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Preservation of Stochastic Orders under the Formation of Generalized Distorted Distributions. Applications to Coherent Systems

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Abstract The preservation of stochastic orders under the formation of coherent systems is a relevant topic in the reliability theory. Several properties have been obtained under the assumption of identically distributed components. In this paper we obtain ordering preservation results for generalized distorted distributions (GDD) which, in particular, can be used to obtain preservation results for coherent systems with non-identically distributed components. We consider both the cases of independent and dependent components. The preservation results obtained here for GDD can also be applied to other statistical concepts.

Keywords Stochastic orders · Coherent systems · Order statistics · Copulas

Mathematics Subject Classification (2010) 62K10 · 60E15 · 90B25

1 Introduction

The distorted distributions were firstly introduced in the dual theory of choice under risk (see Quiggin 1982 and Yaari 1987). Then they were applied to different economic models, order statistics and coherent systems (see Wang 1996; Wang and Young 1998; Belzunce et al. 2001; Hürlimann 2004; Khaledi and Shaked 2010;

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Sordo and Suarez-Llorens 2011; Navarro et al. 2013, 2014, 2015; Gupta and Kumar 2014). The *Distorted Distribution* (DD) associated to a distribution function (DF) F and to a nondecreasing continuous *distortion function* $q : [0, 1] \to [0, 1]$ such that $q(0) = 0$ and $q(1) = 1$, is defined by

$$F_q(t) = q(F(t)). \tag{1.1}$$

If q is strictly increasing, then F and F_q have the same support. For the reliability functions (RF) $\overline{F} = 1 - F$, $\overline{F}_q = 1 - F_q$, we have a similar expression

$$\overline{F}_q(t) = \overline{q}(\overline{F}(t)), \tag{1.2}$$

where $\overline{q}(u) = 1 - q(1 - u)$ is the *dual distortion function*, see Hürlimann (2004). The function $\overline{q}$ is also a distortion function, that is, it satisfies the same properties as q.

The distorted distribution model is a very flexible model which can be used to study different concepts in a unified way. Conditions for the preservation of stochastic orders under the formation of distorted distributions were obtained in Navarro et al. (2013) and Gupta and Kumar (2014). The preservation of stochastic aging classes was studied in Navarro et al. (2014). There, the distorted distributions were extended to the concept of Generalized Distorted Distributions (GDD) which are univariate distribution functions obtained by *distorting n* distribution functions (see next section).

In this paper, we extend the preservation results for stochastic orders given in Navarro et al. (2013) to GDD (Section 2). These results are used to obtain preservation properties for general coherent systems and order statistics (Section 3). Some conclusions are given in Section 4 where we remark that the results obtained here can also be applied to other statistical concepts.

Throughout the paper, increasing and decreasing mean nondecreasing and nonincreasing, respectively.

2 Preservation Results for Generalized Distorted Distributions

The *generalized distorted distribution* (GDD) associated to n distribution functions (DF) $F_1, \ldots, F_n$ and to an increasing continuous *multivariate distortion function* (MDF) $Q : [0, 1]^n \to [0, 1]$ such that $Q(0, \ldots, 0) = 0$ and $Q(1, \ldots, 1) = 1$ is defined by

$$F_Q(t) = Q(F_1(t), \ldots, F_n(t)). \tag{2.1}$$

Obviously, the function F_Q defined above is always a proper (univariate) distribution function (it is increasing, right-continuous and satisfies $F_Q(-\infty) = Q(0, \ldots, 0) = 0$ and $F_Q(\infty) = Q(1, \ldots, 1) = 1$). Moreover, if Q is strictly increasing in each variable and $F_1, \ldots, F_n$ have the same support S, then F_Q also has the same support S. We have a similar expression for the respective reliability functions (RF)

$$\overline{F}_Q(t) = \overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t)), \tag{2.2}$$

where $\overline{F}_i = 1 - F_i$, $\overline{F}_Q = 1 - F_Q$ and where

$$\overline{Q}(u_1, \ldots, u_n) = 1 - Q(1 - u_1, \ldots, 1 - u_n)$$

is the *multivariate dual distortion function*. The function $\overline{Q}$ is also a multivariate distortion function, that is, it satisfies the same properties as Q. Hence the function $\overline{F}_Q$ defined above is always a proper (univariate) reliability function. Note that Q determines $\overline{Q}$ and vice versa. These two representations are equivalent but, sometimes, it is better to use Eq. 2.2 instead of Eq. 2.1.

The concept of multivariate distortion function is related to the concepts of aggregation function, triangular-norm and semi-copula. In the multivariate case, distortions of probability measures can give rise to capacities and they are related to multi-attribute target-based utility functions as can be seen in Fantozzi and Spizzichino (2015) and the references therein.

Of course, if $n = 1$ (or if $F_1 = \cdots = F_n$), then we obtain the DD defined in Eqs. 1.1 and 1.2. Note that representations in Eqs. 2.1 and 2.2 are similar to copula representations for multivariate distributions. Actually the copulas are valid distortion functions. However, note that representations in Eqs. 2.1 and 2.2 define univariate distribution and reliability functions, respectively, and that the distortion functions are not necessarily copulas. We will see some examples in the next section.

We are going to study two kinds of preservation results. The first ones are for GDD based on the same baseline DF $F_1, \ldots, F_n$ and on different MDF Q_1 and Q_2, that is, we study conditions on Q_1 and Q_2 to get

$$F_{Q_1} \leq_{ORD} F_{Q_2}$$

for a given stochastic order ORD and for any DF $F_1, \ldots, F_n$. The second ones are for GDD based on the same MDF Q and different (ordered) baseline DF $F_1, \ldots, F_n$ and $G_1, \ldots, G_n$, that is, we study conditions on Q to get

$$F_Q \leq_{ORD} G_Q,$$

where F_Q is defined by Eq. 2.1 and $G_Q(t) = Q(G_1(t), \ldots, G_n(t))$, for a given stochastic order ORD and for any DF $F_1, \ldots, F_n, G_1, \ldots, G_n$ such that $F_i \leq_{ORD} G_i$ for $i = 1, \ldots, n$. Of course, these two kinds of results can be combined to study ordering properties for GDD based on different MDF and different baseline DF.

We shall study the following stochastic orders. Their basic properties can be seen in Shaked and Shanthikumar (2007). Let F and G be the DF of two random variables X and Y with respective reliability functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$. Then:

(i) F is said to be smaller than G in the usual stochastic order (denoted by $F \leq_{ST} G$ or by $X \leq_{ST} Y$) if $\overline{F}(t) \leq \overline{G}(t)$ for all t.
(ii) F is said to be smaller than G in the hazard rate order (denoted by $F \leq_{HR} G$ or by $X \leq_{HR} Y$) if $\overline{G}(t)/\overline{F}(t)$ is increasing in t.
(iii) F is said to be smaller than G in the reversed hazard rate order (denoted by $F \leq_{RHR} G$ or by $X \leq_{RHR} Y$) if $G(t)/F(t)$ is increasing in t.
(iv) If F and G are absolutely continuous with respective probability density functions (PDF) f and g, then F is said to be smaller than G in the likelihood ratio order (denoted by $F \leq_{LR} G$ or by $X \leq_{LR} Y$) if $g(t)/f(t)$ is increasing in t in the union of their supports.

