Methodol Comput Appl Probab (2016) 18:419–440
DOI 10.1007/s11009-014-9429-0

Analysis of $BMAP/MSP/1$ Queue

S. K. Samanta · M. L. Chaudhry · A. Pacheco

Received: 31 January 2014 / Revised: 14 October 2014 /
Accepted: 9 November 2014 / Published online: 25 November 2014

Abstract The analysis for the $BMAP/MSP/1$ queueing system is based on roots of the associated characteristic equation of the vector-generating function of system-length distribution at random epoch. We obtain the steady-state system-length distributions at various epochs as well as of the actual sojourn-time distribution of an arbitrary customer in an arriving batch.

Keywords Batch Markovian arrival process (BMAP) · Markovian service process (MSP) · Roots · Sojourn time · System length

Mathematics Subject Classification (2010) 60K25 · 68M20 · 90B22

1 Introduction

Queueing models with non-renewal arrivals and/or service processes are often used to model complex computer and communication systems. Several connections (data, voice, video, etc.) generate traffic streams with very different characteristics (required bandwidth, burstiness, correlation, etc.). Traditional teletraffic analysis based on Poisson arrival/service process is not powerful enough to capture this correlative and bursty feature of traffic streams in high-speed packet/cell based networks. These correlated and bursty non-renewal

S. K. Samanta · A. Pacheco (✉)
CEMAT and Department of Mathematics, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal
e-mail: apacheco@math.tecnico.ulisboa.pt

S. K. Samanta
e-mail: sujit.samanta@tecnico.ulisboa.pt

M. L. Chaudhry
Department of Mathematics and Computer Science, Royal Military College of Canada, P.O. Box 17000, STN Forces, Kingston Ontario, Canada K7K 7B4
e-mail: chaudhry-ml@rmc.ca

arrival and/or service processes in queueing systems have been shown empirically and theoretically to have a significant impact on queueing behaviour. The versatile Markovian point process or Neuts (N) process known as Markovian arrival process (MAP) has been introduced by Neuts (1979) due to limitations of the Poisson process in modelling correlated arrivals. Later (Lucantoni 1991) has shown that batch Markovian arrival process (BMAP) generalizes the MAP by allowing batch arrivals. The BMAP is a convenient representation of the N-process that includes several known processes such as Markovian arrival process (MAP), Markov-modulated Poisson process (MMPP), Phase-type renewal process, and superposition of such processes. The concepts of MAP and BMAP have been explained in the past at several places by many researchers and are available in the literature, see, e.g., Pacheco et al. (2009), Chakravarthy (1993), and Machihara (1999). The Markovian service process (MSP) is independent of the arrival process and can capture the correlation that exists among successive service times. Note that the MSP generates real service completions only when the server is busy. For details on the MSP, readers are referred to Bocharov et al. (2003), Albores-Velasco and Tajonar-Sanabria (2004), Gupta and Banik (2007), and Chaudhry et al. (2012) and Alfa et al. (2000). For early works on correlated arrivals and departures, see Chaudhry (1965,1966).

Many researchers have analyzed several queueing models with various types of service as well as arrival processes and such results are available in the literature. The asymptotic behaviour of stationary distributions of $MAP/MSP/1$ queue is discussed in Abate et al. (1994) and Alfa et al. (2000). Ozawa (2006) carried out the analysis of $MAP/MSP/1$ queue and derived the stationary sojourn-time distribution as well as its asymptotic properties. Horváth et al. (2010) proposed an approximation for the output process of $MAP/MSP/1$ queue based on the moments of the inter-departure time and the joint moments of two consecutive inter-departure times. Zhang et al. (2005) analyzed the departure process of $BMAP/MSP/1$ queue based on an Efficient Technique for the Analysis of QBD processes by Aggregation (ETAQA) developed in Riska and Smirni (2002). The interest to analyze $BMAP/MSP/1$ queue with batch non-Poisson traffic and non-exponential service time distributions has been mainly due to the fact that the bursty and correlated nature of traffic arising in modelling telecommunication systems have significant impact on queueing behaviour. For analytic and numerical purposes, we start our work with the vector-generating function (VGF) of system-length distribution at random epochs, evaluate the unknown vector $\boldsymbol{\pi}(0)$ (see Section 3.1) using the roots inside and on the unit circle of the associated characteristic equation. Once the roots are found, it becomes easy to obtain probability distributions of the number in system at random, arrival and post-departure epochs as well as of the actual sojourn-time distribution and other performance measures. We can also get the actual sojourn-time distribution without evaluating the steady-state distribution of the number in system. In this connection, see Chaudhry and Templeton (1983), Tijms HC (2003), and Janssen and Leeuwaarden (2005), and Chaudhry et al. (1990), who have used the roots method. The roots (including repeated roots) can be easily found using one of the several commercially available packages such as MAPLE and MATHEMATICA. At an early stage, Chaudhry (1991) developed a package called QROOT and used it to solve several queueing problems.

This paper is organized as follows. In Section 2, we give the description of the model and introduce the notations to describe the model parameters. The steady-state system-length distributions at various epochs and the actual sojourn-time distribution of an arbitrary customer in an arriving batch are analyzed in Section 3. Numerical results are presented in Section 4. Section 5 concludes the paper.

2 Model Description

We consider a continuous-time single-server queue wherein customers arrive according to a batch Markovian arrival process and are served in accordance with a Markovian service process. Let us first introduce the notation for the $BMAP/MSP/1$ queueing system presented in this paper.

2.1 Batch Markovian Arrival Process

The batch Markovian arrival process is represented with parameter matrices $\mathbf{D}_k$, $k \geq 0$, of order $m_1 \times m_1$, where the matrix $\mathbf{D}_0 = [(D_0)_{i,j}]$ has negative diagonal elements and non-negative off-diagonal elements, and the matrices $\mathbf{D}_k = [(D_k)_{i,j}]$, $k \geq 1$, have non-negative elements. We define the matrix-generating function $\mathbf{D}(z) = \sum_{k=0}^{\infty} \mathbf{D}_k z^k$, for $|z| \leq 1$. The matrix $\mathbf{D} \equiv \mathbf{D}(1) = \sum_{k=0}^{\infty} \mathbf{D}_k$ with $\mathbf{De} = \mathbf{0}$, where $\mathbf{e}$ is a column vector of ones with an appropriate dimension, is an infinitesimal generator corresponding to an irreducible Markov chain underlying the BMAP. Let $\overline{\boldsymbol{\pi}}_a$ be the stationary probability vector of the Markov process with generator $\mathbf{D}$, i.e., $\overline{\boldsymbol{\pi}}_a$ satisfies $\overline{\boldsymbol{\pi}}_a \mathbf{D} = \mathbf{0}$ with $\overline{\boldsymbol{\pi}}_a \mathbf{e} = 1$. Let $N_a(t)$ denote the number of arrivals in $(0, t]$ and $J_a(t)$ the state of the underlying Markov chain (called arrival phase) at time t. To accomplish this, we consider a two-dimensional Markov process $\{N_a(t), J_a(t)\}_{t \geq 0}$ on the state space $\{(n, i) : n \geq 0, 1 \leq i \leq m_1\}$ with an infinitesimal generator

$$\mathbf{Q}^{\text{BMAP}} = \begin{pmatrix} \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \cdots \\ & \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \cdots \\ & & \mathbf{D}_0 & \mathbf{D}_1 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Let $\{\mathbf{P}_a(n, t) : n \geq 0, t \geq 0\}$ denote the matrix of order $m_1 \times m_1$ whose (i, j)th element $((P_a)_{i,j}(n, t))$ represents the probability that n arrivals occur in $(0, t]$ with the arrival process being in phase j at time t, given that the arrival process was in phase i at time zero. Then, the probabilities

$$(P_a)_{i,j}(n, t) = Pr\left\{N_a(t) = n, J_a(t) = j | N_a(0) = 0, J_a(0) = i\right\}, \quad 1 \leq i, j \leq m_1$$

lead to the following equations (in matrix notation)

$$\frac{d}{dt}\mathbf{P}_a(n, t) = \sum_{k=0}^{n} \mathbf{P}_a(k, t)\mathbf{D}_{n-k}, \quad n \geq 0, \quad t > 0, \tag{1}$$

with $\mathbf{P}_a(0, 0) = \mathbf{I}_{m_1}$ and $\mathbf{P}_a(n, 0) = \mathbf{0}$, $n \geq 1$, where $\mathbf{I}_r$ is the identity matrix of order $r \times r$.

Let us define the matrix-generating function $\mathbf{P}_a^*(z, t)$ as

$$\mathbf{P}_a^*(z, t) = \sum_{n=0}^{\infty} \mathbf{P}_a(n, t) z^n, \quad |z| \leq 1, \quad t \geq 0. \tag{2}$$

Multiplying Eq. 1 by z^n and summing over $n = 0$ to ∞, after using Eq. 2, we get

$$\frac{d}{dt}\mathbf{P}_a^*(z, t) = \mathbf{P}_a^*(z, t)\mathbf{D}(z), \quad t > 0,$$

with $\mathbf{P}_a^*(z,0) = \mathbf{I}_{m_1}$. Solving the above matrix-differential equations, we get

$$\mathbf{P}_a^*(z,t) = e^{\mathbf{D}(z)t}, \quad |z| \le 1,\ t \ge 0. \tag{3}$$

The first moment in matrix form [differentiation of Eq. 3 w.r.t. z and setting $z = 1$] is given by

$$\mathbf{M}_a(t) = te^{\mathbf{D}t}\sum_{k=1}^{\infty} k\mathbf{D}_k.$$

The mean number of arrivals during a time of length t is

$$\begin{aligned}\lambda^\star(t) &= \overline{\boldsymbol{\pi}}_a \mathbf{M}_a(t)\mathbf{e} \\ &= t\overline{\boldsymbol{\pi}}_a \sum_{k=1}^{\infty} k\mathbf{D}_k\mathbf{e},\end{aligned}$$

where we are assuming that $J_a(0)$ has distribution $\overline{\boldsymbol{\pi}}_a$.

