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Discrete-time Insurance Model with Capital Injections and Reinsurance

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Abstract A periodic-review insurance model is considered under the following assumptions. In order to avoid ruin the insurer maintains the company surplus above a chosen level a by capital injections at the end of each period. One-period insurance claims form a sequence of independent identically distributed nonnegative random variables with finite mean. A nonproportional reinsurance is applied for minimization of total expected discounted injections during a given planning horizon of n periods. Insurance and reinsurance premiums are calculated using the expected value principle. Optimal reinsurance strategy is established. Numerical results illustrating the theoretical ones are provided for three claims distributions.

Keywords Discrete-time insurance model · Capital injection ·
Nonproportional reinsurance · Optimal strategy

Mathematics Subject Classifications (2010) 91B30 · 90C46 · 90C39

1 Introduction

We begin by recalling the following well known facts. In order to study a real-life system it is desirable to construct its mathematical model. There exist a lot of models describing the system more or less precisely. The same mathematical model can arise in various applications. Thus, the methods employed in one research domain may be useful in other ones (see, e.g., Prabhu (1998)).

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To describe a so-called input-output applied probability model one has to specify, along with the system structure and its operation mode, input, output and control processes, the planning horizon and objective function. The last element called also valuation criterion, target or risk measure evaluates the system performance quality, for details see, e.g., Bulinskaya (2012) .

The control providing the extremum of objective function for a fixed planning horizon is called optimal. The collection of optimal controls for all the planning horizons is called the optimal strategy.

We are interested in obtaining the optimal reinsurance strategy in a discrete-time insurance model with capital injections. Nowadays, reinsurance is an important tool for financial risks management. Any insurance company, notwithstanding its size, has to use reinsurance (risk transfer to other insurer) for stable performance. The definitions can be found in the classical books Bühlmann (1970), Gerber (1980).

The problem of establishing the optimal reinsurance strategy was treated by many researchers during the last decades and it is impossible even to mention all of them, so our list of references does not pretend on completeness. One of the first papers is Dayananda (1970). Then follow the papers Dickson and Waters (1996, 2006) Højgaard and Taksar (1998), Schmidli (2002), Hipp and Vogt (2003), Kaishev (2005) and many others, studying various continuous-time models and different objective functions. The most popular models are the classical Cramér-Lundberg model and its diffusion approximations. The target is minimization of ruin probability, the control being reinsurance and investment.

However, the ruin probability does not take into account the time of ruin and its severity. So, Dickson and Waters (2004), in context of expected discounted dividends maximization, proposed a classical model modification. Namely, they assumed that in case of ruin the shareholders must cover the company deficit to enable the further functioning. The classical and diffusion models with capital injections (if surplus becomes negative) and reinsurance were studied in Eisenberg and Schmidli (2011). The aim was minimization of the expected discounted capital injections over all admissible reinsurance strategies. It was supposed that reinsurance parameter could be changed continuously.

Since reinsurance treaties are usually negotiated at the end of financial year it seems more realistic to study discrete-time insurance models. Exponential utility and ruin probability minimization by means of discrete-time dynamic programming were treated in Schäl (2004). The optimal dynamic control for optimization of various utility functions was established in Irgens (2005). The further examples of discrete-time models are Chan and Zhang (2006) dealing with finite-time ruin probabilities for special cases of claim distributions, Wei and Hu (2008) treating the ruin probabilities under assumption of stochastic interest rates and Diasparra and Romera (2010) providing inequalities for ruin probability in a controlled risk process. Li and Cong (2008) established the necessary conditions for optimality of proportional reinsurance strategy, whereas Cong et al. (2011) further developed the multiperiod risk model with proportional reinsurance. Yartseva (2009) obtained upper and lower bounds for dividends in the discrete model using proportional reinsurance. Optimal investment policy in a model with capital injections (maintaining the company surplus above some prefixed level) was studied in Gromov (2013). It is clear that discrete-time models may be used to investigate various aspects of insurance company activity.

Below we consider a periodic-review insurance model with capital injections and non-proportional reinsurance. In contrast with previous papers, where the ruin probability was chosen as objective function, we use a so-called cost approach introduced in Bulinskaya (2003). That means, our aim is minimization of expected total discounted capital injections during the n-period planning horizon, $n \geq 1$, by an appropriate choice of retention levels.

So we treat a new aspect of insurance and reinsurance not studied previously and interesting for applications.

The paper is organized as follows. In Section 2 we give the model description. One-period case is considered in Section 3. At first, the properties of some auxiliary functions are studied. Moreover, we introduce three sets D_i, $i = 1, 2, 3$, of parameters values (insurer's and reinsurer's safety loadings) playing crucial role in the choice of retention level. In particular, the possibility to eliminate the additional costs in case of special relationship between the safety loadings (set D_1) is established. On the other hand, if the parameters belong to the set D_3 there is no need of reinsurance for the initial surplus lying below some threshold u_*. The properties of the optimal retention level $z_1(u)$ as a function of initial insurance company surplus u are obtained.

Multi-step case is studied in Section 4. A useful mathematical tool here is Bellman's optimality principle. It is interesting to mention that dynamic programming was successfully used in inventory theory and other applications five decades ago. The optimal retention levels $z_n(u)$ providing the objective function minimum are investigated as functions of initial surplus u and planning horizon length n. Asymptotic behaviour of minimal expected n-period capital injections $h_n(u)$ is established, as $n \to \infty$.

Theoretical results are illustrated by numerical examples presented in Section 5. We treat three claim distributions (exponential, uniform and Pareto). For the first two cases (distributions with light tails) it turned out that the domains D_i, $i = 1, 2, 3$, do not depend on distribution parameters. On the contrary, there is a strong dependence of the above mentioned sets on the shape parameter b of the Pareto distribution (having a heavy tail). All the graphics are obtained using Wolfram Mathematica software. In conclusion (Section 6) we discuss the obtained results and further research directions.