If F and G are absolutely continuous distributions, then $F \leq_{HR} G$ is equivalent to the ordering of their respective hazard (or failure) rate (HR) functions $h_F \geq h_G$, where $h_F = f/\overline{F}$ and $h_G = g/\overline{G}$. Analogously, $F \leq_{RHR} G$ is equivalent to the ordering of their respective reversed hazard rate (RHR) functions $\overline{h}_F \leq \overline{h}_G$, where $\overline{h}_F = f/F$ and $\overline{h}_G = g/G$. These functions are used to define the following aging classes. A DF F is said to be IHR or IFR (DHR or DFR) if h_F is increasing (decreasing). A DF F is said to be IRHR (DRHR) if $\overline{h}_F$ is increasing (decreasing). The stochastic orders defined above are related as follows:

$$\begin{array}{ccc} X \leq_{LR} Y & \Rightarrow & X \leq_{HR} Y \\ \Downarrow & & \Downarrow \\ X \leq_{RHR} Y & \Rightarrow & X \leq_{ST} Y. \end{array} \tag{2.3}$$

Now we are ready to obtain the new properties. First we need the following lemma. Throughout the paper we use the notation

$$D_i\Phi(x_1,\ldots,x_n) = \frac{\partial}{\partial x_i}\Phi(x_1,\ldots,x_n)$$

for the partial derivative of any differentiable function $\Phi : S \subseteq \mathbb{R}^n \to \mathbb{R}$. Also we define the associated functions

$$\alpha_i^{\Phi}(u_1,\ldots,u_n) = \frac{u_i D_i\Phi(u_1,\ldots,u_n)}{\Phi(u_1,\ldots,u_n)} \tag{2.4}$$

for $i = 1,\ldots,n$ and the following class of distortion functions:

$$\mathcal{D}_1 = \{\Phi : \alpha_i^{\Phi}(u_1,\ldots,u_n) \text{ is decreasing in } (0,1)^n, i = 1,\ldots,n\}.$$

Lemma 2.1 *Let F_Q be a GDD based on a differentiable MDF Q and on absolutely continuous distribution functions $F_1,\ldots,F_n$ with PDF $f_1,\ldots,f_n$. Then:*

(i) *F_Q is absolutely continuous with PDF f_Q given by*

$$f_Q(t) = \sum_{i=1}^n f_i(t)D_iQ(F_1(t),\ldots,F_n(t)) = \sum_{i=1}^n f_i(t)D_i\overline{Q}(\overline{F}_1(t),\ldots,\overline{F}_n(t)). \tag{2.5}$$

(ii) *The hazard rate function h_Q of F_Q is given by*

$$h_Q(t) = \sum_{i=1}^n h_i(t)\alpha_i^{\overline{Q}}(\overline{F}_1(t),\ldots,\overline{F}_n(t)), \tag{2.6}$$

where $h_i = f_i/\overline{F}_i$ is the hazard rate of F_i for $i = 1,\ldots,n$.

(iii) *The reversed hazard rate function $\overline{h}_Q = f_Q/F_Q$ of F_Q is given by*

$$\overline{h}_Q(t) = \sum_{i=1}^n \overline{h}_i(t)\alpha_i^{Q}(F_1(t),\ldots,F_n(t)), \tag{2.7}$$

where $\overline{h}_i = f_i/F_i$ is the reversed hazard rate of F_i for $i = 1,\ldots,n$.

The proof is straightforward. Throughout the paper, every time we use f_Q, h_Q or $\overline{h}_Q$, we shall assume that $F_1,\ldots,F_n$ are absolutely continuous and that Q is differentiable.

Let us start with the conditions for the preservation properties for GDD based on the same baseline DF. These properties generalize the results obtained in Theorem 2.4 of Navarro et al. (2013).

Proposition 2.2 *Let* $F_{Q_1} = Q_1(F_1, \ldots, F_n)$ *and* $F_{Q_2} = Q_2(F_1, \ldots, F_n)$ *be two GDD based on* $F_1, \ldots, F_n$*. Then:*

(i) $F_{Q_1} \leq_{ST} F_{Q_2}$ *for all* $F_1, \ldots, F_n$ *if and only if* $Q_1 \geq Q_2$ *in* $(0, 1)^n$.
(ii) $F_{Q_1} \leq_{HR} F_{Q_2}$ *for all* $F_1, \ldots, F_n$ *if and only if* $\overline{Q}_2/\overline{Q}_1$ *is decreasing in* $(0, 1)^n$.
(iii) $F_{Q_1} \leq_{HR} F_{Q_2}$ *for all* $F_1, \ldots, F_n$ *if* $\alpha_i^{\overline{Q}_1} \geq \alpha_i^{\overline{Q}_2}$ *in* $(0, 1)^n$ *for* $i = 1, \ldots, n$.
(iv) $F_{Q_1} \leq_{RHR} F_{Q_2}$ *for all* $F_1, \ldots, F_n$ *if and only if* Q_2/Q_1 *is increasing in* $(0, 1)^n$.
(v) $F_{Q_1} \leq_{RHR} F_{Q_2}$ *for all* $F_1, \ldots, F_n$ *if* $\alpha_i^{Q_1} \leq \alpha_i^{Q_2}$ *in* $(0, 1)^n$ *for* $i = 1, \ldots, n$.

The proofs are straightforward from Eqs. 2.1, 2.2, 2.4, 2.6 and 2.7. Next we obtain preservation properties for GDD based on the same MDF and on different (ordered) baseline DF. These properties generalize the results obtained in Theorem 2.6 of Navarro et al. (2013).

Proposition 2.3 *Let* $F_Q = Q(F_1, \ldots, F_n)$ *and* $G_Q = Q(G_1, \ldots, G_n)$ *be two GDD based on the same MDF* Q *and on the DF* $F_1, \ldots, F_n$ *and* $G_1, \ldots, G_n$*, respectively.*

(i) *If* $F_i \leq_{ST} G_i$ *for* $i = 1, \ldots, n$*, then* $F_Q \leq_{ST} G_Q$ *for all MDF* Q.
(ii) *If* $F_i \leq_{HR} G_i$ *for* $i = 1, \ldots, n$*, then* $F_Q \leq_{HR} G_Q$ *for all MDF* $\overline{Q} \in \mathcal{D}_1$.
(iii) *If* $F_i \leq_{RHR} G_i$ *for* $i = 1, \ldots, n$*, then* $F_Q \leq_{RHR} G_Q$ *for all MDF* $Q \in \mathcal{D}_1$.

The proofs are straightforward from Eqs. 2.1, 2.3, 2.6 and 2.7. Note that the ST order is always preserved but that we need some (quite strong) conditions for the preservation of the HR and RHR orders. Moreover, note that if we just want to compare $F_Q = Q(F_1, F_2, \ldots, F_n)$ and $G_Q = Q(G_1, F_2, \ldots, F_n)$ in the HR order, then we just need $F_1 \leq_{HR} G_1$ and that $\alpha_i^{\overline{Q}}$ is decreasing in u_1 in the set $(0, 1)^n$ for $i = 1, \ldots, n$. A similar property holds if we just change the distribution in the position j or the distributions in some specific positions.

The conditions for the LR order are more complicated. They are stated in the following proposition. Also similar properties are obtained for the HR and RHR orders. First, we introduce some notation. If $\Phi : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, then we define the associated functions

$$\beta^{\Phi}(u_1, \ldots, u_n, v_1, \ldots, v_n) = \frac{\Phi(u_1 v_1, \ldots, u_n v_n)}{\Phi(u_1, \ldots, u_n)} \tag{2.8}$$

and

$$\begin{aligned} &\gamma^{\Phi}(u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_n, z_1, \ldots, z_n) \\ &= \frac{w_1 z_1 u_1 D_1 \Phi(u_1 v_1, \ldots, u_n v_n) + \ldots + w_n z_n u_n D_n \Phi(u_1 v_1, \ldots, u_n v_n)}{z_1 u_1 D_1 \Phi(u_1, \ldots, u_n) + \ldots + z_n u_n D_n \Phi(u_1, \ldots, u_n)}. \end{aligned} \tag{2.9}$$

Moreover we define the following classes of distortion functions:

$$\mathcal{D}_2 = \{\Phi : \beta^{\Phi}(u_1, \ldots, u_n, v_1, \ldots, v_n) \text{ is decreasing for } u_i \in (0, 1) \text{ and } v_i \in (1, \infty), i = 1, \ldots, n\}$$

and

$$\mathcal{D}_3 = \{\Phi : \beta^{\Phi}(u_1, \ldots, u_n, v_1, \ldots, v_n) \text{ is increasing for } u_i \in (0, 1) \text{ and } v_i \in (0, 1), i = 1, \ldots, n\}.$$

Proposition 2.4 *Let* $F_Q = Q(F_1, \ldots, F_n)$ *and* $G_Q = Q(G_1, \ldots, G_n)$ *be two GDD based on the MDF* Q *and on the DF* $F_1, \ldots, F_n$ *and* $G_1, \ldots, G_n$*, respectively.*

(i) *If* $F_i \leq_{HR} G_i$ *for* $i = 1, \ldots, n$*, then* $F_Q \leq_{HR} G_Q$ *for all MDF* $\overline{Q} \in \mathcal{D}_2$.