The fundamental arrival rate for the arrival process is then given by

$$\lambda^\star = \frac{\lambda^\star(t)}{t} = \overline{\boldsymbol{\pi}}_a \sum_{k=1}^{\infty} k\mathbf{D}_k\mathbf{e}.$$

2.2 Markovian Service Process

The Markovian service process is represented with parameter matrices $\mathbf{L}_0$ and $\mathbf{L}_1$ of order $m_2 \times m_2$. Note that the matrix $\mathbf{L}_0 = [(L_0)_{i,j}]$ has non-negative off-diagonal and negative diagonal elements, and the matrix $\mathbf{L}_1 = [(L_1)_{i,j}]$ has non-negative elements. Let us denote $\mathbf{L}(z) = \mathbf{L}_0 + \mathbf{L}_1 z$ with $\mathbf{L} \equiv \mathbf{L}(1) = \mathbf{L}_0 + \mathbf{L}_1$ being an infinitesimal generator matrix corresponding to an irreducible Markov chain underlying the MSP. Let $\overline{\boldsymbol{\pi}}_s$ be the stationary probability vector of the Markov process with generator $\mathbf{L}$, i.e., $\overline{\boldsymbol{\pi}}_s$ satisfies $\overline{\boldsymbol{\pi}}_s\mathbf{L} = \mathbf{0}$ with $\overline{\boldsymbol{\pi}}_s\mathbf{e} = 1$. Let $N_s(t)$ denote the number of customers served in $(0, t]$ and $J_s(t)$ the state of the underlying Markov chain (called service phase) at time t. Then $\{N_s(t), J_s(t)\}_{t\ge0}$ is a two-dimensional Markov process with state space $\{(n, i) : n \ge 0, 1 \le i \le m_2\}$ and infinitesimal generator

$$\mathbf{Q}^{\text{MSP}} = \begin{pmatrix} \mathbf{L}_0 & \mathbf{L}_1 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{L}_0 & \mathbf{L}_1 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_0 & \mathbf{L}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $\{\mathbf{P}_s(n,t) : n \ge 0, t \ge 0\}$ denote the matrix of order $m_2 \times m_2$ whose (i, j)th element $((P_s)_{i,j}(n,t))$ represents the probability that n customers are served in $(0, t]$ with the service process being in phase j at time t, given that the service process was in phase i at time zero. Then, the probabilities

$$(P_s)_{i,j}(n,t) = Pr\{N_s(t) = n, J_s(t) = j | N_s(0) = 0, J_s(0) = i\}, \quad 1 \le i, j \le m_2$$

lead to the following equations (in matrix notation)

$$\frac{d}{dt}\mathbf{P}_s(0,t) = \mathbf{P}_s(0,t)\mathbf{L}_0,\ t > 0 \tag{4}$$

$$\frac{d}{dt}\mathbf{P}_s(n,t) = \mathbf{P}_s(n,t)\mathbf{L}_0 + \mathbf{P}_s(n-1,t)\mathbf{L}_1,\ n \ge 1,\ t > 0, \tag{5}$$

with $\mathbf{P}_s(0,0) = \mathbf{I}_{m_2}$ and $\mathbf{P}_s(n,0) = \mathbf{0}$, $n \ge 1$.

Let us define the matrix-generating function $\mathbf{P}_s^*(z,t)$ as

$$\mathbf{P}_s^*(z,t) = \sum_{n=0}^{\infty} \mathbf{P}_s(n,t) z^n, \quad |z| \le 1, \; t \ge 0. \tag{6}$$

Multiplying Eq. 4 by z^0 and Eq. 5 by z^n and summing over $n = 0$ to ∞, after using Eq. 6, we get

$$\frac{d}{dt}\mathbf{P}_s^*(z,t) \; = \; \mathbf{P}_s^*(z,t)\mathbf{L}(z), \quad t > 0,$$

with $\mathbf{P}_s^*(z,0) = \mathbf{I}_{m_2}$. Solving the above matrix-differential equations, we get

$$\mathbf{P}_s^*(z,t) = e^{\mathbf{L}(z)t}, \quad |z| \le 1, \; t \ge 0. \tag{7}$$

The first moment in matrix form [differentiation of Eq. 7 w.r.t. z and setting $z = 1$] is given by

$$\mathbf{M}_s(t) = t e^{\mathbf{L}t}\mathbf{L}_1.$$

The mean number of service completions during a time of length t is

$$\begin{aligned} \mu^\star(t) &= \overline{\boldsymbol{\pi}}_s \mathbf{M}_s(t)\mathbf{e} \\ &= t\overline{\boldsymbol{\pi}}_s \mathbf{L}_1 \mathbf{e}, \end{aligned}$$

where we are assuming that $J_s(0)$ has distribution $\overline{\boldsymbol{\pi}}_s$. The average service rate $\mu^\star$ (the so-called fundamental service rate) of the stationary MSP is given by $\mu^\star = \frac{\mu^\star(t)}{t} = \overline{\boldsymbol{\pi}}_s \mathbf{L}_1 \mathbf{e}$. The traffic intensity is given by $\rho = \lambda^\star/\mu^\star < 1$.

3 Analysis of the Model

In this section, we carry out the analysis of the distributions of system-length at random, arrival and post-departure epochs as well as of the actual sojourn-time distribution of an arbitrary customer in an arriving batch.

3.1 System-Length Distribution at Random Epoch

We consider the steady-state system-length distribution at random epoch. Let $N(t)$ denote the number of customers in the system at time t, $I(t)$ the phase of the BMAP and $J(t)$ the phase of the MSP at the same instant. We define the state of the system at time t by $Y(t) = (N(t), I(t), J(t))$. Then $\{Y(t)\}_{t \ge 0}$ is a continuous-time Markov chain on the state space $\{(n,i,j) : n \ge 0, 1 \le i \le m_1, 1 \le j \le m_2\}$. The infinitesimal generator $\mathbf{Q}$ for the $BMAP/MSP/1$ queue has the following structure:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 & \cdots \\ \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{A}_4 & \cdots \\ & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \cdots \\ & & \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \cdots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{8}$$

The matrices $\mathbf{B}_n$, $n \ge 1$, of order $m \times m$, where $m = m_1 m_2$, are said to increase the level of the chain by n, while the matrix $\mathbf{B}_0$ of order $m \times m$ remains at the zero level. The matrix

$\mathbf{A}_0$ of order $m \times m$ decreases the level of the chain by one. The matrices $\mathbf{A}_n$, $n \geq 2$, of order $m \times m$ are said to increase the level of the chain by $(n-1)$, while the matrix $\mathbf{A}_1$ of order $m \times m$ remains at the same level. The elements of $\mathbf{A}_n$ and $\mathbf{B}_n$ represent the rate of change of both arrival and service phases of the chain. There are two situations to consider based on the assumption that the MSP generates real service completions only when the server is busy. One situation is that the service process is interrupted during idle periods of the system, implying, in particular, that the service phase does not change during idle periods of the system (Model 1). Other one is that the service process runs during idle periods of the system without generating any real service completions (Model 2). Therefore, the block matrices of Eq. 8 can be expressed using the Kronecker product $\otimes$ operation as

$$\begin{aligned}
\mathbf{B}_0 &= \mathbf{D}_0 \otimes \mathbf{I}_{m_2} + \mathbf{I}_{m_1} \otimes \boldsymbol{\Phi} \\
\mathbf{B}_n &= \mathbf{D}_n \otimes \mathbf{I}_{m_2}, \quad n \geq 1 \\
\mathbf{A}_0 &= \mathbf{I}_{m_1} \otimes \mathbf{L}_1 \\
\mathbf{A}_1 &= \mathbf{D}_0 \otimes \mathbf{I}_{m_2} + \mathbf{I}_{m_1} \otimes \mathbf{L}_0 \\
\mathbf{A}_n &= \mathbf{D}_{n-1} \otimes \mathbf{I}_{m_2}, \quad n \geq 2,
\end{aligned}$$

where $\boldsymbol{\Phi} = \mathbf{0}$ for Model 1, while $\boldsymbol{\Phi} = \mathbf{L}$ for Model 2.

Let $\mathbf{A}(z) = [A_{i,j}(z)]$ be the matrix-generating function of a sequence $\{\mathbf{A}_n\}_0^\infty$. Then, we have

$$\mathbf{A}(z) = \sum_{n=0}^{\infty} \mathbf{A}_n z^n = z\left(\mathbf{D}(z) \otimes \mathbf{I}_{m_2} + \mathbf{I}_{m_1} \otimes \mathbf{L}\left(z^{-1}\right)\right). \tag{9}$$

Similarly, let $\mathbf{B}(z)$ be the matrix-generating function of a sequence $\{\mathbf{B}_n\}_0^\infty$. Then, we have

$$\mathbf{B}(z) = \sum_{n=0}^{\infty} \mathbf{B}_n z^n = \mathbf{D}(z) \otimes \mathbf{I}_{m_2} + \mathbf{I}_{m_1} \otimes \boldsymbol{\Phi}.$$

Let $\boldsymbol{\pi}(n) = \left[\pi_{11}(n), \ldots, \pi_{1m_2}(n), \ldots, \pi_{ij}(n), \ldots, \pi_{m_1 1}(n), \ldots, \pi_{m_1 m_2}(n)\right]$, $n \geq 0$, denote the row vector according to the block structure of the generator $\mathbf{Q}$, where $\pi_{ij}(n)$ represents the steady-state probability that there are n customers in the system with the arrival process being in phase i $(1 \leq i \leq m_1)$ and the service process being in phase j $(1 \leq j \leq m_2)$. Now we obtain the VGF of the distribution of number of customers in the system at random epoch. Writing $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$, where $\boldsymbol{\pi}=[\boldsymbol{\pi}(0), \boldsymbol{\pi}(1), \boldsymbol{\pi}(2), \ldots]$, we have

$$\boldsymbol{\pi}(0)\mathbf{B}_n + \sum_{r=1}^{n+1} \boldsymbol{\pi}(r)\mathbf{A}_{n+1-r} = \mathbf{0}, \quad n \geq 0. \tag{10}$$

Multiplying Eq. 10 by z^n and summing them, and using $\boldsymbol{\pi}^*(z) = \sum_{n=0}^{\infty} \boldsymbol{\pi}(n) z^n$, we get

$$\boldsymbol{\pi}^*(z) = \frac{\boldsymbol{\pi}(0)\left[\mathbf{A}(z) - z\mathbf{B}(z)\right] adj[\mathbf{A}(z)]}{det[\mathbf{A}(z)]}, \tag{11}$$

where $adj[\mathbf{T}]$ is the adjoint matrix of a square matrix $\mathbf{T}$ and $det[\mathbf{T}]$ is the determinant of $\mathbf{T}$.

