varepsilon_i(u)=i\}}\mathrm{d}u$ the total time between 0 and t spent by the semi-Markov process ε_i in the state i. Process ε_i is carried out in the opposite state $1 - i$ during time $t - \mathscr{T}_i(t)$.

For the fixed t, $t \in [0, U]$, let $g_i(\cdot, t; n)$, $n \geq 0, i \in \{0, 1\}$, be the density function of the random variable $\mathbb{1}_{\{N_i(t)=n\}}\mathscr{T}_i(t)$:

$$g_i(\tau, t; n) = \mathbb{Q}\{N_i(t) = n,\ \mathscr{T}_i(t) \in \mathrm{d}\tau\}/\mathrm{d}\tau, \quad \tau \in (0, t).$$

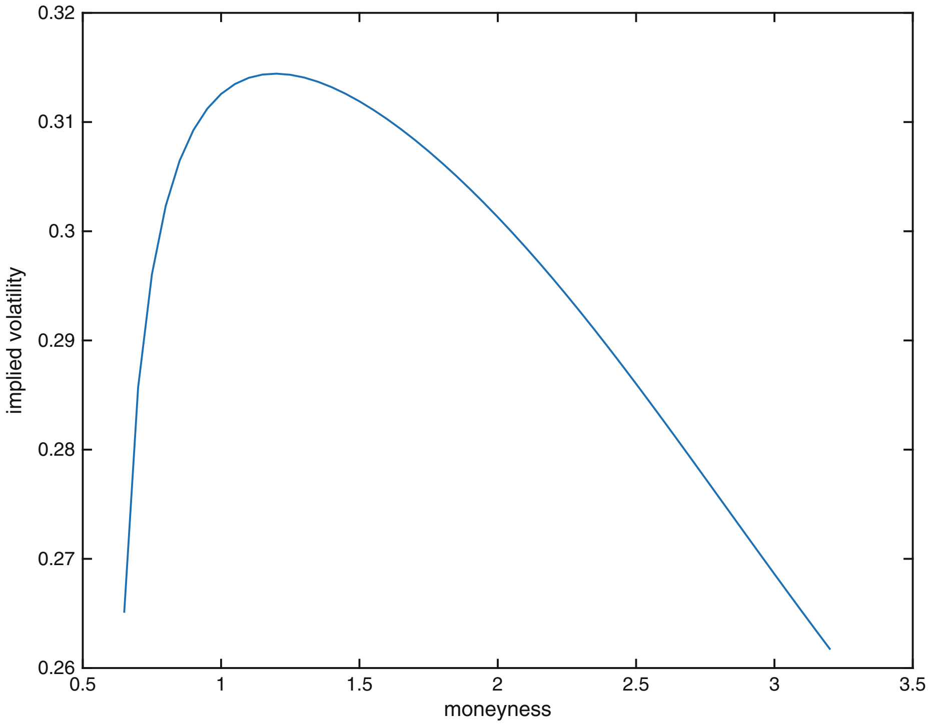

Fig. 2 $\lambda_1^* = \lambda_2^* = 10$, $c_1 = -c_2 = -1$, $h_1 = -h_2 = 0.1$, $\sigma_1 = \sigma_2 = 0.1$; $\delta = 0.1828$

It's clear that $g_i(\tau, t; 0) = \mathrm{e}^{-\lambda_i^* t}\delta(t-\tau)$. Conditioning on the first switching time we obtain the sequence of the equations

$$g_i(\tau, t; n) = \int_0^{\tau} \lambda_i^* \mathrm{e}^{-\lambda_i^* u} g_{1-i}(t-\tau, t-u; n-1)\mathrm{d}u, \qquad \tau \in (0,t),\ n \geq 1,\ i \in \{0,1\}. \tag{5.1}$$