2 Model Description

Thus, we make the following assumptions. Let u be the initial insurer's surplus and X_i the claims amount during the ith period (usually a year). The sequence $\{X_i\}_{i\ge 1}$ consists of independent identically distributed random variables with distribution function $F(x)$ having density $f(x)$ and finite mean γ. Each year, before claims payment, insurer acquires premium c calculated according to the expected value principle with safety loading $\lambda > 0$, i.e. $c = (1+\lambda)\gamma$. In order to avoid ruin insurer chooses some surplus level a which is maintained by capital injections at the end of each period. Nonproportional insurance is used for minimization of expected total discounted injections during the planning horizon of n periods. In other words, after claims payment and capital injection (if any) a retention level z is fixed for the next period. Thus, insurer pays $\min(X, z)$ if the next period claim is X, whereas reinsurer pays $(X - z)^+ = \max(0, X - z)$. The reinsurance premium is equal to $(1 + \mu)\mathsf{E}(X - z)^+$, that is, calculated also on the base of the expected value principle. Furthermore, it is natural to suppose that $\mu > \lambda$, because the direct insurer is more risk averse than reinsurer. That means, we consider a non-arbitrage situation. Hence, insurer's premium after reinsurance is given by

$$c(z) = (1+\lambda)\gamma - (1+\mu)\mathsf{E}(X-z)^+ = l\gamma - m\int_z^\infty S(x)\,dx \qquad (1)$$

where $l = 1 + \lambda$, $m = 1 + \mu$ and $S(x) = 1 - F(x)$ is survival function corresponding to $F(x)$. The discount factor is denoted by α.

Note that due to capital injection insurer's surplus at the beginning of the ith period, $i \geq 2$, cannot be less than a. So, it is natural to assume that $u \geq a$ as well. We do not specify here the source of capital needed for injection. However there exist different possibilities. We can assume, following Dickson and Waters (2004), that shareholders take care of deficit at ruin (that is, $a = 0$). To maintain the surplus at the beginning of each period above some level $a > 0$ one can use either bank loans or additional reinsurance treaties.

3 One-Step Model

We begin by treating a one-period model. Let u be the initial insurer's surplus, X the claim amount and z retention level. Then insurer's premium after reinsurance is $c(z)$ given by Eq. 1. The objective function $H_1(u, z) = \mathsf{E}J(u, z)$ with

$$J(u, z) = (\min(X, z) - e(u, z))^+, \quad e(u, z) = u - a + c(z),$$

represents the expected capital injection at the end of period in order to raise the surplus to a fixed level a. Moreover, putting $g(z) = z - c(z)$ it is possible to write

$$H_1(u, z) = \int_0^z (x - e(u, z))^+ f(x)\, dx + (a - u + g(z))^+ S(z). \tag{2}$$

Denote by $h_1(u)$ the minimal expected capital injection at the end of the period, then

$$h_1(u) = \inf_{z>0} H_1(u, z). \tag{3}$$

Since it is impossible to transfer all the risk to reinsurer we have to choose $z > 0$ in Eq. 3.

Introduce also the following auxiliary functions

$$r(z) = \int_z^\infty S(x)\, dx \quad \text{and} \quad k(z) = z + mr(z).$$

Then $c(z) = l\gamma - mr(z)$, whereas $g(z) = k(z) - l\gamma$.

3.1 Properties of auxiliary functions

For further investigation we need the following results.

Lemma 1 *Function $c(z)$ is concave and increases from $(l - m)\gamma$ to $l\gamma$, as z grows from 0 to ∞.*

Proof It is obvious that $c'(z) = -mr'(z) = mS(z) \geq 0$ and $c''(z) = -mf(z) \leq 0$, while $c(0) = (l - m)\gamma$ and $c(z) \to l\gamma$, $c'(z) \to 0$, as $z \to \infty$. □

Lemma 2 *Functions $k(z)$ and $g(z)$ are convex attaining their minimum at the point $z_* = S^{inv}(m^{-1})$ where S^{inv} is the function inverse to S. Moreover, $k(z_*) = z_* + mr(z_*)$, $g(z_*) = k(z_*) - l\gamma$ and $g'(z) = k'(z) \to 1$, as $z \to \infty$.*

Proof Since $k'(z) = g'(z) = 1 - mS(z)$, the derivatives tend to 1, as $z \to \infty$, and $k''(z) = g''(z) = mf(z)$ are nonnegative. Thus, both functions k and g are convex. Moreover, their minimum is attained for z_* satisfying $1 - mS(z_*) = 0$. Hence, $z_* = S^{inv}(m^{-1})$ and $g(z_*) = k(z_*) - l\gamma$. □

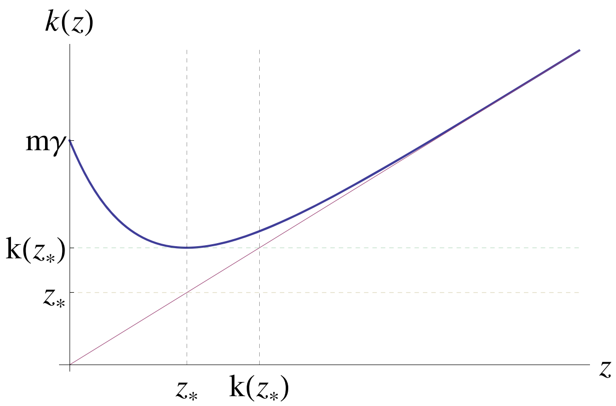


Fig. 1 Function $k(z)$

Remark 1 It is important to underline that $k(z_*) > z_*$, see Fig. 1 as well. Note also that $g(0) = -c(0)$ and $g(z) > -c(z)$ for $z > 0$.

Next we introduce the sets $D_1 = \{m\gamma > l\gamma > k(z_*)\}$, $D_2 = \{k(z_*) \geq l\gamma > z_*\}$ and $D_3 = \{z_* \geq l\gamma > \gamma\}$. It is obvious that $g(z_*) < 0$ in D_1, $g(z_*) \geq 0$ in $D_2 \cup D_3$ and $z_* - c(\infty) \geq 0$ in D_3. Put also $u_* = a + z_* - l\gamma$ and $u_1^* = a + k(z_*) - l\gamma = a + g(z_*)$.

There exist three different situations as stated below.

Lemma 3 *Inequalities $a > u_1^*$, $u_* < a \leq u_1^*$ and $a \leq u_*$ are equivalent to the relations $(l, m) \in D_1$, $(l, m) \in D_2$ and $(l, m) \in D_3$ respectively.*

Proof The desired result follows immediately from definition of D_i, $i = 1, 2, 3$, u_* and u_1^*. □

3.2 Optimal Reinsurance

Now we are able to obtain the optimal one-step reinsurance policy.