To evaluate the system-length distribution $\boldsymbol{\pi}(n)$, $n \geq 0$, we apply the method of roots which involves the determination of roots of the so-called characteristic equation (c.e.) of the vector-generating function $\boldsymbol{\pi}^*(z)$. Our main task is to first calculate the unknown vector $\boldsymbol{\pi}(0)$ accurately. For this, the knowledge of the zeros of $det[\mathbf{A}(z)]$ in the unit disk is required. Thus, we first show that if $\rho < 1$ then $det[\mathbf{A}(z)] = 0$ has exactly $(m-1)$ roots in $|z| < 1$, one root at $z = 1$ and other mb roots in $|z| > 1$ (including multiplicity), where b is the maximum size of batches, see Appendix A. We call the roots whose absolute value is less than one as $\gamma_1, \gamma_2, \ldots, \gamma_{m-1}$ and the roots whose absolute value is greater than one as $\alpha_1, \alpha_2, \ldots, \alpha_{mb}$. We assume that all roots are distinct. Each component $\pi^*_{ij}(z)$ defined as $\pi^*_{ij}(z) = \sum_{n=0}^{\infty} \pi_{ij}(n) z^n$ of the VGF $\boldsymbol{\pi}^*(z)$ given in Eq. 11 being convergent in $|z| \leq 1$ implies that $\boldsymbol{\pi}^*(z)$ is convergent in $|z| \leq 1$. As each component $\pi^*_{ij}(z)$ is convergent in $|z| \leq 1$, the zeros of $det[\mathbf{A}(z)]$ whose absolute value is less or equal to one must be the zeros of the numerator of each component of Eq. 11. This shows that we can determine the unknown vector $\boldsymbol{\pi}(0)$ by considering any one component of $\boldsymbol{\pi}^*(z)$. Therefore, we rewrite the right-hand side of $\boldsymbol{\pi}^*(z)$ in Eq. 11 as

$$\boldsymbol{\pi}^*(z) = \left[\frac{G_{11}(z)}{G(z)}, \frac{G_{12}(z)}{G(z)}, \ldots, \frac{G_{ij}(z)}{G(z)}, \ldots, \frac{G_{m_1 m_2}(z)}{G(z)}\right], \tag{12}$$

where $G_{ij}(z)$ is the ij-th component of the vector $\boldsymbol{\pi}(0)\,[\mathbf{A}(z) - z\mathbf{B}(z)]\,adj[\mathbf{A}(z)]$, and $G(z) = det[\mathbf{A}(z)]$.

Now, since each $\pi^*_{ij}(z)$ is convergent in $|z| \leq 1$ and $\gamma_1, \gamma_2, \ldots, \gamma_{m-1}$ are the zeros of $G(z)$, we have

$$G_{ij}(\gamma_k) = 0, \quad k = 1, 2, \ldots, m-1 \tag{13}$$

and using the normalization condition $\boldsymbol{\pi}^*(1)\mathbf{e} = 1$, we have

$$1 = \lim_{z \to 1} \frac{\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} G_{ij}(z)}{G(z)} = \frac{\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} G'_{ij}(1)}{G'(1)}, \tag{14}$$

where $h'(\xi)$ is the first derivative of $h(z)$ at $z = \xi$.

Equations 13 and 14 give m linearly independent simultaneous equations in m unknowns, $\pi_{ij}(0)$'s $(1 \leq i \leq m_1, 1 \leq j \leq m_2)$. Solving these m equations, we get the m unknowns $\pi_{ij}(0)$'s. This determines the vector $\boldsymbol{\pi}(0)$.

Once $\boldsymbol{\pi}(0)$ is known, we can obtain mean system-length L directly using Eq. 12 by differentiating and taking the limit as $z \to 1$. Using the fact that $G(1) = 0$, we obtain the mean system-length as

$$L = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \frac{G''_{ij}(1)G'(1) - G'_{ij}(1)G''(1)}{2[G'(1)]^2}, \tag{15}$$

where $h''(\xi)$ is the second derivative of $h(z)$ at $z = \xi$.

Now, after substituting the value of $\boldsymbol{\pi}(0)$ in Eq. 12, and let $\overline{\boldsymbol{\pi}} = \boldsymbol{\pi}^*(1)$, we have

$$\overline{\boldsymbol{\pi}} = \left[\frac{G'_{11}(1)}{G'(1)}, \frac{G'_{12}(1)}{G'(1)}, \ldots, \frac{G'_{ij}(1)}{G'(1)}, \ldots, \frac{G'_{m_1 m_2}(1)}{G'(1)}\right]. \tag{16}$$

Having found $\boldsymbol{\pi}(0)$, we now give our attention to calculate the remaining state probabilities $\boldsymbol{\pi}(n)$, $n \geq 1$. As each element of $\boldsymbol{\pi}^*(z)$, i.e., $\pi^*_{ij}(z)$ in Eq. 11, is a rational function with completely known polynomials both in the numerator (after substituting the value of $\boldsymbol{\pi}(0)$) and the denominator, we proceed to find its partial fractions. To do this we need to have the knowledge of the zeros of $det[\mathbf{A}(z)]$. The zeros of $det[\mathbf{A}(z)]$ with absolute value less or equal to one are also the zeros of the numerator of $\pi^*_{ij}(z)$. Therefore in making partial fractions of $\pi^*_{ij}(z)$, these zeros do not play any further role. So, we need to have the knowledge of the zeros of $det[\mathbf{A}(z)]$ whose absolute value is greater than one. Since it is shown that the equation $det[\mathbf{A}(z)] = 0$ has exactly mb roots each with absolute value greater than one, applying the partial-fraction method on the ij-th component $\pi^*_{ij}(z)$ of $\boldsymbol{\pi}^*(z)$, we have

$$\pi^*_{ij}(z) = \sum_{k=1}^{mb} \frac{c_{k,ij}}{\alpha_k - z}, \quad 1 \le i \le m_1, \ \ 1 \le j \le m_2, \tag{17}$$

where

$$c_{k,ij} = -\frac{G_{ij}(\alpha_k)}{G'(\alpha_k)}, \quad k = 1, 2, \ldots, mb.$$

Now, collecting the coefficient of z^n from both sides of Eq. 17, we have

$$\pi_{ij}(n) = \sum_{k=1}^{mb} \frac{c_{k,ij}}{\alpha_k^{n+1}}, \quad 1 \le i \le m_1, \ \ 1 \le j \le m_2, \ \ n \ge 0,$$

and hence

$$\boldsymbol{\pi}(n) = \left[\sum_{k=1}^{mb} \frac{c_{k,11}}{\alpha_k^{n+1}}, \sum_{k=1}^{mb} \frac{c_{k,12}}{\alpha_k^{n+1}}, \ldots, \sum_{k=1}^{mb} \frac{c_{k,m_1m_2}}{\alpha_k^{n+1}} \right], \quad n \ge 0. \tag{18}$$

Clearly, it is shown that $\pi_{ij}(n)$, $n \ge 0$, is a sum of geometric terms.

The mean system-length can be obtained from Eq. 18 as

$$L = \sum_{n=1}^{\infty} n\boldsymbol{\pi}(n)\mathbf{e}. \tag{19}$$

Although one can get the mean system-length L from either Eq. 15 or Eq. 19, it is easier to get the mean system-length from Eq. 15. From the Little's law, we also have mean sojourn-time W as

$$W = \frac{L}{\lambda^\star}.$$

Remark 1 We assumed that the roots of the c.e. are distinct. Many authors have shown in queueing theory that the roots are generally distinct and follow a nice pattern, see, e.g., Tijms HC (2003) and Janssen and Leeuwaarden (2005), and Chaudhry et al. (1990). Our computational experience also indicates that, in general, the roots happen to be distinct. One may note here that if some roots of $det[\mathbf{A}(z)] = 0$ inside and on the unit circle ($|z| = 1$) are repeated, Eqs. 13 and 14 need to be modified slightly to get the vector $\boldsymbol{\pi}(0)$. Again, if some roots of $det[\mathbf{A}(z)] = 0$ outside the unit circle ($|z| = 1$) are repeated, Eq. 17 needs to be modified slightly to get the unknown constants.

Remark 2 It may be pointed out that $\overline{\boldsymbol{\pi}} = \overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s$, when the service process runs during idle periods of the system without generating any real service completions (Model 2). One can check the accuracy of $\boldsymbol{\pi}(0)$'s with the relation $\overline{\boldsymbol{\pi}} = \overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s$, which is independent of the roots. When the service process is interrupted during idle periods of the system (Model 1), we have $\overline{\boldsymbol{\pi}} \neq \overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s$. For this case, one can also check the accuracy of $\boldsymbol{\pi}(0)$'s with the relation $\boldsymbol{\pi}(0)\mathbf{e} = 1 - \rho$, which is also independent of the roots. It is noted here that $\overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s$ is the stationary probability vector of the Markov process with generator $\mathbf{D} \otimes \mathbf{L}$.

Remark 3 For the $BMAP/M/1$ case, Model 1 and Model 2 become identical. Hence, $\overline{\boldsymbol{\pi}} = \overline{\boldsymbol{\pi}}_a$ and $\boldsymbol{\pi}(0)\mathbf{e} = 1 - \rho$ are valid for both the models.

3.2 System-Length Distribution at Arrival Epoch

Let $\boldsymbol{\pi}^{-}(n) = \left[\pi^{-}_{11}(n), \ldots, \pi^{-}_{1m_2}(n), \ldots, \pi^{-}_{ij}(n), \ldots, \pi^{-}_{m_1 1}(n), \ldots, \pi^{-}_{m_1 m_2}(n)\right]$, $n \geq 0$, denote the row vector according to the block structure of the generator $\mathbf{Q}$, where $\pi^{-}_{ij}(n)$ represents the arrival epoch probability that an arbitrary customer of an arriving batch finds n customers in the system with the arrival process being in phase i and the service process being in phase j. Then, we have

$$\boldsymbol{\pi}^{-}(n) = \sum_{r=0}^{n} \boldsymbol{\pi}(r) \left(\mathbf{H}_{n+1-r} \otimes \mathbf{I}_{m_2}\right), \quad n \geq 0, \tag{20}$$

where $\mathbf{H}_k$, $k \geq 1$, is a matrix of order $m_1 \times m_1$ whose (i, j)-th element $[H_k]_{ij}$ represents the probability that the position of an arbitrary customer in an arriving batch is k with phase changes from state i to j. In this connection, the interested reader is referred to Samanta et al. (2007).

