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The Markov Additive Risk Process Under an Erlangized Dividend Barrier Strategy

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Abstract In this paper, we consider a Markov additive insurance risk process under a randomized dividend strategy in the spirit of Albrecher et al. (ASTIN Bull 41(2):645–672, 2011). Decisions on whether to pay dividends are only made at a sequence of dividend decision time points whose intervals are Erlang(n) distributed. At a dividend decision time, if the surplus level is larger than a predetermined dividend barrier, then the excess is paid as a dividend as long as ruin has not occurred. In contrast to Albrecher et al. (ASTIN Bull 41(2):645–672, 2011), it is assumed that the event of ruin is monitored continuously Avanzi et al. (Insur. Math Econ. 52(1):98–113, 2013) and Zhang (J Ind. Manag. Optim. 10(4):1041–1058, 2014), i.e. the surplus process is stopped immediately once it drops below zero. The quantities of our interest include the Gerber-Shiu expected discounted penalty function and the expected present value of dividends paid until ruin. Solutions are derived with the use of Markov renewal equations. Numerical examples are given, and the optimal dividend barrier is identified in some cases.

Keywords Markov additive process · Barrier strategy · Inter-dividend-decision times · Gerber-Shiu function · Dividends · Markov renewal equation · Erlangization

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1 Introduction

In this paper, we model the surplus of an insurance company via a Markov additive process (MAP) with downward jumps (e.g. Asmussen (2003, Chapter XI)). Let $J = \{J_t\}_{t\geq 0}$ be the underlying environment process, which is a homogeneous irreducible continuous-time Markov chain with finite state space $\mathcal{E} = \{1, 2, \ldots, m\}$ and representation $(\boldsymbol{\alpha}, \boldsymbol{D}_0, \boldsymbol{D}_1)$. Here $\boldsymbol{\alpha}$ is the initial probability row vector and $\boldsymbol{D}_0 + \boldsymbol{D}_1$ is the intensity matrix. We shall write $\boldsymbol{D}_0 = (D_{0,ij})_{i,j=1}^m$ and $\boldsymbol{D}_1 = (D_{1,ij})_{i,j=1}^m$. The claim number process $N = \{N_t\}_{t\geq 0}$ of a MAP is controlled by J as follows:

(1) transition of J from state i to state j without any accompanying claim (where $i \neq j$) occurs at rate $D_{0,ij} \geq 0$; and
(2) transition of J from state i to state j with an accompanying claim (with the possibility that $i = j$) occurs at rate $D_{1,ij} \geq 0$.

Note that for $\boldsymbol{D}_0 + \boldsymbol{D}_1$ to be an intensity matrix, each diagonal element of $\boldsymbol{D}_0$ has to be negative and is such that the sum of the elements on each row of $\boldsymbol{D}_0 + \boldsymbol{D}_1$ is zero. The bivariate Markov process (N, J) is called Markovian arrival process. Although in the literature of applied probability, the abbreviation 'MAP' is also used for Markovian arrival process, we will be using it to refer to the Markov additive process that will be introduced below.

Let $\{X_k\}_{k=1}^\infty$ be the sequence of individual claim severities which are positive continuous random variables. It is assumed that the distribution of the claim severity is dependent on the states of the environment process J immediately before and after transition of type (2). More precisely, whenever a transition from i to j is accompanied by a claim, the resulting claim severity has density f_{ij} with mean μ_{ij}. For later use, we let $\boldsymbol{f}(x) = (f_{ij}(x))_{i,j=1}^m$. In order to account for small fluctuations of the insurer's surplus, we shall use a Brownian motion with zero mean as perturbation. Whenever J is in state i, we assume that the insurer collects premium at rate $c_i > 0$ and the diffusion volatility is $\sigma_i > 0$. Under these assumptions, the surplus process $U^\infty = \{U_t^\infty\}_{t\geq 0}$ is defined as

$$U_t^\infty = u + \int_0^t c_{J_s} ds - \sum_{k=1}^{N_t} X_k + \int_0^t \sigma_{J_s} dB_s, \qquad t \geq 0. \tag{1.1}$$

Here $u \geq 0$ is the initial surplus, and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion starting at zero which is independent of other processes. The process U^∞ is a spectrally negative Markov additive process (MAP). For notational convenience, we write $\mathbb{P}_{u,i}\{\cdot\} = \mathbb{P}\{\cdot | U_0^\infty = u, J_0 = i\}$ and $\mathbb{E}_{u,i}[\cdot] = \mathbb{E}[\cdot | U_0^\infty = u, J_0 = i]$ for $i \in \mathcal{E}$ and $u \geq 0$. The time of ruin of the surplus process U^∞ is defined by $\tau^\infty = \inf\{t > 0 : U_t^\infty < 0\}$ with the convention $\inf\{\emptyset\} = \infty$. The net profit condition is given by

$$\sum_{i=1}^m \pi_i \left(c_i - \sum_{j=1}^m D_{1,ij}\mu_{ij} \right) > 0, \tag{1.2}$$

where $(\pi_1, \pi_2, \ldots, \pi_m)$ is the stationary probability row vector of J. Condition (1.2) ensures that the process (1.1) drifts to infinity in the long run (see e.g. Asmussen (2003), Corollaries 2.7 and 2.9)). Throughout this paper, it is assumed that Eq. 1.2 holds.

The class of MAP risk processes (1.1) is known to be very general as it includes the classical compound Poisson risk model (e.g. Asmussen and Albrecher (2010, Section IV)), the Markov-modulated risk process (e.g. Asmussen (1989) and Lu and Tsai (2007)), the semi-Markovian model by Albrecher and Boxma (2005), and renewal risk process with phase-type inter-arrival times (e.g. Feng (2009a,b)) as special cases. Recently, a lot of contributions have been made to the MAP risk model (with or without diffusion). For example, Cheung and Landriault (2009, Section 4) studied a dividend barrier strategy in which the barrier is allowed to depend on J; whereas Zhang et al. (2011) investigated the absolute ruin problem under debit interest. Moreover, Salah and Morales (2012) studied the Gerber-Shiu expected discounted penalty function (Gerber and Shiu (1998)) in a more general spectrally negative MAP risk process; whereas generalizations of the Gerber-Shiu function were analyzed by Cheung and Landriault (2010), Cheung and Feng (2013), and Feng and Shimizu (2014). While the afore-mentioned papers involve analytic derivations of the quantities of interest, we remark that MAP risk processes may also be studied using a more probabilistic approach via connection to Markov-modulated fluid flow (MMFF) processes (e.g. Badescu et al. (2005,2007), Ahn and Badescu (2007), Ahn et al. (2007)).

In this paper, we shall implement a barrier type dividend strategy in the MAP risk process described above. Recall that in the traditional dividend barrier strategy, the insurer pays dividends to its shareholders immediately whenever the surplus process reaches a fixed barrier level if ruin has not yet occurred (e.g. Gerber (1979), Lin et al. (2003) and Gerber and Shiu (2004)). However, when the surplus process contains a diffusion component, dividend payments may occur many times in a small time interval due to the existence of small fluctuations. Following the ideas as in Albrecher et al. (2011), one way to get around this problem is to assume that decisions are only made at discrete time points on whether lump sum dividend payments are paid. More specifically, we let $\{Z_i\}_{i=1}^{\infty}$ be the sequence of dividend decision times. At time Z_i, if the surplus level x is larger than a given barrier $b > 0$, then a lump sum dividend payment of size $x - b$ is paid to the shareholders of the insurance company. To give the mathematical descriptions of the modified surplus process $U^b = \{U_t^b\}_{t\geq 0}$ with dividends, the auxiliary process $U_i^* = \{U_i^*(t)\}_{t\geq 0}$ is introduced for $i = 1, 2, \ldots$. The dynamics of U^b and U_i^* can be jointly described recursively via

$$U_i^*(t) = \begin{cases} U_t^{\infty}, & i = 1; \quad t \geq 0, \\ U_{Z_{i-1}}^b + \int_{Z_{i-1}}^t c_{J_s} ds - \sum_{k=N_{Z_{i-1}}+1}^{N_t} X_k + \int_{Z_{i-1}}^t \sigma_{J_s} dB_s, & i = 2, 3, \ldots; \quad t \geq Z_{i-1}, \end{cases}$$

and for $i = 1, 2, \ldots$,

$$U_t^b = \begin{cases} U_i^*(t), & Z_{i-1} < t < Z_i, \\ \min(U_i^*(Z_i), b), & t = Z_i. \end{cases}$$

Without loss of generality, it is assumed that $Z_0 = 0-$ in the above definition, and therefore $U_0^b = u$ even if $U_0^{\infty} = u > b$. This means that time 0 is not assumed to be a dividend decision time. Unlike Albrecher et al. (2011,2013) who assumed that the event of ruin is only checked at the times $\{Z_i\}_{i=1}^{\infty}$, Zhang (2014) studied a variant of the model where solvency is monitored continuously as in the traditional case (see also Avanzi et al. (2013) for the corresponding variant in a dual risk model). We shall adopt the traditional definition of ruin in the sense that the surplus process is stopped immediately once it drops below zero. Hence, the time of ruin of U^b is defined by $\tau^b = \inf\{t > 0 : U_t^b < 0\}$. Let $T_1 = Z_1$ be the first dividend decision time, and $T_i = Z_i - Z_{i-1}$ be the ith inter-dividend-decision time (i.e. the interval between the $(i-1)$th and the ith dividend decision times) for $i = 2, 3, \ldots$. For the rest of the paper, it is assumed that $\{T_i\}_{i=1}^{\infty}$ forms a sequence of independent and identically distributed random variables distributed as T with the Erlang(n) density

$$f_T(t) = \frac{\beta^n t^{n-1} e^{-\beta t}}{(n-1)!}, \qquad t > 0.$$

Here n is the shape parameter which is a positive integer, and $\beta > 0$ is the scale parameter. It is assumed that $\{T_i\}_{i=1}^{\infty}$ is independent of all the attributes of the barrier-free process U^{∞}. The choice of the Erlang(n) distribution is motivated by the Erlangization techniques proposed by Asmussen et al. (2002) in solving finite-time ruin problems (see also e.g. Stanford et al. (2005, 2011) and Ramaswami et al. (2008)). Indeed, if we fix the mean $\mathbb{E}[T] = n/\beta = h$ and increase n (and β as well), then T converges in distribution to a point mass at h. Hence, one can approximate the situation where the inter-dividend-decision times are deterministic.

The Gerber-Shiu expected discounted penalty function, or Gerber-Shiu function in short, has been analyzed extensively in increasingly complex risk models since its introduction by Gerber and Shiu (1998). It unifies the study of various ruin-related quantities such as the time of ruin and the deficit at ruin. In this paper, we are interested in the Gerber-Shiu function pertaining to U^b defined as (given initial state $i \in \mathcal{E}$ and initial surplus $u \geq 0$)

$$\phi_i(u; b) = \mathbb{E}_{u,i}[e^{-\delta\tau^b} w(|U^b_{\tau^b}|)]. \tag{1.3}$$

Here $\delta \geq 0$ can be interpreted as the force of interest or the Laplace transform argument with respect to τ^b, and $w : [0, \infty) \to [0, \infty)$ is the so-called penalty function that depends on the deficit at ruin $|U^b_{\tau^b}|$. It is assumed that w satisfies some mild integrability conditions. Note that the indicator of the event $\{\tau^b < \infty\}$ is not necessary in the definition (1.3), since ruin occurs almost surely as the surplus can never exceed level b at the dividend decision times. Because of the perturbation, ruin may occur due to a claim or by diffusion. Thus, one may rewrite Eq. 1.3 as (e.g. Gerber and Landry (1998) and Tsai and Willmot (2002))

$$\phi_i(u; b) = w(0)\mathbb{E}_{u,i}[e^{-\delta\tau^b}\mathbf{1}_{\{U^b_{\tau^b}=0\}}] + \mathbb{E}_{u,i}[e^{-\delta\tau^b} w(|U^b_{\tau^b}|)\mathbf{1}_{\{U^b_{\tau^b}<0\}}], \tag{1.4}$$

where $\mathbf{1}_A$ stands for the indicator function of the event A. It is clear from Eq. 1.4 that if we are only interested in the contribution by ruin due to diffusion, one can simply let $w(y) = 0$ for $y > 0$. In contrast, the case where ruin is caused by a claim can be retrieved by letting $w(0) = 0$. For later use, we also let $\phi_i(u; \infty)$ be the Gerber-Shiu function associated with the barrier-free model U^{∞}. Another quantity of interest in this paper is the expected discounted dividends paid until ruin defined by (for a force of interest of $\delta > 0$)

$$V_i(u; b) = \mathbb{E}_{u,i}\left[\sum_{j=1}^{\infty} e^{-\delta Z_j}(U^b_{Z_j-} - b)_+\mathbf{1}_{\{Z_j<\tau^b\}}\right], \tag{1.5}$$

where $a_+ = \max(a, 0)$. In the corporate finance literature, the expectation of the present value of dividends represents the value of the firm. Therefore, under the current barrier type dividend strategy, the shareholders' interest would be to find the optimal barrier b^* that maximizes $V_i(u; b)$ with respect to b.

The remainder of this paper is structured as follows. In Section 2, some preliminary results and notations that will be used throughout are presented. Expressions for the Gerber-Shiu function (1.3) and the expected discounted dividends (1.5) are derived in Sections 3 and 4 respectively using Markov renewal equations. Examples along with numerical illustrations are then given in Section 5. The Appendix is concerned with the proofs of the continuity and smooth pasting conditions required in the derivations.

2 Preliminaries

In this paper, matrix notations will be used extensively. We shall use $\boldsymbol{O}$ to denote the zero matrix or vector with appropriate dimension known from the context. For a positive integer k, let $\boldsymbol{E}_k$ be the identity matrix of dimension k, and $\boldsymbol{e}_k$ be a column vector of ones with length k. For two arbitrary square matrices $\boldsymbol{A} = (a_{ij})_{i,j=1}^{k}$ and $\boldsymbol{B} = (b_{ij})_{i,j=1}^{k}$, the Hadamard product (i.e. entrywise multiplication) is defined as $\boldsymbol{A} \circ \boldsymbol{B} = (a_{ij}b_{ij})_{i,j=1}^{k}$. The notation $\boldsymbol{A}^\top$ denotes the transpose of a matrix or vector $\boldsymbol{A}$. In addition, we denote the Laplace transform of a function defined on $[0, \infty)$ (which is not necessarily a probability density) by adding a hat on it. For example, for $\Re(s) \geq 0$, one has $\widehat{f_{ij}}(s) = \int_0^\infty e^{-sx} f_{ij}(x)dx$. Any integral with respect to a matrix-valued function is taken element-wise. For example, $\widehat{\boldsymbol{f}}(s) = \int_0^\infty e^{-sx} \boldsymbol{f}(x)dx = (\widehat{f_{ij}}(s))_{i,j=1}^{m}$.

The notion of the matrix Dickson-Hipp operator plays an important part in our analysis. The matrix version of the Dickson-Hipp operator was first introduced by Feng (2009b) as an extension of the classical scalar counterpart proposed by Dickson and Hipp (2001). For a square matrix $\boldsymbol{A}$ having eigenvalues on the right-half of the complex plane, the matrix Dickson-Hipp operator $\mathcal{T}_{\boldsymbol{A}}$ is defined as

$$\mathcal{T}_{\boldsymbol{A}}\boldsymbol{h}(x) = \int_x^\infty e^{-\boldsymbol{A}(y-x)}\boldsymbol{h}(y)dy = \int_0^\infty e^{-\boldsymbol{A}y}\boldsymbol{h}(x+y)dy, \qquad x \geq 0, \tag{2.1}$$

where $\boldsymbol{h}$ is a matrix-valued function with appropriate dimension (such that the multiplication $\boldsymbol{A}\boldsymbol{h}$ makes sense) satisfying some integrability conditions (such that the above integral exists). When $\boldsymbol{A}$ reduces to a scalar r with non-negative real part, then $\mathcal{T}_r$ is the classical Dickson-Hipp operator. If $x = 0$ and $\boldsymbol{A} = s\boldsymbol{E}_k$ for $\Re(s) \geq 0$ and some positive integer k such that $\boldsymbol{E}_k\boldsymbol{h}$ makes sense, then Eq. 2.1 is equivalent to the Laplace transform of $\boldsymbol{h}$ with argument s, namely $\widehat{\boldsymbol{h}}(s)$. An appealing property of the (matrix) Dickson-Hipp operator is the commutative property. In particular, if the square matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ commute (i.e. $\boldsymbol{A}_1\boldsymbol{A}_2 = \boldsymbol{A}_2\boldsymbol{A}_1$) and $\boldsymbol{A}_1 - \boldsymbol{A}_2$ is nonsingular, then Feng (2009b, Lemma 2.1) showed that

$$\mathcal{T}_{\boldsymbol{A}_1}\mathcal{T}_{\boldsymbol{A}_2}\boldsymbol{h}(x) = \mathcal{T}_{\boldsymbol{A}_2}\mathcal{T}_{\boldsymbol{A}_1}\boldsymbol{h}(x) = (\boldsymbol{A}_1 - \boldsymbol{A}_2)^{-1}(\mathcal{T}_{\boldsymbol{A}_2}\boldsymbol{h}(x) - \mathcal{T}_{\boldsymbol{A}_1}\boldsymbol{h}(x)), \qquad x \geq 0. \tag{2.2}$$

Whenever a function under consideration has two arguments u and b, any derivative, Laplace transform or Dickson-Hipp operator is assumed to be taken with respect to the first argument u by default.