By induction on n, the solution of Eq. 5.1, $\{g_i(\tau, t; n)\},\ n \geq 1,\ i \in \{0,1\}$, takes the following explicit form: for $k \geq 1$

$$g_0(\tau, t; 2k) = (\lambda_0^*)^k(\lambda_1^*)^k \frac{(t-\tau)^{k-1}\tau^k}{(k-1)!k!} \mathrm{e}^{-\lambda_0^*\tau - \lambda_1^*(t-\tau)} \mathbb{1}_{\{0\leq\tau\leq t\}},$$

$$g_1(\tau, t; 2k) = (\lambda_0^*)^k(\lambda_1^*)^k \frac{(t-\tau)^{k}\tau^{k-1}}{(k-1)!k!} \mathrm{e}^{-\lambda_0^*(t-\tau) - \lambda_1^*\tau} \mathbb{1}_{\{0\leq\tau\leq t\}},$$

and for $k \geq 0$

$$g_0(\tau, t; 2k+1) = (\lambda_0^*)^{k+1}(\lambda_1^*)^k \frac{(t-\tau)^{k}\tau^k}{(k!)^2} \mathrm{e}^{-\lambda_0^*\tau - \lambda_1^*(t-\tau)} \mathbb{1}_{\{0\leq\tau\leq t\}},$$

$$g_1(\tau, t; 2k+1) = (\lambda_0^*)^{k}(\lambda_1^*)^{k+1} \frac{(t-\tau)^{k}\tau^k}{(k!)^2} \mathrm{e}^{-\lambda_0^*(t-\tau) - \lambda_1^*\tau} \mathbb{1}_{\{0\leq\tau\leq t\}},$$

cf. (Ratanov 2010, eqs (2.17)-(2.20)), where the erroneous formulae have been presented.

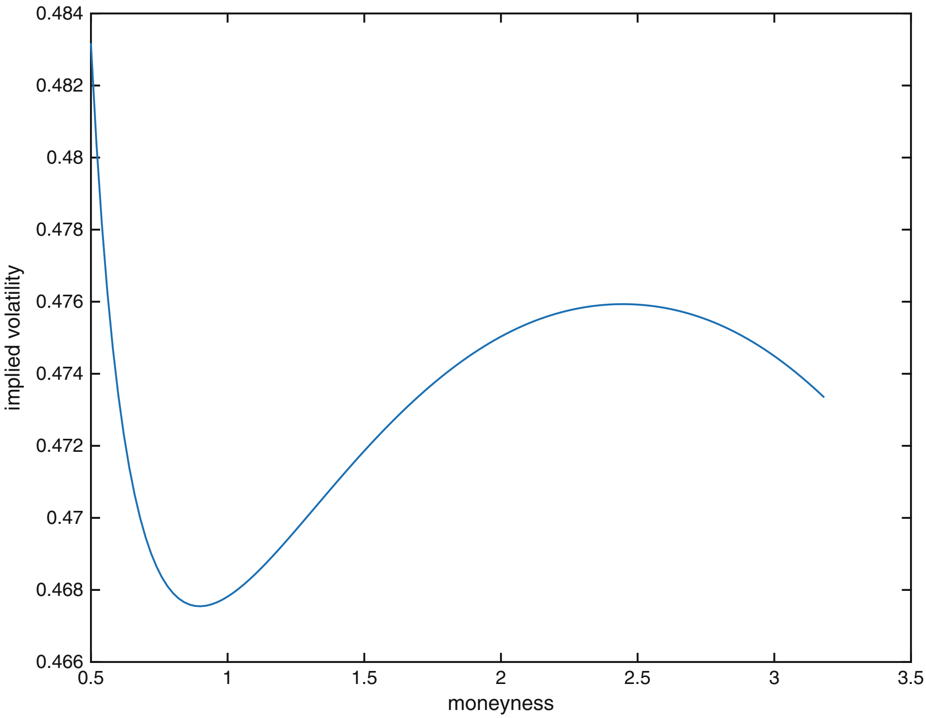

Fig. 3 $\lambda_1^* = \lambda_2^* = 10,\ c_1 = 0.3, c_2 = 1.9,\ h_1 = -0.03, h_2 = -0.19,\ \sigma_1 = \sigma_2 = 0.1;\ \delta = 0.0329$

In the framework of model (2.8)-(2.9) (with zero interest rates) consider the option with the payoff function $\mathcal{H} = \mathcal{H}(S(U))$. Denote by $\varphi(x, \sigma)$ the following expectation

$$\varphi(x, \sigma) = \mathbb{E}\left[\mathscr{H}\left(x e^{Y_\sigma - \sigma^2/2}\right)\right], \tag{5.2}$$

where r.v. Y_σ is normally distributed, $\mathscr{N}(0, \sigma^2)$. Notice that $\varphi(x, \sigma\sqrt{U})$ is the Black-Scholes option price with maturity at U and with volatility σ.