For the sake of completeness, the procedure is briefly described here. The probability that an arbitrary customer belongs to a batch of size n is

$$Pr\{BS = n\} = \frac{\overline{\boldsymbol{\pi}} n \mathbf{D}_n \mathbf{e}}{\lambda^*}, \quad n = 1, 2, 3, \ldots,$$

where 'BS' represents the batch size.

The probability that an arbitrary customer occupies a position, say k-th position, in a batch of size n and phase changes from state i to j is

$$[Pr\{PA = k | BS = n\}]_{ij} = \frac{1}{n} \cdot \frac{(D_n)_{i,j}}{\overline{\boldsymbol{\pi}} \mathbf{D}_n \mathbf{e}}, \quad 1 \leq k \leq n,$$

where 'PA' represents the position of an arbitrary customer.

The probability that the position of an arbitrary customer in an arriving batch of size n is k with phase changes from state i to j is given by

$$[Pr\{BS = n, PA = k\}]_{ij} = \frac{\overline{\boldsymbol{\pi}} n \mathbf{D}_n \mathbf{e}}{\lambda^*} \cdot \frac{1}{n} \cdot \frac{(D_n)_{i,j}}{\overline{\boldsymbol{\pi}} \mathbf{D}_n \mathbf{e}} = \frac{(D_n)_{i,j}}{\lambda^*}, \quad 1 \leq k \leq n.$$

Hence the probability that the position of an arbitrary customer in an arriving batch is k with phase changes from state i to j is given by

$$[H_k]_{ij} = \sum_{n=k}^{\infty} [Pr\{BS = n, PA = k\}]_{ij} = \frac{1}{\lambda^*} \sum_{n=k}^{\infty} (D_n)_{i,j}, \quad k = 1, 2, 3, \ldots.$$

Therefore, in matrix notation, we have

$$\mathbf{H}_k = \frac{1}{\lambda^*} \sum_{n=k}^{\infty} \mathbf{D}_n, \quad k = 1, 2, 3, \ldots.$$

3.3 System-Length Distribution at Post-Departure Epoch

Let $\boldsymbol{\pi}^+(n) = \left[\pi_{11}^+(n), \ldots, \pi_{1m_2}^+(n), \ldots, \pi_{ij}^+(n), \ldots, \pi_{m_1 1}^+(n), \ldots, \pi_{m_1 m_2}^+(n)\right]$, $n \geq 0$, denote the row vector according to the block structure of the generator $\mathbf{Q}$, where $\pi_{ij}^+(n)$ represents the post-departure epoch probability that there are n customers in the system immediately after a service completion with the arrival process being in phase i and the service process being in phase j. The post-departure epoch thus occurs immediately after the server has either reduced the queue or has become idle. Hence, using level-crossing arguments, we have

$$\boldsymbol{\pi}^+(n) = \frac{1}{\lambda^\star} \boldsymbol{\pi}(n+1) \left(\mathbf{I}_{m_1} \otimes \mathbf{L}_1\right), \quad n \geq 0.$$

It is noted that the relation $\boldsymbol{\pi}^+(n)\mathbf{e} = \boldsymbol{\pi}^-(n)\mathbf{e}$, $n \geq 0$, also holds, as it should.

3.4 Actual Sojourn-Time Distribution

In this section, we obtain the actual sojourn time of an arbitrary customer in an arriving batch. By the sojourn time we mean the total time spent by an arbitrary customer of a batch in the system (from its arrival until departure). Let $\mathbf{W}(x) = \left[W_{11}(x), \ldots, W_{1m_2}(x), \ldots, W_{ij}(x), \ldots, W_{m_1 1}(x), \ldots, W_{m_1 m_2}(x)\right]$, $x \geq 0$, denote the row vector according to the block structure of the generator $\mathbf{Q}$, where $W_{ij}(x)$ represents the stationary joint probability that the sojourn time is less than or equal to x, the arrival phase is in i *at time* x and the service phase is in j *at time* x, given that arbitrary customer arrived *at time* 0. Further, let $\widetilde{\mathbf{W}}(\theta) = \left[\widetilde{W}_{11}(\theta), \ldots, \widetilde{W}_{1m_2}(\theta), \ldots, \widetilde{W}_{ij}(\theta), \ldots, \widetilde{W}_{m_1 1}(\theta), \ldots, \widetilde{W}_{m_1 m_2}(\theta)\right]$, where $\widetilde{W}_{ij}(\theta)$ is the Laplace-Stieltjes transform (LST) of $W_{ij}(x)$, i.e.,

$$\widetilde{W}_{ij}(\theta) = \int_0^\infty e^{-\theta x} dW_{ij}(x), \quad \Re(\theta) \geq 0.$$

Here, let $\boldsymbol{\Psi}_n(x)$ denote the matrix of order $m_2 \times m_2$ whose (i, j)th element represents the probability that n customers will be served within time x and the service phase is j at time x, provided at the initial instant of arrival epoch there were n customers in the system and the service process was in phase i. Let $\widetilde{\boldsymbol{\Psi}}_n(\theta)$ denote the LST of the matrix $\boldsymbol{\Psi}_n(x)$. Since the probability that the service of a customer is completed in the time interval $(x, x+dx]$ is given by the matrix $e^{\mathbf{L}_0 x}\mathbf{L}_1 dx + o(dx)$ and the total service time of n customers is the sum of their service times, we have

$$\widetilde{\boldsymbol{\Psi}}_n(\theta) = [\widetilde{\boldsymbol{\Psi}}_1(\theta)]^n, \; n \geq 1,$$

where

$$\begin{aligned} \widetilde{\boldsymbol{\Psi}}_1(\theta) &= \int_0^\infty e^{-\theta x} e^{\mathbf{L}_0 x} \mathbf{L}_1 dx \\ &= (\theta \mathbf{I}_{m_2} - \mathbf{L}_0)^{-1} \mathbf{L}_1. \end{aligned} \tag{21}$$

Hence, the LST of the stationary distribution of the sojourn time of an arbitrary customer in an arriving batch is given by

$$\widetilde{\mathbf{W}}(\theta) = \sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n) \left(\mathbf{I}_{m_1} \otimes [\widetilde{\boldsymbol{\Psi}}_1(\theta)]^{n+1}\right). \tag{22}$$

Using Eq. 20 in Eq. 22, after simplification, we obtain

$$\widetilde{\mathbf{W}}(\theta) = \frac{1}{\lambda^{\star}} \left(\sum_{n=0}^{\infty} \boldsymbol{\pi}(n) \left(\mathbf{I}_{m_1} \otimes \widetilde{\boldsymbol{\Psi}}_1(\theta)\right)^n\right) \left(\sum_{k=1}^{\infty} \mathbf{D}_k \otimes \sum_{r=1}^{k} [\widetilde{\boldsymbol{\Psi}}_1(\theta)]^r\right). \tag{23}$$

Using Eq. 11 and 21 in Eq. 23, after simplification, we obtain

$$\widetilde{\mathbf{W}}(\theta) = \frac{\boldsymbol{\pi}(0) \left[\theta \left(\mathbf{I}_{m_1} \otimes \mathbf{I}_{m_2}\right) - \left(\mathbf{I}_{m_1} \otimes \boldsymbol{\Phi}\right)\right] adj\, [\boldsymbol{\Omega}(\theta)] \left(\sum_{k=1}^{\infty} \mathbf{D}_k \otimes \sum_{r=1}^{k} [\widetilde{\boldsymbol{\Psi}}_1(\theta)]^r\right)}{\lambda^{\star} det\, [\boldsymbol{\Omega}(\theta)]}, \tag{24}$$

where

$$\boldsymbol{\Omega}(\theta) = \theta \left(\mathbf{I}_{m_1} \otimes \mathbf{I}_{m_2}\right) + \sum_{k=0}^{\infty} \mathbf{D}_k \otimes [\widetilde{\boldsymbol{\Psi}}_1(\theta)]^k.$$

Proceeding in a similar manner as we did in Subsection 3.1, the equation $det[\boldsymbol{\Omega}(\theta)] = 0$ has $(m-1)$ roots with positive real parts, one root at $\theta = 0$, while other mb roots each with negative real part. We call these roots whose real part is negative as β_k $(1 \leq k \leq mb)$. We assume that all β_k $(1 \leq k \leq mb)$ are distinct. We can get the vector $\boldsymbol{\pi}(0)$ from Eq. 24 in a similar way as we did before in Subsection 3.1. However, since we already know the vector $\boldsymbol{\pi}(0)$, we use its value to calculate the actual sojourn-time distribution.

Now each element of $\widetilde{\mathbf{W}}(\theta)$, i.e., $\widetilde{W}_{ij}(\theta)$ in Eq. 24 is a rational function with completely known polynomials both in the numerator (after substituting the value of $\boldsymbol{\pi}(0)$) and the denominator. Therefore, we rewrite the right-hand side of $\widetilde{\mathbf{W}}(\theta)$ in Eq. 24 as

$$\widetilde{\mathbf{W}}(\theta) = \left[\frac{F_{11}(\theta)}{F(\theta)}, \frac{F_{12}(\theta)}{F(\theta)}, \ldots, \frac{F_{ij}(\theta)}{F(\theta)}, \ldots, \frac{F_{m_1 m_2}(\theta)}{F(\theta)}\right], \tag{25}$$

where $F_{ij}(\theta)$ is the ij-th component of the vector

$$\frac{1}{\lambda^{\star}} \boldsymbol{\pi}(0) \left[\theta \left(\mathbf{I}_{m_1} \otimes \mathbf{I}_{m_2}\right) - \left(\mathbf{I}_{m_1} \otimes \boldsymbol{\Phi}\right)\right] adj\, [\boldsymbol{\Omega}(\theta)] \left(\sum_{k=1}^{\infty} \mathbf{D}_k \otimes \sum_{r=1}^{k} [\widetilde{\boldsymbol{\Psi}}_1(\theta)]^r\right)$$

and $F(\theta) = det[\boldsymbol{\Omega}(\theta)]$.