Theorem 1 *If $(l, m) \in D_1$ then $h_1(u) = 0$ for all $u \geq a$. The optimal retention level $z_1(u)$ is the maximal root $z_{r1}(u)$ of the equation $g(z) = u - a$. Moreover, $z_1(u)$ is concave increasing function and $z_1'(u) \to 1$, as $u \to \infty$.*

Proof Since $x - e(u, z) \leq a - u + g(z)$ for $x \in [0, z]$ it follows from Eq. 2 that $H_1(u, z) = 0$ for $g(z) \leq u - a$. It is clear that for $(l, m) \in D_1$ we have $k(z_*) < l\gamma$, in other words, $g(z_*) < 0$. That means, for all $u \geq a$ equation $g(z) = u - a$ has a root $z_{r1}(u) \geq z_*$, whereas for $u < a + (m - l)\gamma$ there exists another root $z_{l1}(u) \leq z_*$. Obviously, $H_1(u, z) = 0$ for $z \in [z_{l1}(u), z_{r1}(u)]$ and $H_1(u, z) \geq (a - u + g(z))S(z) > 0$ for $z > z_{r1}(u)$. We take $z_{r1}(u)$ as optimal retention level $z_1(u)$ since it gives the maximal insurance premium after reinsurance (the function $c(z)$ is increasing). Furthermore, using the rules of differentiation of implicit functions, one easily obtains that

$$\begin{aligned} z_1'(u) &= (1 - mS(z_1(u)))^{-1} > 0 \text{ and} \\ z_1''(u) &= -mf(z_1(u))(1 - mS(z_1(u)))^{-3} < 0. \end{aligned} \tag{4}$$

Thus, $z_1(u)$ is concave increasing function and $z_1'(u) \to 1$, since $z_1(u) \to \infty$, as $u \to \infty$. □

Theorem 2 *If $(l, m) \in D_2$ then $h_1(u) = 0$ for $u \geq u_1^*$. The optimal retention level $z_1(u) = z_{r1}(u)$ is defined as in Theorem 1 and has the same properties. Furthermore, $z_1(u_1^*) = z_*$ and $z_1'(u) \to \infty$ as $u \searrow u_1^*$.*

For $u \in [a, u_1^*)$ the function $z_1(u) = z_0(u)$, the unique solution, for a fixed u, of the equation $e(u, z) = z_*$. The function $z_0(u)$ is convex decreasing, $z_0(u) \to z_*$ and $z_0'(u) \to -1$, as $u \nearrow u_1^*$.

Proof For $(l, m) \in D_2$ we have $u_* < a \leq u_1^*$. Thus, $g(z_*) \geq 0$ and for $u \geq u_1^*$ equation $g(z) = u - a$ has two solutions $0 \leq z_{l1}(u) \leq z_* \leq z_{r1}(u)$ if $g(z_*) \leq u - a \leq (m - l)\gamma$, and only one solution $z_{r1}(u) > z_*$ if $u - a > (m - l)\gamma$. Reasoning as in the proof of Theorem 1 we take $z_1(u) = z_{r1}(u)$ for $u \geq u_1^*$. Obviously, $z_{l1}(u_1^*) = z_{r1}(u_1^*) = z_*$. Thus, it is easily seen from Eq. 4 that $z_1'(u_1^*) = +\infty$.

Now let $a \leq u < u_1^*$, that is, $u - a < g(z_*) \leq g(z)$ for all $z > 0$. In this case we can rewrite Eq. 2 as follows

$$H_1(u, z) = \int_{e(u,z)^+}^{z} (x - e(u, z)) f(x)\, dx + (a - u + g(z))S(z). \tag{5}$$

Hence, if $e(u, z) \leq 0$ then

$$\frac{\partial H_1}{\partial z}(u, z) = S(z)(1 - m),$$

whereas for $e(u, z) > 0$

$$\frac{\partial H_1}{\partial z}(u, z) = S(z)(1 - mS(e(u, z))).$$

Thus, it is possible to write, for all $z > 0$,

$$\frac{\partial H_1}{\partial z}(u, z) = S(z)G_1(u, z) \quad \text{with} \quad G_1(u, z) = 1 - mS(e(u, z)).$$

It follows immediately that $G_1(u, z) = 0$ if

$$e(u, z) = z_*. \tag{6}$$

This equation has a unique solution for each $u \in [a, u_1^*]$ since $l\gamma > z_*$ for $(l, m) \in D_2$. Hence, it defines implicitly a function $z_0(u)$. For $u = u_1^*$ we get $z_* = c(z_0(u_1^*)) + u_1^* - a = c(z_0(u_1^*)) + g(z_*)$ therefore $c(z_0(u_1^*)) = c(z_*)$ and $z_0(u_1^*) = z_*$. Using Eq. 6 and the rules of implicit functions differentiation one obtains

$$z_0'(u) = -(c'(z_0(u)))^{-1} < 0, \quad z_0''(u) = m^{-2}S^{-3}(z_0(u)) f(z_0(u)) > 0.$$

It follows immediately that $z_0(u)$ is convex decreasing and $z_0'(u_1^*) = -1$. Since $z_0(u)$ provides minimum of $H_1(u, z)$ for $u \in [a, u_1^*]$ we take $z_1(u) = z_0(u)$ in this interval. □

Theorem 3 *If $(l, m) \in D_3$ then for $u > u_*$ the results coincide with those of Theorem 2. Moreover, $z_1(u) \to \infty$ and $z_1'(u) \to -\infty$, as $u \searrow u_*$, whereas for $u \in [a, u_*]$ it is optimal to use no reinsurance, that is, to take $z_1(u) = \infty$.*

Proof For $(l, m) \in D_3$ one has $u_* \geq a$. Hence, we have only to deal with $u = u_*$. In other words, it follows $e(u_*, z_1(u_*)) = z_*$, that is, $c(z_1(u_*)) = l\gamma$ implying $z_1(u_*) = \infty$. Since $c'(z) \to 0$, as $z \to \infty$, we get $z_1'(u) \to -\infty$, as $u \searrow u_*$. Furthermore, if $u \in [a, u_*)$ the inequality $e(u, z) < z_*$ is valid for all $z > 0$. The function $c(z)$ being increasing that entails $z_1(u) = \infty$. Thus, it is optimal to use no reinsurance for the mentioned values of u. □

Corollary 1 *For* $(l, m) \in D_3$ *one has*

$$h_1'(u) = \begin{cases} -S(l\gamma + u - a), & u \in [a, u_*), \\ -m^{-1}, & u \in [u_*, u_1^*), \\ 0, & u > u_1^*. \end{cases}$$

Proof We have established in Theorem 3 the form of optimal retention level $z_1(u)$. Therefore $h_1(u) = H_1(u, z_1(u)) = H_1(u, \infty)$ for $u \le u_*$. Under assumption $\gamma < \infty$ one gets $zS(z) \to 0$, as $z \to \infty$, thus, $g(z)S(z) \to 0$ as well and $H_1(u, \infty) = \int_{l\gamma+u-a}^{\infty}(a - u - l\gamma + x) f(x)\, dx$. That gives immediately $h_1'(u) = -S(l\gamma + u - a)$ for $u \in [a, u_*]$.