The viable option-price is delivered by $\mathbb{Q}$-expectation of the form (see Eqs. 4.1 and 3.4)

$$\begin{aligned}\Phi(x; U) &= \mathbb{E}_i^{\mathbb{Q}}\left[\mathscr{H}\left(x \exp\left(T^{c+\sigma\sigma^*-\sigma^2/2}(t) + \widetilde{W}^\sigma(t)\right)\prod_{n=1}^{N(t)}\left(1 + h_{\varepsilon(\tau_n)}(T_n)\right)\right)\right] \\ &= \mathbb{E}_i^{\mathbb{Q}}\left[\mathscr{H}\left(x \exp\left(T^{\tilde{c}}(U)\right) M^{\widetilde{W}}(U)\prod_{n=1}^{N(t)}\left(1 + h_{\varepsilon(\tau_n)}(T_n)\right)\right)\right],\end{aligned} \tag{5.3}$$

where $\widetilde{W}^\sigma$ is the Wiener part of X defined by the Itô integral with respect to the $\mathbb{Q}$-Brownian motion $\widetilde{B}$, the exponential $M^{\widetilde{W}}(U) = \exp\left(\widetilde{W}^\sigma(U) + T^{-\sigma^2/2}(U)\right)$ is the positive $\mathbb{Q}$-martingale, $\tilde{c}_i = c_i + \sigma_i\sigma_i^*$ and the subscript $i \in \{0, 1\}$ refers to the initial market state, $\varepsilon(0) = i$.

Note that if the time $\mathscr{T}_i(t) = \tau,\ 0 < \tau < U$, is given, then

$$T^{\tilde{c}}(U) \overset{d}{=} \tilde{c}_i\tau + \tilde{c}_{1-i}(U - \tau),$$

and the Wiener part $\widetilde{W}^\sigma(U)$ is distributed as the sum of two independent Gaussian parts,

$$\widetilde{W}^\sigma(U) \overset{d}{=} \sigma_i Z(\tau) + \sigma_{1-i} Z'(U - \tau), \tag{5.4}$$

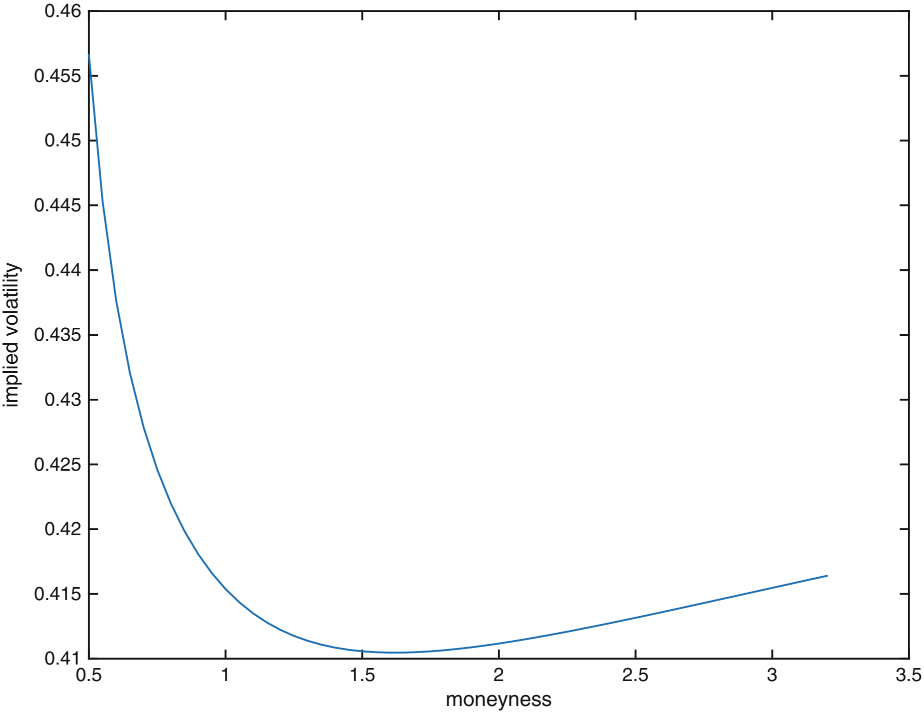

Fig. 4 $\lambda_1^* = 10, \lambda_2^* = 10,\ c_1 = -0.3, c_2 = -1.9,\ h_1 = 0.03, h_2 = 0.19,\ \sigma_1 = \sigma_2 = 0.1;\ \delta = 0.1064$

where $Z(\tau)$, $Z'(U-\tau)$ are independent and normally distributed, $Z(s), Z'(s) \sim \mathscr{N}(0, s)$, $s \in [0, U]$.