Once $\boldsymbol{\pi}(0)$ is known, we can obtain mean actual sojourn-time W directly using Eq. 25 by differentiating and taking the limit as $\theta \to 0$. Using the fact that $F(0) = 0$, we obtain the mean actual sojourn-time as

$$W = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \frac{F'_{ij}(0) F''(0) - F''_{ij}(0) F'(0)}{2[F'(0)]^2}. \tag{26}$$

Let $\overline{\mathbf{W}} = [\overline{W}_{11}, \overline{W}_{12}, \ldots, \overline{W}_{ij}, \ldots, \overline{W}_{m_1 m_2}]$ be the stationary probability vector of the actual sojourn time of an arbitrary customer in an arriving batch. From Eq. 25, using $\overline{\mathbf{W}} =$

$\widetilde{\mathbf{W}}(0)$, we have

$$\overline{\mathbf{W}} = \left[\frac{F'_{11}(0)}{F'(0)}, \frac{F'_{12}(0)}{F'(0)}, \ldots, \frac{F'_{ij}(0)}{F'(0)}, \ldots, \frac{F'_{m_1 m_2}(0)}{F'(0)} \right]. \tag{27}$$

Now applying the partial-fraction method on the ij-th component $\widetilde{W}_{ij}(\theta)$ of $\widetilde{\mathbf{W}}(\theta)$, we have

$$\widetilde{W}_{ij}(\theta) = \sum_{k=1}^{mb} \frac{d_{k,ij}}{\theta - \beta_k}, \quad 1 \le i \le m_1, \ 1 \le j \le m_2, \tag{28}$$

where

$$d_{k,ij} = \frac{F_{ij}(\beta_k)}{F'(\beta_k)}, \quad k = 1, 2, \ldots, mb.$$

Now, taking the inverse Laplace-transform of Eq. 28, component-wise explicit closed-form expression of the probability density function (p.d.f.) $w_{ij}(x) = \frac{d}{dx} W_{ij}(x)$ is given by

$$w_{ij}(x) = \sum_{k=1}^{mb} d_{k,ij} e^{\beta_k x}, \quad 1 \le i \le m_1, \ 1 \le j \le m_2, \ x \ge 0. \tag{29}$$

Clearly, it is shown that $w_{ij}(x)$, $x \ge 0$, is a sum of exponential terms.

From Eq. 29, we can get $W_{ij}(x)$ as

$$\begin{aligned} W_{ij}(x) &= \int_0^x w_{ij}(t)dt \\ &= \sum_{k=1}^{mb} \frac{d_{k,ij}}{\beta_k} e^{\beta_k x} - \sum_{k=1}^{mb} \frac{d_{k,ij}}{\beta_k} \\ &= \overline{W}_{ij} + \sum_{k=1}^{mb} \frac{d_{k,ij}}{\beta_k} e^{\beta_k x}, \quad x \ge 0, \end{aligned} \tag{30}$$

where

$$\overline{W}_{ij} = -\sum_{k=1}^{mb} \frac{d_{k,ij}}{\beta_k}$$

is obtained by taking $\theta \to 0$ in Eq. 28.

Hence, the actual sojourn-time distribution is obtained as

$$\mathbf{W}(x) = \overline{\mathbf{W}} + \sum_{k=1}^{mb} \frac{1}{\beta_k} \mathbf{d}_k e^{\beta_k x}, \quad x \ge 0, \tag{31}$$

where $\mathbf{d}_k = \left[d_{k,11}, d_{k,12}, \ldots, d_{k,ij}, \ldots, d_{k,m_1 m_2} \right]$.

The mean sojourn-time from Eq. 28 is given by

$$W = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \sum_{k=1}^{mb} \frac{d_{k,ij}}{\beta_k^2}. \tag{32}$$

Although one can get the mean sojourn-time from either Eq. 26 or Eq. 32, it is easier to get the mean sojourn-time from Eq. 26.

Table 1 System-length distribution at random epoch (Model 1)

n	$\pi_{11}(n)$	$\pi_{12}(n)$	$\pi_{13}(n)$	$\pi_{21}(n)$	$\pi_{22}(n)$	$\pi_{23}(n)$	$\boldsymbol{\pi}(n)\mathbf{e}$
0	0.106975	0.030037	0.023295	0.151447	0.042515	0.032965	0.387234
1	0.008489	0.002604	0.001902	0.012947	0.003960	0.002893	0.032794
2	0.009025	0.002771	0.002023	0.014212	0.004347	0.003177	0.035555
3	0.009522	0.002925	0.002135	0.015657	0.004794	0.003506	0.038539
4	0.007947	0.002442	0.001783	0.011382	0.003501	0.002557	0.029612
5	0.008402	0.002581	0.001885	0.011893	0.003659	0.002672	0.031092
10	0.006889	0.002134	0.001558	0.011398	0.003510	0.002563	0.028052
50	0.000193	0.000061	0.000044	0.000320	0.000100	0.000073	0.000792
100	0.000002	0.000001	0.000001	0.000004	0.000001	0.000001	0.000010
115	0.000001	0.000000	0.000000	0.000001	0.000000	0.000000	0.000003
130	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001
sum	0.261298	0.077863	0.058219	0.396138	0.118221	0.088261	1.000000
$L = 8.071949$, $W = 0.877024$							

4 Numerical Results

We have carried out numerical work based on the procedure discussed in this paper. During the computational work, several outputs were generated for testing the procedure but only a few of them are appended here. All the calculations were performed using MAPLE and some sample outputs are shown in the following tables. We use the command $fsolve(G(z), z, complex)$ to find all roots of the characteristic equation $G(z) = 0$. The equation $G(z) = 0$ has $m_1 m_2(b + 1) = 78$ $(m_1 = 2, m_2 = 3, b = 12)$ roots in total. Out of these 78 roots, $m_1 m_2 - 1 = 5$ roots $\gamma_1, \gamma_2, \ldots, \gamma_5$ are inside the unit circle, one root at $z = 1$ and all other $m_1 m_2 b = 72$ roots are outside the unit circle and we call these roots as $\alpha_1, \alpha_2, \ldots, \alpha_{72}$. These 72 outside roots are used in Eq. 17 for the partial-fractions. Using the inside roots $\gamma_1, \gamma_2, \ldots, \gamma_5$ in Eq. 13 and $z = 1$ in Eq. 14, and then solving these

Table 2 System-length distribution at arrival epoch (Model 1)

n	$\pi^-_{11}(n)$	$\pi^-_{12}(n)$	$\pi^-_{13}(n)$	$\pi^-_{21}(n)$	$\pi^-_{22}(n)$	$\pi^-_{23}(n)$	$\boldsymbol{\pi}^-(n)\mathbf{e}$
0	0.010952	0.003075	0.002384	0.024767	0.006953	0.005392	0.053524
1	0.011848	0.003349	0.002585	0.026815	0.007581	0.005850	0.058029
2	0.012814	0.003645	0.002801	0.029033	0.008260	0.006347	0.062900
3	0.010532	0.003032	0.002311	0.021516	0.006212	0.004725	0.048328
4	0.011074	0.003199	0.002433	0.022549	0.006530	0.004958	0.050744
5	0.011637	0.003372	0.002560	0.023608	0.006857	0.005196	0.053230
10	0.007614	0.002282	0.001696	0.022948	0.006785	0.005085	0.046411
50	0.000203	0.000064	0.000046	0.000564	0.000177	0.000129	0.001184
100	0.000003	0.000001	0.000001	0.000007	0.000002	0.000002	0.000015
115	0.000001	0.000000	0.000000	0.000002	0.000001	0.000001	0.000004
130	0.000000	0.000000	0.000000	0.000001	0.000000	0.000000	0.000001
sum	0.187326	0.055848	0.041737	0.470125	0.140216	0.104747	1.000000

Table 3 System-length distribution at post-departure epoch (Model 1)

n	$\pi^+_{11}(n)$	$\pi^+_{12}(n)$	$\pi^+_{13}(n)$	$\pi^+_{21}(n)$	$\pi^+_{22}(n)$	$\pi^+_{23}(n)$	$\boldsymbol{\pi}^+(n)\mathbf{e}$
0	0.014149	0.003975	0.003085	0.021570	0.006053	0.004692	0.053524
1	0.015043	0.004227	0.003282	0.023680	0.006645	0.005152	0.058029
2	0.015873	0.004462	0.003465	0.026094	0.007324	0.005683	0.062900
3	0.013249	0.003724	0.002893	0.018978	0.005337	0.004147	0.048328
4	0.014006	0.003937	0.003058	0.019831	0.005577	0.004335	0.050744
5	0.014837	0.004170	0.003239	0.020657	0.005810	0.004516	0.053230
10	0.011171	0.003150	0.002458	0.019757	0.005555	0.004319	0.046411
50	0.000296	0.000084	0.000066	0.000490	0.000139	0.000109	0.001184
100	0.000004	0.000001	0.000001	0.000006	0.000002	0.000001	0.000015
115	0.000001	0.000000	0.000000	0.000002	0.000000	0.000000	0.000004
130	0.000000	0.000000	0.000000	0.000001	0.000000	0.000000	0.000001
sum	0.257599	0.072632	0.056654	0.408357	0.115067	0.089690	1.000000

$m_1 m_2 = 6$ linearly independent simultaneous equations, we get the vector $\boldsymbol{\pi}(0)$. In the case of waiting-time analysis, we also use the same command for the characteristic equation $F(\theta) = 0$ which has 5 roots with positive real parts, one root at $\theta = 0$, while other 72 roots each with negative real part. These 72 roots each with negative real part are used in Eq. 28 for the partial-fractions. Numerical results for Model 1 have been presented in Tables 1, 2, 3 and 4. Similarly, numerical results for Model 2 have also been given in Tables 5, 6, 7 and 8. It is found that the mean sojourn time W using Little's rule given in Tables 1 and 5 match with the results obtained from the actual sojourn-time distribution. Moreover, it is found in Table 1 for Model 1 that $\boldsymbol{\pi}(0)\mathbf{e} = 1 - \rho$ which represents the probability that the server is idle. While for Model 2 it is found in Table 5 that $\overline{\boldsymbol{\pi}} = \overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s$. These are internal checks

