Methodol Comput Appl Probab (2015) 17:761–780
DOI 10.1007/s11009-013-9392-1

Probability Law and Flow Function of Brownian Motion Driven by a Generalized Telegraph Process

Antonio Di Crescenzo · Shelemyahu Zacks

Received: 30 November 2012 / Revised: 3 December 2013 /
Accepted: 9 December 2013 / Published online: 28 December 2013

Abstract We consider a standard Brownian motion whose drift alternates randomly between a positive and a negative value, according to a generalized telegraph process. We first investigate the distribution of the occupation time, i.e. the fraction of time when the motion moves with positive drift. This allows to obtain explicitly the probability law and the flow function of the random motion. We discuss three special cases when the times separating consecutive drift changes have (i) exponential distribution with constant rates, (ii) Erlang distribution, and (iii) exponential distribution with linear rates. In conclusion, in view of an application in environmental sciences we evaluate the density of a Wiener process with infinitesimal moments alternating at inverse Gaussian distributed random times.

Keywords Standard Brownian motion · Alternating drift · Alternating counting process · Exponential random times · Erlang random times · Modified Bessel function · Two-index pseudo-Bessel function

AMS 2000 Subject Classifications 60J65 · 60K15

1 Introduction

The problem of describing random motions which follow a trend subject to occasional changes is not new. We recall for instance Chernoff and Zacks (1964), who developed a

A. Di Crescenzo (✉)
Dipartimento di Matematica, Università di Salerno,
84084 Fisciano, SA, Italy
e-mail: adicrescenzo@unisa.it

S. Zacks
Department of Mathematical Sciences, Binghamton University,
Binghamton, NY 13902-6000, USA
e-mail: shelly@math.binghamton.edu

Bayesian procedure able to estimate the current position on the ground of observations on a motion described by a sequence of Gaussian random variables with alternating means. In this paper we focus on an alternating random motion performed by a Brownian particle moving on the real line and subject to random changes in the trend. More precisely, we deal with a Brownian motion whose drift takes alternately a positive and a negative value at random times. The position of the moving particle is formally given by the sum of a standard Brownian motion and an independent generalized telegraph process. The resulting stochastic process is suitable to describe the motion of a particle that is subject to two streams of perturbations: the first is a random noise responsible of the Brownian fluctuations, the second is a random sequence of shocks which change the drift's value. We aim to determine the probability law of the particle motion and the related flow function.

Other versions of modified Brownian motion processes have been considered in the past. Keilson and Wellner (1978) treated the case in which the variance parameter alternates when the process crosses state zero. Orsingher (1986) analyzed a Brownian-type particle in a space-time plane with time alternating between forward and backward directions, where the time alternation is governed by a Poisson process. Perry (1997) considered an alternating two-sided regulated Brownian motion in which the drift switches when the process crosses two fixed levels. We further point out that the perturbation of stochastic systems by diffusion processes is often used in financial mathematics and risk theory, when the diffusive term is aimed to describe the random error or random variations due to environmental variables (see, for instance, Zhang and Wu 2002).

It should be mentioned that a topic related to the present framework is the so-called 'Wiener disorder problem'. It is aimed to determine a stopping time which is as close as possible to the (unknown) time when the drift of an observed Wiener process changes from one value to another. In a similar problem, a least-squares approach has been employed by De Gregorio and Iacus (2011) to estimate the change-point and the two intensities of the random telegraph process.

Furthermore, the Brownian motion with alternating drift deserves interest in various applied fields, such as financial mathematics. We mention, for instance, the paper by Esser and Mönch (2007), where a stock price is modeled by a suitable jump-diffusion process whose drift alternates according to a certain recursive rule. Furthermore, Gapeev and Peskir (2006) provide a wide list of applied fields and contexts where the Wiener process with varying drift plays a relevant role. Furthermore, we recall Guo (2001), who considered a Brownian motion whose drift and variance vary according to a continuous-time Markov chain. See also Jobert and Rogers (2006), where Markov-modulated diffusions governed by an irreducible Markov chain are investigated. Other examples of switching processes applied to financial models come from Elliott et al. (2007) and Ratanov (2010), where the dynamics of risky assets are driven by a Markov-modulated jump-diffusion process.