Obviously, for $u \in [u_*, u_1^*)$ one can write

$$h_1'(u) = \frac{\partial H_1}{\partial u}(u, z_1(u)) + \frac{\partial H_1}{\partial z}(u, z_1(u))z_1'(u). \tag{7}$$

By definition of $z_1(u)$ the second term on the right-hand side of relation Eq. 7 is equal to zero. Moreover, equation Eq. 5 gives $h_1'(u) = -m^{-1}$ for $u \in [u_*, u_1^*)$ since

$$\frac{\partial H_1}{\partial u}(u, z_1(u)) = -S(e(u, z_1(u))) = -S(z_*).$$

Finally, $h_1'(u) = 0$ for $u > u_1^*$ because $h_1(u) = 0$ for $u \ge u_1^*$. □

Remark 2 The changes necessary for the sets D_2 and D_1 are obvious.

4 Multi-Step Model

Now we consider the multi-period case, that is, the planning horizon $n \ge 2$. Denote by $U_k(z)$ the surplus at the end of the kth period if z is the retention level assigned for this period. It is clear that $U_k(z) = u_{k-1} + c(z) - \min(X_k, z)$ where $c(z)$ is defined by formula Eq. 1, X_k is the claims amount in the kth period and u_{k-1} is the surplus at the end of the previous period.

Our aim is to obtain the minimal expected n-period discounted costs $h_n(u)$ and establish the optimal reinsurance strategy. In other words, we want to find the sequence of functions $z_k(u)$, $k = \overline{1, n}$, representing the optimal retention levels. More precisely, $z_k(u)$ is the retention level to use at the first step of the k-step process with initial surplus u.

The objective function is calculated as follows

$$\mathcal{L}_n(u, z_1, \ldots, z_n) = \mathsf{E}\left(\sum_{k=0}^{n-1} \alpha^k J(U_k(z_{n-k}), z_{n-k})/U_0 = u\right),$$

hence,

$$h_n(u) = \inf_{z_i > 0, i=\overline{1,n}} \mathcal{L}_n(u, z_1, \ldots, z_n).$$

Using the Bellman optimality principle (see, e.g., Bellman (1957)) we get the following recurrent relation

$$h_n(u) = \inf_{z>0}[H_1(u, z) + \alpha \mathsf{E} h_{n-1}(\max(a, u + c(z) - \min(z, X)))] \tag{8}$$

where X has the same distribution function $F(x)$ as all X_k, $k \geq 1$. Obviously, $h_0(u) \equiv 0$. Thus, $h_1(u) = \inf_{z>0} H_1(u, z)$ was already obtained in previous Section 3.

In order to investigate the multi-period case we have to use the following representation $H_n(u, z) = H_1(u, z) + \alpha d_{n-1}(u, z)$ where

$$d_k(u, z) = \mathsf{E} h_k(\max(a, u + c(z) - \min(z, X))).$$

It is easy to see that putting $e(u, z) = u + c(z) - a$ we get

$$d_k(u,z) = \begin{cases} h_k(a), & e(u,z) < 0 \\ \int_0^{e(u,z)} h_k(u + c(z) - x) f(x)\,dx + h_k(a) S(e(u,z)), & e(u,z) \in [0, z], \\ \int_0^{z} h_k(u + c(z) - x) f(x)\,dx + h_k(u - g(z)) S(z), & e(u,z) > z. \end{cases} \quad (9)$$

It follows immediately that $\frac{\partial H_{k+1}}{\partial z}(u, z) = S(z) G_{k+1}(u, z)$ where $G_{k+1}(u, z)$ is equal to

$$1 - m S(e(u,z)) + m\alpha \int_0^{e(u,z)} h'_k(u + c(z) - x) f(x)\,dx$$

for $e(u, z) \leq z$ whereas for $e(u, z) > z$ it has the form

$$\alpha[m \int_0^{z} h'_k(u + c(z) - x) f(x)\,dx - h'_k(u - g(z)) g'(z)].$$

In particular, $G_{k+1}(u, z) = 1 - m$ if $e(u, z) < 0$.

Now we can prove the following results.

Theorem 4 *If $(l, m) \in D_1$ one gets $h_n(u) = 0$ for any $u \geq a$ and $n \geq 1$. The optimal retention level $z_n(u) = z_{r1}(u)$ for all $n \geq 1$.*

Proof One has $a \geq u_1^*$ for $(l, m) \in D_1$ according to Lemma 3. It was also established in Theorem 1 that $h_1(u) = 0$ for all $u \geq a$. It follows immediately that $H_2(u, z) = H_1(u, z)$ because $d_1(u, z) = 0$ for all $u \geq a$ and $z > 0$. Moreover, $H_1(u, z) = 0$ for $g(z) \leq u - a$. As previously, we choose the maximal z for which $H_1(u, z) = 0$, namely, put $z_2(u) = z_{r1}(u)$.

We proceed by mathematical induction. Assume that $h_k(u) = 0$ for $u \geq a$ and $k \leq n-1$. Then $d_{n-1}(u, z) = 0$ for all $u \geq a$ and $z > 0$. That means $H_n(u, z) = H_1(u, z)$. So it is optimal to take $z_n(u) = z_1(u)$ getting $h_n(u) = 0$. □

For other values of l and m the optimal behaviour is more complicated.

Theorem 5 *For $(l, m) \in D_2 \cup D_3$ one gets $h_n(u) = 0$ if $u \geq u_n^* = a + n g(z_*)$. The optimal retention at the beginning of the n-step process $z_n(u) = z_1(u - (n-1) g(z_*))$ for $u \geq u_n^*$.*

Proof As in the proof of Theorem 4, we use the mathematical induction. Clearly, for $(l, m) \in D_2 \cup D_3$, it follows from Theorems 2 and 3 that $h_1(u) = 0$ if $u \geq u_1^*$. Assume now that the desired result is proved for $k \leq n - 1$ and establish its validity for $k = n$. According to our assumption $d_{n-1}(u, z) = 0$ for $u - g(z) \geq u_{n-1}^* = a + (n-1) g(z_*)$. Rewriting this inequality in the form $u - (n-1) g(z_*) - a \geq g(z)$ we see immediately, that $d_{n-1}(u, z) = 0$ for $z \in [z_{l1}(u - (n-1) g(z_*)), z_{r1}(u - (n-1) g(z_*))]$. Since $H_1(u, z) = 0$ for $z \in [z_{l1}(u), z_{r1}(u)]$ and function $z_{r1}(u)$ is increasing in u one can take $z_n(u) = z_{r1}(u - (n-1) g(z_*))$ and obtain $h_n(u) = 0$ for $u \geq u_n^*$. □

Theorem 6 *If* $(l, m) \in D_2 \cup D_3$ *the optimal retention level* $z_n(u)$ *is a convex decreasing function for* $u \in (\max(a, u_*), u_1^*)$*, moreover,* $z_n(u) > z_1(u)$ *and* $z_n'(u) = -(c'(z_n(u)))^{-1}$.