We express $\Phi(x; U)$ by using the density functions $g_i(\tau, t; n)$ of $\mathscr{T}_i(t)$.

First, note that if there are no switchings, we have

$$\begin{aligned} V_i^0 :&= \mathbb{E}_i^{\mathbb{Q}}\left[\mathbb{1}_{\{N(U)=0\}}\mathscr{H}\left(x \exp\left(T^{\tilde{c}}(U)\right) M^{\widetilde{W}}(U)\right)\right] \\ &= \mathrm{e}^{-\lambda_i^* U}\mathbb{E}^{\mathbb{Q}}\left[\mathscr{H}\left(x \exp\left(\tilde{c}_i U + \sigma_i \widetilde{B}(U) - \sigma_i^2 U/2\right)\right)\right] = \mathrm{e}^{-\lambda_i^* U}\varphi(x\mathrm{e}^{\tilde{c}_i U}, \sigma_i\sqrt{U}). \end{aligned}$$

We continue conditioning on $\{N(U) = n\}$, $n \geq 1$. From Eq. 5.3 we have

$$\begin{aligned} \Phi(x; U) &= V_i^0 + \sum_{n=1}^{\infty}\int_0^U g_i(\tau, t; n)\mathbb{E}^{\mathbb{Q}}\left[\mathscr{H}\left(x\kappa_{in}\exp\left(\tilde{c}_i\tau + \tilde{c}_{1-i}(U-\tau)\right)\mathrm{e}^{Z_i(\tau)-\Sigma_i(\tau)^2/2}\right)\right]\mathrm{d}\tau \\ &= V_i^0 + \sum_{n=1}^{\infty}\int_0^U g_i(\tau, U; n)\mathbb{E}^{\mathbb{Q}}\left[\mathscr{H}\left(\tilde{x}_{in}(\tau)\exp\left(Z_i(\tau) - \Sigma_i(\tau)^2/2\right)\right)\right]\mathrm{d}\tau. \end{aligned} \tag{5.5}$$

Here $\tau \in (0, U)$ corresponds to the time spent by the process ε_i in the state i, $i \in \{0, 1\}$, $n = N_i(U)$ is the number of jumps, $\kappa_{in} = (1+h_i)^{[(n+1)/2]}(1+h_{1-i})^{[n/2]}$ is the displacement of the asset price after n alternated jumps and $\tilde{x}_{in}(\tau) = x\kappa_{in}\exp\left(\tilde{c}_i\tau + \tilde{c}_{1-i}(U-\tau)\right)$.

Due to Eq. 5.4 the random variable $Z_i(\tau)$ is normally distributed $\mathscr{N}(0, \Sigma_i(\tau)^2)$ with the variance $\Sigma_i(\tau)^2 = \tau\sigma_i^2 + (U-\tau)\sigma_{1-i}^2$, $i \in \{0, 1\}$.