Next, we introduce some preliminaries on MAP. Let $\boldsymbol{\Delta}_{\sigma^2} = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2)$ and $\boldsymbol{\Delta}_c = \text{diag}(c_1, \ldots, c_m)$. From Asmussen (2003), Proposition XI.2.2), the matrix cumulant generating function of U^∞ is given by

$$\boldsymbol{G}(s) = \frac{1}{2}s^2\boldsymbol{\Delta}_{\sigma^2} + s\boldsymbol{\Delta}_c + \boldsymbol{D}_0 + \boldsymbol{D}_1 \circ \widehat{\boldsymbol{f}}(s)$$

for $s \in \mathbb{C}$ such that the integral in the last term exists. Note that $\boldsymbol{G}(s)$ is well defined at least for $\Re(s) \geq 0$. There exists a matrix $\boldsymbol{Q}_\gamma$ that satisfies, for a given $\gamma \geq 0$,

$$\frac{1}{2}\boldsymbol{Q}_\gamma^2\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{Q}_\gamma\boldsymbol{\Delta}_c - \gamma\boldsymbol{E}_m + \boldsymbol{D}_0 + \int_0^\infty e^{-\boldsymbol{Q}_\gamma x}(\boldsymbol{D}_1 \circ \boldsymbol{f}(x))dx = \boldsymbol{O}. \tag{2.3}$$

The existence of $\boldsymbol{Q}_\gamma$ is known from Breuer (2008, Theorems 1 and 2) and Feng and Shimizu (2014, Lemma 3.2); whereas the relation of $\boldsymbol{Q}_\gamma$ to the intensity matrix of the time-reversed version of the MAP risk model has been discussed by e.g. Zhang et al. (2011, Section 3) and Salah and Morales (2012, Section 4). In particular, the matrix $\boldsymbol{Q}_\gamma$ can be computed using either an iterative approach (Breuer (2008, Theorem 2)) or the more well-known eigenvalue/eigenvector method (e.g. Zhang et al. (2011, Lemma 1), and Cheung and Feng (2013, Appendix)). We shall describe the latter method which is indeed linked to the more classical form of the Lundberg's equation (in ξ), namely

$$\det(\boldsymbol{G}(\xi) - \gamma \boldsymbol{E}_m) = 0. \tag{2.4}$$

It follows from Feng and Shimizu (2014, Lemma 3.2) that the above equation has exactly m roots with non-negative real parts. These roots are denoted by $\rho_{\gamma,1}, \ldots, \rho_{\gamma,m}$. Throughout this paper, we suppose that $\boldsymbol{Q}_\gamma$ is diagonalizable. A sufficient condition for $\boldsymbol{Q}_\gamma$ to be diagonalizable is that $\rho_{\gamma,1}, \ldots, \rho_{\gamma,m}$ are distinct. Then $\boldsymbol{Q}_\gamma$ admits the representation $\boldsymbol{Q}_\gamma = \boldsymbol{B}_\gamma^{-1} \boldsymbol{\Delta}_{\rho_\gamma} \boldsymbol{B}_\gamma$ (see Zhang et al. (2011, Lemma 1)). Here $\boldsymbol{\Delta}_{\rho_\gamma} = \text{diag}(\rho_{\gamma,1}, \ldots, \rho_{\gamma,m})$ is the matrix of eigenvalues and $\boldsymbol{B}_\gamma = (\boldsymbol{b}_{\gamma,1}^\top, \ldots, \boldsymbol{b}_{\gamma,m}^\top)^\top$ is the matrix containing the corresponding eigenvectors. In particular, for each fixed $i = 1, 2, \ldots, m$, the left eigenvector $\boldsymbol{b}_{\gamma,i}$ is a non-trivial solution of the equation (in $\boldsymbol{b}$)

$$\boldsymbol{b}[\boldsymbol{G}(\rho_{\gamma,i}) - \gamma \boldsymbol{E}_m] = \boldsymbol{O}.$$

It is instructive to note that since the situations in which there are multiple roots to the Lundberg's equation (2.4) are rare, the diagonalizability assumption imposed on $\boldsymbol{Q}_\gamma$ is not restrictive. Interested readers are referred to Ji and Zhang (2012) for the treatment of risk models with multiple Lundberg's roots.

3 The Gerber-Shiu Function

This section aims at deriving the solution to the Gerber-Shiu function. Note that the Erlang(n) inter-dividend-decision time T can be regarded as the sum of n independent and identically distributed exponential variables. To ease our analysis, for $k = 1, 2, \ldots, n$ we define $\phi_{k,i}(u; b)$ to be the Gerber-Shiu function under the same conditions as $\phi_i(u; b)$, except that the time until the first (not between all) dividend decision time is Erlang($n-k+1$) distributed. Obviously, one has that $\phi_i(u; b) = \phi_{1,i}(u; b)$. The introduction of these auxiliary functions will enable us to capture the underlying phase-type structure of the problem.

3.1 System of Integro-Differential Equations

We can start by considering the competition between the state transition of J and the phase transition of the first dividend decision time over a very small time interval $[0, h]$ for $i \in \mathcal{E}$ and $u > 0$. For $k = 1, 2, \ldots, n-1$, no dividends will be payable within the interval and one has

$$\begin{aligned} \phi_{k,i}(u; b) = \; & (1 - (-D_{0,ii} + \beta)h)e^{-\delta h}\mathbb{E}\left[\phi_{k,i}(u + c_i h + \sigma_i B_h; b)\right] \\ & + \sum_{j=1, j\neq i}^{m} D_{0,ij} h e^{-\delta h}\mathbb{E}\left[\phi_{k,j}(u + c_i h + \sigma_i B_h; b)\right] \\ & + \sum_{j=1}^{m} D_{1,ij} h e^{-\delta h}\mathbb{E}\left[\gamma_{k,ij}(u + c_i h + \sigma_i B_h; b) + \omega_{ij}(u + c_i h + \sigma_i B_h)\right] \\ & + \beta h e^{-\delta h}\mathbb{E}\left[\phi_{k+1,i}(u + c_i h + \sigma_i B_h; b)\right] + o(h), \end{aligned} \tag{3.1}$$

where $\gamma_{k,ij}(u;b) = \int_0^u \phi_{k,j}(u-x;b) f_{ij}(x)dx$ and $\omega_{ij}(u) = \int_u^\infty w(x-u) f_{ij}(x)dx$. Applying Taylor's expansion to Eq. 3.1, dividing by h, letting $h \to 0$ and rearranging terms, we obtain

$$\begin{aligned} 0 = \ & \frac{\sigma_i^2}{2}\phi''_{k,i}(u;b) + c_i\phi'_{k,i}(u;b) - (\delta+\beta)\phi_{k,i}(u;b) \\ & + \sum_{j=1}^m D_{0,ij}\phi_{k,j}(u;b) + \sum_{j=1}^m D_{1,ij}(\gamma_{k,ij}(u;b) + \omega_{ij}(u)) \\ & + \beta\phi_{k+1,i}(u;b), \qquad k = 1,2,\ldots,n-1. \end{aligned} \tag{3.2}$$

For $k = n$, the analysis is essentially the same, except that dividends will be paid if the surplus is above b when the first dividend decision time occurs. This leads us to

$$\begin{aligned} 0 = \ & \frac{\sigma_i^2}{2}\phi''_{n,i}(u;b) + c_i\phi'_{n,i}(u;b) - (\delta+\beta)\phi_{n,i}(u;b) + \sum_{j=1}^m D_{0,ij}\phi_{n,j}(u;b) \\ & + \sum_{j=1}^m D_{1,ij}(\gamma_{n,ij}(u;b) + \omega_{ij}(u)) \\ & + \beta(\phi_{1,i}(u;b)\mathbf{1}_{\{0<u\le b\}} + \phi_{1,i}(b;b)\mathbf{1}_{\{u>b\}}). \end{aligned} \tag{3.3}$$

Define $\boldsymbol{\phi}_k(u;b) = (\phi_{k,1}(u;b),\ldots,\phi_{k,m}(u;b))^\top$ for $k = 1,2,\ldots,n$. The integro-differential equations (3.2) and (3.3) can then be rewritten in matrix form as

$$\begin{aligned} \boldsymbol{O} = & \left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\frac{d^2}{du^2} + \boldsymbol{\Delta}_c\frac{d}{du} - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0\right)\boldsymbol{\phi}_k(u;b) + \int_0^u (\boldsymbol{D}_1 \circ \boldsymbol{f}(x))\boldsymbol{\phi}_k(u-x;b)dx \\ & + \beta\boldsymbol{\phi}_{k+1}(u;b) + \boldsymbol{\zeta}(u), \qquad k = 1,2,\ldots,n-1, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \boldsymbol{O} = & \left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\frac{d^2}{du^2} + \boldsymbol{\Delta}_c\frac{d}{du} - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0\right)\boldsymbol{\phi}_n(u;b) + \int_0^u (\boldsymbol{D}_1 \circ \boldsymbol{f}(x))\boldsymbol{\phi}_n(u-x;b)dx \\ & + \beta\boldsymbol{\phi}_1(u;b)\mathbf{1}_{\{0<u\le b\}} + \beta\boldsymbol{\phi}_1(b;b)\mathbf{1}_{\{u>b\}} + \boldsymbol{\zeta}(u), \end{aligned} \tag{3.5}$$

where $\boldsymbol{\zeta}(u) = (\boldsymbol{D}_1 \circ \boldsymbol{\omega}(u))\boldsymbol{e}_m$ and $\boldsymbol{\omega}(u) = (\omega_{ij}(u))_{i,j=1}^m$.

A trivial boundary condition for the system comprising Eqs. 3.4 and 3.5 is given by

$$\boldsymbol{\phi}_k(0;b) = w(0)\boldsymbol{e}_m, \qquad k = 1,2,\ldots,n, \tag{3.6}$$

since ruin occurs immediately with zero initial surplus. In addition, we assert that the continuity condition

$$\boldsymbol{\phi}_k(b-;b) = \boldsymbol{\phi}_k(b+;b), \qquad k = 1,2,\ldots,n, \tag{3.7}$$

and the smooth pasting condition

$$\boldsymbol{\phi}'_k(b-;b) = \boldsymbol{\phi}'_k(b+;b), \qquad k = 1,2,\ldots,n, \tag{3.8}$$

hold at the barrier b . See Appendix for further discussions of Eqs. 3.7 and 3.8.

3.2 The Case $0 < u < b$

In this subsection, we will solve Eqs. 3.4 and 3.5 when $0 < u < b$ apart from some unknown constants. Using the notion of Kronecker product, we define the square matrices

$\tilde{\mathbf{\Delta}}_{\sigma^2} = \mathbf{E}_n \otimes \mathbf{\Delta}_{\sigma^2}$, $\tilde{\mathbf{\Delta}}_c = \mathbf{E}_n \otimes \mathbf{\Delta}_c$, $\tilde{\mathbf{D}}_1 = \mathbf{E}_n \otimes \mathbf{D}_1$, $\tilde{\mathbf{f}}(x) = \mathbf{E}_n \otimes \mathbf{f}(x)$, and

$$\tilde{\mathbf{D}}_0 = \begin{pmatrix} \mathbf{D}_0 - \beta \mathbf{E}_m & \beta \mathbf{E}_m & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{D}_0 - \beta \mathbf{E}_m & \beta \mathbf{E}_m & \cdots & \mathbf{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \beta \mathbf{E} \\ \beta \mathbf{E}_m & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{D}_0 - \beta \mathbf{E}_m \end{pmatrix},$$

all of dimension mn. (The above matrix is understood to be $\mathbf{D}_0$ when $n = 1$.) Further define the column vectors $\underline{\boldsymbol{\phi}}(u; b) = (\boldsymbol{\phi}_1^\top(u; b), \ldots, \boldsymbol{\phi}_n^\top(u; b))^\top$ and $\underline{\boldsymbol{\zeta}}(u) = \mathbf{e}_n \otimes \boldsymbol{\zeta}(u)$. Then Eqs. 3.4 and 3.5 can be neatly combined to yield

$$\left(\frac{1}{2}\tilde{\mathbf{\Delta}}_{\sigma^2}\frac{d^2}{du^2} + \tilde{\mathbf{\Delta}}_c\frac{d}{du} - \delta \mathbf{E}_{mn} + \tilde{\mathbf{D}}_0\right)\underline{\boldsymbol{\phi}}(u; b) + \int_0^u (\tilde{\mathbf{D}}_1 \circ \tilde{\mathbf{f}}(x))\underline{\boldsymbol{\phi}}(u - x; b)dx + \underline{\boldsymbol{\zeta}}(u) = \mathbf{O}, \qquad 0 < u < b, \tag{3.9}$$

which is a non-homogeneous matrix integro-differential equation. From the theory of integro-differential equations, the general solution of Eq. 3.9 can be expressed in terms of one of its particular solution plus a fundamental set of solutions of the homogeneous system. Hence we first identify a particular solution as follows. Define $\boldsymbol{\phi}(u; \infty) = (\phi_1(u; \infty), \ldots, \phi_m(u; \infty))^\top$ for the barrier-free model U^∞. Note that the integro-differential equation satisfied by $\boldsymbol{\phi}(u; \infty)$ can be obtained from Eq. 3.5 by setting $n = 1$ and taking the limit $b \to \infty$. Thus, we have

$$\left(\frac{1}{2}\mathbf{\Delta}_{\sigma^2}\frac{d^2}{du^2} + \mathbf{\Delta}_c\frac{d}{du} - \delta \mathbf{E}_m + \mathbf{D}_0\right)\boldsymbol{\phi}(u; \infty) + \int_0^u (\mathbf{D}_1 \circ \mathbf{f}(x))\boldsymbol{\phi}(u - x; \infty)dx + \boldsymbol{\zeta}(u) = \mathbf{O}, \qquad u > 0,$$

from which one can easily deduce that $\underline{\boldsymbol{\phi}}(u; \infty) = \mathbf{e}_n \otimes \boldsymbol{\phi}(u; \infty)$ is a particular solution of Eq. 3.9, i.e.

$$\left(\frac{1}{2}\tilde{\mathbf{\Delta}}_{\sigma^2}\frac{d^2}{du^2} + \tilde{\mathbf{\Delta}}_c\frac{d}{du} - \delta \mathbf{E}_{mn} + \tilde{\mathbf{D}}_0\right)\underline{\boldsymbol{\phi}}(u; \infty) + \int_0^u (\tilde{\mathbf{D}}_1 \circ \tilde{\mathbf{f}}(x))\underline{\boldsymbol{\phi}}(u - x; \infty)dx + \underline{\boldsymbol{\zeta}}(u) = \mathbf{O}, \qquad u > 0. \tag{3.10}$$

Next, let $\underline{\mathbf{v}}_\delta(u)$ be a vector valued function of length mn such that $\underline{\mathbf{v}}_\delta(0) = \mathbf{O}$ and it satisfies the homogeneous version of Eq. 3.9, namely

$$\left(\frac{1}{2}\tilde{\mathbf{\Delta}}_{\sigma^2}\frac{d^2}{du^2} + \tilde{\mathbf{\Delta}}_c\frac{d}{du} - \delta \mathbf{E}_{mn} + \tilde{\mathbf{D}}_0\right)\underline{\mathbf{v}}_\delta(u) + \int_0^u (\tilde{\mathbf{D}}_1 \circ \tilde{\mathbf{f}}(x))\underline{\mathbf{v}}_\delta(u - x)dx = \mathbf{O}, \qquad u > 0. \tag{3.11}$$

By taking Laplace transforms on both sides of Eq. 3.11, we obtain

$$\left(\frac{1}{2}s^2\tilde{\mathbf{\Delta}}_{\sigma^2} + s\tilde{\mathbf{\Delta}}_c - \delta \mathbf{E}_{mn} + \tilde{\mathbf{D}}_0 + \tilde{\mathbf{D}}_1 \circ \widehat{\tilde{\mathbf{f}}}(s)\right)\widehat{\underline{\mathbf{v}}}_\delta(s) = \frac{1}{2}\tilde{\mathbf{\Delta}}_{\sigma^2}\underline{\mathbf{v}}'_\delta(0),$$

leading to

$$\underline{\mathbf{v}}_\delta(u) = \underline{\mathbf{L}}_\delta(u)\left(\frac{1}{2}\tilde{\mathbf{\Delta}}_{\sigma^2}\underline{\mathbf{v}}'_\delta(0)\right), \qquad u \geq 0, \tag{3.12}$$

where

$$\underline{\boldsymbol{L}}_{\delta}(u)=\begin{pmatrix}\boldsymbol{L}_{\delta,1,1}(u) & \cdots & \boldsymbol{L}_{\delta,1,n}(u)\\ \vdots & \ddots & \vdots\\ \boldsymbol{L}_{\delta,n,1}(u) & \cdots & \boldsymbol{L}_{\delta,n,n}(u)\end{pmatrix}=\mathcal{L}^{-1}\left(\left(\frac{1}{2}s^2\tilde{\boldsymbol{\Delta}}_{\sigma^2}+s\tilde{\boldsymbol{\Delta}}_c-\delta\boldsymbol{E}_{mn}+\tilde{\boldsymbol{D}}_0+\tilde{\boldsymbol{D}}_1\circ\widehat{\tilde{\boldsymbol{f}}}(s)\right)^{-1}\right). \tag{3.13}$$

Here each sub-matrix $\boldsymbol{L}_{\delta,i,j}(u)$ is a square matrix of dimension m, and $\mathcal{L}^{-1}$ represents the inverse Laplace transform operator.