Table 4 Actual sojourn-time distribution (Model 1)

x	$W_{11}(x)$	$W_{12}(x)$	$W_{13}(x)$	$W_{21}(x)$	$W_{22}(x)$	$W_{23}(x)$	$\mathbf{W}(x)\mathbf{e}$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.017372	0.004784	0.003642	0.039090	0.010766	0.008196	0.083851
0.5	0.081906	0.022949	0.017743	0.183301	0.051342	0.039675	0.396917
1.0	0.132876	0.037375	0.029057	0.319557	0.089860	0.069832	0.678557
1.5	0.159626	0.044951	0.035003	0.392762	0.110591	0.086105	0.829038
2.0	0.173731	0.048945	0.038138	0.431811	0.121648	0.094782	0.909055
2.5	0.181225	0.051067	0.039803	0.452595	0.127533	0.099400	0.951624
3.0	0.185210	0.052196	0.040689	0.463652	0.130663	0.101857	0.974267
4.0	0.188457	0.053115	0.041411	0.472662	0.133215	0.103859	0.992719
5.0	0.189376	0.053375	0.041615	0.475211	0.133937	0.104426	0.997940
6.0	0.189636	0.053449	0.041673	0.475933	0.134141	0.104586	0.999417
7.0	0.189710	0.053469	0.041689	0.476137	0.134199	0.104632	0.999835
10.0	0.189738	0.053478	0.041695	0.476215	0.134221	0.104649	0.999996
11.5	0.189739	0.053478	0.041695	0.476217	0.134221	0.104649	0.999999
20.0	0.189739	0.053478	0.041695	0.476217	0.134222	0.104649	1.000000

Table 5 System-length distribution at random epoch (Model 2)

n	$\pi_{11}(n)$	$\pi_{12}(n)$	$\pi_{13}(n)$	$\pi_{21}(n)$	$\pi_{22}(n)$	$\pi_{23}(n)$	$\boldsymbol{\pi}(n)\mathbf{e}$
0	0.104765	0.032089	0.023438	0.148329	0.045408	0.033168	0.387198
1	0.008473	0.002608	0.001912	0.012917	0.003967	0.002911	0.032789
2	0.008993	0.002785	0.002040	0.014145	0.004376	0.003212	0.035551
3	0.009475	0.002963	0.002148	0.015548	0.004885	0.003532	0.038551
4	0.007940	0.002443	0.001788	0.011372	0.003503	0.002563	0.029609
5	0.008384	0.002586	0.001896	0.011872	0.003666	0.002685	0.031089
10	0.006873	0.002140	0.001568	0.011365	0.003521	0.002583	0.028050
50	0.000193	0.000061	0.000044	0.000320	0.000100	0.000073	0.000792
100	0.000002	0.000001	0.000001	0.000004	0.000001	0.000001	0.000010
115	0.000001	0.000000	0.000000	0.000001	0.000000	0.000000	0.000003
130	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001
sum	0.258761	0.080111	0.058508	0.392407	0.121487	0.088726	1.000000
$L = 8.072680$, $W = 0.877103$							

which are independent of the roots. We choose the following matrices $\mathbf{D}_n$, $n \geq 0$, of order 2 of the arrival process (BMAP):

$$\mathbf{D}_0 = \begin{bmatrix} -4.70 & 3.17 \\ 2.46 & -3.55 \end{bmatrix}, \quad \mathbf{D}_3 = \begin{bmatrix} 0.13 & 0.40 \\ 0.11 & 0.32 \end{bmatrix}, \quad \mathbf{D}_7 = \begin{bmatrix} 0.23 & 0.27 \\ 0.16 & 0.09 \end{bmatrix},$$

$$\mathbf{D}_{12} = \begin{bmatrix} 0.20 & 0.30 \\ 0.00 & 0.41 \end{bmatrix}.$$

This leads to

$$\overline{\boldsymbol{\pi}}_a = \begin{bmatrix} 0.397380 & 0.602620 \end{bmatrix}$$

with $\lambda^{\star} = 9.203799$.

Table 6 System-length distribution at arrival epoch (Model 2)

n	$\pi^-_{11}(n)$	$\pi^-_{12}(n)$	$\pi^-_{13}(n)$	$\pi^-_{21}(n)$	$\pi^-_{22}(n)$	$\pi^-_{23}(n)$	$\boldsymbol{\pi}^-(n)\mathbf{e}$
0	0.010726	0.003285	0.002399	0.024257	0.007427	0.005425	0.053519
1	0.011620	0.003560	0.002601	0.026300	0.008056	0.005886	0.058023
2	0.012582	0.003857	0.002819	0.028508	0.008739	0.006387	0.062894
3	0.010362	0.003185	0.002326	0.021182	0.006513	0.004756	0.048325
4	0.010905	0.003352	0.002448	0.022215	0.006832	0.004989	0.050741
5	0.011467	0.003525	0.002575	0.023273	0.007158	0.005228	0.053227
10	0.00755	0.002333	0.001705	0.022703	0.007000	0.005115	0.046409
50	0.000203	0.000064	0.000047	0.000564	0.000177	0.000129	0.001184
100	0.000003	0.000001	0.000001	0.000007	0.000002	0.000002	0.000015
115	0.000001	0.000000	0.000000	0.000002	0.000001	0.000001	0.000004
130	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001
sum	0.185526	0.057438	0.041949	0.465643	0.144160	0.105285	1.000000

Table 7 System-length distribution at post-departure epoch (Model 2)

n	$\pi^+_{11}(n)$	$\pi^+_{12}(n)$	$\pi^+_{13}(n)$	$\pi^+_{21}(n)$	$\pi^+_{22}(n)$	$\pi^+_{23}(n)$	$\boldsymbol{\pi}^+(n)\mathbf{e}$
0	0.014137	0.003975	0.003095	0.021549	0.006052	0.004711	0.053519
1	0.015016	0.004231	0.003304	0.023624	0.006651	0.005197	0.058023
2	0.015820	0.004480	0.003498	0.025967	0.007369	0.005760	0.062894
3	0.013244	0.003724	0.002897	0.018971	0.005336	0.004153	0.048325
4	0.013994	0.003937	0.003069	0.019815	0.005577	0.004349	0.050741
5	0.014806	0.004174	0.003264	0.020623	0.005815	0.004544	0.053227
10	0.011149	0.003154	0.002477	0.019702	0.005562	0.004365	0.046409
50	0.000296	0.000084	0.000066	0.000490	0.000139	0.000109	0.001184
100	0.000004	0.000001	0.000000	0.000006	0.000002	0.000001	0.000015
115	0.000001	0.000000	0.000000	0.000002	0.000000	0.000000	0.000004
130	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001
sum	0.257297	0.072705	0.056888	0.407776	0.115209	0.090124	1.000000

We choose the following matrices $\mathbf{L}_0$ and $\mathbf{L}_1$ of order 3 of the service process (MSP):

$$\mathbf{L}_0 = \begin{bmatrix} -15.42 & 0.11 & 0.10 \\ 0.19 & -18.05 & 5.06 \\ 0.00 & 5.08 & -22.30 \end{bmatrix}, \quad \mathbf{L}_1 = \begin{bmatrix} 13.06 & 2.15 & 0.00 \\ 0.02 & 7.04 & 5.74 \\ 10.15 & 0.00 & 7.07 \end{bmatrix}.$$

This leads to

$$\overline{\boldsymbol{\pi}}_s = \begin{bmatrix} 0.651169 & 0.201597 & 0.147234 \end{bmatrix}$$

with $\mu^\star = 15.020090$, and hence $\rho = 0.612766$.

Table 8 Actual sojourn-time distribution (Model 2)

x	$W_{11}(x)$	$W_{12}(x)$	$W_{13}(x)$	$W_{21}(x)$	$W_{22}(x)$	$W_{23}(x)$	$\mathbf{W}(x)\mathbf{e}$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.017197	0.004831	0.003750	0.038694	0.010873	0.008440	0.083785
0.5	0.081626	0.023011	0.017944	0.182670	0.051483	0.040126	0.396861
1.0	0.132598	0.037439	0.029261	0.318929	0.090005	0.070290	0.678523
1.5	0.159351	0.045016	0.035209	0.392142	0.110738	0.086564	0.829020
2.0	0.173458	0.049010	0.038344	0.431196	0.121796	0.095242	0.909046
2.5	0.180952	0.051133	0.040009	0.451982	0.127681	0.099861	0.951618
3.0	0.184938	0.052261	0.040895	0.463039	0.130812	0.102318	0.974264
4.0	0.188185	0.053181	0.041616	0.472051	0.133364	0.104321	0.992718
5.0	0.189104	0.053441	0.041821	0.474600	0.134086	0.104887	0.997940
6.0	0.189364	0.053514	0.041879	0.475322	0.134290	0.105048	0.999417
7.0	0.189438	0.053535	0.041895	0.475526	0.134348	0.105093	0.999835
10.0	0.189466	0.053543	0.041901	0.475605	0.134370	0.105111	0.999996
11.5	0.189467	0.053543	0.041901	0.475606	0.134371	0.105111	0.999999
20.0	0.189467	0.053544	0.041901	0.475606	0.134371	0.105111	1.000000

Hence, we have

$$\overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s = \left[\, 0.258761 \;\; 0.080111 \;\; 0.058508 \;\; 0.392407 \;\; 0.121487 \;\; 0.088726 \,\right].$$

5 Conclusion

In this paper, we have presented a procedure to evaluate the steady-state system-length distributions at random, arrival and post-departure epochs, and the actual sojourn-time distribution of the $BMAP/MSP/1$ queue. The analysis is based on roots of the associated characteristic equation of the vector-generating function (VGF) of system-length distribution at random epoch. The proposed method can be applied to get computational results of more complex models such as bulk service $BMAP/BMSP/1$ and multi-server $BMAP/MSP/c$ queues.

Acknowledgements The authors are thankful to the referee whose valuable comments and suggestions led to improvement in the paper. The first author was supported by the Foundation for Science and Technology (FCT) research grant SFRH/BPD/64372/2009. The second author was supported partially by NSERC. The third author received partial support from FCT.