Proof According to Lemma 3, for $(l, m) \in D_2$ we have $u_* < a \le u_1^*$, whereas $a \le u_*$ for $(l, m) \in D_3$. We begin by treating the case $n = 2$. Since $u < u_1^*$, we have the following relation

$$\frac{\partial H_2}{\partial z}(u, z) = S(z)G_2(u, z)$$

with

$$G_2(u, z) = 1 - mS(e(u, z)) + \alpha m \int_0^{e(u,z)} h_1'(a + e(u, z) - x) f(x)\, dx. \qquad (10)$$

It is clear that

$$\frac{\partial G_2}{\partial z}(u, z) = c'(z)\frac{\partial G_2}{\partial u}(u, z).$$

Furthermore, the optimal retention level $z_2(u) = z_0^{(2)}(u)$ defined implicitly by the following relation $G_2\left(u, z_0^{(2)}(u)\right) = 0$. Hence

$$z_2'(u) = -\left(\frac{\partial G_2}{\partial u} / \frac{\partial G_2}{\partial z}\right)(u, z_2(u)) = -(c'(z_2(u)))^{-1} < 0.$$

Obviously,

$$z_2''(u) = f(z_2(u))m^{-2}S^{-3}(u, z_2(u)) > 0.$$

Recalling that $e(u, z_1(u)) = z_*$ for $u < u_1^*$ we get

$$G_2(u, z_1(u)) = \alpha m \int_0^{z_*} h_1'(a + z_* - x) f(x)\, dx < 0.$$

That means $z_1(u) < z_2(u)$ for $u < u_1^*$. So, we have established the desired result for $n = 2$. The results for $n > 2$ can be obtained by induction. Due to the lack of space they are omitted. □

For the same reason we formulate the next result only for $n = 2$.

Theorem 7 *If* $(l, m) \in D_2$ *then* $z_2(u) = \min(z_0(u - g(z_*)), \max(z_{r1}(u), z_0(u)))$ *for* $u \in (u_1^*, u_2^*)$.

Proof Recall that $h_1'(u) = -m^{-1}$ for $u \in (\max(a, u_*), u_1^*)$ and $h_1'(u) = 0$ for $u > u_1^*$ according to Corollary 1. Moreover, for $z \in A(u) = (z_{l1}(u), z_{r1}(u))$ we have

$$G_2(u, z) = \alpha[m \int_0^z h_1'(a + e(u, z) - x) f(x)\, dx - g'(z)h_1'(u - g(z))]. \qquad (11)$$

Using the form of $h_1'(u)$ and equivalency of the inequalities

$$a + e(u, z) - x > u_1^* \Leftrightarrow x < a + e(u, z) - u_1^* = e(u, z) - g(z_*) = e(u - g(z_*), u),$$

we can rewrite Eq. 11 as follows

$$G_2(u, z) = \alpha m^{-1}[1 - mS(e(u - g(z_*), z))].$$

Thus, solution $z_0^{(2)}(u)$ of equation $G_2(u, z) = 0$, for a fixed u, is given implicitly by the relation $e(u - g(z_*), z_0^{(2)}(u)) = z_*$. That means, $z_0^{(2)}(u) = z_0(u - g(z_*))$, therefore $z_0^{(2)}(u_2^*) = z_0(u_1^*) = z_*$ and $z_0^{(2)}(u) \to \infty$, as $u \searrow u_* + g(z_*) > u_1^*$. It follows immediately that curves $z_0^{(2)}(u)$ and $z_{r1}(u)$ will cross for some $\bar{u} \in (u_* + g(z_*), u_2^*)$. So, it is impossible to choose $z_0^{(2)}(u)$ as optimal retention level $z_2(u)$ for $u < \bar{u}$. Moreover, $G_2(u, z_{r1}(u)) < 0$ for $u < \bar{u}$, whereas it is positive for $u > \bar{u}$. Hence, it is reasonable to take $z_2(u) = \min\left(z_0^{(2)}(u), z_{r1}(u)\right)$.

However another adjustment is needed in the neighbourhood of the point u_1^*. We know that for $z \notin A(u)$

$$G_2(u, z) = g'(e(u, z)) + \alpha m \int_0^{e(u,z)} h_1'(a + e(u, z) - x) f(x)\, dx.$$

This expression can be rewritten as $1 - (m - \alpha)S(e(u, z)) - \alpha S(e(u - g(z_*), z))$, making obvious that it is an increasing function in z. Proving Theorem 6 we denoted by $z_0(u)$ the root of equation $G_2(u, z) = 0$ in this case. It was established that the function $z_0(u)$ is decreasing and $z_0(u_1^*) > z_* = z_{r1}(u_1^*)$. Therefore one has to take $\max(z_0(u), z_{r1}(u))$ obtaining the desired form of $z_2(u)$. □

The last result we present here is

Theorem 8 *For $(l, m) \in D_2$ minimal expected discounted costs $h_n(u)$ converge uniformly in u, as $n \to \infty$.*

Proof After establishing the optimal retention levels $z_n(u)$ we have the representation $h_n(u) = H_n(u, z_n(u))$, $n \geq 1$. This entails the following inequality

$$|h_{n+1}(u) - h_n(u)| \leq \max_{z = z_n(u), z_{n+1}(u)} |H_{n+1}(u, z) - H_n(u, z)|.$$

Introducing $\delta_n = \max_{u \geq a} |h_{n+1}(u) - h_n(u)|$ it is easy to obtain $\delta_n \leq C\alpha^n$ with $C = h_1(a)$. Hence, there exists $h(u) = \lim_{n\to\infty} h_n(u)$ and convergence is uniform in u. □

5 Examples

Now we consider three examples. Two distributions (exponential and uniform) have light tails, the third one (Pareto) is heavy-tailed. All the numerical results were obtained by means of program Wolfram Mathematica.

Exponential Distribution Let $f(x) = b \exp\{-bx\}\mathbb{I}_{[0,\infty)}(x)$, $b > 0$, then $S(x) = \exp\{-bx\}$ for $x \geq 0$ and $S(x) = 1$ for $x < 0$.