This is the plan of the paper. In Section 2 we introduce the stochastic model and the necessary notation. Section 3 is devoted to the determination of the formal distribution of the occupation time, which is the fraction of time that the particle moved with positive drift. In Section 4 the distribution of the particle position is expressed in terms of a suitable mixture of Gaussian densities, where the mixing law is the distribution of the occupation time. The mean, the variance and the flow function of the process are also investigated. Some special cases are developed in Section 5, where the random times separating consecutive drift changes have (i) exponential distribution with constant rates, (ii) Erlang distribution, and (iii) exponential distribution with linear rates. We also discuss some comparisons between the densities of the occupation time for such three cases. In conclusion, in Section 6 we consider an application in environmental sciences, for which we apply the foregoing approach to evaluate the probability density of a Wiener process with infinitesimal moments alternating at inverse Gaussian distributed random times.

We point out that the methods used in this paper might be adopted also to study similar kinds of motions subject to alternating behaviour. We recall for instance Lagache and Holcman (2008), where a stochastic model is employed to describe a viral movement which alternates between purely Brownian motion and a deterministic motion with a constant drift.

Throughout the paper we write $[X \mid B]$ to refer to a random variable having the same distribution of X conditional on B.

2 The Stochastic Model

Consider the random motion model based on the stochastic process $\{X(t); t \geq 0\}$, defined as

$$X(t) = Y(t) + B(t), \qquad t \geq 0, \tag{1}$$

where $\{Y(t)\}$ is a generalized (integrated) telegraph process, and $\{B(t)\}$ is a standard Brownian motion (Wiener process), i.e. $B(t) \sim \mathcal{N}(0, t)$. Assume that $Y(t)$ and $B(t)$ are independent processes, and let $Y(0) = B(0) = 0$, so that $X(0) = 0$, i.e. the motion starts from the origin. It should be remarked that the model (1) with the standard Brownian motion B by a suitable scaling can be modified to $X(t) = Y(t) + \sigma B(t)$, with $\sigma > 0$.

The telegraph process $Y(t)$ is characterized by velocities c and $-v$ (with $c > 0$ and $-v < 0$), which vary according to an independent alternating counting process $N(t)$. The latter process is governed by the sequences of positive independent random times $\{U_1, U_2, \ldots\}$ and $\{D_1, D_2, \ldots\}$, which in turn are assumed to be independent. The random variable U_i (resp. D_i) describes the i-th random period during which $Y(t)$ has positive (resp. negative) velocity, i.e., the motion has positive (resp. negative) drift. Denoting by $V(t)$ the velocity of $Y(t)$ at time $t \geq 0$, we assume that initially it is random, i.e.,

$$\mathbb{P}\{V(0) = c\} = \mathbb{P}\{V(0) = -v\} = \frac{1}{2}, \tag{2}$$

with $V(0)$ independent from $X(t)$. Hence, the following equations hold:

$$Y(t) = \int_0^t V(s)\,\mathrm{d}s, \qquad V(t) = \frac{c-v}{2} + \mathrm{sgn}(V(0))\,\frac{c+v}{2}\,(-1)^{N(t)}, \qquad t > 0. \tag{3}$$

We remark that $N(t)$ denotes the number of velocity changes of $Y(t)$ occurred in $[0, t]$, and thus $N(t)$ depends on $V(0)$ because the sequence of alternating periods is $U_1, D_1, U_2, D_2, \ldots$ if $V(0) = c$, and $D_1, U_1, D_2, U_2, \ldots$ if $V(0) = -v$. Hence, from the previous assumptions we have

$$[V(t) \mid V(0) = c] = \begin{cases} c & \text{if } 0 \leq t < U_1,\ U_1 + D_1 \leq t < U_1 + D_1 + U_2,\ \ldots \\ -v & \text{if } U_1 \leq t < U_1 + D_1,\ U_1 + D_1 + U_2 \leq t < U_1 + D_1 + U_2 + D_2,\ \ldots, \end{cases}$$

and

$$[V(t) \mid V(0) = -v] = \begin{cases} -v & \text{if } 0 \leq t < D_1,\ D_1 + U_1 \leq t < D_1 + U_1 + D_2,\ \ldots \\ c & \text{if } D_1 \leq t < D_1 + U_1,\ D_1 + U_1 + D_2 \leq t < D_1 + U_1 + D_2 + U_2,\ \ldots. \end{cases}$$

Figures 1 and 2 show suitable realizations of $Y(t)$ and $X(t)$, respectively, with indication of the random times U_i and D_i; in both cases the initial velocity is positive.