(ii) *If $F_i \le_{RHR} G_i$ for $i = 1, \ldots, n$, then $F_Q \le_{RHR} G_Q$ for all MDF $Q \in \mathcal{D}_3$.*

(iii) *If $F_i \le_{LR} G_i$ and F_i is IHR (DHR) for $i = 1, \ldots, n$, then $F_Q \le_{LR} G_Q$ for all MDF Q such that $\gamma^{\overline{Q}}$ defined in* Eq. 2.9 *is decreasing in $u_1, \ldots, u_n$, is increasing in $v_1, \ldots, v_n, w_1, \ldots, w_n$ and is increasing (decreasing) in z_i for $i = 1, \ldots, n$ in the set* $(0,1)^n \times (1,\infty)^n \times (0,\infty)^{2n}$.

(iv) *If $F_i \le_{LR} G_i$ and G_i is IHR (DHR) for $i = 1, \ldots, n$, then $F_Q \le_{LR} G_Q$ for all MDF Q such that $\gamma^{\overline{Q}}$ defined in* Eq. 2.9 *is increasing in $u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_n$ and is decreasing (increasing) in z_i for $i = 1, \ldots, n$ in the set* $(0,1)^{2n} \times (0,\infty)^{2n}$.

(v) *If $F_i \le_{LR} G_i$ and F_i is DRHR for $i = 1, \ldots, n$, then $F_Q \le_{LR} G_Q$ for all MDF Q such that γ^{Q} defined in* Eq. 2.9 *is increasing in $u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_n$ and is decreasing in z_i for $i = 1, \ldots, n$ in the set* $(0,1)^{2n} \times (0,\infty)^{2n}$.

Proof To prove (i) note that if $F_i \le_{HR} G_i$, then $\overline{G}_i/\overline{F}_i$ is increasing for $i = 1, \ldots, n$. Then $\overline{G}_i/\overline{F}_i \ge 1$ for $i = 1, \ldots, n$. Therefore, from Eq. 2.8, we have

$$\frac{\overline{G}_Q(t)}{\overline{F}_Q(t)} = \beta^{\overline{Q}}\left(\overline{F}_1(t), \ldots, \overline{F}_n(t), \frac{\overline{G}_1(t)}{\overline{F}_1(t)}, \ldots, \frac{\overline{G}_n(t)}{\overline{F}_n(t)}\right)$$

and, as $\overline{F}_i$ is decreasing and $\overline{Q} \in \mathcal{D}_2$, then $\overline{G}_Q/\overline{F}_Q$ is increasing and $F_Q \le_{HR} G_Q$ holds. Note that $G_i/F_i \le 1$ for $i = 1, \ldots, n$.

To prove (ii) note that if $F_i \le_{RHR} G_i$, then G_i/F_i is increasing for $i = 1, \ldots, n$. Then $G_i/F_i \le 1$ for $i = 1, \ldots, n$. Therefore, from Eq. 2.8, we have

$$\frac{G_Q(t)}{F_Q(t)} = \beta^{Q}\left(F_1(t), \ldots, F_n(t), \frac{G_1(t)}{F_1(t)}, \ldots, \frac{G_n(t)}{F_n(t)}\right)$$

and, as F_i is increasing and $Q \in \mathcal{D}_3$, then G_Q/F_Q is increasing and $F_Q \le_{RHR} G_Q$ holds.

Finally, to prove (iii) note that if $F_i \le_{LR} G_i$, then g_i/f_i and $\overline{G}_i/\overline{F}_i$ are increasing for $i = 1, \ldots, n$. Then $\overline{G}_i/\overline{F}_i \ge 1$ for $i = 1, \ldots, n$. Therefore, from Eqs. 2.5 and 2.9, we have

$$\frac{g_Q(t)}{f_Q(t)} = \gamma^{\overline{Q}}\left(\overline{F}_1(t), \ldots, \overline{F}_n(t), \frac{\overline{G}_1(t)}{\overline{F}_1(t)}, \ldots, \frac{\overline{G}_n(t)}{\overline{F}_n(t)}, \frac{g_1(t)}{f_1(t)}, \ldots, \frac{g_n(t)}{f_n(t)}, h_1^F(t), \ldots, h_n^F(t)\right)$$

and, as $\overline{F}_i$ is decreasing and $h_i^F = f_i/\overline{F}_i$ is increasing (decreasing) for $i = 1, \ldots, n$, then g_Q/f_Q is increasing and $F_Q \le_{LR} G_Q$ holds.

The proofs of (iv) and (v) are similar to that of (iii). □

Remark 2.5 Note that in (i) it is enough to have the property stated in $\mathcal{D}_2$ for $u_i \in (0, 1/v_i)$ and $v_i \in (1,\infty)$. Note that the assumptions about aging classes IHR, DHR and DRHR in (iii)-(v) are crucial for the preservation of the LR order since the hazard rate functions appear in the representation of the ratio g_Q/f_Q in term of the gamma functions (as variables $z_1, \ldots, z_n$). So, if $F_1, \ldots, F_n$ are exponential distributions, then we do not need the conditions about $z_1, \ldots, z_n$ for the preservation of the LR order (since the hazard rate functions are constant). These facts will be used to obtain Propositions 3.1 and 3.2.

Finally, we obtain specific conditions to compare GDD when we only change one of the baseline distributions. Without loss of generality, we can assume that we change the first ones. Some applications of these results are given in the next section. Again we need some previous notation. If $\Phi : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, then we define the associated functions

$$\delta^{\Phi}(u_1, v_1, \ldots, v_n) = \frac{\Phi(u_1 v_1, u_1 v_2, \ldots, u_1 v_n)}{\Phi(u_1, u_1 v_2, \ldots, u_1 v_n)} \tag{2.10}$$

and

$$\lambda^{\Phi}(u_1, v_1, \ldots, v_n, w_1, \ldots, w_n) = \frac{w_1 D_1 \Phi(u_1 v_1, \ldots, u_1 v_n) + \ldots + w_n D_n \Phi(u_1 v_1, \ldots, u_1 v_n)}{D_1 \Phi(u_1, u_1 v_2, \ldots, u_1 v_n) + \cdots + w_n D_n \Phi(u_1, u_1 v_2, \ldots, u_1 v_n)}. \tag{2.11}$$

Proposition 2.6 *Let $F_Q = Q(F_1, F_2, \ldots, F_n)$ and $G_Q = Q(G_1, F_2, \ldots, F_n)$ be two GDD based on the MDF Q and on the DF $F_1, F_2, \ldots, F_n$ and $G_1, F_2, \ldots, F_n$, respectively.*

- (i) *If $F_1 \leq_{HR} G_1$ and $F_1 \geq_{HR} F_i$ (resp. $\leq_{HR}$) for $i = 2, \ldots, n$, then $F_Q \leq_{HR} G_Q$ for all MDF Q such that $\delta^{\overline{Q}}$ is decreasing in u_1 and is decreasing (resp. increasing) in v_i for $i = 2, \ldots, n$ in the set $(0, 1) \times (1, \infty) \times (0, 1)^{n-1}$ (resp. $(0, 1) \times (1, \infty)^n$).*
- (ii) *If $F_1 \leq_{RHR} G_1$ and $F_1 \leq_{RHR} F_i$ (resp. $\geq_{RHR}$) for $i = 2, \ldots, n$, then $F_Q \leq_{RHR} G_Q$ for all MDF Q such that δ^{Q} is increasing in u_1 and is increasing (resp. decreasing) in v_i for $i = 2, \ldots, n$ in the set $(0, 1)^{n+1}$ (resp. $(0, 1)^2 \times (1, \infty)^{n-1}$).*
- (iii) *If $F_1 \leq_{LR} G_1$ and $F_1 \leq_{LR} F_i$ (resp. $\geq_{LR}$) for $i = 2, \ldots, n$, then $F_Q \leq_{LR} G_Q$ for all MDF Q such that $\lambda^{\overline{Q}}$ is decreasing in u_1, is increasing in v_1 and is increasing (resp. decreasing) in v_i and w_i for $i = 2, \ldots, n$ in the set $(0, 1) \times (1, \infty)^n \times (0, \infty)^n$ (resp. $(0, 1) \times (1, \infty) \times (0, 1)^{n-1} \times (0, \infty)^n$).*
- (iv) *If $F_1 \leq_{LR} G_1$ and $F_1 \leq_{LR} F_i$ (resp. $\geq_{LR}$) for $i = 2, \ldots, n$, then $F_Q \leq_{LR} G_Q$ for all MDF Q such that λ^{Q} is increasing in u_1 and v_1 and is increasing (resp. decreasing) in v_i and w_i for $i = 2, \ldots, n$ in the set $(0, 1)^{n+1} \times (0, \infty)^n$ (resp. $(0, 1)^2 \times (1, \infty)^{n-1} \times (0, \infty)^n$).*