Appendix A

Theorem A.1 *Every function $A_{k,k}(z)$, $1 \le k \le m$, has exactly one zero inside or on the unit circle, and b zeros outside the unit circle.*

Proof From Eq. 9, we have

$$A_{k,k}(z) = \sum_{n=0}^{b} (D_n)_{i,i} z^{n+1} + z(L_0)_{j,j} + (L_1)_{j,j}, \qquad (33)$$
$$k = j + (i-1)m_2;\; i = 1, 2, \ldots, m_1,\; j = 1, 2, \ldots, m_2.$$

Each $A_{k,k}(z)$ is a polynomial function of degree $b+1$. Consider absolute values of $f(z) = z\left((D_0)_{i,i} + (L_0)_{j,j}\right)$ and $g(z) = \sum_{n=1}^{b} (D_n)_{i,i} z^{n+1} + (L_1)_{j,j}$ on the circle $|z| = 1+\delta$, where δ is positive and sufficiently small. Now, on the circle $|z| = 1+\delta$, using the Taylor series expansion, we have

$$\begin{aligned}
|g(z)| &\le \sum_{n=1}^{b} (D_n)_{i,i} (|z|)^{n+1} + (L_1)_{j,j} \\
&= \sum_{n=1}^{b} (D_n)_{i,i}\,(1 + (n+1)\delta) + (L_1)_{j,j} + o(\delta) \\
&= \sum_{n=1}^{b} (D_n)_{i,i}(1+\delta) + \delta \sum_{n=1}^{b} n (D_n)_{i,i} + (L_1)_{j,j} + o(\delta).
\end{aligned}$$

Since $\sum_{n=1}^{b}(D_n)_{i,i} \leq |(D_0)_{i,i}|$ and $(L_1)_{j,j} \leq |(L_0)_{j,j}|$, we have

$$|g(z)| \leq |(D_0)_{i,i}|(1+\delta) + \delta \sum_{n=1}^{b} n(D_n)_{i,i} + |(L_0)_{j,j}| + o(\delta). \tag{34}$$

Differentiating Eq. 9 w.r.t. z and setting $z = 1$, we have

$$\mathbf{A}'(1) = \mathbf{D} \otimes \mathbf{I}_{m_2} + \sum_{k=1}^{b} k\mathbf{D}_k \otimes \mathbf{I}_{m_2} + \mathbf{I}_{m_1} \otimes \mathbf{L}_0.$$

Since $(\overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s)\left(\mathbf{D} \otimes \mathbf{I}_{m_2}\right)\mathbf{e} = \mathbf{0}$, $(\overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s)\left(\sum_{k=1}^{b} k\mathbf{D}_k \otimes \mathbf{I}_{m_2}\right)\mathbf{e} = \lambda^\star$ and $(\overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s)\left(\mathbf{I}_{m_1} \otimes \mathbf{L}_0\right)\mathbf{e} = -\mu^\star$, we have

$$(\overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s)\,\mathbf{A}'(1)\mathbf{e} = \lambda^\star - \mu^\star. \tag{35}$$

Now assume the following inequality holds for some $i \in \{1, 2, \ldots, m\}$:

$$\sum_{j=1}^{m} A'_{i,j}(1) = \mathbf{A}'(1)\mathbf{e} \geq 0. \tag{36}$$

Pre-multiplying Eq. 36 by i-th component of $(\overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s)$ and then summing over i, we get

$$(\overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s)\,\mathbf{A}'(1)\mathbf{e} \geq 0 \Rightarrow \lambda^\star - \mu^\star \geq 0 \Rightarrow \rho \geq 1.$$

This contradicts $\rho < 1$, and hence

$$\begin{aligned}
&\mathbf{A}'(1)\mathbf{e} < \mathbf{0} \\
&\Rightarrow \sum_{n=1}^{b} n \sum_{k=1}^{m_1}(D_n)_{i,k} + \sum_{k=1}^{m_2}(L_0)_{j,k} < 0, \quad i = 1, 2, \ldots, m_1;\ j = 1, 2, \ldots, m_2 \\
&\Rightarrow \sum_{n=1}^{b} n(D_n)_{i,i} < |(L_0)_{j,j}|.
\end{aligned} \tag{37}$$

Hence, using Eq. 37 in Eq. 34, we have

$$|g(z)| < |f(z)|.$$

Hence, using the well-known Rouché's theorem, $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside the circle $|z| = 1 + \delta$. It is obvious that $f(z) = 0$ has exactly one zero inside the circle $|z| = 1 + \delta$. Thus, $f(z) + g(z) = A_{k,k}(z)$ has exactly one zero inside or on the unit circle. As $A_{k,k}(z)$ is a polynomial function of degree $b + 1$, other b zeros are outside the unit circle. □

Theorem A.2 *The following inequalities hold on the circle* $|z| = 1 + \delta$*:*

$$|A_{i,i}(z)| > \sum_{j=1,\ j\neq i}^{m} |A_{i,j}(z)|, \quad 1 \leq i \leq m. \tag{38}$$

Proof On the circle $|z| = 1 + \delta$, using the Taylor series expansion, we have

$$\begin{aligned} |A_{i,i}(z)| &\geq -A_{i,i}(|z|) \\ &= -A_{i,i}(1) - \delta A'_{i,i}(1) + o(\delta) \\ &= \sum_{j=1,\ j\neq i}^{m} A_{i,j}(1) - \delta A'_{i,i}(1) + o(\delta), \quad 1 \leq i \leq m, \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1,\ j\neq i}^{m} |A_{i,j}(z)| &\leq \sum_{j=1,\ j\neq i}^{m} A_{i,j}(|z|) \\ &= \sum_{j=1,\ j\neq i}^{m} A_{i,j}(1) + \delta \sum_{j=1,\ j\neq i}^{m} A'_{i,j}(1) + o(\delta), \quad 1 \leq i \leq m. \end{aligned}$$

Now assume the following inequality holds for some $i \in \{1, 2, \ldots, m\}$:

$$|A_{i,i}(z)| \leq \sum_{j=1,\ j\neq i}^{m} |A_{i,j}(z)|.$$

This implies

$$\begin{aligned} \sum_{j=1,\ j\neq i}^{m} A_{i,j}(1) - \delta A'_{i,i}(1) + o(\delta) &\leq \sum_{j=1,\ j\neq i}^{m} A_{i,j}(1) + \delta \sum_{j=1,\ j\neq i}^{m} A'_{i,j}(1) + o(\delta) \\ \Rightarrow -\delta A'_{i,i}(1) + o(\delta) &\leq \delta \sum_{j=1,\ j\neq i}^{m} A'_{i,j}(1) + o(\delta). \end{aligned}$$

When $\delta \to 0$, we have

$$\begin{aligned} 0 &\leq \sum_{j=1}^{m} A'_{i,j}(1) = \mathbf{A}'(1)\mathbf{e} \\ \Rightarrow 0 &\leq (\overline{\boldsymbol{\pi}}_a \otimes \overline{\boldsymbol{\pi}}_s)\, \mathbf{A}'(1)\mathbf{e} \\ \Rightarrow \lambda^\star &\geq \mu^\star \Rightarrow \rho \geq 1. \end{aligned}$$

This contradicts the system stability condition $\rho < 1$, and hence Eq. 38 is satisfied. □

Theorem A.3 *The determinant $det[\mathbf{A}(z)]$ has exactly $(m-1)$ zeros in $|z| < 1$, one zero at $z = 1$ and mb zeros in $|z| > 1$ (including multiplicity).*

Proof Mathematical induction is used to prove this theorem. Let us denote

$$R_n(z) = det[\mathbf{A}(z)] \tag{39}$$

represents the determinant of the underlying square matrix of order n. □

1. We show that the statement is true for $n = 1$.

   For $n = 1$, Eq. 39 becomes $R_1(z) = A(z)$, where $A(z) \equiv A_{1,1}(z) = z(-\lambda + \lambda X(z)) - \mu z + \mu$ with $X(z)$ being the probability generating function of batch size. The arrival process is a compound Poisson process with mean arrival rate $\lambda X'(1)$. Service time is exponential with rate μ. That is, $\mu = (L_1)_{1,1}$, $-\lambda = (D_0)_{1,1}$, $-\mu = (L_0)_{1,1}$, $\lambda g_n = (D_n)_{1,1}$, $n \geq 1$, where g_n represents the batch size with p.g.f. $X(z)$. Thus,

$A(z)$ is the characteristic function for the classical $M^X/M/1$ queue. This polynomial function is of degree $b+1$. Consider absolute values of $f(z) = -(\lambda+\mu)z$ and $g(z) = \lambda z X(z) + \mu$ on the circle $|z| = 1+\delta$, where δ is positive and sufficiently small. Now, on the circle $|z| = 1+\delta$, using the Taylor series expansion, we have

$$\begin{aligned} |g(z)| &\leq \lambda|z|X(|z|) + \mu \\ &= \lambda(1+\delta)(1+\delta X'(1)) + \mu + o(\delta) \\ &= \lambda(1+\delta) + \delta\lambda X'(1) + \mu + o(\delta). \end{aligned} \tag{40}$$

Since $\rho = \frac{\lambda X'(1)}{\mu} < 1$, we have $\lambda X'(1) < \mu$ which leads (40) that $|g(z)| < |f(z)|$. Hence, using the well-known Rouché's theorem, $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside the circle $|z| = 1+\delta$. It is obvious that $f(z) = 0$ has exactly one zero inside the circle $|z| = 1+\delta$. Thus, $f(z)+g(z) = A(z)$ has exactly one zero inside the circle $|z| = 1+\delta$. It is cleared that $A(z) = 0$ has one root at $z = 1$. Thus, $A(z)$ has no zeros in $|z| < 1$. As $A(z)$ is a polynomial function of degree $b+1$, other b zeros are outside the unit circle.