It is easy to calculate that $\gamma = b^{-1}$, $z_* = b^{-1} \ln m$ and $\int_{z_*}^{\infty} S(x)\, dx = (mb)^{-1}$. Thus, the regions D_i, $i = 1, 2, 3$, are given by the following relations between insurer's and reinsurer's loadings, more precisely between l and m,

$$D_1 = \{m > l > 1 + \ln m\}, \quad D_2 = \{1 + \ln m \geq l > \ln m\}, \quad D_3 = \{\ln m \geq l > 1\}.$$

Recall that $D_1 = \{(l, m) : g(z_*) < 0\}$, $D_3 = \{(l, m) : z_* - c(\infty) \geq 0\}$ and $D_2 \cup D_3 = \{(l, m) : g(z_*) \geq 0\}$, as depicted in Fig. 2. It is important to underline that sets D_i, $i = 1, 2, 3$, do not depend on distribution parameter b.

For illustration of theoretical results obtained in previous sections we consider several special cases fixing different pairs (l, m).

1. At first we take $l = 2$, $m = 2.1$. That means $(l, m) \in D_1$, hence, $g(z_*) < 0$. According to Theorem 1, the minimal expected costs $h_n(u) = 0$ for all $n \geq 1$ and $u \geq a$, whereas all the retention levels $z_n(u)$ are equal to $z_{r1}(u)$.

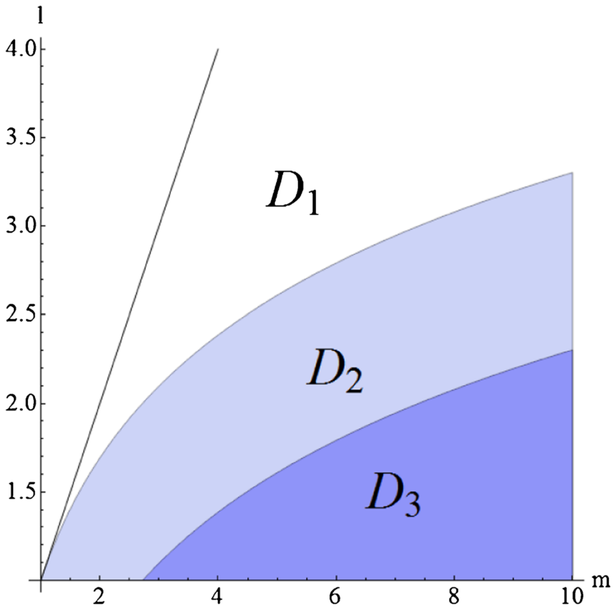


Fig. 2 Sets D_i for exponential distribution

Using the explicit form of function $g(z) = z - b^{-1}(l - me^{-bz})$ we obtain the maximal solution of the equation $g(z) = u - a$ by means of function FindRoot of the program Wolfram Mathematica. Since $g(z)$ is convex having a unique minimum attained at z_*, taking the initial value greater than z_* implies that the procedure FindRoot based on the Newton method converges to $z_{r1}(u)$ for any $u \geq a$.

Thus, the graphics of $z_n(u)$, $n \geq 1$, have the form given by Fig. 3(a). We have taken $b = 0.5$, $b = 1$ and $b = 4$ and put $a = 1$ for simplicity. Note that the retention levels corresponding to smaller values of parameter b are situated above.

2. Now take $l = 2$, $m = 5$, that is, $(l, m) \in D_2$. So, $g(z_*) \geq 0$ and $z_* - c(\infty) < 0$. In this case the conditions of Theorem 2 are fulfilled for $n = 1$ and we obtain $z_n(u)$ and $h_n(u)$ as follows.

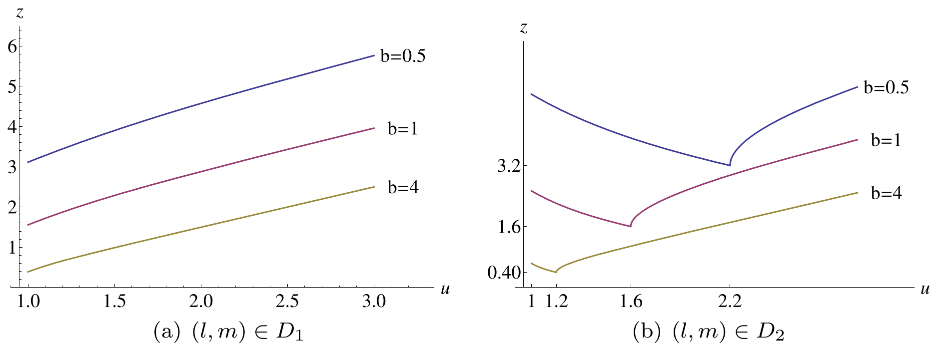


Fig. 3 Optimal retention levels for exponential distribution

Let $n = 1$. For $u \ge u_1^* = a + g(z_*)$ we have $h_1(u) = 0$ and $z_1(u) = z_{r1}(u)$ is calculated as described above.

For $a \le u < u_1^*$ we can easily obtain the explicit form of $z_1(u) = z_0(u)$ and $h_1(u)$. Namely, they are given by the following relations

$$z_0(u) = c^{inv}(z_* + a - u) = b^{-1}[\ln m - \ln(l - \ln m - ba + bu)],$$

$$h_1(u) = b^{-1}(e^{-bz_*} - e^{-bz_0(u)}) = \frac{1 - l + \ln m + ba - bu}{bm}.$$

For the same values of parameter b the optimal retention level $z_1(u)$ is depicted by (Fig. 3(b))

Now turn to the case $n > 1$. We have proved that $h_n(u) = 0$ for $u \ge u_n^* = a + ng(z_*)$. Moreover, $z_n(u) = z_1(u - (n-1)g(z_*))$, that is, the maximal root of equation $u - a = g(z) + (n-1)g(z_*)$. It can be calculated using the subprogram FindRoot.

If $a \le u < u_n^*$ to obtain $h_n(u)$ we calculated $h_n(u_k) = \min_z H_n(u_k, z)$ for $u_k = a + kg(z_*)/20$, $k \ge 0$, and then made the interpolation which was used at the next step $n+1$. To get $H_n(u, z)$ we used the numerical integration. The subprogram FindMinimum provided not only the desired minimum but the value of argument $z_n(u)$ for which this minimum is attained. In particular, for $b = 1$ the functions $h_n(u)$, $1 \le n \le 7$, have the form presented by Fig. 4(a). In all the calculations we put $\alpha = 0.5$.