Let $F_{U_i}(\cdot)$ and $F_{D_i}(\cdot)$ be respectively the absolutely continuous distribution functions of U_i and D_i, $i = 1, 2, \ldots$, with densities $f_{U_i}(\cdot)$ and $f_{D_i}(\cdot)$. Moreover, the complementary distribution functions will be denoted as $\overline{F}_{U_i}(\cdot) = 1 - F_{U_i}(\cdot)$ and $\overline{F}_{D_i}(\cdot) = 1 - F_{D_i}(\cdot)$, respectively. We denote as follows the sums of the first n random periods during which $Y(t)$ has positive and negative velocity, respectively:

$$U^{(n)} := U_1 + \ldots + U_n, \qquad D^{(n)} := D_1 + \ldots + D_n, \qquad n = 1, 2, \ldots. \tag{4}$$

Their distribution functions will be denoted $F_U^{(n)}(\cdot)$ and $F_D^{(n)}(\cdot)$, with densities $f_U^{(n)}(\cdot)$ and $f_D^{(n)}(\cdot)$, respectively. Clearly, if the random variables U_i are i.i.d. for $i = 1, 2, \ldots, n$, then $F_U^{(n)}(\cdot)$ is the n-fold convolution of $F_{U_1}(\cdot)$, and similarly for $F_D^{(n)}(\cdot)$. Moreover, we set $F_U^{(0)}(x) = F_D^{(0)}(x) = 1$ for $x \geq 0$.

From Eq. 1 it follows that $X(t)$ describes a Brownian motion whose drift alternates randomly according to $Y(t)$. The first-passage-time problem for such process through a single constant boundary was investigated by Di Crescenzo et al. (2005) who provided bounds to the first-passage-time density and distribution functions, and constructed a simulation procedure to estimate first-passage-time densities. Furthermore, Guo (2001) obtained the Laplace transform of the first-passage-time distribution through a constant level for the more general model given by a Brownian motion whose drift and variance vary according to a continuous-time Markov chain. We also mention Buonocore et al. (2002), where a Brownian motion with alternating drift is studied as an integrate-and-fire neuronal model subject to fluctuating currents, and its simulating results are compared with those of a Wiener-type neuronal model with sinusoidal input. Moreover, the optimal stopping problem of detecting the first time when the drift changes has been recently faced by Dayanik (2010) and by Sezer (2010) within a Bayesian framework, when the changing time has exponential distribution or zero-modified exponential distribution.

We remark that the drift values of process $X(t)$ have been taken with opposite sign by analogy with the telegraph process. However, this assumption could be relaxed anyway. The essential part of this theory is that the drift is an alternating renewal process.

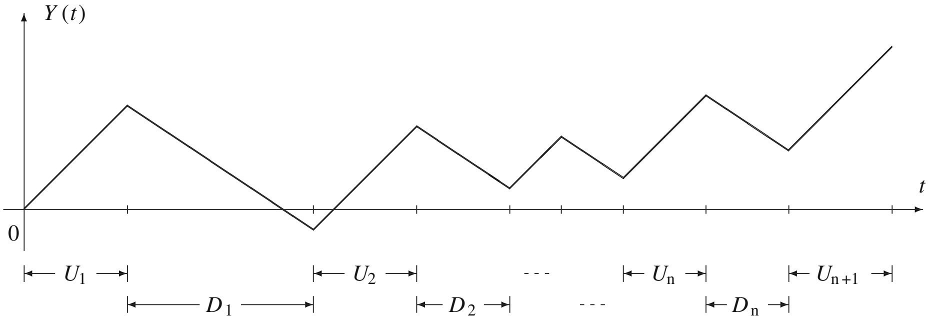


Fig. 1 A sample path of $Y(t)$, with $V(0) = c$

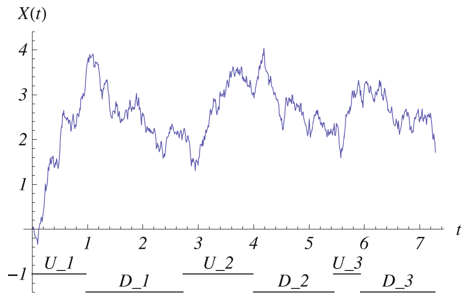


Fig. 2 A simulated sample path of $X(t)$, with $c = 2$ and $v = 1$

In order to obtain the probability law of $X(t)$ we follow the general method presented by Zacks (2004). The key to the solution is to determine, for each fixed $t > 0$, the distribution of the following occupation time:

$$W(t) := \int_0^t \mathbf{1}_{\{V(s)=c\}} \mathrm{d}s, \, t > 0, \tag{5}$$

which is the fraction of time that the motion moved with positive drift (i.e., $Y(t)$ had velocity c) in $[0, t]$. In the following section we shall obtain the distribution of $W(t)$.