2. For $n = 2$, Eq. 39 becomes

$$\begin{aligned} R_2(z) &= \begin{vmatrix} A_{1,1}(z) & A_{1,2}(z) \\ A_{2,1}(z) & A_{2,2}(z) \end{vmatrix} \\ &= A_{2,2}(z)R_1(z) + A_{2,1}(z)C_{2,1}(z), \end{aligned} \tag{41}$$

where $C_{2,1}(z) = -A_{1,2}(z)$ is the cofactor of $A_{2,1}(z)$, and $R_1(z) = A_{1,1}(z)$,

Again, we can write Eq. 41 as

$$\begin{aligned} \left|\frac{R_2(z) - A_{2,2}(z)R_1(z)}{A_{2,2}(z)R_1(z)}\right| &= \left|\frac{A_{2,1}(z)C_{2,1}(z)}{A_{2,2}(z)R_1(z)}\right| \\ &= \frac{\left|A_{2,1}(z)\right|\left|y_{2,1}(z)\right|}{\left|A_{2,2}(z)\right|} < 1, \end{aligned}$$

where

$$|y_{2,1}(z)| = \frac{|C_{2,1}(z)|}{|R_1(z)|} = \frac{|A_{1,2}(z)|}{|A_{1,1}(z)|} < 1$$

and

$$\frac{\left|A_{2,1}(z)\right|}{\left|A_{2,2}(z)\right|} < 1$$

by Theorem A.2 for $m = 2$.

Hence, we have $|g(z)| < |f(z)|$, where $f(z) = A_{2,2}(z)R_1(z)$ and $g(z) = R_2(z) - A_{2,2}(z)R_1(z)$. By Rouché's theorem $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside the circle $|z| = 1+\delta$. By Theorem A.1, $f(z)$ has two zeros inside the circle $|z| = 1+\delta$. Since $A_{1,2}(1) = -A_{1,1}(1)$ and $A_{2,1}(1) = -A_{2,2}(1)$, $R_2(z) = 0$ has one root at $z = 1$. Thus, $R_2(z) = f(z)+g(z)$ has two zeros inside and on the unit circle. As $R_2(z)$ is a polynomial function of degree $2(b+1)$, other $2b$ zeros are outside the unit circle.

3. We assume that the statement is true for $n = m-1$. It must then be shown that the statement holds for $n = m$.

The determinant $R_n(z)$ for $n = m$ is given by

$$R_m(z) = \begin{vmatrix} A_{1,1}(z) & A_{1,2}(z) & \cdots & A_{1,m-1}(z) & A_{1,m}(z) \\ A_{2,1}(z) & A_{2,2}(z) & \cdots & A_{2,m-1}(z) & A_{2,m}(z) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m-1,1}(z) & A_{m-1,2}(z) & \cdots & A_{m-1,m-1}(z) & A_{m-1,m}(z) \\ A_{m,1}(z) & A_{m,2}(z) & \cdots & A_{m,m-1}(z) & A_{m,m}(z) \end{vmatrix}. \tag{42}$$

Now we can rewrite Eq. 42 in the following way

$$R_m(z) = \sum_{j=1}^{m-1} A_{m,j}(z)C_{m,j}(z) + A_{m,m}(z)R_{m-1}(z), \tag{43}$$

where $C_{m,j}(z)$ is the cofactor of $A_{m,j}(z)$.

Again, we can write Eq. 43 as

$$\begin{aligned} \left|\frac{R_m(z) - A_{m,m}(z)R_{m-1}(z)}{A_{m,m}(z)R_{m-1}(z)}\right| &= \left|\frac{\sum_{j=1}^{m-1} A_{m,j}(z)C_{m,j}(z)}{A_{m,m}(z)R_{m-1}(z)}\right| \\ &\leq \frac{\sum_{j=1}^{m-1} \left|A_{m,j}(z)\right| \left|y_{m,j}(z)\right|}{\left|A_{m,m}(z)\right|}, \end{aligned} \tag{44}$$

where $|y_{m,j}(z)| = \frac{|C_{m,j}(z)|}{|R_{m-1}(z)|}$ is the unique solution (by Cramer's rule, provided $R_{m-1}(z) \neq 0$) of the system of equations

$$\begin{pmatrix} A_{1,1}(z) & A_{1,2}(z) & \cdots & A_{1,m-1}(z) \\ A_{2,1}(z) & A_{2,2}(z) & \cdots & A_{2,m-1}(z) \\ \vdots & \vdots & \vdots & \vdots \\ A_{m-1,1}(z) & A_{m-1,2}(z) & \cdots & A_{m-1,m-1}(z) \end{pmatrix} \cdot \begin{pmatrix} y_{m,1}(z) \\ y_{m,2}(z) \\ \vdots \\ y_{m,m-1}(z) \end{pmatrix} = \begin{pmatrix} A_{1,m}(z) \\ A_{2,m}(z) \\ \vdots \\ A_{m-1,m}(z) \end{pmatrix}. \tag{45}$$

The k-th equation of Eq. 45 is given by

$$A_{k,k}(z)y_{m,k}(z) + \sum_{j=1, j\neq k}^{m-1} A_{k,j}(z)y_{m,j}(z) = A_{k,m}(z), \quad 1 \leq k \leq m-1. \tag{46}$$

Now, we assume the contrary that

$$\text{Max}_j |y_{m,j}(z)| = |y_{m,k}(z)| \geq 1.$$

Because of our assumption $\left|\frac{y_{m,j}(z)}{y_{m,k}(z)}\right| \leq 1$ and $\left|\frac{1}{y_{m,k}(z)}\right| \leq 1$, we can rewrite Eq. 46 in the form

$$\begin{aligned} |A_{k,k}(z)| &\leq \sum_{j=1, j\neq k}^{m-1} \left|A_{k,j}(z)\right| \left|\frac{y_{m,j}(z)}{y_{m,k}(z)}\right| + \left|A_{k,m}(z)\right| \left|\frac{1}{y_{m,k}(z)}\right| \\ &\leq \sum_{j=1, j\neq k}^{m} \left|A_{k,j}(z)\right|. \end{aligned}$$

This contradicts Theorem A.2. Thus we have $|y_{m,j}(z)| < 1$.

Hence, using Theorem A.2 and $|y_{m,j}(z)| < 1$, the right-hand side expression of Eq.44 is less than one, and therefore $|g(z)| < |f(z)|$, where $f(z) = A_{m,m}(z)R_{m-1}(z)$ and $g(z) = R_m(z) - A_{m,m}(z)R_{m-1}(z)$. By Rouché's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside the circle $|z| = 1+\delta$. By our assumption $R_{m-1}(z)$

has $(m-1)$ zeros, and $A_{m,m}(z)$ has one zero inside and on the unit circle, $|z|=1$, by Theorem A.1. This implies $f(z)$ has m zeros inside and on the unit circle. Hence, it has now been proved by mathematical induction that $f(z)+g(z)=R_m(z)$ has exactly m zeros inside and on the unit circle. As $R_m(z)$ is a polynomial function of degree $m(b+1)$, other mb zeros are outside the unit circle.

Remark 4 For alternative proofs of this theorem, one may see references such as Gail et al. (1996), and Dudin and Klimenok (1996).

References

Abate J, Choudhury GL, Whitt W (1994) Asymptotics for steady-state tail probabilities in structured Markov queueing models. Stoch Model 10(1):99–143

Albores-Velasco FJ, Tajonar-Sanabria FS (2004) Analysis of the $GI/MSP/c/r$ queueing system. Information Theory and Information Processing 4(1):46–57

Alfa AS, Xue J, Ye Q (2000) Perturbation theory for the asymptotic decay rates in the queues with Markovian arrival process and/or Markovian service process. Queueing Systems 36(4):287–301

Bocharov PP, D'Apice C, Pechinkin A, Salerno S (2003) The stationary characteristics of the $G/MSP/1/r$ queueing system. Autom Remote Control 64(2):288–301

Chakravarthy S (1993) Analysis of a finite $MAP/G/1$ queue with group services. Queueing Systems 13(4):385–407

Chaudhry ML (1965) Correlated queueing. J Canadian Operational Research Society 3:142–151

Chaudhry ML (1966) Limited space queueing with arrivals correlated. Journal of the Royal Naval Scientific Service 21:272–278

Chaudhry ML (1991) QPACK Software Package. A & A Publications, 395 Carrie Crescent, Kingston, Ontario, K7M 5X7, Canada

Chaudhry ML, Harris CM, Marchal WG (1990) Robustness of rootfinding in single-server queueing models. INFORMS J Comput 2:273–286

Chaudhry ML, Templeton JGC (1983) A First Course in Bulk Queues. John Wiley & Sons, New York

Chaudhry ML, Samanta SK, Pacheco A (2012) Analytically explicit results for the $GI/C\text{-}MSP/1/\infty$ queueing system using roots. Probability in the Engineering and Informational Sciences 26(2):221–244

Dudin AN, Klimenok VI (1996) Queueing system with passive servers. J Appl Math Stoch Anal 9(2):185–204

Gail HR, Hantler SL, Taylor BA (1996) Spectral analysis of $M/G/1$ and $G/M/1$ type Markov chains. Adv Appl Probab 28:114–165

Gupta UC, Banik AD (2007) Complete analysis of finite and infinite buffer $GI/MSP/1$ queue - A computational approach. Oper Res Lett 35:273–280

Horváth A, Horváth G, Telek M (2010) A joint moments based analysis of networks of $MAP/MAP/1$ queues. Perform Eval 67(9):759–778

Janssen A, Leeuwaarden J (2005) Analytic computation schemes for the discrete time bulk service queue. Queueing Systems 50:141–163

Lucantoni DM (1991) New results on the single server queue with a batch Markovian arrival process. Stoch Model 7:1–46

Machihara F (1999) A $BMAP/SM/1$ queue with service times depending on the arrival process. Queueing Systems 33:277–291

Neuts MF (1979) A versatile Markovian point process. J Appl Probab 16:764–779

Ozawa T (2006) Sojourn time distributions in the queue defined by a general QBD process. Queueing Systems 53(4):203–211

Pacheco A, Tang LC, Prabhu NU (2009) Markov-Modulated Processes and Semiregenerative Phenomena. World Scientific, Singapore

Riska A, Smirni E (2002) Exact aggregate solutions for $M/G/1$-type Markov processes. In: Proceedings Int. Conf. on Measurement and Modeling of Computer Systems (ACM SIGMETRICS 2002). ACM Press, Marina Del Rey, pp 86-96

Samanta SK, Gupta UC, Sharma RK (2007) Analyzing discrete-time $D\text{-}BMAP/G/1/N$ queue with single and multiple vacations. Eur J Oper Res 182(1):321–339

Zhang Q, Heindl A, Smirni E (2005) Characterizing the $BMAP/MAP/1$ departure process via the ETAQA truncation. Stoch Model 21(2-3):821–846

Tijms HC (2003) A First Course in Stochastic Models, John Wiley and Sons