Proof To prove (i) we first note that

$$\frac{\overline{G}_Q(t)}{\overline{F}_Q(t)} = \delta^{\overline{Q}}\left(\overline{F}_1(t), \frac{\overline{G}_1(t)}{\overline{F}_1(t)}, \frac{\overline{F}_2(t)}{\overline{F}_1(t)}, \ldots, \frac{\overline{F}_n(t)}{\overline{F}_1(t)}\right).$$

Also note that $\delta^{\overline{Q}}(u_1, v_1, \ldots, v_n)$ is increasing in v_1. Hence, as $F_1 \leq_{HR} G_1$ implies that $\overline{G}_1/\overline{F}_1$ is increasing, then $\delta^{\overline{Q}}(u_1, \overline{G}_1(t)/\overline{F}_1(t), \ldots, v_n)$ is increasing in t.

Analogously, if we assume that $F_1 \geq_{HR} F_i$ ($\leq_{HR}$) and that $\delta^{\overline{Q}}(u_1, v_1, \ldots, v_n)$ is decreasing (increasing) in v_i for a fixed $i \in \{2, \ldots, n\}$, then $\overline{F}_i/\overline{F}_1$ is decreasing (increasing) and $\delta^{\overline{Q}}(u_1, v_1, \ldots, v_{i-1}, \overline{F}_i(t)/\overline{F}_1(t), v_{i+1}, \ldots, v_n)$ is increasing in t. This last property is true for all $i = 2, \ldots, n$.

Finally, if $\delta^{\overline{Q}}(u_1, v_1, \ldots, v_n)$ is decreasing in u_1, then $\delta^{\overline{Q}}(\overline{F}_1(t), v_1, \ldots, v_n)$ is increasing in t. From this last property and the preceding ones, it follows that $\overline{G}_Q/\overline{F}_Q$ increasing in t, that is, $F_Q \leq_{HR} G_Q$.

The proof of (ii) is similar to that of (i).

To prove (iii) we first note that, from Eqs. 2.5 and 2.11, we have

$$\frac{g_Q(t)}{f_Q(t)} = \lambda^{\overline{Q}}\left(\overline{F}_1(t), \frac{\overline{G}_1(t)}{\overline{F}_1(t)}, \frac{\overline{F}_2(t)}{\overline{F}_1(t)}, \ldots, \frac{\overline{F}_n(t)}{\overline{F}_1(t)}, \frac{g_1(t)}{f_1(t)}, \frac{f_2(t)}{f_1(t)}, \ldots, \frac{f_n(t)}{f_1(t)}\right).$$

Then, if $\lambda^{\overline{Q}}(u_1, v_1, \ldots, v_n, w_1, \ldots, w_n)$ is decreasing in u_1, then

$$\lambda^{\overline{Q}}(\overline{F}_1(t), v_1, \ldots, v_n, w_1, \ldots, w_n)$$

is increasing in t. Also note that if $F_1 \leq_{LR} G_1$, then g_1/f_1 and $\overline{G}_1/\overline{F}_1$ are increasing, then

$$\lambda^{\overline{Q}}\left(u_1, \frac{\overline{G}_1(t)}{\overline{F}_1(t)}, v_2, \dots, v_n, \frac{g_1(t)}{f_1(t)}, w_2, \dots, w_n\right)$$

is increasing in t whenever $\lambda^{\overline{Q}}$ is increasing in v_1 ($\lambda^{\overline{Q}}$ is always increasing in w_1).

Analogously, if $F_1 \leq_{LR} F_i$ ($\geq_{LR}$) for a fixed $i \in \{2, \dots, n\}$, then f_i/f_1 and $\overline{F}_i/\overline{F}_1$ are increasing (decreasing). Therefore

$$\lambda^{\overline{Q}}\left(u_1, v_1, \dots, v_{i-1}, \frac{\overline{F}_i(t)}{\overline{F}_1(t)}, v_{i+1}, \dots, v_n, w_1, \dots, w_{i-1}, \frac{f_i(t)}{f_1(t)}, w_{i+1} \dots, w_n\right)$$

is increasing in t whenever $\lambda^{\overline{Q}}$ is increasing (decreasing) in v_i and w_i. This last property is true for all $i = 2, \dots, n$. From this last property and the preceding ones, it follows that g_Q/f_Q increasing in t, that is, $F_Q \leq_{LR} G_Q$.

The proof of (iv) is similar to that of (iii). □

Remark 2.7 Note that, in the preceding proposition, if $\delta^{\overline{Q}}(u_1, v_1, \dots, v_n)$ is constant in v_i for a given $i \in \{2, \dots, n\}$, then we do not need the condition about F_i (i.e., $F_1 \geq_{HR} F_i$ or $F_1 \leq_{HR} F_i$). Also note that $\delta^{\overline{Q}}(u_1, v_1, \dots, v_n)$ is decreasing in v_i for a $i \in \{2, \dots, n\}$ if, and only if,

$$\frac{\overline{Q}(u_1, u_1v_2, \dots, u_1v_n)}{\overline{Q}(u_1v_1, u_1v_2, \dots, u_1v_n)} \frac{D_i\overline{Q}(u_1v_1, u_1v_2, \dots, u_1v_n)}{D_i\overline{Q}(u_1, u_1v_2, \dots, u_1v_n)} \leq 1. \tag{2.12}$$

If $v_1 > 1$ and $u_1 \in (0, 1)$, then $u_1 < u_1v_1$ and

$$\overline{Q}(u_1, u_1v_2, \dots, u_1v_n) \leq \overline{Q}(u_1v_1, u_1v_2, \dots, u_1v_n).$$

Analogously, if we assume that $D_i\overline{Q}(u_1, \dots, u_n)$ is decreasing in u_1, then

$$D_i\overline{Q}(u_1v_1, u_1v_2, \dots, u_1v_n) \leq D_i\overline{Q}(u_1, u_1v_2, \dots, u_1v_n)$$

and Eq. 2.12 holds. Hence $\delta^{\overline{Q}}(u_1, v_1, \dots, v_n)$ is decreasing in v_i. Analogously, it can be proved that if $D_iQ(u_1, \dots, u_n)$ is decreasing in u_1, then $\delta^Q(u_1, v_1, \dots, v_n)$ is increasing in v_i. These results will be used in Example 3.6.

3 Applications to Coherent Systems and Order Statistics

Let us consider a coherent system with lifetime $T = \psi(X_1, \dots, X_n)$ based on possibly dependent components with lifetimes $X_1, \dots, X_n$. Let us assume that $X_1, \dots, X_n$ have DF $F_i(t) = \Pr(X_i \leq t)$ and RF $\overline{F}_i(t) = \Pr(X_i > t)$ for $i = 1, \dots, n$. The k-out-of-n systems are systems which work when at least k of their n component work. Their lifetimes are the order statistics $X_{1:n}, \dots, X_{n:n}$ obtained from $X_1, \dots, X_n$. In particular, $X_{1:n} = \min(X_1, \dots, X_n)$ and $X_{n:n} = \max(X_1, \dots, X_n)$ represent the series and parallel system lifetimes, respectively.

We assume that the component lifetimes $X_1, \dots, X_n$ can be dependent or independent and that this possible dependence will be represented by the joint reliability (or survival) function of $(X_1, \dots, X_n)$

$$\overline{F}(t_1, \dots, t_n) = \Pr(X_1 > t_1, \dots, X_n > t_n).$$

This function can be written using its copula representation as

$$\overline{F}(t_1,\ldots,t_n) = K(\overline{F}_1(t_1),\ldots,\overline{F}_n(t_n)), \tag{3.1}$$

where K is the *survival copula* (K is a multivariate distribution function with uniform marginals in (0, 1)). If the components are independent, then K is the product copula. This representation is very convenient in this context since the different kinds of components are represented by the component reliability functions $\overline{F}_i$ and the dependence between the components is represented by the survival copula K. A representation similar to Eq. 3.1 holds for the distribution functions (with the usual or distribution copula C).