Now, by taking the difference of Eqs. 3.9 and 3.10, we note that $\underline{\boldsymbol{\phi}}(u;b)-\underline{\boldsymbol{\phi}}(u;\infty)$ satisfies (3.11) for $0<u<b$. Moreover, using $\underline{\boldsymbol{\phi}}(0;\infty)=w(0)\boldsymbol{e}_{mn}$ and Eq. 3.6, it is clear that the condition $\underline{\boldsymbol{\phi}}(0;b)-\underline{\boldsymbol{\phi}}(0;\infty)=\boldsymbol{O}$ holds true. Thus, Eq. 3.12 implies that we must have

$$\underline{\boldsymbol{\phi}}(u;b)=\underline{\boldsymbol{\phi}}(u;\infty)+\underline{\boldsymbol{L}}_{\delta}(u)(\boldsymbol{k}_1^{\top},\ldots,\boldsymbol{k}_n^{\top})^{\top},\qquad 0\le u\le b, \tag{3.14}$$

where $\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n$ are unknown column vectors of constants, each of length m, that are to be determined later (as in Eq. 3.48). It remains to derive exact expressions for $\underline{\boldsymbol{\phi}}(u;\infty)$ and $\underline{\boldsymbol{L}}_{\delta}(u)$. The derivation relies on the fact that

$$\frac{1}{2}s^2\tilde{\boldsymbol{\Delta}}_{\sigma^2}+s\tilde{\boldsymbol{\Delta}}_c+\tilde{\boldsymbol{D}}_0+\tilde{\boldsymbol{D}}_1\circ\widehat{\tilde{\boldsymbol{f}}}(s)$$

is the matrix cumulant generating function of a certain MAP with intensity matrix $\tilde{\boldsymbol{D}}_0+\tilde{\boldsymbol{D}}_1$. Hence, it follows from Section 2 (see Eq. 2.3) that there exists a matrix $\tilde{\boldsymbol{Q}}_{\delta}$ (assumed to be diagonalizable) such that

$$\frac{1}{2}\tilde{\boldsymbol{Q}}_{\delta}^2\tilde{\boldsymbol{\Delta}}_{\sigma^2}+\tilde{\boldsymbol{Q}}_{\delta}\tilde{\boldsymbol{\Delta}}_c-\delta\boldsymbol{E}_{mn}+\tilde{\boldsymbol{D}}_0+\int_0^{\infty}e^{-\tilde{\boldsymbol{Q}}_{\delta}x}(\tilde{\boldsymbol{D}}_1\circ\tilde{\boldsymbol{f}}(x))dx=\boldsymbol{O}, \tag{3.15}$$

and the eigenvalues of $\tilde{\boldsymbol{Q}}_{\delta}$ are all on the right-half of the complex plane. The solution to $\underline{\boldsymbol{L}}_{\delta}(u)$ is first given in the next Proposition.

Proposition 1 *Let*

$$\tilde{\boldsymbol{M}}_{\delta}(u)=\int_0^u 2\tilde{\boldsymbol{\Delta}}_{\sigma^2}^{-1}e^{-(\tilde{\boldsymbol{Q}}_{\delta}+2\tilde{\boldsymbol{\Delta}}_c\tilde{\boldsymbol{\Delta}}_{\sigma^2}^{-1})(u-x)}e^{\tilde{\boldsymbol{Q}}_{\delta}x}dx \tag{3.16}$$

and

$$\tilde{\boldsymbol{g}}_{\delta}(x)=\int_0^x 2\tilde{\boldsymbol{\Delta}}_{\sigma^2}^{-1}e^{-(\tilde{\boldsymbol{Q}}_{\delta}+2\tilde{\boldsymbol{\Delta}}_c\tilde{\boldsymbol{\Delta}}_{\sigma^2}^{-1})(x-y)}\mathcal{T}_{\tilde{\boldsymbol{Q}}_{\delta}}(\tilde{\boldsymbol{D}}_1\circ\tilde{\boldsymbol{f}}(y))dy. \tag{3.17}$$

Then we have

$$\underline{\boldsymbol{L}}_{\delta}(u)=\tilde{\boldsymbol{M}}_{\delta}(u)+\int_0^u\tilde{\boldsymbol{S}}_{\delta}(x)\tilde{\boldsymbol{M}}_{\delta}(u-x)dx,\qquad u\ge 0, \tag{3.18}$$

where

$$\tilde{\boldsymbol{S}}_{\delta}(x)=\sum_{i=1}^{\infty}\tilde{\boldsymbol{g}}_{\delta}^{*i}(x). \tag{3.19}$$

Here the i-fold convolution is defined recursively as $\tilde{\boldsymbol{g}}_{\delta}^{*i}(x)=\int_0^x\tilde{\boldsymbol{g}}_{\delta}^{*(i-1)}(x-y)\tilde{\boldsymbol{g}}_{\delta}(y)dy$ for $i=2,3,\ldots$, with the starting point $\tilde{\boldsymbol{g}}_{\delta}^{*1}(x)=\tilde{\boldsymbol{g}}_{\delta}(x)$.

Proof Because the left-hand side of Eq. 3.15 represents a zero matrix by definition, by subtraction we obtain

$$\begin{aligned}
&\frac{1}{2}s^2\tilde{\mathbf{\Delta}}_{\sigma^2} + s\tilde{\mathbf{\Delta}}_c - \delta\boldsymbol{E}_{mn} + \tilde{\boldsymbol{D}}_0 + \tilde{\boldsymbol{D}}_1 \circ \widehat{\tilde{\boldsymbol{f}}}(s) \\
&= \frac{1}{2}((s\boldsymbol{E}_{mn})^2 - \tilde{\boldsymbol{Q}}_\delta^2)\tilde{\mathbf{\Delta}}_{\sigma^2} + (s\boldsymbol{E}_{mn} - \tilde{\boldsymbol{Q}}_\delta)\tilde{\mathbf{\Delta}}_c + \int_0^\infty (e^{-s\boldsymbol{E}_{mn}x} - e^{-\tilde{\boldsymbol{Q}}_\delta x})(\tilde{\boldsymbol{D}}_1 \circ \tilde{\boldsymbol{f}}(x))dx \\
&= (s\boldsymbol{E}_{mn} - \tilde{\boldsymbol{Q}}_\delta)\left(\frac{1}{2}(s\boldsymbol{E}_{mn} + \tilde{\boldsymbol{Q}}_\delta)\tilde{\mathbf{\Delta}}_{\sigma^2} + \tilde{\mathbf{\Delta}}_c - \mathcal{T}_{s\boldsymbol{E}_{mn}}\mathcal{T}_{\tilde{\boldsymbol{Q}}_\delta}(\tilde{\boldsymbol{D}}_1 \circ \tilde{\boldsymbol{f}}(0))\right), \qquad (3.20)
\end{aligned}$$

where the second step follows from the commutative property (2.2) of Dickson-Hipp operators. Note from Eq. 3.13 that the matrix inverse of the above expression is the Laplace transform of $\underline{\boldsymbol{L}}_\delta(u)$, namely $\widehat{\underline{\boldsymbol{L}}}_\delta(s)$. Hence, by simple manipulations we have that

$$\begin{aligned}
&\left(\boldsymbol{E}_{mn} - \left(\frac{1}{2}(s\boldsymbol{E}_{mn} + \tilde{\boldsymbol{Q}}_\delta)\tilde{\mathbf{\Delta}}_{\sigma^2} + \tilde{\mathbf{\Delta}}_c\right)^{-1}\mathcal{T}_{s\boldsymbol{E}_{mn}}\mathcal{T}_{\tilde{\boldsymbol{Q}}_\delta}(\tilde{\boldsymbol{D}}_1 \circ \tilde{\boldsymbol{f}}(0))\right)\widehat{\underline{\boldsymbol{L}}}_\delta(s) \\
&= \left(\frac{1}{2}(s\boldsymbol{E}_{mn} + \tilde{\boldsymbol{Q}}_\delta)\tilde{\mathbf{\Delta}}_{\sigma^2} + \tilde{\mathbf{\Delta}}_c\right)^{-1}(s\boldsymbol{E}_{mn} - \tilde{\boldsymbol{Q}}_\delta)^{-1}. \qquad (3.21)
\end{aligned}$$

Because

$$\left(\frac{1}{2}(s\boldsymbol{E}_{mn} + \tilde{\boldsymbol{Q}}_\delta)\tilde{\mathbf{\Delta}}_{\sigma^2} + \tilde{\mathbf{\Delta}}_c\right)^{-1} = 2\tilde{\mathbf{\Delta}}_{\sigma^2}^{-1}\left(s\boldsymbol{E}_{mn} + \tilde{\boldsymbol{Q}}_\delta + 2\tilde{\mathbf{\Delta}}_c\tilde{\mathbf{\Delta}}_{\sigma^2}^{-1}\right)^{-1},$$

inverting the Laplace transforms with respect to s in Eq. 3.21 yields the Markov renewal equation

$$\underline{\boldsymbol{L}}_\delta(u) = \int_0^u \tilde{\boldsymbol{g}}_\delta(x)\underline{\boldsymbol{L}}_\delta(u-x)dx + \tilde{\boldsymbol{M}}_\delta(u), \qquad u \geq 0, \qquad (3.22)$$

where $\tilde{\boldsymbol{M}}_\delta(u)$ and $\tilde{\boldsymbol{g}}_\delta(x)$ are defined in Eqs. 3.16 and 3.17, respectively. The matrix $\int_0^\infty \tilde{\boldsymbol{g}}_\delta(x)dx$ is known to be strictly substochastic (see Feng and Shimizu (2014, Appendix D)), and therefore Eq. 3.22 can be regarded as a matrix version of defective renewal equation. By Markov renewal theory (e.g. Cinlar (1969, Section 3a) or Asmussen (2003, Section VII.4)), the solution of Eq. 3.22 is given by Eq. 3.18. □

We remark that the Gerber-Shiu function $\underline{\boldsymbol{\phi}}(u;\infty)$ in the absence of dividends can in principle be obtained from Feng and Shimizu (2014, Theorem 3.1 and Remark 5.1) via some tedious algebra. Nonetheless, the solution to $\underline{\boldsymbol{\phi}}(u;\infty)$ is given in the next Proposition to keep this paper self-contained. We provide a direct proof because some of the techniques will be used later on as well.

Proposition 2 *Let*

$$\tilde{\boldsymbol{Z}}_\delta(u) = w(0)\tilde{\mathbf{\Delta}}_{\sigma^2}^{-1}e^{-(\tilde{\boldsymbol{Q}}_\delta + 2\tilde{\mathbf{\Delta}}_c\tilde{\mathbf{\Delta}}_{\sigma^2}^{-1})u}\tilde{\mathbf{\Delta}}_{\sigma^2}\boldsymbol{e}_{mn} + \int_0^u 2\tilde{\mathbf{\Delta}}_{\sigma^2}^{-1}e^{-(\tilde{\boldsymbol{Q}}_\delta + 2\tilde{\mathbf{\Delta}}_c\tilde{\mathbf{\Delta}}_{\sigma^2}^{-1})(u-x)}\mathcal{T}_{\tilde{\boldsymbol{Q}}_\delta}\underline{\boldsymbol{\zeta}}(x)dx. \qquad (3.23)$$

Then we have

$$\underline{\boldsymbol{\phi}}(u;\infty) = \tilde{\boldsymbol{Z}}_\delta(u) + \int_0^u \tilde{\boldsymbol{S}}_\delta(x)\tilde{\boldsymbol{Z}}_\delta(u-x)dx, \qquad u \geq 0, \qquad (3.24)$$

where $\tilde{\boldsymbol{S}}_\delta(x)$ is defined in Eq. 3.19.

Proof Taking Laplace transforms in Eq. 3.10 along with the use of $\underline{\boldsymbol{\phi}}(0;\infty) = w(0)\boldsymbol{e}_{mn}$ gives

$$\left(\frac{1}{2}s^2\tilde{\boldsymbol{\Delta}}_{\sigma^2}+s\tilde{\boldsymbol{\Delta}}_c-\delta\boldsymbol{E}_{mn}+\tilde{\boldsymbol{D}}_0+\tilde{\boldsymbol{D}}_1\circ\widehat{\boldsymbol{f}}(s)\right)\widehat{\underline{\boldsymbol{\phi}}}(s;\infty)=\frac{1}{2}\tilde{\boldsymbol{\Delta}}_{\sigma^2}\underline{\boldsymbol{\phi}}'(0;\infty)+w(0)\left(\frac{1}{2}s\tilde{\boldsymbol{\Delta}}_{\sigma^2}+\tilde{\boldsymbol{\Delta}}_c\right)\boldsymbol{e}_{mn}-\widehat{\underline{\boldsymbol{\zeta}}}(s). \tag{3.25}$$

Note that the term $\underline{\boldsymbol{\phi}}'(0;\infty)$ appearing in the above equation is unknown. Following the same arguments as in the proof of Theorem 2 in Zhang et al. (2011), we omit the straightforward algebra and obtain

$$\boldsymbol{O}=\frac{1}{2}\tilde{\boldsymbol{\Delta}}_{\sigma^2}\underline{\boldsymbol{\phi}}'(0;\infty)+w(0)\left(\frac{1}{2}\tilde{\boldsymbol{Q}}_\delta\tilde{\boldsymbol{\Delta}}_{\sigma^2}+\tilde{\boldsymbol{\Delta}}_c\right)\boldsymbol{e}_{mn}-\int_0^\infty e^{-\tilde{\boldsymbol{Q}}_\delta u}\underline{\boldsymbol{\zeta}}(u)du. \tag{3.26}$$

By subtraction and property (2.2) of Dickson-Hipp operators, the right-hand side of Eq. 3.25 can be represented as

$$\begin{aligned}&\frac{1}{2}\tilde{\boldsymbol{\Delta}}_{\sigma^2}\underline{\boldsymbol{\phi}}'(0;\infty)+w(0)\left(\frac{1}{2}s\tilde{\boldsymbol{\Delta}}_{\sigma^2}+\tilde{\boldsymbol{\Delta}}_c\right)\boldsymbol{e}_{mn}-\widehat{\underline{\boldsymbol{\zeta}}}(s)\\&=\frac{1}{2}w(0)(s\boldsymbol{E}_{mn}-\tilde{\boldsymbol{Q}}_\delta)\tilde{\boldsymbol{\Delta}}_{\sigma^2}\boldsymbol{e}_{mn}+\int_0^\infty(e^{-\tilde{\boldsymbol{Q}}_\delta u}-e^{-s\boldsymbol{E}_{mn}u})\underline{\boldsymbol{\zeta}}(u)du\\&=(s\boldsymbol{E}_{mn}-\tilde{\boldsymbol{Q}}_\delta)\left(\frac{1}{2}w(0)\tilde{\boldsymbol{\Delta}}_{\sigma^2}\boldsymbol{e}_{mn}+\mathcal{T}_{s\boldsymbol{E}_{mn}}\mathcal{T}_{\tilde{\boldsymbol{Q}}_\delta}\underline{\boldsymbol{\zeta}}(0)\right).\end{aligned} \tag{3.27}$$