For a fixed u the functions $h_n(u)$ increase in n. We can distinguish only three functions, since $h_n(u)$, $n = 4, 5, 6, 7$, practically coincide with $h_3(u)$.

It is interesting to see (Fig. 4(b)) that function $z_n(u)$ has n local minima the last one attained at $u = u_n^*$.

Uniform distribution In this case the density $f(x) = b^{-1}\mathbb{I}_{[0,b]}(x)$, $b > 0$. Hence, $S(x) = 1$ for $x \le 0$, $S(x) = 1 - xb^{-1}$ for $x \in [0, b]$ and $S(x) = 0$ for $x \ge b$.

That gives $\gamma = b/2$, $z_* = b(1 - m^{-1})$ and $\int_{z_*}^b S(x)\,dx = \gamma m^{-2}$. Then it easily follows that $D_1 = \{m > l > 2 - m^{-1}\}$, $D_2 = \{2 - m^{-1} \ge l > 2 - 2m^{-1}\}$, $D_3 = \{2 - 2m^{-1} \ge l > 1\}$.

Hence, we get Fig. 5, where as in previous example, $(l, m) \in D_1$ means $g(z_*) < 0$, $(l, m) \in D_3$ signifies that $z_* - c(\infty) \ge 0$ and $g(z_*) \ge 0$ for $(l, m) \in D_2 \cup D_3$.

As in previous case of exponential distribution the sets D_i, $i = 1, 2, 3$, have the same form for all values of parameter b.

The procedure of obtaining the functions $z_n(u)$ and $h_n(u)$ for uniform distribution of claims is similar to that for exponential one.

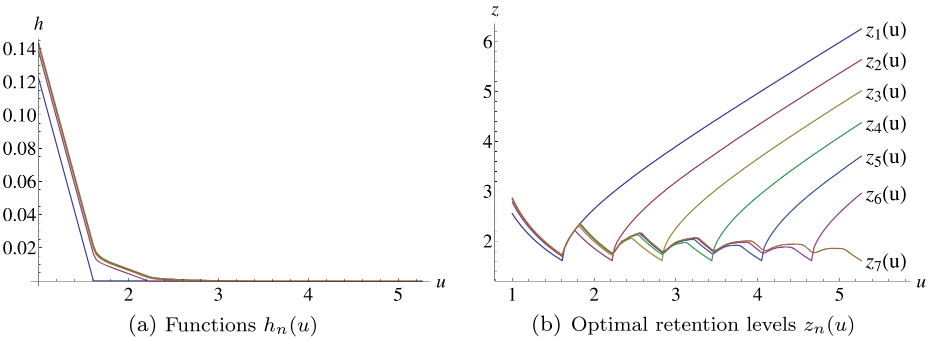


Fig. 4 Exponential distribution with $b = 1$

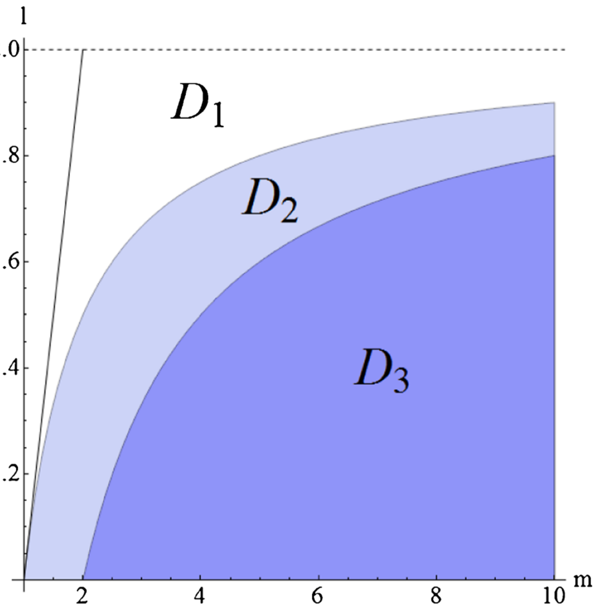


Fig. 5 Sets D_i for uniform distribution

1. Let us take $l = 1.7$ and $m = 3$. For such values of parameters $g(z_*) < 0$. Hence, $h_n(u) = 0$ for all $n \geq 1$ and $u \geq a$. It is convenient to set $a = 1$. The optimal retention levels $z_n(u)$ are shown by Fig. 6(a) for $b = 2$, $b = 3$, $b = 4$, $b = 5$ and $b = 6$. The levels corresponding to larger values of b are lying above.
2. Now set $l = 1.5$, $m = 3$, it follows that $g(z_*) \geq 0$ and $z_* - c(\infty) < 0$.

Consider $n = 1$. If $u \geq u_1^*$ then $h_1(u) = 0$ and optimal retention level $z_1(u) = z_{r1}(u)$, the maximal root of the equation $u - a = g(z)$.

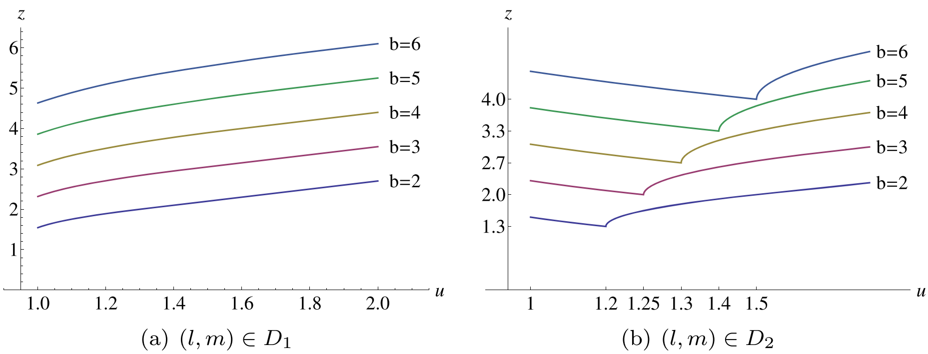
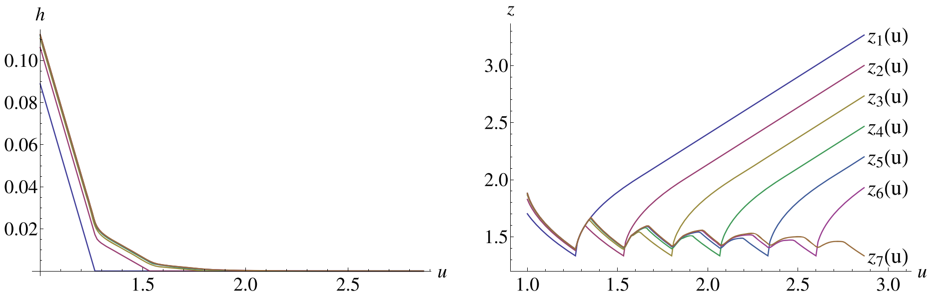