It is well known (see, e.g., Barlow and Proschan 1975, p. 12) that the lifetime of a coherent system can be written as $T = \max_{j=1,\ldots,r} X_{P_j}$, where $X_P = \min_{i\in P} X_i$ is the lifetime of the series system with components in P for all $P \subseteq \{1,\ldots,n\}$ and $P_1,\ldots,P_r$ are the minimal path sets of the system. A *path set* is a set of indices P such that if all the components in P work, then the system works. A *minimal path set* is a path set which does not contain other path sets. For example, the minimal path sets of the coherent system with lifetime $T = \min(X_1, \max(X_2, X_3))$ are $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$. The minimal path sets only depend on the system structure function.

The system reliability function $\overline{F}_T(t) = \Pr(T > t)$ can be obtained by using the above minimal path set representation, the inclusion-exclusion formula and the copula representation given in Eq. 3.1 as

$$\overline{F}_T(t) = H(\overline{F}_1(t),\ldots,\overline{F}_n(t)) \tag{3.2}$$

(see, e.g., Navarro and Spizzichino 2010; Navarro et al. 2011, 2014), where H is a function which depends on the minimal path sets $P_1,\ldots,P_r$ (the system structure) and on the survival copula K (the dependence between the components' lifetimes) but it does not depend on $\overline{F}_1,\ldots,\overline{F}_n$. Moreover, H is an increasing continuous function from $[0, 1]^n$ to $[0, 1]$ such that $H(0,\ldots,0) = 0$ and $H(1,\ldots,1) = 1$. Therefore, the system distribution is a GDD from the components' distributions with multivariate dual distortion function $\overline{Q} = H$. In the case of independent components, K is the product copula and the function H is a multivariate polynomial called *reliability function of the structure* (see Barlow and Proschan 1975, p. 21) or *domination polynomial* (see Satyanarayana and Prabhakar 1978). In the general case, the function H was called *domination function* in Navarro and Spizzichino (2010) and *structure-dependence function* in Navarro et al. (2011). A representation similar to Eq. 3.2 holds for the distribution functions with a different distortion function (see, e.g., Navarro et al. 2015).

The representation formula presented here is based on the minimal path sets of the system but the same object can be described in terms of different notions (such as aggregation functions, utilities or capacities) as can be seen in the alternative expressions for this representation obtained in Dukhovny and Marichal (2012), Gandy (2013) and Fantozzi and Spizzichino (2015). For an interesting study concerning aggregation functions, one can see, e.g., the paper Kolesarova et al. (2012) and references given therein.

In particular, for the series system $X_{1:n}$, we have $H = \overline{Q} = K$, that is,

$$\overline{F}_{1:n}(t) = K(\overline{F}_1(t),\ldots,\overline{F}_n(t)). \tag{3.3}$$

Analogously, for the parallel system $X_{n:n}$, we have $Q = C$, where C is the usual (distribution) copula, that is,

$$F_{n:n}(t) = C(F_1(t),\ldots,F_n(t)). \tag{3.4}$$

Hence

$$\overline{F}_{n:n}(t) = 1 - F_{n:n}(t) = 1 - C(1-\overline{F}_1(t),\ldots,1-\overline{F}_n(t)),$$

that is, its domination function is $H(u_1,\dots,u_n)=1-C(1-u_1,\dots,1-u_n)$.

Let us see how to obtain the domination function H for other coherent systems. For example, let us consider the system with lifetime $T=\min(X_1,\max(X_2,X_3))$. Then the minimal path sets are $\{1,2\}$ and $\{1,3\}$ and we have

$$\begin{aligned}\overline{F}_T(t) &= \Pr(\{X_{\{1,2\}}>t\}\cup\{X_{\{1,3\}}>t\})\\ &= \Pr(X_{\{1,2\}}>t)+\Pr(X_{\{1,3\}}>t)-P(X_{\{1,2,3\}}>t)\\ &= \overline{F}(t,t,0)+\overline{F}(t,0,t)-\overline{F}(t,t,t)\\ &= K(\overline{F}_1(t),\overline{F}_2(t),1)+K(\overline{F}_1(t),1,\overline{F}_3(t))-K(\overline{F}_1(t),\overline{F}_2(t),\overline{F}_3(t))\\ &= H(\overline{F}_1(t),\overline{F}_2(t),\overline{F}_3(t)),\end{aligned} \tag{3.5}$$

where the domination function H is given by

$$H(u_1,u_2,u_3)=K(u_1,u_2,1)+K(u_1,1,u_3)-K(u_1,u_2,u_3).$$

Clearly, H is a multivariate distortion function (i.e., it is increasing and continuous in $[0,1]^3$ and satisfies $H(0,0,0)=0$ and $H(1,1,1)=1$). In particular, in the independent case, the domination polinomial (reliability function of the structure) is

$$H(u_1,u_2,u_3)=u_1u_2+u_1u_3-u_1u_2u_3.$$

Now we can use the results obtained in the preceding section to study preservation properties for coherent systems (order statistics). For example, in the independent case, for the series system $X_{1:n}=\min(X_1,\dots,X_n)$, we have

$$\overline{Q}(u_1,\dots,u_n)=K(u_1,\dots,u_n)=u_1\dots u_n,$$

and then $\alpha_i^K(u_1,\dots,u_n)=1$ for $i=1,\dots,n$. Therefore, from Proposition 2.3 (ii), the HR order is preserved (a well known property). In the next proposition we study the preservation properties for the LR order in series systems with independent components.

Proposition 3.1 *Let $X_{1:n}=\min(X_1,\dots,X_n)$ and $Y_{1:n}=\min(Y_1,\dots,Y_n)$ be the lifetimes of two series systems with independent components having distributions $F_1,F_2,\dots,F_n$ and $G_1,F_2,\dots,F_n$, respectively, such that $F_1\le_{LR}G_1$.*

(i) *If F_1 is DHR and F_i is IHR for $i=2,\dots,n$, then $X_{1:n}\le_{LR}Y_{1:n}$.*
(ii) *If G_1 is DHR and F_i is IHR for $i=2,\dots,n$, then $X_{1:n}\le_{LR}Y_{1:n}$.*

Proof To prove (i), note that from Eq. 3.3, the reliability function of $X_{1:n}$ is the GDD

$$\overline{F}_{1:n}(t)=\overline{Q}(\overline{F}_1(t),\overline{F}_2(t),\dots,\overline{F}_n(t))$$

with $\overline{Q}(u_1,\dots,u_n)=u_1\dots u_n$. Analogously, for $Y_{1:n}$ we have

$$\overline{G}_{1:n}(t)=\overline{Q}(\overline{G}_1(t),\overline{F}_2(t),\dots,\overline{F}_n(t)).$$

Therefore, the function $\gamma^{\overline{Q}}$ defined in Eq. 2.9 is given by

$$\begin{aligned}\gamma^{\overline{Q}}x(u_1,\dots,u_n,v_1,\dots,v_n,w_1,\dots,w_n,z_1,\dots,z_n)\\ =v_1\dots v_n\frac{(w_1/v_1)z_1+\dots+(w_n/v_n)z_1}{z_1+\dots+z_n}.\end{aligned}$$