Substitution of Eqs. 3.20 and 3.27 into Eq. 3.25 yields

$$\begin{aligned}&\left(\boldsymbol{E}_{mn}-\left(\frac{1}{2}(s\boldsymbol{E}_{mn}+\tilde{\boldsymbol{Q}}_\delta)\tilde{\boldsymbol{\Delta}}_{\sigma^2}+\tilde{\boldsymbol{\Delta}}_c\right)^{-1}\mathcal{T}_{s\boldsymbol{E}_{mn}}\mathcal{T}_{\tilde{\boldsymbol{Q}}_\delta}(\tilde{\boldsymbol{D}}_1\circ\tilde{\boldsymbol{f}}(0))\right)\widehat{\underline{\boldsymbol{\phi}}}(s;\infty)\\&=\left(\frac{1}{2}(s\boldsymbol{E}_{mn}+\tilde{\boldsymbol{Q}}_\delta)\tilde{\boldsymbol{\Delta}}_{\sigma^2}+\tilde{\boldsymbol{\Delta}}_c\right)^{-1}\left(\frac{1}{2}w(0)\tilde{\boldsymbol{\Delta}}_{\sigma^2}\boldsymbol{e}_{mn}+\mathcal{T}_{s\boldsymbol{E}_{mn}}\mathcal{T}_{\tilde{\boldsymbol{Q}}_\delta}\underline{\boldsymbol{\zeta}}(0)\right).\end{aligned}$$

Upon Laplace transform inversion, this leads to the (defective) Markov renewal equation

$$\underline{\boldsymbol{\phi}}(u;\infty)=\int_0^u\tilde{\boldsymbol{g}}_\delta(x)\underline{\boldsymbol{\phi}}(u-x;\infty)dx+\tilde{\boldsymbol{Z}}_\delta(u),\qquad u\geq 0,$$

with $\tilde{\boldsymbol{Z}}_\delta(u)$ defined in Eq. 3.23. Then the solution (3.24) follows immediately. □

Remark 1 For a Markov additive risk process under the traditional dividend barrier strategy, Cheung and Landriault (2009) provided the representations of the expected discounted dividends, the higher moments of discounted dividends and the Gerber-Shiu function in their equations (11), (22) and (37), respectively. These formulas were expressed in terms of a homogeneous solution (which the authors denoted by $\boldsymbol{v}^B(u)$) and the barrier-free Gerber-Shiu function. However, general solutions for these two components were not given. While the barrier-free Gerber-Shiu function can be obtained from our Proposition 2 (with $n=1$), the quantity $\boldsymbol{v}^B(u)$ is related to the results in Proposition 1 (under $n=1$) via $\boldsymbol{v}^B(u)=(1/2)\underline{\boldsymbol{L}}_\delta(u)\boldsymbol{\Delta}_{\sigma^2}$. □

3.3 The Case $u > b$

In this subsection, we consider the case $u > b$ for Eqs. 3.4 and 3.5, which will eventually lead to the full solution to $\underline{\boldsymbol{\phi}}(u;b)=(\boldsymbol{\phi}_1^\top(u;b),\ldots,\boldsymbol{\phi}_n^\top(u;b))^\top$ for $u\geq 0$ as in Theorem 1.

When $u > b$, we note that for $k = 1, 2, \ldots, n-1$ the equation (3.4) involves both $\boldsymbol{\phi}_k(u; b)$ and $\boldsymbol{\phi}_{k+1}(u; b)$; whereas Eq. 3.5 only involves $\boldsymbol{\phi}_n(u; b)$ as the unknown function. Therefore, our solution procedure is to solve (3.4) for $\boldsymbol{\phi}_k(u; b)$ in terms of $\boldsymbol{\phi}_{k+1}(u; b)$ recursively for $k = 1, 2, \ldots, n-1$, with the starting point $\boldsymbol{\phi}_n(u; b)$ obtained as the solution of Eq. 3.5.

First, by some straightforward calculations, we obtain (for $\Re(s) > 0$)

$$\begin{aligned}
&\int_b^\infty e^{-s\boldsymbol{E}_m(u-b)} \left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\frac{d^2}{du^2} + \boldsymbol{\Delta}_c\frac{d}{du} - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0\right)\boldsymbol{\phi}_k(u; b)du \\
&= \left(\frac{1}{2}s^2\boldsymbol{\Delta}_{\sigma^2} + s\boldsymbol{\Delta}_c - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0\right)\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_k(b; b) \\
&\quad - \frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_k'(b; b) - \left(\frac{1}{2}s\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)\boldsymbol{\phi}_k(b; b).
\end{aligned}$$

Moreover, by a change of order of integrations, it can be shown that

$$\begin{aligned}
&\int_b^\infty e^{-s\boldsymbol{E}_m(u-b)}\int_0^u (\boldsymbol{D}_1 \circ \boldsymbol{f}(x))\boldsymbol{\phi}_k(u-x; b)dxdu \\
&= (\boldsymbol{D}_1 \circ \widehat{\boldsymbol{f}}(s))\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_k(b; b) + \int_0^b \left(\mathcal{T}_{s\boldsymbol{E}_m}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x))\right)\boldsymbol{\phi}_k(x; b)dx.
\end{aligned}$$

Hence, multiplying both sides of Eq. 3.4 by $e^{-s\boldsymbol{E}_m(u-b)}$ and performing integration with respect to u from b to ∞ , we arrive at

$$\begin{aligned}
&\left(\frac{1}{2}s^2\boldsymbol{\Delta}_{\sigma^2} + s\boldsymbol{\Delta}_c - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0 + \boldsymbol{D}_1 \circ \widehat{\boldsymbol{f}}(s)\right)\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_k(b; b) \\
&= \frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_k'(b; b) + \left(\frac{1}{2}s\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)\boldsymbol{\phi}_k(b; b) - \beta\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_{k+1}(b; b) - \mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\zeta}(b) \\
&\quad - \int_0^b \left(\mathcal{T}_{s\boldsymbol{E}_m}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x))\right)\boldsymbol{\phi}_k(x; b)dx, \qquad k = 1, 2, \ldots, n-1. \qquad (3.28)
\end{aligned}$$

With the use of the matrix $\boldsymbol{Q}_\gamma$ defined in Section 2 (under $\gamma = \delta + \beta$), analogous to (3.20) one has

$$\begin{aligned}
&\left(\frac{1}{2}s^2\boldsymbol{\Delta}_{\sigma^2} + s\boldsymbol{\Delta}_c - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0 + \boldsymbol{D}_1 \circ \widehat{\boldsymbol{f}}(s)\right) \\
&= (s\boldsymbol{E}_m - \boldsymbol{Q}_{\delta+\beta})\left(\frac{1}{2}(s\boldsymbol{E}_m + \boldsymbol{Q}_{\delta+\beta})\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c - \mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(0))\right). \qquad (3.29)
\end{aligned}$$

Similar to Eq. 3.26, the matrix $\boldsymbol{Q}_{\delta+\beta}$ can also be used to determine the condition

$$\begin{aligned}
\boldsymbol{O} &= \frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_k'(b; b) + \left(\frac{1}{2}\boldsymbol{Q}_{\delta+\beta}\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)\boldsymbol{\phi}_k(b; b) - \beta\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\phi}_{k+1}(b; b) - \mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\zeta}(b) \\
&\quad - \int_0^b \left(\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x))\right)\boldsymbol{\phi}_k(x; b)dx, \qquad k = 1, 2, \ldots, n-1.
\end{aligned}$$

Thus, as in Eq. 3.27, the right-hand side of Eq. 3.28 can be expressed as

$$\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_k'(b;b) + \left(\frac{1}{2}s\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)\boldsymbol{\phi}_k(b;b) - \beta\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_{k+1}(b;b) - \mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\zeta}(b)$$
$$- \int_0^b \left(\mathcal{T}_{s\boldsymbol{E}_m}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x))\right)\boldsymbol{\phi}_k(x;b)dx$$
$$= (s\boldsymbol{E}_m - \boldsymbol{Q}_{\delta+\beta})\left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_k(b;b) + \beta\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\phi}_{k+1}(b;b) + \mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\zeta}(b)\right.$$
$$\left. + \int_0^b (\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x)))\boldsymbol{\phi}_k(x;b)dx\right), \qquad k = 1, 2, \ldots, n-1. \tag{3.30}$$

Plugging Eqs. 3.29 and 3.30 into Eq. 3.28 gives

$$\left(\boldsymbol{E}_m - \left(\frac{1}{2}(s\boldsymbol{E}_m + \boldsymbol{Q}_{\delta+\beta})\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)^{-1}\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(0))\right)\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_k(b;b)$$
$$= \left(\frac{1}{2}(s\boldsymbol{E}_m + \boldsymbol{Q}_{\delta+\beta})\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)^{-1}\left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_k(b;b) + \beta\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\phi}_{k+1}(b;b) + \mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\zeta}(b)\right.$$
$$\left. + \int_0^b (\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x)))\boldsymbol{\phi}_k(x;b)dx\right), \qquad k=1, 2, \ldots, n-1. \tag{3.31}$$

Using the fact that the Dickson-Hipp transform $\mathcal{T}_s h(b)$ can be regarded as the Laplace transform (with argument s) of the shifted function $h(b+\cdot)$ (which extends to matrix quantities), one can perform Laplace transform inversion in the above equation. Together with the application of Eq. 3.14 and the continuity condition (3.7), this leads to

$$\boldsymbol{\phi}_k(b+u;b) = \int_0^u \boldsymbol{g}_{\delta+\beta}(x)\boldsymbol{\phi}_k(b+u-x;b)dx$$
$$+ \ \boldsymbol{W}_{\phi,k}(u) \ + \ \boldsymbol{R}_{\phi,k}(u) \ + \ \sum_{j=1}^n \boldsymbol{H}_{k,j}(u)\boldsymbol{k}_j, \qquad k = 1, 2, \ldots, n-1; u \geq 0, \tag{3.32}$$

where
$$\boldsymbol{g}_{\delta+\beta}(x) = 2\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^x e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(x-y)}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(y))dy,$$

$$\boldsymbol{W}_{\phi,k}(u) = 2\beta\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(u-x)}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\phi}_{k+1}(b+x;b)dx,$$

$$\boldsymbol{R}_{\phi,k}(u) = \boldsymbol{\Delta}_{\sigma^2}^{-1}e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})u}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}(b;\infty) + 2\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(u-x)}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\zeta}(b+x)dx$$
$$+ 2\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(u-y)}\int_0^b (\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b+y-x)))\boldsymbol{\phi}(x;\infty)dxdy, \tag{3.33}$$

and

$$\boldsymbol{H}_{k,j}(u) = \boldsymbol{\Delta}_{\sigma^2}^{-1} e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})u} \boldsymbol{\Delta}_{\sigma^2} \boldsymbol{L}_{\delta,k,j}(b) + 2\boldsymbol{\Delta}_{\sigma^2}^{-1} \int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(u-y)} \int_0^b (\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b+y-x)))\boldsymbol{L}_{\delta,k,j}(x)dxdy, \quad j=1,2\ldots,n. \tag{3.34}$$

Clearly, Eq. 3.32 is a Markov renewal equation satisfied by $\boldsymbol{\phi}_k(b+\cdot;b)$ (and again $\int_0^\infty \boldsymbol{g}_{\delta+\beta}(x)dx$ is strictly substochastic). Note that the non-homogeneous term depends on $\boldsymbol{\phi}_{k+1}(b+x;b)$ for $x>0$ via $\boldsymbol{W}_{\phi,k}(u)$. Upon defining the quantity $\boldsymbol{S}_{\delta+\beta}(x) = \sum_{i=1}^\infty \boldsymbol{g}_{\delta+\beta}^{*i}(x)$ with the obvious definition of i-fold convolution, by renewal theory and some tedious but straightforward calculations, we arrive at

$$\begin{aligned}\boldsymbol{\phi}_k(b+u;b) &= \boldsymbol{W}_{\phi,k}(u) + \boldsymbol{R}_{\phi,k}(u) + \sum_{j=1}^n \boldsymbol{H}_{k,j}(u)\boldsymbol{k}_j \\ &\quad + \int_0^u \boldsymbol{S}_{\delta+\beta}(u-z)\left(\boldsymbol{W}_{\phi,k}(z) + \boldsymbol{R}_{\phi,k}(z) + \sum_{j=1}^n \boldsymbol{H}_{k,j}(z)\boldsymbol{k}_j\right)dz \\ &= \int_0^\infty \boldsymbol{Z}_{\delta,\beta}(u,y)\boldsymbol{\phi}_{k+1}(b+y;b)dy + \boldsymbol{K}_{\phi,k}(u) + \sum_{j=1}^n \boldsymbol{P}_{k,j}(u)\boldsymbol{k}_j, \quad k=1,2,\ldots,n-1;\ u\geq 0,\end{aligned} \tag{3.35}$$

where

$$\boldsymbol{Z}_{\delta,\beta}(u,y) = 2\beta\boldsymbol{\Delta}_{\sigma^2}^{-1} \int_0^{\min(u,y)} e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(u-x)} e^{-\boldsymbol{Q}_{\delta+\beta}(y-x)}dx + 2\beta \int_0^u \int_0^{\min(z,y)} \boldsymbol{S}_{\delta+\beta}(u-z)\boldsymbol{\Delta}_{\sigma^2}^{-1} e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(z-x)} e^{-\boldsymbol{Q}_{\delta+\beta}(y-x)}dxdz, \tag{3.36}$$

$$\boldsymbol{K}_{\phi,k}(u) = \boldsymbol{R}_{\phi,k}(u) + \int_0^u \boldsymbol{S}_{\delta+\beta}(u-z)\boldsymbol{R}_{\phi,k}(z)dz, \tag{3.37}$$

and

$$\boldsymbol{P}_{k,j}(u) = \boldsymbol{H}_{k,j}(u) + \int_0^u \boldsymbol{S}_{\delta+\beta}(u-z)\boldsymbol{H}_{k,j}(z)dz, \qquad j=1,2\ldots,n. \tag{3.38}$$

Note from Eq. 3.33 that $\boldsymbol{R}_{\phi,1} \equiv \boldsymbol{R}_{\phi,2} \equiv \cdots \equiv \boldsymbol{R}_{\phi,n-1}$ and therefore Eq. 3.37 implies $\boldsymbol{K}_{\phi,1} \equiv \boldsymbol{K}_{\phi,2} \equiv \cdots \equiv \boldsymbol{K}_{\phi,n-1}$. We adopt the seemingly redundant subscript to ease presentation later on, as we consider the case $k=n$ next.

For $k=n$, multiplying both sides of Eq. 3.5 by $e^{-s\boldsymbol{E}_m(u-b)}$ and integrating from b to ∞ yields

$$\begin{aligned}&\left(\frac{1}{2}s^2\boldsymbol{\Delta}_{\sigma^2} + s\boldsymbol{\Delta}_c - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0 + \boldsymbol{D}_1 \circ \widehat{\boldsymbol{f}}(s)\right)\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_n(b;b) \\ &= \frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_n'(b;b) + \left(\frac{1}{2}s\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)\boldsymbol{\phi}_n(b;b) - \beta s^{-1}\boldsymbol{\phi}_1(b;b) - \mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\zeta}(b) \\ &\quad - \int_0^b \left(\mathcal{T}_{s\boldsymbol{E}_m}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x))\right)\boldsymbol{\phi}_n(x;b)dx.\end{aligned} \tag{3.39}$$

We now look at the right-hand side of the above equation. Using the same arguments leading to Eq. 3.30 and omitting the details, we obtain

$$\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_n'(b;b) + \left(\frac{1}{2}s\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)\boldsymbol{\phi}_n(b;b) - \beta s^{-1}\boldsymbol{\phi}_1(b;b) - \mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\zeta}(b)$$
$$- \int_0^b \left(\mathcal{T}_{s\boldsymbol{E}_m}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x))\right)\boldsymbol{\phi}_n(x;b)dx$$
$$= (s\boldsymbol{E}_m - \boldsymbol{Q}_{\delta+\beta})\left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_n(b;b) + \beta(s\boldsymbol{Q}_{\delta+\beta})^{-1}\boldsymbol{\phi}_1(b;b) + \mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\zeta}(b)\right.$$
$$\left. + \int_0^b (\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x)))\boldsymbol{\phi}_n(x;b)dx\right).$$

This helps us convert (3.39) into

$$\left(\boldsymbol{E}_m - \left(\frac{1}{2}(s\boldsymbol{E}_m + \boldsymbol{Q}_{\delta+\beta})\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)^{-1}\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(0))\right)\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_n(b;b)$$
$$= \left(\frac{1}{2}(s\boldsymbol{E}_m + \boldsymbol{Q}_{\delta+\beta})\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)^{-1}\left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_n(b;b) + \beta(s\boldsymbol{Q}_{\delta+\beta})^{-1}\boldsymbol{\phi}_1(b;b) + \mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\zeta}(b)\right.$$
$$\left. + \int_0^b (\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x)))\boldsymbol{\phi}_n(x;b)dx\right). \tag{3.40}$$

Similar to Eq. 3.32, inversion of Laplace transforms yields the defective Markov renewal equation

$$\boldsymbol{\phi}_n(b+u;b) = \int_0^u \boldsymbol{g}_{\delta+\beta}(x)\boldsymbol{\phi}_n(b+u-x;b)dx + \boldsymbol{R}_{\phi,n}(u) + \sum_{j=1}^n \boldsymbol{H}_{n,j}(u)\boldsymbol{k}_j, \qquad u \geq 0, \tag{3.41}$$

where

$$\boldsymbol{R}_{\phi,n}(u) = \boldsymbol{\Delta}_{\sigma^2}^{-1}e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})u}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}(b;\infty) + 2\beta\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})x}\boldsymbol{Q}_{\delta+\beta}^{-1}\boldsymbol{\phi}(b;\infty)dx$$
$$+ 2\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(u-x)}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{\zeta}(b+x)dx$$
$$+ 2\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(u-y)}\int_0^b (\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b+y-x)))\boldsymbol{\phi}(x;\infty)dxdy, \tag{3.42}$$

and

$$\boldsymbol{H}_{n,j}(u) = \boldsymbol{\Delta}_{\sigma^2}^{-1}e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})u}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{L}_{\delta,n,j}(b) + 2\beta\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})x}\boldsymbol{Q}_{\delta+\beta}^{-1}\boldsymbol{L}_{\delta,1,j}(b)dx$$
$$+ 2\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(u-y)}\int_0^b (\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b+y-x)))\boldsymbol{L}_{\delta,n,j}(x)dxdy, \qquad j = 1, 2\ldots, n. \tag{3.43}$$

As a result, the application of Markov renewal theory gives

$$\boldsymbol{\phi}_n(b+u;b) = \boldsymbol{K}_{\phi,n}(u) + \sum_{j=1}^n \boldsymbol{P}_{n,j}(u)\boldsymbol{k}_j, \qquad u \geq 0, \tag{3.44}$$

where the definitions (3.37) and (3.38) are now extended to $k = n$.