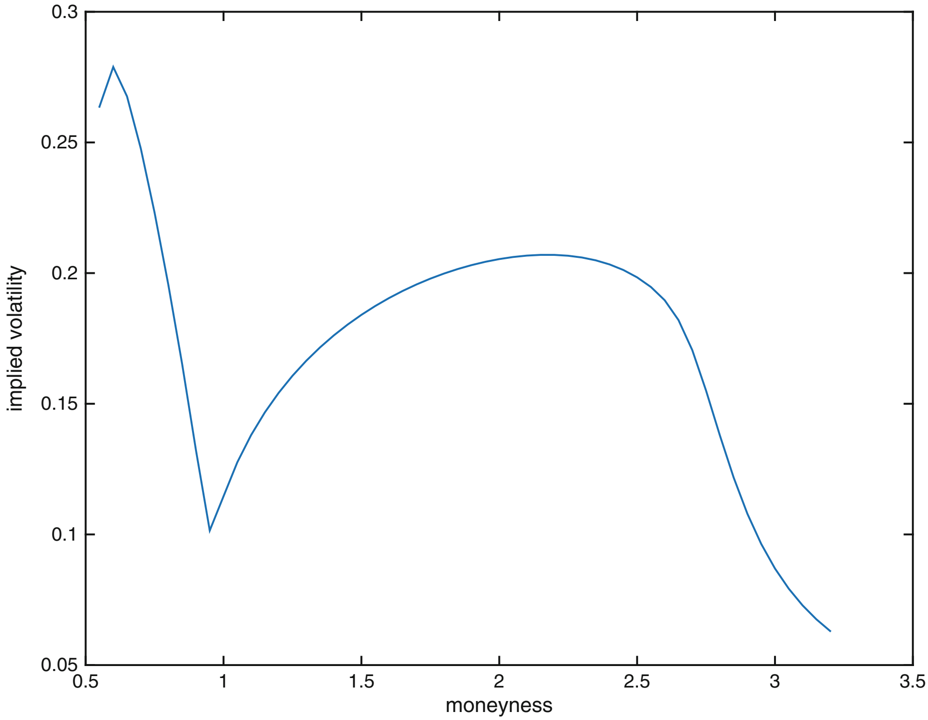

Fig. 5 $\lambda_1^* = 1, \lambda_2^* = 10,\ c_1 = -2, c_2 = 0.5,\ h_1 = 0.05, h_2 = -0.1,\ \sigma_1 = \sigma_2 = 0.01;\ \delta = 1.2626$

For the call option with strike K the payoff function is $\mathscr{H} = (x - K)^+$. The Black-Scholes call-option price is given by the formula

$$\varphi^K(x, \sigma\sqrt{U}) = x\Psi(z^+) - K\Psi(z^-), \tag{5.6}$$

where $z^{\pm} = \dfrac{\ln(x/K) \pm \sigma^2 U/2}{\sigma\sqrt{U}}$ and $\Psi(\cdot)$ is the distribution function of the standard normal distribution $\mathscr{N}(0, 1)$.

We arrive to the following result.

Theorem 3 *The call-option price is given by*

$$\Phi^K(x; U) = \mathrm{e}^{-\lambda_i^* U}\varphi^K(x\mathrm{e}^{\tilde{c}_i U}, \sigma_i\sqrt{U}) + \sum_{n=1}^{\infty}\int_0^U g_i(\tau, U; n)\varphi^K\left(\tilde{x}_{in}(\tau),\ \Sigma_i(\tau)\right)\mathrm{d}\tau, \tag{5.7}$$

where function φ^K *is defined by* Eq. 5.6.

By easy computing we can compare pricing formula (5.7) with the standard Black-Scholes pricing. Let $\mathcal{H} = (x - K)^+$ be the payoff function of the call option with strike K and $\varphi^K(x, \sigma\sqrt{U})$ be the corresponding option price with the time to maturity U, see Eq. 5.6. The implied volatility $\mathbb{IV} = \mathbb{IV}(K/x, U)$ of model (2.8) is defined as the solution with respect to σ of the equation