Hence, $\gamma^{\overline{Q}}$ is constant in $u_1,\dots,u_n$ and increasing in $v_1,\dots,v_n,w_1,\dots,w_n$ in the set $D=(0,1)^n\times(1,\infty)^n\times(0,\infty)^{2n}$. Moreover, $\gamma^{\overline{Q}}$ is decreasing in z_1 if, and only if

$$\frac{w_1}{v_1}(z_2+\dots+z_n)\le\frac{w_2}{v_2}z_2+\dots+\frac{w_n}{v_n}z_n$$

holds. This property is satisfied if

$$\frac{w_1}{v_1} - \frac{w_i}{v_i} \le 0, \quad i = 2, \ldots, n.$$

This property is not true in D and so Proposition 2.4 (iii) cannot be applied directly to this case. However, by using a procedure similar to that used in the proof of that proposition and taking into account the fact that, in the present case, $w_1 = g_1/f_1$, $v_1 = \overline{G}_1/\overline{F}_1$, $w_i = g_i/f_i = 1$ and $v_i = \overline{G}_i/\overline{F}_i = 1$ for $i = 2, \ldots, n$, we see that the function $\gamma^{\overline{Q}}$ can be written as

$$\gamma^{\overline{Q}}(u_1, \ldots, u_n, v_1, \ldots, v_n, w_1, \ldots, w_n, z_1, \ldots, z_n) \\ = v_1 \ldots v_n \frac{(w_1/v_1)z_1 + z_2 + \cdots + z_1}{z_1 + z_2 + \cdots + z_n}$$

where w_1/v_1 will be replaced with $(g_1/\overline{G}_1)/(f_1/\overline{F}_1) = h_1^G/h_1^F$. As $F_1 \le_{LR} G_1$ implies $F_1 \le_{HR} G_1$, then we have $h_1^G \le h_1^F$ and so we can assume $w_1 \le v_1$. A straightforward calculation shows that $\gamma^{\overline{Q}}$ is decreasing in z_1 and increasing in $z_2, \ldots, z_n$ whenever $w_1 \le v_1$. Therefore, as F_1 is DHR and F_i is IHR for $i = 2, \ldots, n$, from Proposition 2.4 (iii) , we have $X_{1:n} \le_{LR} Y_{1:n}$.

The proof of (ii) can be obtained by using a procedure similar to that used in the proof of Proposition 2.4 (iv). □

Proposition 3.2 *Let $X_{1:n} = \min(X_1, \ldots, X_n)$ and $Y_{1:n} = \min(Y_1, \ldots, Y_n)$ be the lifetimes of two series systems with independent components having distributions $F_1, \ldots, F_n$ and $G_1, \ldots, G_n$, respectively, such that $F_i \le_{LR} G_i$ for $i = 1, \ldots, n$. If $F_1, \ldots, F_n$ (or $G_1, \ldots, G_n$) are exponential, then $X_{1:n} \le_{LR} Y_{1:n}$.*

The proof is immediate from the first part of the preceding proof and Remark 2.5. However, the next example shows that the LR order is not necessarily preserved for other distributions.

Example 3.3 Let us consider the series systems $X_{1:2} = \min(X_1, X_2)$ and $Y_{1:2} = \min(Y_1, Y_2)$ with independent components and Weibull reliability functions $\overline{F}_1(t) = \exp(-2t^a)$, $\overline{G}_1(t) = \exp(-t^a)$ and $\overline{F}_2(t) = \overline{G}_2(t) = \exp(-t^b)$ for $a, b > 0$. Note that $F_1 \le_{LR} G_1$ and F_1 and G_1 are IHR when $a \ge 1$. If $a = 2$ and $b = 1$ (i.e., F_1, G_1 are strictly IHR and F_2 is exponential), then the ratio $r(t) = g_{1:2}(t)/f_{1:2}(t)$ is decreasing in $(0, 0.291578)$ and increasing in $(0.291578, \infty)$. Hence, $X_{1:2}$ and $Y_{1:2}$ are not LR-ordered. If $a = b = 1$ (i.e. F_1, G_1, F_2 are exponential), then $r(t)$ is increasing and $X_{1:2} \le_{LR} Y_{1:2}$. If $a = b = 2$ (i.e. F_1, G_1, F_2 are strictly IHR), then $r(t)$ is strictly increasing and $X_{1:2} \le_{LR} Y_{1:2}$. The case $a = b = 2$ can also be solved from the results included in Example 1.C.36 of Shaked and Shanthikumar (2007).

The following example shows that Proposition 3.2 is not necessarily true for other coherent systems (parallel systems).

Example 3.4 Let us consider the parallel systems $X_{2:2} = \max(X_1, X_2)$ and $Y_{2:2} = \max(Y_1, Y_2)$ with independent components and exponential reliability functions $\overline{F}_1(t) = \exp(-at)$, $\overline{G}_1(t) = \exp(-bt)$ and $\overline{F}_2(t) = \overline{G}_2(t) = \exp(-ct)$ for $a, b, c > 0$. Note that $F_1 \le_{LR} G_1$ if $a \ge b$. If $a = 2$ and $b = c = 1$, then the ratio $r(t) = g_{2:2}(t)/f_{2:2}(t)$ is increasing and $X_{2:2} \le_{LR} Y_{2:2}$ holds. However, if $a = 3$, $b = 2$ and $c = 1$, then by plotting the ratio $r(t) = g_{2:2}(t)/f_{2:2}(t)$ we see that it is first increasing and then decreasing. Hence $X_{2:2} \leq_{LR} Y_{2:2}$ does not hold. Therefore, the LR order is not preserved and Proposition 3.2 is not true for parallel systems (non-series systems). The case $a = 2$ and $b = c = 1$, can also be solved from the results included in Example 1.C.36 of Shaked and Shanthikumar (2007).

Proposition 2.2 can be used to obtain comparison results for this system. For example, if we have two independent components and we want to compare $X_{2:2} = \max(X_1, X_2)$ and $X_{1:2} = \min(X_1, X_2)$ in the HR order, then we should study the ratio

$$\frac{H_{2:2}(u_1, u_2)}{H_{1:2}(u_1, u_2)} = \frac{u_1 + u_2 - u_1 u_2}{u_1 u_2} = \frac{1}{u_2} + \frac{1}{u_1} - 1.$$

As it is decreasing in both u_1 and u_2, then, from Proposition 2.2 (ii) , we have $X_{1:2} \leq_{HR} X_{2:2}$ for all F_1, F_2. However, if we want to compare $X_{2:2} = \max(X_1, X_2)$ and X_1, then we should study the ratio

$$\frac{H_{2:2}(u_1, u_2)}{H_{1:1}(u_1, u_2)} = \frac{u_1 + u_2 - u_1 u_2}{u_1} = 1 + \frac{u_2}{u_1} - u_2.$$

As it is decreasing in u_1 and increasing in u_2 in the set $(0, 1)^2$, then, surprisingly, $X_{2:2}$ and X_1 are not HR ordered for all F_1, F_2. For example, they are not HR ordered if the independent components have exponential distributions with means 1 and $1/2$, respectively.

Next we study preservation properties for other coherent system structures and coherent systems with dependent components. In the first one, we study the preservation of the HR order in series systems with dependent components having an Archimedean survival copula.

Example 3.5 Let $X_{1:2} = \min(X_1, X_2)$ be the lifetime of a series system with two dependent components having an Archimedean survival copula

$$K(u_1, u_2) = g(g^{-1}(u_1) + g^{-1}(u_2)),$$

where g is a non-negative decreasing real-valued function (called *generator*) satisfying some properties (see Theorem 4.1.4 in Nelsen 2006, p. 111) and g^{-1} is its inverse function. Recall that for series systems we have $\overline{Q} = H = K$. We want to use Proposition 2.3 (ii), to study if the HR order is preserved under the formation of this kind of systems. By the symmetry of the copula, from Proposition 2.3, it is enough to study the monotonicity of

$$\alpha_1^K(u_1, u_2) = \frac{u_1 D_1 K(u_1, u_2)}{K(u_1, u_2)}$$

in $(0, 1)^2$. For the Archimedean copulas, this function can be written as

$$\alpha_1^K(u_1, u_2) = u_1 \frac{g'(g^{-1}(u_1) + g^{-1}(u_2))}{g(g^{-1}(u_1) + g^{-1}(u_2))} (g^{-1})'(u_1).$$

Therefore, if α_1^K is decreasing in $(0, 1)^2$, then the HR order is preserved.

For example, the Gumbel-Barnett copula is obtained with $g(x) = \exp(-(\exp(x)-1)/\theta)$, where $\theta \in (0, 1]$ (see copula number 9 of Table 4.1 in Nelsen 2006, p. 116). Then a straightforward calculation gives

$$\alpha_1^{K_{GB}}(u_1, u_2) = 1 - \theta \ln u_2.$$

Therefore, $\alpha_1^{K_{GB}}$ is decreasing in $(0, 1)^2$ and the HR order is preserved, that is, if $X_i \leq_{HR} Y_i$ for $i = 1, 2$, then $X_{1:2} \leq_{HR} Y_{1:2}$ for this copula and any F_1, F_2. It is easy to see that this order is also preserved in the case of independent components (a well known property).