Fig. 6 Optimal retention levels for uniform distribution



(a) Optimal discounted cost (b) Optimal retention levels, $n \geq 1$

Fig. 7 Uniform distribution in [0, 2]

On the other hand, if $a \leq u < u_1^*$ we can obtain the explicit form of $z_1(u) = z_0(u)$, namely,

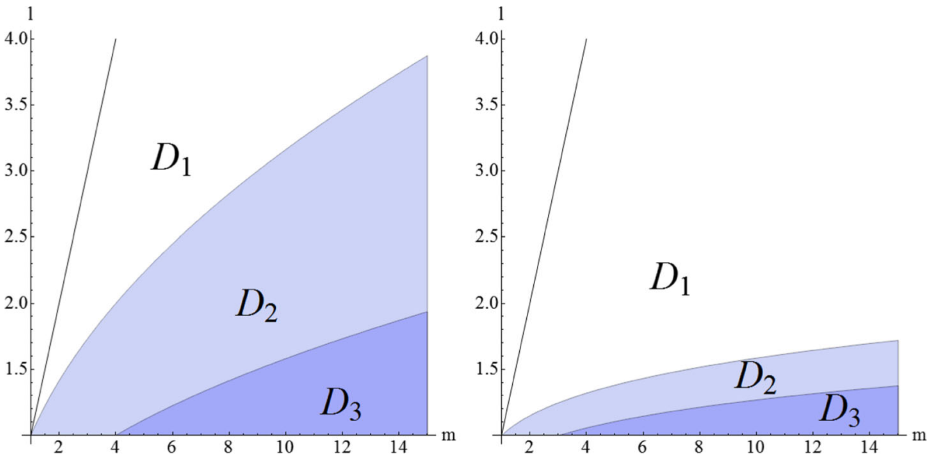
$$z_0(u) = c^{inv}(z_* + a - u) = b - b\sqrt{\left(\frac{l}{2} - 1 + \frac{1}{m} + \frac{u-a}{b}\right)\frac{2}{m}},$$

whereas $h_1(u)$ is calculated as follows

$$h_1(u) = \int_{e(u,z_0(u))}^{z_0(u)} (x - e(u, z_0(u))) f(x)\, dx + (a - u + g(z_0(u)))S(z_0(u)).$$

Taking $b = 2$, $b = 3$, $b = 4$, $b = 5$ and $b = 6$ we get the Fig. 6(b) representing $z_1(u)$ for these parameter values.

For $n > 1$ we use once more the program Wolfram Mathematica to calculate $h_n(u)$ and $z_n(u)$. Thus, for $b = 2$ and $1 \leq n \leq 7$ we get the results given by Fig. 7(a).



(a) $b = 2$ (b) $b = 5$

Fig. 8 Domains D_i for Pareto distribution

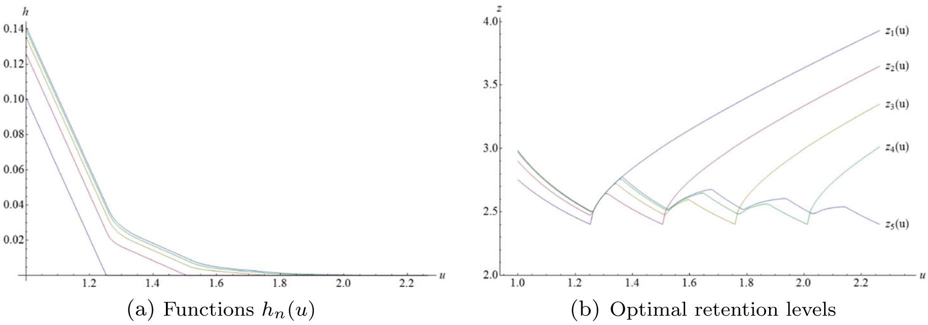


Fig. 9 Domains D_i for Pareto distribution with $b = 5, d = 2$

Here we cannot distinguish the functions $h_n(u)$, $n \geq 5$, because they almost coincide with $h_4(u)$.

The optimal retention levels have the form depicted by Fig. 7(b).

Pareto distribution This distribution depends on two parameters d and b in the following way: $f(x) = bd^b x^{-b-1}\mathbb{I}_{[d,\infty)}(x)$. Here, $d > 0$ and $b > 1$, since we assumed that the expected value $\gamma < \infty$. Thus, $S(x) = 1$ for $x \leq d$ and $S(x) = d^b x^{-b}$ for $x \geq d$. It is easy to obtain that $\gamma = bd(b-1)^{-1}$, $z_* = dm^{1/b}$ and $m\int_{z_*}^{\infty} S(x)\,dx = dm^{1/b}(b-1)^{-1}$. That means $D_1 = \{m > l > m^{1/b}\}$, $D_2 = \{m^{1/b} \geq l > (1-b^{-1})m^{1/b}\}$, $D_3 = \{(1-b^{-1})m^{1/b} \geq l > 1\}$.

We see that sets D_i depend heavily on parameter b. So, it is interesting to compare the graphics for $b = 2$ and $b = 5$.

We see that the set D_1 grows, whereas D_2 and D_3 diminish, as $b \to \infty$. It is clearly shown by Fig. 8(a) and Fig. 8(b).

We have also calculated the minimal expected costs (Fig. 9(a)) and optimal retention levels (Fig. 9(b)).

6 Conclusion and Further Research Directions

We have considered a periodic-review insurance model with capital injections. The main tool for minimizing the discounted expected injections was the nonproportional reinsurance of stop loss type. The optimal reinsurance strategy was established under different assumptions on system parameters. Numerical results were provided for three distributions of claim amounts, namely, exponential, uniform and Pareto.

It is interesting to stress that choosing a special relation between insurer's and reinsurer's safety loadings one can guarantee the expected discounted costs equal to zero for any planning horizon n and initial surplus $u \geq a$ (see Theorems1 and 4). Furthermore, under other assumptions about safety loadings, for each planning horizon n there exists a threshold u_n^* such that $h_n(u) = 0$ for $u \geq u_n^*$ (see Theorems 2, 3 and 5).

The results established for the multi-period model (under conditions of Theorem 3 proved for one-step model) will be published in a forthcoming paper.

The next problem to solve is how to choose an appropriate reinstatement level a and the source of capital injection. We plan also to investigate the asymptotic behaviour of the insurer's surplus under optimal reinsurance strategy.

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