Note that Eq. 3.35 for $k = 1, 2, \ldots, n-1$ forms an iterative system with the starting value given by Eq. 3.44. By some straightforward algebra, one can put the iteration in nicer form as

$$\boldsymbol{\phi}_{n-k+1}(b+u; b) = \boldsymbol{B}_{\phi,n-k+1}(u) + \sum_{j=1}^{n} \boldsymbol{C}_{n-k+1,j}(u)\boldsymbol{k}_j, \qquad k = 1, 2, \ldots, n; u \geq 0, \tag{3.45}$$

where $\boldsymbol{B}_{\phi,n-k+1}(u)$ and $\boldsymbol{C}_{n-k+1,j}(u)$ are evaluated recursively (for increasing k) as

$$\begin{cases} \boldsymbol{B}_{\phi,n}(u) = \boldsymbol{K}_{\phi,n}(u), \\ \boldsymbol{B}_{\phi,n-k+1}(u) = \boldsymbol{K}_{\phi,n-k+1}(u) + \int_0^\infty \boldsymbol{Z}_{\delta,\beta}(u,y)\boldsymbol{B}_{\phi,n-k+2}(y)dy,\ k = 2, 3, \ldots, n, \end{cases} \tag{3.46}$$

and for each fixed $j = 1, 2, \ldots, n$,

$$\begin{cases} \boldsymbol{C}_{n,j}(u) = \boldsymbol{P}_{n,j}(u). \\ \boldsymbol{C}_{n-k+1,j}(u) = \boldsymbol{P}_{n-k+1,j}(u) + \int_0^\infty \boldsymbol{Z}_{\delta,\beta}(u,y)\boldsymbol{C}_{n-k+2,j}(y)dy,\ k = 2, 3, \ldots, n. \end{cases} \tag{3.47}$$

Furthermore, setting $\underline{\boldsymbol{B}}_\phi(u) = (\boldsymbol{B}_{\phi,1}^\top(u), \ldots, \boldsymbol{B}_{\phi,n}^\top(u))^\top$ and

$$\underline{\boldsymbol{C}}(u) = \begin{pmatrix} \boldsymbol{C}_{1,1}(u) & \cdots & \boldsymbol{C}_{1,n}(u) \\ \vdots & \ddots & \vdots \\ \boldsymbol{C}_{n,1}(u) & \cdots & \boldsymbol{C}_{n,n}(u) \end{pmatrix},$$

Eq. 3.45 can be rewritten as

$$\underline{\boldsymbol{\phi}}(b+u; b) = \underline{\boldsymbol{B}}_\phi(u) + \underline{\boldsymbol{C}}(u)(\boldsymbol{k}_1^\top, \ldots, \boldsymbol{k}_n^\top)^\top, \qquad u \geq 0.$$

Except for $(\boldsymbol{k}_1^\top, \ldots, \boldsymbol{k}_n^\top)^\top$, the above equation gives the exact solution for the Gerber-Shiu function in the upper layer. The unknown vector $(\boldsymbol{k}_1^\top, \ldots, \boldsymbol{k}_n^\top)^\top$ can be obtained using the smooth pasting condition (3.8) together with Eq. 3.14, giving

$$(\boldsymbol{k}_1^\top, \ldots, \boldsymbol{k}_n^\top)^\top = \left(\underline{\boldsymbol{L}}_\delta'(b) - \underline{\boldsymbol{C}}'(0)\right)^{-1}\left(\underline{\boldsymbol{B}}_\phi'(0) - \underline{\boldsymbol{\phi}}'(b;\infty)\right). \tag{3.48}$$

The main results of this section are summarized in the following Theorem.

Theorem 1 *Suppose that the matrices $\tilde{\boldsymbol{Q}}_\delta$ and $\boldsymbol{Q}_{\delta+\beta}$ are diagonalizable. Then*

$$\underline{\boldsymbol{\phi}}(u;b) = \underline{\boldsymbol{\phi}}(u;\infty) + \underline{\boldsymbol{L}}_\delta(u)\left(\underline{\boldsymbol{L}}_\delta'(b) - \underline{\boldsymbol{C}}'(0)\right)^{-1}\left(\underline{\boldsymbol{B}}_\phi'(0) - \underline{\boldsymbol{\phi}}'(b;\infty)\right), \qquad 0 \leq u \leq b, \tag{3.49}$$

and

$$\underline{\boldsymbol{\phi}}(u;b) = \underline{\boldsymbol{B}}_\phi(u-b) + \underline{\boldsymbol{C}}(u-b)\left(\underline{\boldsymbol{L}}_\delta'(b) - \underline{\boldsymbol{C}}'(0)\right)^{-1}\left(\underline{\boldsymbol{B}}_\phi'(0) - \underline{\boldsymbol{\phi}}'(b;\infty)\right), \qquad u > b. \tag{3.50}$$

In particular, the Gerber-Shiu function $\underline{\boldsymbol{\phi}}(u;b)$ can be computed by the following procedure.

- *Step 1: Compute the matrices $\tilde{\boldsymbol{Q}}_\delta$ and $\boldsymbol{Q}_{\delta+\beta}$ using one of the methods discussed in Section 2.*
- *Step 2: Compute $\underline{\boldsymbol{L}}_\delta(u)$ and $\underline{\boldsymbol{\phi}}(u;\infty)$ by Propositions 1 and 2, respectively.*
- *Step 3: Compute $\boldsymbol{R}_{\phi,k}(u)$ by Eqs. 3.33 and 3.42; and $\boldsymbol{H}_{k,j}(u)$ by Eqs. 3.34 and 3.43.*
- *Step 4: Compute $\boldsymbol{Z}_{\delta,\beta}(u,y)$ by Eq. 3.36; and $\boldsymbol{K}_{\phi,k}(u)$ and $\boldsymbol{P}_{k,j}(u)$ by Eqs. 3.37 and 3.38, respectively.*
- *Step 5: Compute $\boldsymbol{B}_{\phi,k}(u)$ and $\boldsymbol{C}_{k,j}(u)$ recursively via Eqs. 3.46 and 3.47, respectively.*

- *Step 6: Compute $\underline{\boldsymbol{\phi}}(u;b)$ via Eqs. 3.49 and 3.50.*

Remark 2 If one is interested in the limit behavior of the Gerber-Shiu function as $u \to \infty$, it suffices to consider the case $u > b$. Applying the Final Value Theorem for Laplace transforms, we have

$$\lim_{u\to\infty} \boldsymbol{\phi}_k(u;b) = \lim_{u\to\infty} \boldsymbol{\phi}_k(b+u;b) = \lim_{s\to 0} s\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_k(b;b).$$

Hence, application of Eq. 3.28 gives the iterative expression

$$\begin{aligned}\lim_{s\to 0} s\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_k(b;b) &= \left(\lim_{s\to 0}\left(\frac{1}{2}s^2\boldsymbol{\Delta}_{\sigma^2} + s\boldsymbol{\Delta}_c - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0 + \boldsymbol{D}_1 \circ \widehat{\boldsymbol{f}}(s)\right)^{-1}\right)\\ &\quad\times\left(\lim_{s\to 0} s\left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{\phi}_k'(b;b) + \left(\frac{1}{2}s\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)\boldsymbol{\phi}_k(b;b) - \beta\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_{k+1}(b;b) - \mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\zeta}(b)\right.\right.\\ &\quad\left.\left. - \int_0^b \left(\mathcal{T}_{s\boldsymbol{E}_m}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x))\right)\boldsymbol{\phi}_k(x;b)dx\right)\right)\\ &= \beta\left((\delta+\beta)\boldsymbol{E}_m - \boldsymbol{D}_0 - \boldsymbol{D}_1\right)^{-1} \lim_{s\to 0} s\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_{k+1}(b;b), \qquad k = 1,2,\ldots,n-1.\end{aligned}$$

Similarly, Eq. 3.39 leads to the starting point

$$\lim_{s\to 0} s\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{\phi}_n(b;b) = \beta\left((\delta+\beta)\boldsymbol{E}_m - \boldsymbol{D}_0 - \boldsymbol{D}_1\right)^{-1}\boldsymbol{\phi}_1(b;b).$$

Combining all the above, we arrive at the asymptotic formula, for each $k = 1,2,\ldots,n$,

$$\boldsymbol{\phi}_k(u;b) \sim \beta^{n-k+1}\left((\delta+\beta)\boldsymbol{E}_m - \boldsymbol{D}_0 - \boldsymbol{D}_1\right)^{-(n-k+1)}\boldsymbol{\phi}_1(b;b) \qquad \text{as } u \to \infty. \quad (3.51)$$

It can be verified that

$$\int_0^\infty e^{-\delta t} f_T(t) e^{(\boldsymbol{D}_0+\boldsymbol{D}_1)t} dt = \beta^n \left((\delta+\beta)\boldsymbol{E}_m - \boldsymbol{D}_0 - \boldsymbol{D}_1\right)^{-n}.$$

The (i,j)th element of the above expression is $\mathbb{E}[e^{-\delta T}\mathbf{1}_{\{J_T=j\}}|J_0=i]$, namely the expected present value of a dollar payable at the first dividend decision time T_1 if J is in state j at time T_1, given that J starts in state i. Then Eq. 3.51 at $k=1$ can be interpreted probabilistically as follows. Suppose $J_0 = i$. When the initial surplus $U_0^b = u$ is very large, it is highly likely that the surplus process U^b is above b at time T_1 (before dividends) and ruin has not occurred in the interim, regardless of the initial environmental state $J_0 = i$. If J is in state j at time T_1, this first gives rise to the discount factor $\mathbb{E}[e^{-\delta T}\mathbf{1}_{\{J_T=j\}}|J_0=i]$ from time T_1 to time 0. Then the payment of dividend will cause the surplus to drop to the level b, from which the expected discounted penalty onward is $\phi_{1,j}(b;b)$. Since the state j is arbitrary, summing over j explains (3.51) at $k=1$. Similarly, (3.51) for $k=2,3,\ldots$ can be interpreted by replacing T_1 with an Erlang$(n-k+1)$ random variable in the above arguments. See Avanzi et al. (2013, Remark 2.3) for related intuitions in the context of a dual risk model. □

4 Expected Present Value of Dividends Paid Until Ruin

This section is concerned with the full solution to the dividend function $V_i(u;b)$ defined by Eq. 1.5. Since the derivations closely resemble those in Section 3, we mostly present the key steps involved with omission of some algebraic details. As in Section 3, we define $V_{k,i}(u;b)$ (for $k = 1,2,\ldots,n$) to be the expected present value of total dividends paid until ruin, given that the time until the first dividend decision time is distributed as Erlang$(n-k+1)$. Clearly, $V_i(u;b) = V_{1,i}(u;b)$. Let $\boldsymbol{V}_k(u;b) = (V_{k,1}(u;b),\dots,V_{k,m}(u;b))^\top$ for $k = 1,2,\dots,n$. Then, applying the same arguments used to obtain (3.4) and (3.5), we arrive at the matrix integro-differential equations

$$\boldsymbol{O} = \left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\frac{d^2}{du^2} + \boldsymbol{\Delta}_c\frac{d}{du} - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0\right)\boldsymbol{V}_k(u;b) + \int_0^u (\boldsymbol{D}_1 \circ \boldsymbol{f}(x))\boldsymbol{V}_k(u-x;b)dx + \beta\boldsymbol{V}_{k+1}(u;b), \qquad k=1,2,\dots,n-1, \tag{4.1}$$

and

$$\begin{aligned}\boldsymbol{O} = &\left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\frac{d^2}{du^2} + \boldsymbol{\Delta}_c\frac{d}{du} - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0\right)\boldsymbol{V}_n(u;b)\\ &+ \int_0^u (\boldsymbol{D}_1 \circ \boldsymbol{f}(x))\boldsymbol{V}_n(u-x;b)dx + \beta\boldsymbol{V}_1(u;b)\mathbf{1}_{\{0<u\le b\}}\\ &+ \beta\left((u-b)\boldsymbol{e}_m + \boldsymbol{V}_1(b;b)\right)\mathbf{1}_{\{u>b\}}.\end{aligned} \tag{4.2}$$

One has the trivial boundary condition

$$\boldsymbol{V}_k(0;b) = \boldsymbol{O}, \qquad k=1,2,\dots,n, \tag{4.3}$$

as well as the continuity and the smooth pasting conditions given by

$$\boldsymbol{V}_k(b-;b) = \boldsymbol{V}_k(b+;b), \qquad k=1,2,\dots,n, \tag{4.4}$$

and

$$\boldsymbol{V}'_k(b-;b) = \boldsymbol{V}'_k(b+;b), \qquad k=1,2,\dots,n. \tag{4.5}$$

Setting $\underline{\boldsymbol{V}}(u;b) = (\boldsymbol{V}_1^\top(u;b),\dots,\boldsymbol{V}_n^\top(u;b))^\top$, (4.1) and (4.2) in the lower layer can be collectively written as

$$\left(\frac{1}{2}\tilde{\boldsymbol{\Delta}}_{\sigma^2}\frac{d^2}{du^2} + \tilde{\boldsymbol{\Delta}}_c\frac{d}{du} - \delta\boldsymbol{E}_{mn} + \tilde{\boldsymbol{D}}_0\right)\underline{\boldsymbol{V}}(u;b) + \int_0^u (\tilde{\boldsymbol{D}}_1 \circ \tilde{\boldsymbol{f}}(x))\underline{\boldsymbol{V}}(u-x;b)dx = \boldsymbol{O}, \qquad 0<u<b.$$

With the boundary condition (4.3), it follows from Section 3.2 that

$$\underline{\boldsymbol{V}}(u;b) = \underline{\boldsymbol{L}}_\delta(u)(\boldsymbol{y}_1^\top,\dots,\boldsymbol{y}_n^\top)^\top, \qquad 0 \le u \le b, \tag{4.6}$$

where $\boldsymbol{y}_1,\dots,\boldsymbol{y}_n$ are m-dimensional column vectors to be determined. For $u > b$, multiplying Eqs. 4.1 and 4.2 by $e^{-s\boldsymbol{E}_m(u-b)}$ and integrating from b to ∞, we obtain

$$\begin{aligned}&\left(\frac{1}{2}s^2\boldsymbol{\Delta}_{\sigma^2} + s\boldsymbol{\Delta}_c - (\delta+\beta)\boldsymbol{E}_m + \boldsymbol{D}_0 + \boldsymbol{D}_1 \circ \widehat{\boldsymbol{f}}(s)\right)\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{V}_k(b;b)\\ &= \frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{V}'_k(b;b) + \left(\frac{1}{2}s\boldsymbol{\Delta}_{\sigma^2} + \boldsymbol{\Delta}_c\right)\boldsymbol{V}_k(b;b) - \beta\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{V}_{k+1}(b;b)\\ &\quad - \int_0^b \left(\mathcal{T}_{s\boldsymbol{E}_m}(\boldsymbol{D}_1 \circ \boldsymbol{f}(b-x))\right)\boldsymbol{V}_k(x;b)dx, \qquad k=1,2,\dots,n-1,\end{aligned} \tag{4.7}$$

and

$$\left(\frac{1}{2}s^2\boldsymbol{\Delta}_{\sigma^2}+s\boldsymbol{\Delta}_c-(\delta+\beta)\boldsymbol{E}_m+\boldsymbol{D}_0+\boldsymbol{D}_1\circ\widehat{\boldsymbol{f}}(s)\right)\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{V}_n(b;b)$$
$$=\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{V}_n'(b;b)+\left(\frac{1}{2}s\boldsymbol{\Delta}_{\sigma^2}+\boldsymbol{\Delta}_c\right)\boldsymbol{V}_n(b;b)-\beta s^{-1}\boldsymbol{V}_1(b;b)-\beta s^{-2}\boldsymbol{e}_m$$
$$-\int_0^b\left(\mathcal{T}_{s\boldsymbol{E}_m}(\boldsymbol{D}_1\circ\boldsymbol{f}(b-x))\right)\boldsymbol{V}_n(x;b)dx. \tag{4.8}$$