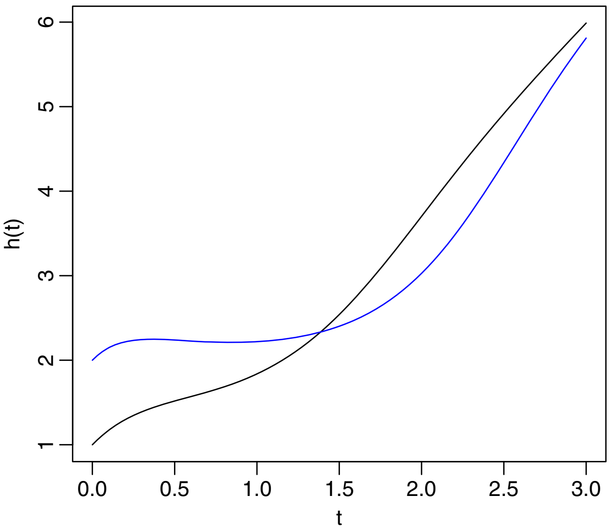

Fig. 1 Plots of the hazard rate functions for the series systems with two dependent components with the Clayton copula considered in Example 3.5 when $\theta = 0.5$, the first components have exponential distributions with hazard rates 2 (blue line) and 1 (black line) and the second components have a common IHR Weibull distribution

However, the HR is not always preserved. For example, if the copula is the Clayton Archimedean copula

$$K_C(u_1, u_2) = (\max(u_1^{-\theta} + u_2^{-\theta} - 1, 0))^{-1/\theta}$$

for $\theta > 0$, which is obtained with the generator $g(x) = (1 + \theta x)^{-1/\theta}$ (see copula number 1 of Table 4.1 in Nelsen 2006, p. 116), then a straightforward calculation gives

$$\alpha_1^{K_C}(u_1, u_2) = \frac{u_1^{-\theta}}{u_1^{-\theta} + u_2^{-\theta} - 1}$$

for all u_1, u_2 such that $u_1^{-\theta} + u_2^{-\theta} - 1 > 0$. This function is decreasing in u_1 but increasing in u_2. Hence the condition for the preservation of the HR order given in Proposition 2.3 (ii) does hold. Then this order is not necessarily preserved. For example, if $\theta = 0.5$, $\overline{F}_1(t) = exp(-2t)$, $\overline{G}_1(t) = exp(-t)$ and $\overline{F}_2(t) = \overline{G}_2(t) = exp(-t^2)$, then the hazard rate functions are not ordered (see Fig. 1). Note that, at the beginning, $h_{1:2}^F(t) > h_{1:2}^G(t)$ (the first system is worse than the second one). However, when $t > 1.38622$, then $h_{1:2}^F(t) < h_{1:2}^G(t)$. Hence,

$$(X_{1:2} - t | X_{1:2} > t) \geq_{ST} (Y_{1:2} - t | Y_{1:2} > t), \quad for\ all\ t > 1.38622,$$

that is, the used series systems with age $t > 1.38622$ obtained from the worse components are ST-better (more reliable) than that obtained from the better components!

In the next example we apply Proposition 2.6 and Remark 2.7 to obtain HR ordering properties for a system with independent components.

Example 3.6 Let us consider the coherent systems with lifetimes $T_{\mathbf{X}} = \min(X_2, \max(X_1, X_3))$ and $T_{\mathbf{Y}} = \min(Y_2, \max(Y_1, Y_3))$ where we assume that

$X_1, X_2, X_3, Y_1, Y_2, Y_3$ are independent with reliability functions $\overline{F}_1, \overline{F}_2, \overline{F}_3, \overline{G}_1, \overline{F}_2, \overline{F}_3$, respectively. From Eq. 3.5 the reliability functions of these systems are

$$\overline{F}_{T_{\mathbf{X}}}(t) = \overline{Q}(\overline{F}_1(t), \overline{F}_2(t), \overline{F}_3(t))$$

and

$$\overline{F}_{T_{\mathbf{Y}}}(t) = \overline{Q}(\overline{G}_1(t), \overline{F}_2(t), \overline{F}_3(t))$$

respectively, where

$$\overline{Q}(u_1, u_2, u_3) = u_1u_2 + u_2u_3 - u_1u_2u_3.$$

To apply Proposition 2.6 (i) , we compute

$$\delta^{\overline{Q}}(u_1, v_1, v_2, v_3) = \frac{v_1 + v_3 - u_1v_1v_3}{1 + v_3 - u_1v_3}.$$

This function is decreasing in u_1 and v_3 and constant with respect to v_2 in the set $(0, 1) \times (1, \infty) \times (0, \infty) \times (1, \infty)$. Hence, from Proposition 2.6 (i) and Remark 2.7, we have $T_{\mathbf{X}} \leq_{HR} T_{\mathbf{Y}}$ whenever $F_1 \leq_{HR} G_1$ and $F_1 \geq_{HR} F_3$. This property is not necessarily true if $F_1 \geq_{HR} F_3$ does not hold. For example, if $\overline{F}_1(t) = \exp(-2t^2)$, $\overline{F}_2(t) = \overline{F}_3(t) = \exp(-t)$ and $\overline{G}_1(t) = \exp(-t^2)$, by plotting the hazard rate functions of these systems we see that they are not HR ordered.

In the following example, we use the results given in Proposition 2.2 to compare systems with different structures. The example shows that these results can also be used to compare systems with the same structure but having components with different levels of dependency.

Example 3.7 Let us consider the series system with lifetime $T_1 = \min(X_1, X_2, X_3)$ and the coherent system with lifetime $T_2 = \min(X_1, \max(X_2, X_3))$. If the components are independent, then the system reliability functions are

$$\overline{F}_{T_1}(t) = \overline{Q}_1(\overline{F}_1(t), \overline{F}_2(t), \overline{F}_3(t))$$

and

$$\overline{F}_{T_2}(t) = \overline{Q}_2(\overline{F}_1(t), \overline{F}_2(t), \overline{F}_3(t))$$

respectively, where $\overline{F}_1, \overline{F}_2, \overline{F}_3$ are the component reliability functions and where

$$\overline{Q}_1(u_1, u_2, u_3) = u_1u_2u_3$$

and

$$\overline{Q}_2(u_1, u_2, u_3) = u_1u_2 + u_1u_3 - u_1u_2u_3$$

from Eq. 3.5. Then, from Proposition 2.2 (ii), $T_1 \leq_{HR} T_2$ holds if and only if $\overline{Q}_2/\overline{Q}_1$ is decreasing in $(0, 1)^3$. As

$$\frac{\overline{Q}_2(u_1, u_2, u_3)}{\overline{Q}_1(u_1, u_2, u_3)} = \frac{1}{u_3} + \frac{1}{u_2} - 1$$

is decreasing, we have $T_1 \leq_{HR} T_2$ for all $\overline{F}_1, \overline{F}_2, \overline{F}_3$.

Let us assume now that the components in both systems are dependent with a common Farlie-Gumbel-Morgenstern (FGM) survival copula

$$K_{FGM}(u_1, u_2, u_3) = u_1u_2u_3 \left(1 + \theta(1 - u_1)(1 - u_2)(1 - u_3)\right), \tag{3.6}$$

where $\theta \in [-1, 1]$. The independence case is obtained when $\theta = 0$. Then, from Eq. 3.5,

$$\overline{Q}_1^{\theta}(u_1, u_2, u_3) = u_1u_2u_3(1 + \theta(1 - u_1)(1 - u_2)(1 - u_3))$$

and

$$\overline{Q}_2^{\theta}(u_1, u_2, u_3) = u_1u_2 + u_1u_3 - u_1u_2u_3(1 + \theta(1 - u_1)(1 - u_2)(1 - u_3))$$

when the copula is given by Eq. 3.6. Hence

$$\frac{\overline{Q}_2^{\theta}(u_1, u_2, u_3)}{\overline{Q}_1^{\theta}(u_1, u_2, u_3)} = \frac{1}{u_3(1 + \theta(1 - u_1)(1 - u_2)(1 - u_3))} + \frac{1}{u_2(1 + \theta(1 - u_1)(1 - u_2)(1 - u_3))} - 1$$

is decreasing in $(0, 1)^3$ if $\theta \leq 0$. Therefore, we have $T_1 \leq_{HR} T_2$ for all $\overline{F}_1, \overline{F}_2, \overline{F}_3$ and all $\theta \leq 0$ (negative dependency). However, these systems are not necessarily ordered if $\theta > 0$ (positive dependency).