$$\Phi^K(x; U) = \varphi^K(x, \sigma\sqrt{U}). \tag{5.8}$$

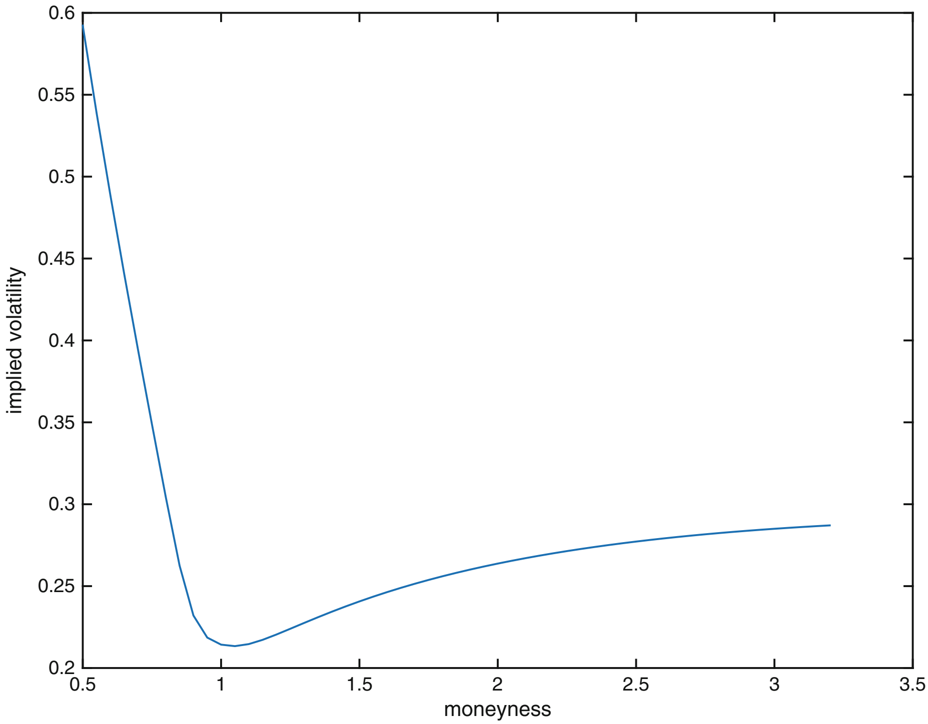

Fig. 6 $\lambda_1^* = 1, \lambda_2^* = 10,\ c_1 = -1, c_2 = 0.1,\ h_1 = 0.1, h_2 = -0.1,\ \sigma_1 = 0.05, \sigma_2 = 0.2;\ \delta = 0.9406$

The implied volatility plots (volatility smiles) with respect to the moneyness, K/x, demonstrate the difference between the jump-diffusion model with switchings and the standard Black-Scholes model.

Equation 5.8 has been resolved numerically with the different sets of the given parameters of the model. We assume the time to maturity $U = 1$, the initial asset price $x = 100$, and the strikes K to be variable, where the moneyness K/x varies from 0.5 to 3.5. It appears that the volatility smiles look different depending on various ratios of the model data.

The numerical results based on the pricing formula (5.7) and on the definition of implied volatility (5.8) are presented by the implied volatility plots, Figs. 1, 2, 3, 4, 5 and 6.

Compare the computation results for different choices of the equivalent martingale measures. First, we consider the case with the switching intensities of the equivalent martingale measure that are defined by $\lambda_i^* = -c_i/h_i$, Figs. 1 – 4, and the Radon-Nikodym derivative defined by Eq. 4.6. Figures 1 and 2 correspond to the symmetric case with the parameters $\lambda_1^* = \lambda_2^* = 10,\ c_1 = -c_2 = -1,\ h_1 = -h_2 = 0.1$ and with different diffusion coefficients. Figures 3 and 4 demonstrate the case of the asymmetric observable parameters.

Figures 5 and 6 describe the case of asymmetric parameters λ_i^* with $\lambda_i^* \neq -c_i/h_i$. We denote $\delta = \dfrac{\mathbf{IV}_{\max} - \mathbf{IV}_{\min}}{m}$, where $\mathbf{IV}_{\max},\ \mathbf{IV}_{\min}$ are the maximal/minimal implied volatility w.r.t. the moneyness and $m = (\mathbf{IV}_{\max} + \mathbf{IV}_{\min})/2$ is the medium.

We observe that in the case of $\lambda_i^* = -c_i/h_i$ (Figs. 1 – 4) the implied volatility varies less than in the case which corresponds to $\lambda_i^* \neq -c_i/h_i$ (Figs. 5 and 6). For example, plots