Remark 3 As a by-product of Eqs. 4.7 and 4.8, analogous to Eq. 3.51 we can obtain the asymptotic result, for each $k=1,2,\ldots,n$,

$$\boldsymbol{V}_k(u;b)\sim\beta^{n-k+1}\left((\delta+\beta)\boldsymbol{E}_m-\boldsymbol{D}_0-\boldsymbol{D}_1\right)^{-(n-k+1)}\boldsymbol{e}_m u\qquad\text{as }u\to\infty. \tag{4.9}$$

The above formula can be interpreted as in Remark 2. See also Avanzi et al. (2013, Remark 3.2). □

Following the same steps used to transform Eqs. 3.28 and 3.39 to Eqs. 3.31 and 3.40, Eqs. 4.7 and 4.8 respectively become

$$\left(\boldsymbol{E}_m-\left(\frac{1}{2}(s\boldsymbol{E}_m+\boldsymbol{Q}_{\delta+\beta})\boldsymbol{\Delta}_{\sigma^2}+\boldsymbol{\Delta}_c\right)^{-1}\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1\circ\boldsymbol{f}(0))\right)\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{V}_k(b;b)$$
$$=\left(\frac{1}{2}(s\boldsymbol{E}_m+\boldsymbol{Q}_{\delta+\beta})\boldsymbol{\Delta}_{\sigma^2}+\boldsymbol{\Delta}_c\right)^{-1}\left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{V}_k(b;b)+\beta\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{V}_{k+1}(b;b)\right.$$
$$\left.+\int_0^b(\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1\circ\boldsymbol{f}(b-x)))\boldsymbol{V}_k(x;b)dx\right),\qquad k=1,2,\ldots,n-1, \tag{4.10}$$

and

$$\left(\boldsymbol{E}_m-\left(\frac{1}{2}(s\boldsymbol{E}_m+\boldsymbol{Q}_{\delta+\beta})\boldsymbol{\Delta}_{\sigma^2}+\boldsymbol{\Delta}_c\right)^{-1}\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1\circ\boldsymbol{f}(0))\right)\mathcal{T}_{s\boldsymbol{E}_m}\boldsymbol{V}_n(b;b)$$
$$=\left(\frac{1}{2}(s\boldsymbol{E}_m+\boldsymbol{Q}_{\delta+\beta})\boldsymbol{\Delta}_{\sigma^2}+\boldsymbol{\Delta}_c\right)^{-1}\left(\frac{1}{2}\boldsymbol{\Delta}_{\sigma^2}\boldsymbol{V}_n(b;b)+\beta(s\boldsymbol{Q}_{\delta+\beta})^{-1}\boldsymbol{V}_1(b;b)+\beta s^{-1}\boldsymbol{Q}_{\delta+\beta}^{-2}\boldsymbol{e}_m\right.$$
$$\left.+\beta s^{-2}\boldsymbol{Q}_{\delta+\beta}^{-1}\boldsymbol{e}_m+\int_0^b(\mathcal{T}_{s\boldsymbol{E}_m}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}(\boldsymbol{D}_1\circ\boldsymbol{f}(b-x)))\boldsymbol{V}_n(x;b)dx\right). \tag{4.11}$$

Upon inversion of Laplace transforms in Eq. 4.10 along with the use of the representation (4.6) and the continuity condition (4.4), we obtain the defective Markov renewal equation

$$\boldsymbol{V}_k(b+u;b)=\int_0^u\boldsymbol{g}_{\delta+\beta}(x)\boldsymbol{V}_k(b+u-x;b)dx$$
$$+\boldsymbol{W}_{V,k}(u)+\sum_{j=1}^n\boldsymbol{H}_{k,j}(u)\boldsymbol{y}_j,\qquad k=1,2,\ldots,n-1;\,u\geq 0, \tag{4.12}$$

where $\boldsymbol{H}_{k,j}(u)$ is defined in Eq. 3.34, and

$$\boldsymbol{W}_{V,k}(u)=2\beta\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(u-x)}\mathcal{T}_{\boldsymbol{Q}_{\delta+\beta}}\boldsymbol{V}_{k+1}(b+x;b)dx.$$

Similarly, Eq. 4.11 leads to

$$\boldsymbol{V}_n(b+u;b)=\int_0^u \boldsymbol{g}_{\delta+\beta}(x)\boldsymbol{V}_n(b+u-x;b)dx+\boldsymbol{R}_V(u)+\sum_{j=1}^{n}\boldsymbol{H}_{n,j}(u)\boldsymbol{y}_j,\qquad u\geq 0, \tag{4.13}$$

where $\boldsymbol{H}_{n,j}(u)$ is given by Eq. 3.43, and

$$\boldsymbol{R}_V(u)=2\beta\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})x}\boldsymbol{Q}_{\delta+\beta}^{-2}\boldsymbol{e}_m dx+2\beta\boldsymbol{\Delta}_{\sigma^2}^{-1}\int_0^u e^{-(\boldsymbol{Q}_{\delta+\beta}+2\boldsymbol{\Delta}_c\boldsymbol{\Delta}_{\sigma^2}^{-1})(u-x)}x\,\boldsymbol{Q}_{\delta+\beta}^{-1}\boldsymbol{e}_m dx. \tag{4.14}$$

It is instructive to note that Eqs. 4.12 and 4.13 are structurally similar to Eqs. 3.32 and 3.41, respectively. Consequently, in parallel to Eqs. 3.35 and 3.44, their solutions are

$$\boldsymbol{V}_k(b+u;b)=\int_0^\infty \boldsymbol{Z}_{\delta,\beta}(u,y)\boldsymbol{V}_{k+1}(b+y;b)dy+\sum_{j=1}^{n}\boldsymbol{P}_{k,j}(u)\boldsymbol{y}_j,\qquad k=1,2,\ldots,n-1;\ u\geq 0, \tag{4.15}$$

and

$$\boldsymbol{V}_n(b+u;b)=\boldsymbol{K}_V(u)+\sum_{j=1}^{n}\boldsymbol{P}_{n,j}(u)\boldsymbol{y}_j\qquad u\geq 0. \tag{4.16}$$

Here the function $\boldsymbol{P}_{k,j}(u)$ has the same definition as in Eq. 3.38, and

$$\boldsymbol{K}_V(u)=\boldsymbol{R}_V(u)+\int_0^u \boldsymbol{S}_{\delta+\beta}(u-x)\boldsymbol{R}_V(x)dx. \tag{4.17}$$

On the grounds of the iterative system that consists of Eqs. 4.15 and 4.16, we obtain

$$\boldsymbol{V}_{n-k+1}(b+u;b)=\boldsymbol{B}_{V,n-k+1}(u)+\sum_{j=1}^{n}\boldsymbol{C}_{n-k+1,j}(u)\boldsymbol{y}_j,\qquad k=1,2,\ldots,n;\ u\geq 0, \tag{4.18}$$

where $\boldsymbol{C}_{n-k+1,j}(u)$ follows the definition (3.47), and $\boldsymbol{B}_{V,n-k+1}(u)$ can be computed recursively via

$$\begin{cases}\boldsymbol{B}_{V,n}(u)=\boldsymbol{K}_V(u).\\ \boldsymbol{B}_{V,n-k+1}(u)=\int_0^\infty \boldsymbol{Z}_{\delta,\beta}(u,y)\boldsymbol{B}_{V,n-k+2}(y)dy,\ k=2,3,\ldots,n.\end{cases} \tag{4.19}$$

Letting $\underline{\boldsymbol{B}}_V(u)=(\boldsymbol{B}_{V,1}^\top(u),\ldots,\boldsymbol{B}_{V,n}^\top(u))^\top$, we can rewrite (4.18) as

$$\underline{\boldsymbol{V}}(b+u;b)=\underline{\boldsymbol{B}}_V(u)+\underline{\boldsymbol{C}}(u)(\boldsymbol{y}_1^\top,\ldots,\boldsymbol{y}_n^\top)^\top,\qquad u\geq 0.$$

Finally, we further apply Eq. 4.6 and the smooth pasting condition (4.5) to determine the unknown vector as

$$(\boldsymbol{y}_1^\top,\ldots,\boldsymbol{y}_n^\top)^\top=\left(\underline{\boldsymbol{L}}_\delta'(b)-\underline{\boldsymbol{C}}'(0)\right)^{-1}\underline{\boldsymbol{B}}_V'(0).$$

We summarize the main results in the following Theorem.

Theorem 2 *Suppose that the matrices $\tilde{\boldsymbol{Q}}_\delta$ and $\boldsymbol{Q}_{\delta+\beta}$ are diagonalizable. Then*

$$\underline{\boldsymbol{V}}(u;b)=\underline{\boldsymbol{L}}_\delta(u)\left(\underline{\boldsymbol{L}}_\delta'(b)-\underline{\boldsymbol{C}}'(0)\right)^{-1}\underline{\boldsymbol{B}}_V'(0),\qquad 0\leq u\leq b, \tag{4.20}$$

and

$$\underline{\boldsymbol{V}}(u;b)=\underline{\boldsymbol{B}}_V(u-b)+\underline{\boldsymbol{C}}(u-b)\left(\underline{\boldsymbol{L}}_\delta'(b)-\underline{\boldsymbol{C}}'(0)\right)^{-1}\underline{\boldsymbol{B}}_V'(0),\qquad u>b. \tag{4.21}$$

In particular, the expected discounted dividends paid until ruin $\underline{\boldsymbol{V}}(u;b)$ can be computed by the following procedure.

- *Step 1: Compute the matrices $\tilde{\boldsymbol{Q}}_{\delta}$ and $\boldsymbol{Q}_{\delta+\beta}$ using one of the methods discussed in Section 2.*
- *Step 2: Compute $\underline{\boldsymbol{L}}_{\delta}(u)$ by Proposition 1.*
- *Step 3: Compute $\boldsymbol{R}_V(u)$ by Eq. 4.14; and $\boldsymbol{H}_{k,j}(u)$ by Eqs. 3.34 and 3.43.*
- *Step 4: Compute $\mathbf{Z}_{\delta,\beta}(u,y)$ by Eq. 3.36; and $\boldsymbol{K}_V(u)$ and $\boldsymbol{P}_{k,j}(u)$ by Eqs. 4.17 and 3.38, respectively.*
- *Step 5: Compute $\boldsymbol{B}_{V,k}(u)$ and $\boldsymbol{C}_{k,j}(u)$ recursively via Eqs. 4.19 and 3.47, respectively.*
- *Step 6: Compute $\underline{\boldsymbol{V}}(u;b)$ via Eqs. 4.20 and 4.21.*

5 Numerical Examples

5.1 Brownian Motion Risk Model

In this subsection, we consider the Brownian motion risk model (i.e. $m=1$ and there are no claims at all). Writing $c=c_1$ and $\sigma_1=\sigma$, the cumulant generating function of the barrier-free process U^{∞} is given by

$$\frac{1}{t}\ln\mathbb{E}[e^{sU_t^{\infty}}]=\frac{1}{2}\sigma^2 s^2+cs.$$

Since there is only one environmental state, we also let $\phi(u;b)=\phi_1(u;b)$ and $V(u;b)=V_1(u;b)$. Note that ruin can only be caused by diffusion because there are no claims. Hence, without loss of generality we let $w(0)=1$, and then $\phi(u;b)$ actually represents the Laplace transform of the time of ruin.

We first assume that the inter-dividend-decision times follow an exponential distribution, i.e. $n=1$. In this simplest case, the results for $\phi(u;b)$, $V(u;b)$ and the optimal dividend barrier are very explicit. For the Gerber-Shiu function $\phi(u;b)$, it follows from Eq. 3.5 that

$$\frac{1}{2}\sigma^2\phi''(u;b)+c\phi'(u;b)-(\delta+\beta)\phi(u;b)+\beta\phi(u;b)\mathbf{1}_{\{0<u\le b\}}+\beta\phi(b;b)\mathbf{1}_{\{u>b\}}=0.$$

Due to the simplicity of the problem, instead of using Theorem 1 one can proceed to solve the above piecewise differential equation subject to the conditions

$$\phi(0;b)=1,\quad \phi(b-;b)=\phi(b+;b),\quad \phi'(b-;b)=\phi'(b+;b),$$

as well as the fact that $\phi(u;b)\le 1$. Following similar notations as in Gerber and Shiu (2004), for $\gamma\ge 0$ we define $s_\gamma<0$ and $r_\gamma\ge 0$ to be the roots of the quadratic equation (in ξ)

$$\frac{1}{2}\sigma^2\xi^2+c\xi-\gamma=0.$$

Omitting the straightforward algebra, we arrive at

$$\phi(u;b)=\frac{(r_\delta-\frac{\delta}{\delta+\beta}s_{\delta+\beta})e^{-s_\delta(b-u)}-(s_\delta-\frac{\delta}{\delta+\beta}s_{\delta+\beta})e^{-r_\delta(b-u)}}{(r_\delta-\frac{\delta}{\delta+\beta}s_{\delta+\beta})e^{-s_\delta b}-(s_\delta-\frac{\delta}{\delta+\beta}s_{\delta+\beta})e^{-r_\delta b}},\qquad 0\le u\le b,\quad (5.1)$$

and

$$\phi(u;b)=\phi(b;b)\left(\frac{\beta}{\delta+\beta}+\frac{\delta}{\delta+\beta}e^{s_{\delta+\beta}(u-b)}\right),\qquad u>b.$$

For the expected discounted dividends paid until ruin $V(u;b)$, we obtain from Eq. 4.2 that

$$\frac{1}{2}\sigma^2V''(u;b)+cV'(u;b)-(\delta+\beta)V(u;b)+\beta V(u;b)\mathbf{1}_{\{0<u\le b\}}+\beta(u-b+V(b;b))\mathbf{1}_{\{u>b\}}=0.$$

It can be solved using the conditions

$$V(0;b)=0,\quad V(b-;b)=V(b+;b),\quad V'(b-;b)=V'(b+;b),$$

and the fact that $V(u;b)$ is asymptotically linear in u (see Remark 3). This yields

$$V(u;b)=\frac{\frac{\beta}{\delta+\beta}(1-\frac{cs_{\delta+\beta}}{\delta+\beta})(e^{r_\delta u}-e^{s_\delta u})}{(r_\delta-\frac{\delta}{\delta+\beta}s_{\delta+\beta})e^{r_\delta b}-(s_\delta-\frac{\delta}{\delta+\beta}s_{\delta+\beta})e^{s_\delta b}},\qquad 0\le u\le b, \tag{5.2}$$

and

$$V(u;b)=V(b;b)\left(\frac{\beta}{\delta+\beta}+\frac{\delta}{\delta+\beta}e^{s_{\delta+\beta}(u-b)}\right)+\frac{\beta}{\delta+\beta}\left(u-b+\frac{c}{\delta+\beta}(1-e^{s_{\delta+\beta}(u-b)})\right),\quad u>b. \tag{5.3}$$

Apart from explicit expressions for $\phi(u;b)$ and $V(u;b)$, we are also interested in the optimal dividend barrier b^* maximizing the dividend function $V(u;b)$ with respect to b. All else being equal, on average a lower (higher) barrier leads to more (less) dividends at early times but less (more) dividends in the long run due to earlier (later) ruin. Hence, choosing b^* can somehow be regarded as striking a balance between the timing of dividend payments (because of discounting) and the total (non-discounted) amount of dividends paid. The value of b^* can be obtained by solving

$$\frac{\partial}{\partial b}V(u;b)=0.$$

From Eq. 5.2, one readily obtains

$$b^*=\frac{1}{r_\delta-s_\delta}\ln\frac{s_\delta(s_\delta-\frac{\delta}{\delta+\beta}s_{\delta+\beta})}{r_\delta(r_\delta-\frac{\delta}{\delta+\beta}s_{\delta+\beta})}, \tag{5.4}$$

which maximizes $V(u;b)$ as long as $0\le u\le b^*$. It can also be checked that the above b^* is also a turning point of the expression (5.3), and therefore it indeed maximizes $V(u;b)$ for all $u\ge 0$. In parallel to Avanzi et al. (2013, Section 4.3) who considered the optimal barrier in a dual risk model with exponential jumps in the absence of diffusion, we can verify that $V'(b^*;b^*)=1$. This leads to

$$\frac{\partial}{\partial b}[u-b+V(b;b)]\Big|_{b=b^*}=0.$$

The above equation means that the optimal barrier is still b^* even if we declare time 0 to be a dividend decision time.