Finally, if we want to compare the series systems with dependent components having a FGM survival copula with different levels of dependency, from Proposition 2.2 (ii), we should compute

$$\frac{\overline{Q}_1^{\theta'}(u_1, u_2, u_3)}{\overline{Q}_1^{\theta}(u_1, u_2, u_3)} = \frac{1 + \theta'(1 - u_1)(1 - u_2)(1 - u_3)}{1 + \theta(1 - u_1)(1 - u_2)(1 - u_3)}.$$

This function is decreasing in $(0, 1)^3$ when $\theta' \geq \theta$. Therefore, we have $T_1^{\theta} \leq_{HR} T_2^{\theta'}$ for all $\overline{F}_1, \overline{F}_2, \overline{F}_3$ and all $-1 \leq \theta \leq \theta' \leq 1$, that is, the performance of the series system is improved (in the HR order) when the components are more dependent (an expectable property).

We conclude this section giving some direct applications of the simple result included in Proposition 2.2 (i). Firstly, we note that many copulas include a dependence parameter and they are ordered with respect to this parameter. Hence, from Proposition 2.2 (i), Eqs. 3.3 and 3.4, the respective series (or parallel) systems will be ST-ordered with respect to the (dependence) copula parameter. For example, we can consider the Cuadras-Augé bivariate family of copulas given by

$$C_{\theta}(u_1, u_2) = [\min(u_1, u_2)]^{\theta}[u_1u_2]^{1-\theta}, \ \theta \in [0, 1], \tag{3.7}$$

(see equation (2.2.10) in Nelsen 2006, p. 15). In this copula, the parameter θ measures the degree of stochastic dependence between the two components. Note that $C_0(u_1, u_2)$ represents the case of independent components and $C_1(u_1, u_2)$ is the Fréchet-Hoeffding upper bound copula. This parametric family of copulas is positively ordered, that is, if $0 \leq \theta_1 \leq \theta_2 \leq 1$, then $C_{\theta_1}(u_1, u_2) \leq C_{\theta_2}(u_1, u_2)$ for $u_1, u_2 \in (0, 1)$ (see Example 2.19 in Nelsen 2006, p. 39). Therefore, under a Cuadras-Augé distribution copula, if $0 \leq \theta_1 \leq \theta_2 \leq 1$, then the respective parallel system lifetimes satisfy $X_{2:2}^{\theta_1} \geq_{ST} X_{2:2}^{\theta_2}$, which means that a greater positive dependence implies a smaller reliability of the parallel system. In a similar way, if we consider the series systems with component lifetimes having the survival copula $K = C_{\theta}$ defined in Eq. 3.7, then we obtain $X_{1:2}^{\theta_1} \leq_{ST} X_{1:2}^{\theta_2}$, which means that a greater positive dependence implies a greater reliability of the series system. Similar results can be obtained, for example, for the Gumbel-Barnett copula considered in Example 3.5 or for the Farlie-Gumbel-Morgenstern (FGM) copula considered in Example 3.7. These results suggest that dependence among components is crucial for the reliability of parallel and series systems.

The following result provides simple stochastic bounds for a large class of GDD in terms of the baseline distribution functions $F_1, \ldots, F_n$.

Proposition 3.8 *Let $F_Q = Q(F_1, \ldots, F_n)$ be a GDD based on the MDF Q and on the distribution functions $F_1, \ldots, F_n$.*

(i) *If $Q(u_1, \ldots, u_n) \leq u_i$ ($\geq$) for some $i \in \{1, \ldots, n\}$, then $F_Q \geq_{ST} F_i$ ($\leq_{ST}$).*
(ii) *If $Q(u_1, \ldots, u_n) \leq u_i$ ($\geq$) for all $i \in \{1, \ldots, n\}$, then $F_Q \geq_{ST} \min(F_1, \ldots, F_n)$ (resp. $F_Q \leq_{ST} \max(F_1, \ldots, F_n)$).*
(iii) *If $\overline{Q}(u_1, \ldots, u_n) \leq u_i$ ($\geq$) for some $i \in \{1, \ldots, n\}$, then $\overline{F}_Q \leq_{ST} \overline{F}_i$ ($\geq_{ST}$).*
(iv) *If $\overline{Q}(u_1, \ldots, u_n) \leq u_i$ ($\geq$) for all $i \in \{1, \ldots, n\}$, then $\overline{F}_Q \leq_{ST} \min\left(\overline{F}_1, \ldots, \overline{F}_n\right)$ (resp. $\overline{F}_Q \geq_{ST} \max\left(\overline{F}_1, \ldots, \overline{F}_n\right)$).*

The proof is immediate. These results can be applied to coherent systems by using the Fréchet-Hoeffding upper bound (UB) copula given by

$$C^{UB}(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n).$$

This copula represents comonotonicity which corresponds to the strongest type of positive dependence. It is well known that any other copula K, satisfies

$$K(u_1, \ldots, u_n) \leq C^{UB}(u_1, \ldots, u_n).$$

Hence, we obtain $X_{1:n}^K \leq_{ST} X_{1:n}^{UB}$, where $X_{1:n}^K$ and $X_{1:n}^{UB}$ represent the series system lifetimes obtained with the arbitrary survival copula K and with the upper bound survival copula C^{UB}, respectively. Moreover, if T is the lifetime of a coherent system whose component lifetimes $X_1, \ldots, X_n$ have the survival copula K and whose domination function H satisfies $H(u_1, \ldots, u_n) \leq u_i$ for $i = 1, \ldots, n$, then

$$X_{1:n}^K \leq_{ST} T \leq_{ST} X_{1:n}^{UB}.$$

The first inequality is obtained from $T = \phi(X_1, \ldots, X_n) \geq \min(X_1, \ldots, X_n) = X_{1:n}$ and the second from Proposition 3.8 (iv). Analogously, for the parallel systems with components having the distributional copulas C and C^{UB}, we obtain $X_{n:n}^C \geq_{ST} X_{n:n}^{UB}$. Hence, if $H(u_1, \ldots, u_n) \geq u_i$ for $i = 1, \ldots, n$, then

$$X_{n:n}^{UB} \leq_{ST} T \leq_{ST} X_{n:n}^C.$$

Moreover, we of course have $X_{1:n}^{UB} \leq_{ST} X_{n:n}^{UB}$.

4 Conclusions

In this paper we give a procedure to study preservation properties for stochastic orders under the formation of generalized distorted distributions. This general approach can be applied to study order statistics and coherent systems with independent or dependent components. We want to point out that, for certain systems, some stochastic orders are not preserved. Thus, for example, in the case of independent components, the LR order is not preserved under the formation of series systems. Then, we obtain some conditions under which this order is preserved. For example, it is preserved if the components in one system have exponential distributions. The tools provided in this paper can be used to obtain more results of this type for other system structures and/or specific dependence models for the components in the system. They can also be used to study other statistical concepts that can be written as generalized distorted distributions. This includes finite mixtures

($F = p_1F_1 + \cdots + p_nF_n$ for $p_1, \ldots, p_n \geq 0$), arithmetic means ($F = (F_1 + \cdots + F_n)/n$), geometric means ($F = (F_1 \ldots F_n)^{1/n}$ and $\overline{F} = (\overline{F}_1 \ldots \overline{F}_n)^{1/n}$), generalized proportional hazard rate models ($\overline{F} = \overline{F}_1^{\alpha_1} \ldots \overline{F}_n^{\alpha_n}$, $\alpha_1, \ldots, \alpha_n \geq 0$), generalized proportional reversed hazard rate models ($F = F_1^{\alpha_1} \ldots F_n^{\alpha_n}$, $\alpha_1, \ldots, \alpha_n \geq 0$), bounds ($F_U = \max(F_1, \ldots, F_n)$ and $F_L = \min(F_1, \ldots, F_n)$), etc.

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