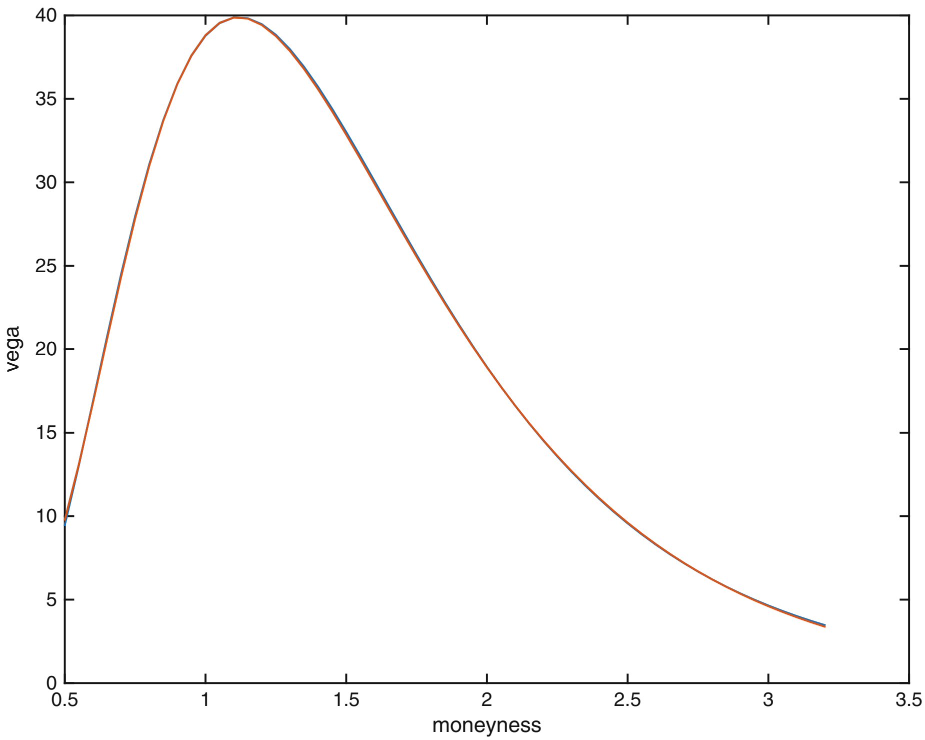

Fig. 7 Vega: $\lambda_1^* = \lambda_2^* = 10,\ c_1 = 0.3, c_2 = 1.9,\ h_1 = -0.03, h_2 = -0.19,\ \sigma_1 = \sigma_2 = 0.1$

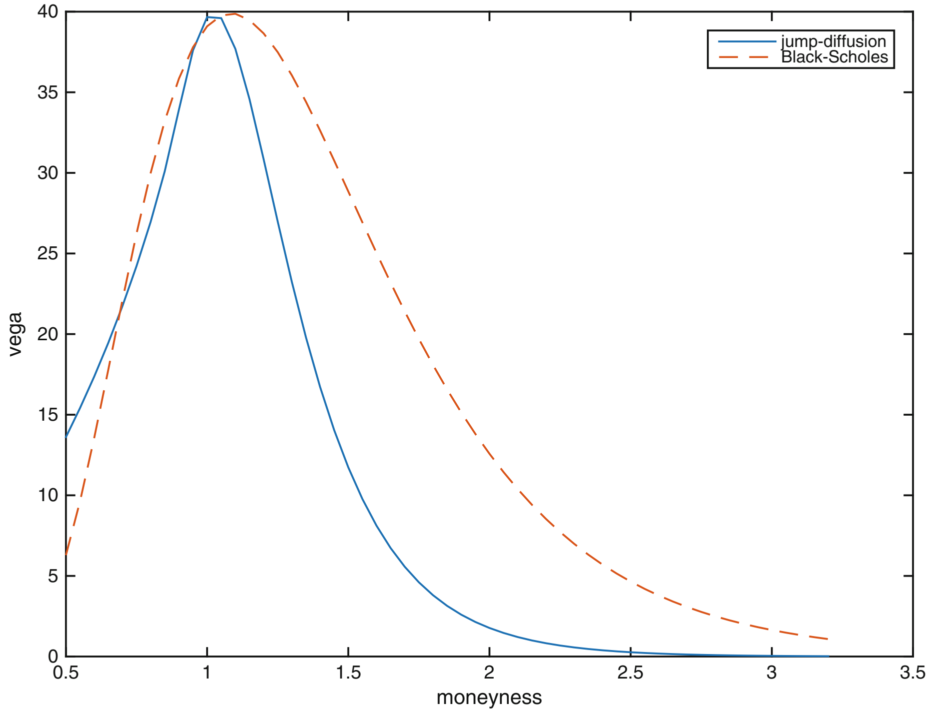

Fig. 8 Vega:$\lambda_1^* = 1, \lambda_2^* = 10,\ c_1 = -1, c_2 = 0.1,\ h_1 = 0.1, h_2 = -0.1,\ \sigma_1 = 0.05, \sigma_2 = 0.2$

of *vega* for the cases of the data of Figs. 3 and 6 are presented in Figs. 7 and 8 respectively. *Vega* for our model is defined as the derivative of Φ with respect to implied volatility, $\mathscr{V}(K/x) = \dfrac{\partial \Phi}{\partial \mathbf{N}}$. To compare we also draw the Black-Scholes *vega* (w.r.t. volatility $\sigma = m$). Observe that in Fig. 7 the curves practically coincide.