Remark 4 Since $s_{\delta+\beta}=(-c-\sqrt{c^2+2\sigma^2(\delta+\beta)})/\sigma^2$, it is clear that $\lim_{\beta\to\infty}s_{\delta+\beta}=-\infty$ and
$\lim_{\beta\to\infty}s_{\delta+\beta}/(\delta+\beta)=0$. Thus, the limits of Eqs. 5.1 and 5.2 as $\beta\to\infty$ are identical to equations (3.7) and (2.11) of Gerber and Shiu (2004), respectively. This is expected because the inter-dividend-decision times tend to zero as $\beta\to\infty$, and we are back to the traditional barrier strategy in which dividend decisions are made continuously. Moreover, the optimal barrier (5.4) reduces to the one from Gerber and Shiu (2004, equation (6.2)) at the limit. It can also be verified analytically that Eq. 5.4 is an increasing concave function of β and the details are omitted here. □

Based on the explicit formulas derived above, we look at a numerical example by setting $c=0.2$, $\sigma^2=0.3$, $\delta=0.01$ and $\beta=0.05$. Figures 1a and b show the behaviour of the Laplace transform of the time of ruin $\phi(u;b)$ when either u or b varies. It can be seen that

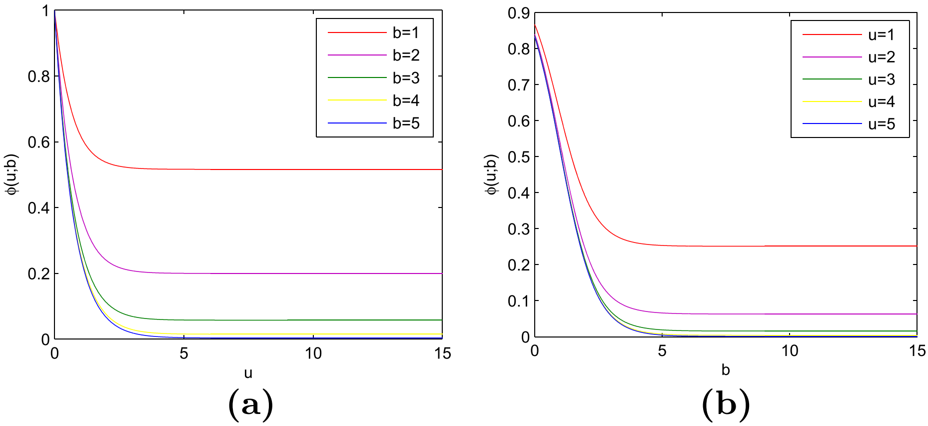

Fig. 1 The Laplace transform of the time of ruin (**a**) as a function of u; and (**b**) as a function of b

$\phi(u; b)$ is decreasing in both u and b. This is because for larger initial surplus u or larger barrier level b, ruin is likely to happen at a later time, and therefore the present value of a dollar payable at ruin is worth less. Note that for each fixed b, the curve in Fig. 1a flattens out as u increases, which can be attributed to the asymptotic formula (3.51). Similarly, the plots of the expected discounted dividends $V(u; b)$ with respect to u and b are depicted in Figs. 2a and b. For each fixed b, Fig 2a illustrates that $V(u; b)$ is an increasing function of u, and the relationship is almost linear as u increases due to Eq. 4.9. However, from Fig. 2b, for each fixed u the dividend function $V(u; b)$ is first increasing and then decreasing in b, and there is a unique optimal barrier b^*. In particular, the value of $b^* = 3.237$ is independent of the initial surplus u, which is consistent with our theoretical findings.

Next, we still retain the same parameters except that the value of β is varied in order to study its impact on the optimal barrier b^*. In accordance with Remark 4, Fig. 3 shows that b^* is increasing concave in β, converging to the classical optimal barrier of 4.692 as $\beta \to \infty$. When β increases, the inter-dividend-decision times are shorter (since $\beta = 1/\mathbb{E}[T]$).

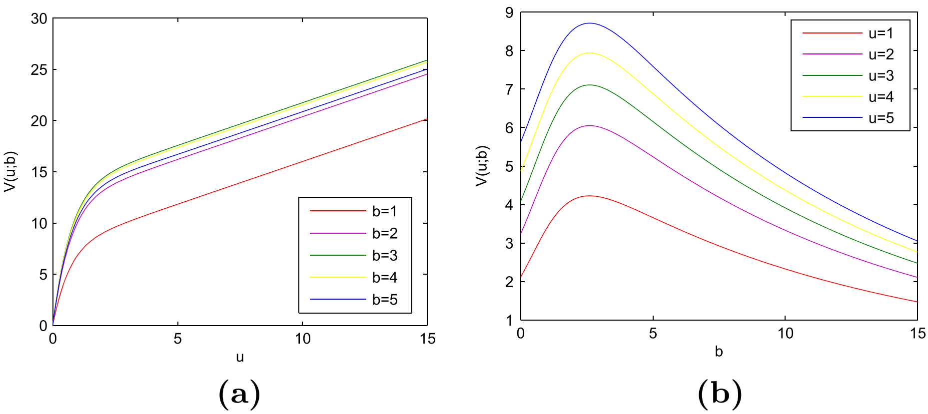

Fig. 2 The expected discounted dividends (**a**) as a function of u; and (**b**) as a function of b

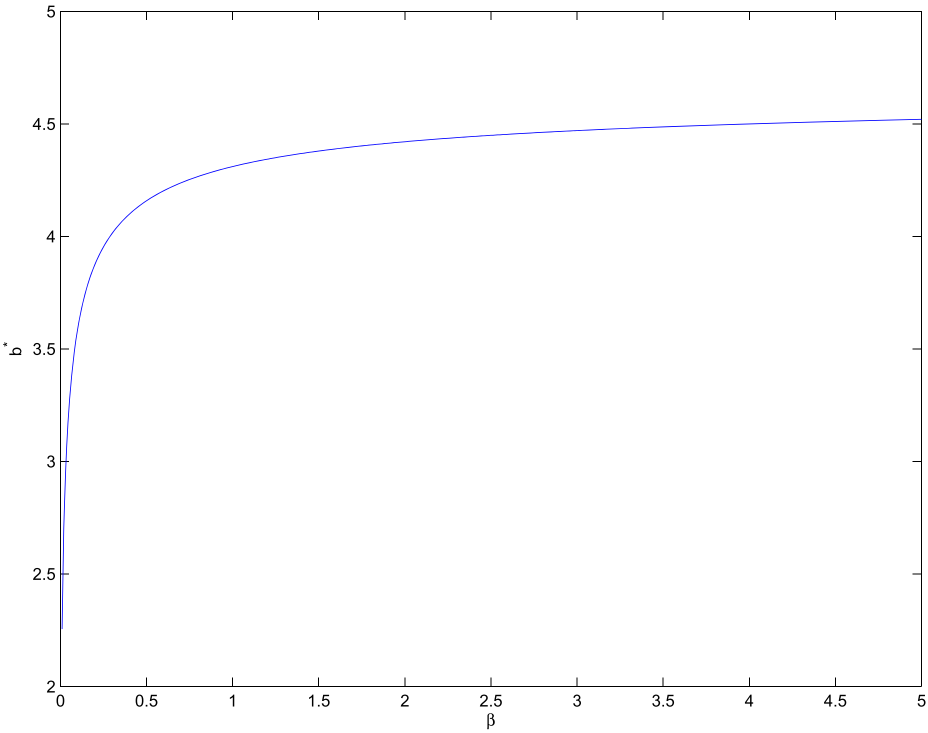

Fig. 3 The optimal dividend barrier b^* as a function of $\beta = 1/\mathbb{E}[T]$

If the barrier value remains the same, a larger amount of early dividends would be paid at the expense of earlier ruin as dividend decisions are made more frequently, shifting the balance between the timing and the total amount of dividend payments to the former. Thus, the optimal barrier b^* should increase to counterbalance the effect of larger β.

So far exponential inter-dividend-decision times have been assumed in this subsection. For general Erlang(n) inter-dividend-decision times with $n > 1$, expressions for $\phi(u; b)$ and $V(u; b)$ still involve exponential and linear functions in u and b only, but the optimal barrier b^* can no longer be represented in explicit form. Instead of directly solving ordinary differential equations, we can use the algorithms provided in Theorems 1 and 2 to get exact values of $\phi(u; b)$ and $V(u; b)$. For numerical illustrations, we only look at $V(u; b)$ since this will give insights to the optimal barrier. We still let $c = 0.2$, $\sigma^2 = 0.3$ and $\delta = 0.01$. To see the effect of Erlangization, we increase n from 1 to 5 while fixing $\mathbb{E}[T] = n/\beta = 20$ (i.e. increasing β). Figs. 4a and b show that $V(u; 3)$ as a function of u and $V(3; b)$ as a function of b are of the same shape as in the case $n = 1$ even if we increase n. In addition, for fixed values of u and b, the dividend function $V(u; b)$ appears to be increasing and converging as n increases. More importantly, we observe from Fig. 4b that for each fixed n the optimal barrier b^* exists. We have further carried out some numerical checking using different values of initial surplus (which is not reproduced here), and found that for each fixed n the optimal barrier b^* is independent of u. The values of b^* are given by 3.237, 3.299, 3.300, 3.331, 3.337 respectively when $n = 1, 2, 3, 4, 5$.

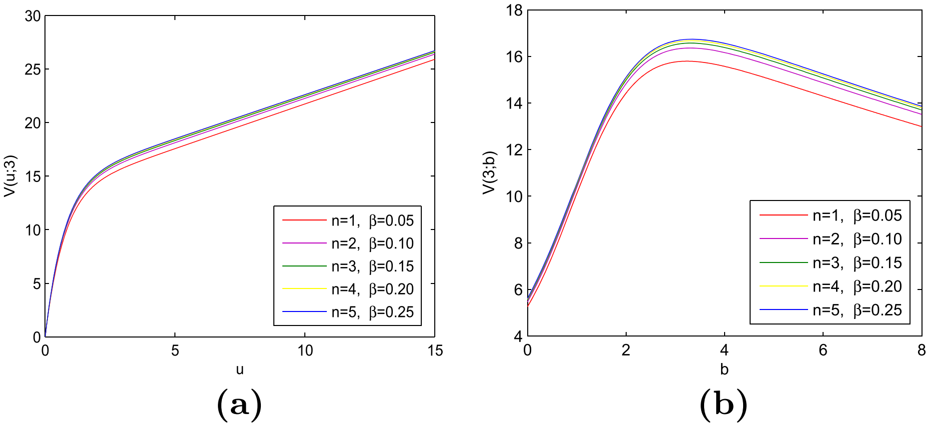

Fig. 4 The expected discounted dividends (**a**) as a function of u when $b = 3$; and (**b**) as a function of b when $u = 3$

5.2 Bivariate MAP Risk Model

This subsection aims at providing further numerical examples for bivariate MAP risk models. First, we consider a bivariate Markov-modulated Brownian risk model (i.e. there are no claims). We set

$$\boldsymbol{\Delta}_c = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.25 \end{pmatrix}, \quad \boldsymbol{\Delta}_{\sigma^2} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad \boldsymbol{D}_0 = \begin{pmatrix} -0.06 & 0.06 \\ 0.03 & -0.03 \end{pmatrix}, \quad \boldsymbol{D}_1 = \boldsymbol{O}.$$

It is assumed that the inter-dividend-decision times follow the Erlang(2) distribution with $\beta = 0.05$ and the force of interest is $\delta = 0.01$. From Fig. 5, the expected present values of dividends given different initial states, namely $V_1(u; b)$ and $V_2(u; b)$, show similar behaviour as in Fig. 2a, i.e. they both increase in u for each fixed b and then essentially grow linearly as u increases further. For each fixed u, Fig. 6 shows that $V_1(u; b)$ and $V_2(u; b)$ first increase and then decrease in b. Interestingly, using the exact dividend values calculated

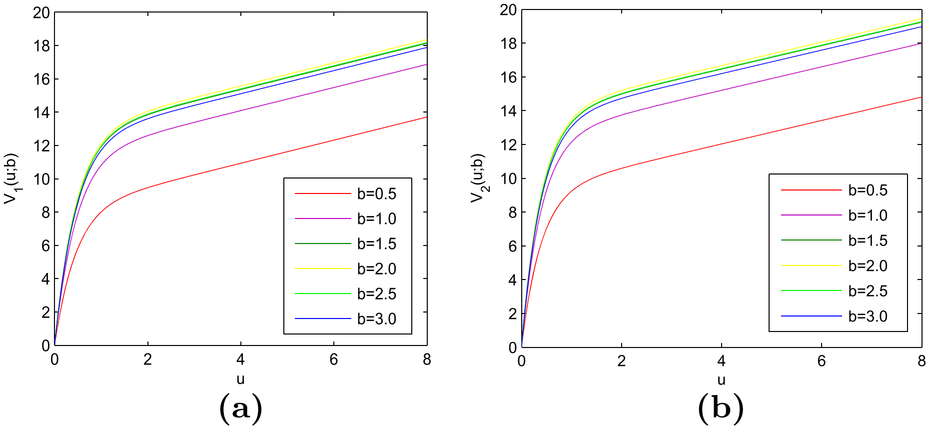

Fig. 5 (**a**) $V_1(u; b)$ as a function of u; and (**b**) $V_2(u; b)$ as a function of u

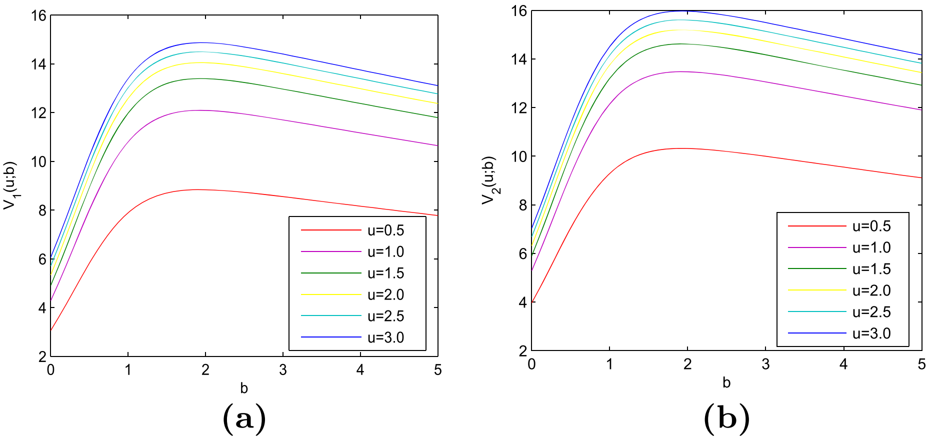

Fig. 6 (**a**) $V_1(u; b)$ as a function of b; and (**b**) $V_2(u; b)$ as a function of b

via Theorem 2, it is found that regardless of the initial surplus level u, the optimal dividend barriers that maximize $V_1(u; b)$ and $V_2(u; b)$ coincide and are both given by $b^* = 1.935$. In other words, the optimal dividend barrier b^* appears to be independent of the initial surplus and the initial environmental state. The latter also implies that b^* is the same for the unconditional process U^b under any initial probability row vector $\boldsymbol{\alpha}$ of the Markov chain J.

Next, we look at a perturbed Markov-modulated risk model with two states. Let

$$\boldsymbol{\Delta}_c = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.03 \end{pmatrix}, \quad \boldsymbol{\Delta}_{\sigma^2} = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.05 \end{pmatrix} \tag{5.5}$$

and

$$\boldsymbol{D}_0 = \begin{pmatrix} -0.06 & 0.03 \\ 0.1 & -0.2 \end{pmatrix}, \quad \boldsymbol{D}_1 = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

The claim severities are assumed to be exponentially distributed such that

$$f_{11}(x) = e^{-x}, \quad f_{22}(x) = 5e^{-5x},$$

so that $\mu_{11} = 1$ and $\mu_{22} = 0.2$ (and the loading condition (1.2) is satisfied). In addition, the inter-dividend-decision times are assumed to be exponential with $\beta = 0.1$. We are interested in the distribution of the deficit at ruin, which is denoted by $\boldsymbol{F}(z|u; b) = (F_1(z|u; b), F_2(z|u; b))$. For $i = 1, 2$, the quantity $F_i(z|u; b)$ can be retrieved from $\phi_i(u; b)$ by letting $\delta = 0$ and $w(y) = \mathbf{1}_{\{y \leq z\}}$. Fig. 7 depicts $\boldsymbol{F}(z|u; b)$ under $b = 4$. Note that both $F_1(z|u; 4)$ and $F_2(z|u; 4)$ have probability masses at $z = 0$ because ruin may be caused by oscillation. Furthermore, we observe that $F_1(z|u; 4)$ and $F_2(z|u; 4)$ tend to 1 as z increases, which is due to the fact that ruin is certain under this Erlangized dividend barrier strategy. Except for small values of u where there is higher chance of early ruin by oscillation, the values of $F_1(z|u; 4)$ and $F_2(z|u; 4)$ appear to be not very sensitive to change in the initial surplus level.

Finally, we study a bivariate MAP risk process in which $\boldsymbol{\Delta}_c$ and $\boldsymbol{\Delta}_{\sigma^2}$ are still given by (5.5) and the inter-dividend-decision times are exponential with $\beta = 0.1$. However, the generators are now assumed to be

$$\boldsymbol{D}_0 = \begin{pmatrix} -0.06 & 0.03 \\ 0.01 & -0.02 \end{pmatrix}, \quad \boldsymbol{D}_1 = \begin{pmatrix} 0.02 & 0.01 \\ 0 & 0.01 \end{pmatrix}.$$

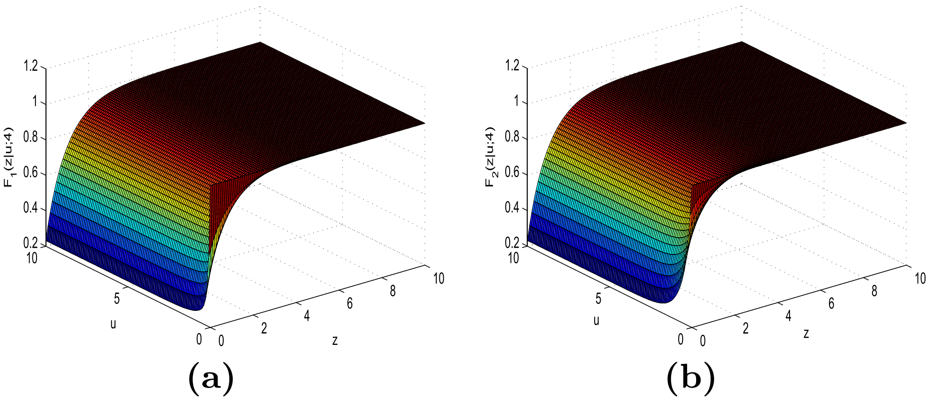

Fig. 7 (**a**) $F_1(z|u; 4)$ as a function of u and z; and (**b**) $F_2(z|u; 4)$ as a function of u and z

Furthermore, the claim densities are

$$f_{11}(x) = 2e^{-2x}, \quad f_{12}(x) = e^{-x}, \quad f_{22}(x) = 0.3(2e^{-2x}) + 0.7(5e^{-5x}).$$

Compared to the previous example, the current specification of $\boldsymbol{D}_1$ allows a transition from state 1 to state 2 to be accompanied by a claim (that follows a mixture of two exponentials). We are interested in the Laplace transform of the ruin time given initial state i, which can be retrieved from the Gerber-Shiu function $\phi_i(u; b)$ by letting $w \equiv 1$. Under a Laplace transform argument of $\delta = 0.01$, the quantities $\phi_1(u; b)$ and $\phi_2(u; b)$ are plotted against the initial surplus u for each fixed $b = 1, 2, 3, 4, 5$ in Fig. 8. Similar behaviour as in Fig. 1 is observed, and the same interpretation therein applies.

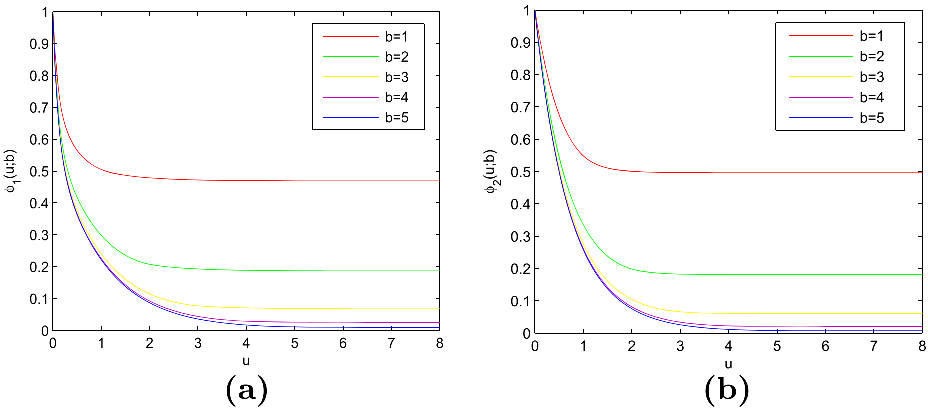

Fig. 8 (**a**) $\phi_1(u; b)$ as a function of u; and (**b**) $\phi_2(u; b)$ as a function of u

6 Appendix on Continuity and Smooth Pasting

In this appendix, we demonstrate how to check the continuity conditions (3.7) and (4.4) as well as the smooth pasting conditions (3.8) and (4.5), which have been used to derive full solutions to the Gerber-Shiu function and the expected discounted dividends until ruin. To begin, we need some auxiliary functions. For $i \in \mathcal{E}$ and $u \geq 0$, we define the stopping time $\tau_i^u = \inf\{t > 0 : u + c_i t + \sigma_i B_t < 0\}$ and the associated resolvent measure, for $q \geq 0$,

$$\mathcal{R}_i^{(q)}(u, dx) = \int_0^\infty e^{-qt} \mathbb{P}\{u + c_i t + \sigma_i B_t \in dx, t < \tau_i^u\} dt.$$

Further let $\eta_{1i}^{(q)} \geq 0$ and $\eta_{2i}^{(q)} < 0$ be the roots of the quadratic equation (in ξ)

$$\frac{1}{2}\sigma_i^2 \xi^2 + c_i \xi - q = 0.$$

It follows from Theorem 8.7 and Corollary 8.8 in Kyprianou (2006) that the above resolvent measure admits a density, which is such that $\mathcal{R}_i^{(q)}(u, dx) = r_i^{(q)}(u, x)dx$ and given by

$$r_i^{(q)}(u, x) = e^{-\eta_{1i}^{(q)} x} W_i^{(q)}(u) - W_i^{(q)}(u - x).$$

Here $W_i^{(q)}$ is a q-scale function defined as $W_i^{(q)}(x) = 0$ for $x < 0$ and

$$W_i^{(q)}(x) = \frac{e^{\eta_{1i}^{(q)} x} - e^{\eta_{2i}^{(q)} x}}{\frac{\sigma_i^2}{2}(\eta_{1i}^{(q)} - \eta_{2i}^{(q)})}, \qquad x \geq 0.$$

More explicitly, we have

$$r_i^{(q)}(u, x) = \begin{cases} \frac{e^{\eta_{2i}^{(q)}(u-x)} - e^{\eta_{2i}^{(q)} u - \eta_{1i}^{(q)} x}}{\frac{\sigma_i^2}{2}(\eta_{1i}^{(q)} - \eta_{2i}^{(q)})}, & 0 \leq x \leq u, \\ \frac{e^{\eta_{1i}^{(q)}(u-x)} - e^{\eta_{2i}^{(q)} u - \eta_{1i}^{(q)} x}}{\frac{\sigma_i^2}{2}(\eta_{1i}^{(q)} - \eta_{2i}^{(q)})}, & x > u. \end{cases} \tag{A.1}$$

It is also well known that the Laplace transform of τ_i^u is (see e.g. Borodin and Salminen (2002, p.295))

$$\mathcal{H}_i^{(q)}(u) = \mathbb{E}[e^{-q\tau_i^u}] = e^{\eta_{2i}^{(q)} u}.$$

The key to proving continuity and smooth pasting is the derivation of appropriate integral equations as follows. Suppose that for the process U^b, the initial surplus is $u \geq 0$, the initial state is $J_0 = i \in \mathcal{E}$, and the time until the next dividend decision time is Erlang$(n - k + 1)$ distributed for some $k = 1, 2, \ldots, n$. Let C_1 be the time until the first phase change of the dividend decision time. Clearly, C_1 is always exponentially distributed with mean $1/\beta$. Define E_i to be the time until the first event of the bivariate Markov process (N, J) occurs. Then E_i is an exponential random variable with mean $-1/D_{0,ii}$. Three situations need to be distinguished: (1) $\tau^b < E_i \wedge C_1$; (2) $E_i < \tau^b \wedge C_1$; and (3) $C_1 < \tau^b \wedge E_i$. Note that under $\mathbb{P}_{u,i}$, U^b is distributed as the process $\{u + c_i t + \sigma_i B_t\}$ for $0 \leq t < E_i \wedge C_1$. In addition, $\{u + c_i t + \sigma_i B_t\}$ and the random variables E_i and C_1 are mutually independent. Therefore, for the Gerber-Shiu function we arrive at

$$\begin{aligned}\phi_{k,i}(u;b) &= w(0)\mathbb{E}[e^{-\delta\tau_i^u}\mathbf{1}_{\{\tau_i^u<E_i,\tau_i^u<C_1\}}]\\ &+\int_0^\infty e^{-(\delta+\beta-D_{0,ii})t}\sum_{j=1,j\neq i}^m D_{0,ij}\int_0^\infty \phi_{k,j}(x;b)\mathbb{P}\{u+c_it+\sigma_iB_t\in dx, t<\tau_i^u\}dt\\ &+\int_0^\infty e^{-(\delta+\beta-D_{0,ii})t}\sum_{j=1}^m D_{1,ij}\int_0^\infty (\gamma_{k,ij}(x;b)+\omega_{ij}(x))\mathbb{P}\{u+c_it+\sigma_iB_t\in dx, t<\tau_i^u\}dt\\ &+\int_0^\infty \beta e^{-(\delta+\beta-D_{0,ii})t}\int_0^\infty \phi_{k+1,i}(x;b)\mathbb{P}\{u+c_it+\sigma_iB_t\in dx, t<\tau_i^u\}dt, \quad k=1,2,\ldots,n-1.\end{aligned} \tag{A.2}$$

Because

$$\begin{aligned}\mathbb{E}[e^{-\delta\tau_i^u}\mathbf{1}_{\{\tau_i^u<E_i,\tau_i^u<C_1\}}] &= \mathbb{E}[\mathbb{E}[e^{-\delta\tau_i^u}\mathbf{1}_{\{\tau_i^u<E_i,\tau_i^u<C_1\}}|\tau_i^u]]\\ &= \mathbb{E}\left[e^{-\delta\tau_i^u}\left(\int_{\tau_i^u}^\infty(-D_{0,ii})e^{D_{0,ii}t}dt\right)\left(\int_{\tau_i^u}^\infty \beta e^{-\beta x}dx\right)\right]\\ &= \mathbb{E}[e^{-(\delta+\beta-D_{0,ii})\tau_i^u}] = \mathcal{H}^{(\delta+\beta-D_{0,ii})}(u),\end{aligned}$$

using the resolvent measure we can rewrite (A.2) as

$$\begin{aligned}\phi_{k,i}(u;b) &= w(0)\mathcal{H}^{(\delta+\beta-D_{0,ii})}(u)+\sum_{j=1,j\neq i}^m D_{0,ij}\int_0^\infty \phi_{k,j}(x;b)r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx\\ &+\sum_{j=1}^m D_{1,ij}\int_0^\infty(\gamma_{k,ij}(x;b)+\omega_{ij}(x))r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx\\ &+\beta\int_0^\infty \phi_{k+1,i}(x;b)r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx, \qquad k=1,2,\ldots,n-1.\end{aligned} \tag{A.3}$$

Similarly, for $k=n$ we have

$$\begin{aligned}\phi_{n,i}(u;b) &= w(0)\mathcal{H}^{(\delta+\beta-D_{0,ii})}(u)+\sum_{j=1,j\neq i}^m D_{0,ij}\int_0^\infty \phi_{n,j}(x;b)r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx\\ &+\sum_{j=1}^m D_{1,ij}\int_0^\infty(\gamma_{n,ij}(x;b)+\omega_{ij}(x))r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx\\ &+\beta\int_0^b \phi_{1,i}(x;b)r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx+\beta\int_b^\infty \phi_{1,i}(b;b)r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx.\end{aligned} \tag{A.4}$$

Concerning the expected present value of dividends paid until ruin, we have

$$\begin{aligned}V_{k,i}(u;b) &= \sum_{j=1,j\neq i}^m D_{0,ij}\int_0^\infty V_{k,j}(x;b)r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx\\ &+\sum_{j=1}^m D_{1,ij}\int_0^\infty\left(\int_0^x V_{k,j}(x-y;b)f_{ij}(y)dy\right)r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx\\ &+\beta\int_0^\infty V_{k+1,i}(x;b)r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx, \qquad k=1,2,\ldots,n-1,\end{aligned} \tag{A.5}$$

and

$$V_{n,i}(u;b) = \sum_{j=1,j\neq i}^{m} D_{0,ij} \int_0^\infty V_{n,j}(x;b) r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx + \sum_{j=1}^{m} D_{1,ij} \int_0^\infty \left(\int_0^x V_{n,j}(x-y;b) f_{ij}(y)dy \right) r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx + \beta \int_0^b V_{1,i}(x;b) r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx + \beta \int_b^\infty (x-b+V_{1,i}(b;b)) r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx. \tag{A.6}$$

Continuity and smooth pasting can be shown based on Eqs. A.3-A.6. For illustrative purposes, we only focus on $V_{n,i}(u;b)$ since the other functions can be checked analogously. Letting

$$\vartheta_i(x) = \sum_{j=1,j\neq i}^{m} D_{0,ij} V_{n,j}(x;b) + \sum_{j=1}^{m} D_{1,ij} \int_0^x V_{n,j}(x-y;b) f_{ij}(y)dy + \beta V_{1,i}(x;b) \mathbf{1}_{\{0<x\leq b\}} + \beta(x-b+V_{1,i}(b;b)) \mathbf{1}_{\{x>b\}}, \tag{A.7}$$

Equation A.6 can be rewritten as

$$V_{n,i}(u;b) = \int_0^\infty \vartheta_i(x) r_i^{(\delta+\beta-D_{0,ii})}(u,x)dx = \int_0^u \vartheta_i(x) \frac{e^{\tilde{\eta}_{2i}(u-x)} - e^{\tilde{\eta}_{2i}u-\tilde{\eta}_{1i}x}}{\frac{\sigma_i^2}{2}(\tilde{\eta}_{1i}-\tilde{\eta}_{2i})} dx + \int_u^\infty \vartheta_i(x) \frac{e^{\tilde{\eta}_{1i}(u-x)} - e^{\tilde{\eta}_{2i}u-\tilde{\eta}_{1i}x}}{\frac{\sigma_i^2}{2}(\tilde{\eta}_{1i}-\tilde{\eta}_{2i})} dx. \tag{A.8}$$

Here Eq. A.1 has been used in the second equality, and we define $\tilde{\eta}_{1i} = \eta_{1i}^{(\delta+\beta-D_{0,ii})}$ and $\tilde{\eta}_{2i} = \eta_{2i}^{(\delta+\beta-D_{0,ii})}$ for convenience. From the above representation, it is clear that $V_{n,i}(0;b) = 0$ and $V_{n,i}(u;b)$ is a continuous function in u for all $u \geq 0$. Similarly, one can deduce from Eq. A.5 that $V_{k,i}(u;b)$ is continuous in u for each $k = 1, 2, \ldots, n-1$. Consequently, $\vartheta_i(x)$ is continuous as well, as evident from Eq. A.7. Hence, taking derivative of Eq. A.8 with respect to u gives

$$V_{n,i'}(u;b) = \int_0^u \vartheta_i(x) \frac{\tilde{\eta}_{2i} e^{\tilde{\eta}_{2i}(u-x)} - \tilde{\eta}_{2i} e^{\tilde{\eta}_{2i}u-\tilde{\eta}_{1i}x}}{\frac{\sigma_i^2}{2}(\tilde{\eta}_{1i}-\tilde{\eta}_{2i})} dx + \int_u^\infty \vartheta_i(x) \frac{\tilde{\eta}_{1i} e^{\tilde{\eta}_{1i}(u-x)} - \tilde{\eta}_{2i} e^{\tilde{\eta}_{2i}u-\tilde{\eta}_{1i}x}}{\frac{\sigma_i^2}{2}(\tilde{\eta}_{1i}-\tilde{\eta}_{2i})} dx.$$

Therefore, $V_{n,i'}(u;b)$ is continuous in u for all $u \geq 0$. Indeed, by further differentiating the above equation with respect to u (or by inspecting (4.2) and using the fact that $V_{n,i}(u;b)$ and $V_{n,i'}(u;b)$ are continuous in u), one can observe that $V_{n,i''}(u;b)$ is also continuous in u for all $u \geq 0$. However, higher order derivatives are in general not continuous at $u = b$.

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