6 Conclusion

We offer some reasonable choice of equivalent martingale measures for option pricing in an incomplete market model based on the jump-diffusion process with simultaneously alternating drift parametrs, variances and jumps.

It is a common mistake to believe that the Esscher transform (4.5) always provides the minimal relative entropy. It is correct for the models based on Lévy processes, see Fujiwara and Miyahara (2003), and for the model based on the diffusion with regime switching, but with missed jump component, Elliott et al. (2005). Nevertheless, for model (2.8)-(2.9) based on Brownian motion with jumps and with switching regimes the Esscher transform does not produce the minimal relative entropy, see the detailed analysis in (Di Crescenzo and Ratanov 2015).

Some numerical verifications of this conclusion are provided on the basis of the explicit option pricing formulae (5.7) obtained in Section 5. Figures 1–6 demonstrate a rich diversity of the volatility smiles. This ensures that the simple model (2.8)-(2.9) might be useful for various practical purposes.

We also compare the plots of *vega* (see Figs. 7 and 8) for model (2.8)–(2.9) and for the Black-Scholes.

References

Bellamy N, Jeanblanc M (2000) Incompleteness of markets driven by a mixed diffusion. Finance Stochast 4:209–222

Delbaen F, Schachermayer W (1994) General version of the fundamental theorem of asset pricing. Mathematische Annalen 300:463–520

Di Crescenzo A, Martinucci B, Zacks S (2014) On the geometric Brownian motion with alternating trend. In: Perna C, Sibillo M (eds) Mathematical and Statistical Methods for Actuarial Sciences and Finance. Springer, pp 81–85

Di Crescenzo A, Ratanov N (2015) On jump-diffusion processes with regime switching: martingale approach. To appear in ALEA: Latin American Journal of Probability and Mathematical Statistics

Dybvig Ph H, Ross SA (2008) arbitrage. In: Durlauf SN, Blume LE (eds) The New Palgrave Dictionary of Economics. 2nd edn. Palgrave Macmillan

Elliott RJ, Chan L, Siu TK (2005) Option pricing and Esscher transform under regime switching. Ann Finance 1:423–432

Frittelli M (2000) The minimal entropy martingale measure and the valuation problem in incomplete markets. Math Finance 10:215–225

Fujiwara T, Miyahara Y (2003) The minimal entropy martingale measure for geometric Lévy processes. Finance Stochast 27:509–531

Jacobsen M (2006) Point process theory and applications. Marked point and piecewise deterministic processes. Birkhäuser, Berlin

Jeanblanc M, Yor M, Chesney M (2009) Mathematical methods for financial markets. Springer, Heidelberg

Kolesnik AD, Ratanov N (2013) Telegraph processes and option pricing. Springer, Heidelberg

Melnikov AV, Ratanov NE (2007) Nonhomogeneous telegraph processes and their application to financial market modeling. Doklady Math 75:115–117

Ratanov N (2007) A jump telegraph model for option pricing. Quant Finan 7:575–583

Ratanov N (2010) Option pricing model based on a Markov-modulated diffusion with jumps. Braz J Probab Stat 24:413–431

Ratanov N (2015) Telegraph processes with random jumps and complete market models. Methodol Comput Appl Prob 17:677–695. doi:10.1007/s11009-013-9388-x

Runggaldier WJ (2003) Jump diffusion models. In : Handbook of Heavy Tailed Distributions in Finance (S.T. Rachev, ed.), Handbooks in Finance, Book 1 (W.Ziemba Series Ed.), Elesevier/North-Holland, pp.169-209

Weiss GH (1994) Aspects and applications of the random walk. North-Holland, Amsterdam