3 Distribution of Occupation Time

Consider the occupation time (5); we note that

$$\mathbb{P}\{0 \le W(t) \le t\} = 1,$$

and that the probability distribution of $W(t)$ has a discrete component on points 0 and t, and an absolutely continuous component over $(0, t)$. For $v_0, v_t \in \{-v, c\}$ we introduce the following function:

$$\psi_{v_0}(x, t; v_t) := \frac{\partial}{\partial x} \mathbb{P}\{W(t) \le x, V(t) = v_t \mid V(0) = v_0\}, \qquad 0 < x < t. \tag{6}$$

This is the joint density that at time t the occupation time is x and the drift of the motion is v_t, given that the initial drift is v_0. Hence, recalling the initial condition (2), for $v_t \in \{-v, c\}$ we have

$$\psi(x, t; v_t) := \frac{\partial}{\partial x} \mathbb{P}\{W(t) \le x, V(t) = v_t\} = \frac{1}{2}\left[\psi_{-v}(x, t; v_t) + \psi_c(x, t; v_t)\right], 0 < x < t. \tag{7}$$

Theorem 3.1 *For all $t > 0$ we have*

$$\mathbb{P}[W(t) = 0] = \frac{1}{2}\overline{F}_{D_1}(t), \qquad \mathbb{P}[W(t) = t] = \frac{1}{2}\overline{F}_{U_1}(t). \tag{8}$$

Moreover, the density $\psi(x, t) := \frac{\partial}{\partial x}\mathbb{P}\{W(t) \le x\}$ is given by

$$\psi(x, t) = \psi(x, t; c) + \psi(x, t; -v), \qquad 0 < x < t, \tag{9}$$

where $\psi(x,t;v_t)$ *is given in* Eq. 7*, and where the densities* (6) *can be expressed as*

$$\begin{aligned}\psi_c(x,t;c) &= \sum_{n=1}^{+\infty}\left[F_U^{(n)}(x)-F_U^{(n+1)}(x)\right]f_D^{(n)}(t-x),\\ \psi_c(x,t;-v) &= \sum_{n=0}^{+\infty}\left[F_D^{(n)}(t-x)-F_D^{(n+1)}(t-x)\right]f_U^{(n+1)}(x),\end{aligned}\tag{10}$$

and, by symmetry,

$$\begin{aligned}\psi_{-v}(x,t;c) &= \sum_{n=0}^{+\infty}\left[F_U^{(n)}(x)-F_U^{(n+1)}(x)\right]f_D^{(n+1)}(t-x),\\ \psi_{-v}(x,t;-v) &= \sum_{n=1}^{+\infty}\left[F_D^{(n)}(t-x)-F_D^{(n+1)}(t-x)\right]f_U^{(n)}(x).\end{aligned}\tag{11}$$

Proof Equalities (8) can be obtained from Eqs. 2 and 5. Density (9) immediately follows from definition (7). The proof of Eqs. 10 and 11 can be stated by resorting to the analysis of suitable compound counting processes. It is omitted, since it goes along the line of similar results shown in Zacks (2012) or Bshouty et al. (2012). □

Notice that the terms appearing in the right-hand-sides of Eqs. 10 and 11 have a probabilistic meaning. For instance the term $\left[F_U^{(n)}(x)-F_U^{(n+1)}(x)\right]f_D^{(n)}(t-x)$, appearing in the first of Eq. 10, refers to the $(n+1)$-th time period (having duration U_{n+1} and elapsing at time t) in which the telegraph process $Y(t)$ has positive velocity, when $V(0)=c$. Indeed,

$$F_U^{(n)}(x)-F_U^{(n+1)}(x)=\mathbb{P}[U_1+\ldots+U_n<x\le U_1+\ldots+U_{n+1}]$$

is the probability that the first n upward periods of $Y(t)$ plus the portion of the $(n+1)$-th upward period contained in $[0,t]$ have total duration x, whereas $f_D^{(n)}(t-x)$ is the density that the first n downward periods of $Y(t)$ have total duration $t-x$, for $0<x<t$.

Remark 3.1 From Theorem 3.1 it is not hard to see that the probability mass of density $\psi(x,t)$ is given by the following mixture:

$$P(t):=\mathbb{P}\{0<W(t)<t\}=\int_0^t\psi(x,t)\,\mathrm{d}x=\frac{1}{2}\left[F_{D_1}(t)+F_{U_1}(t)\right],\qquad t\ge 0.\tag{12}$$

Moreover, the following symmetry property holds:

$$\mathbb{P}[W(t)\le x]\big|_{\{(U^{(n)},D^{(n)});\,n\ge1\}}=\mathbb{P}[W(t)\ge t-x]\big|_{\{(D^{(n)},U^{(n)});\,n\ge1\}}\qquad\text{for all } 0\le x\le t.\tag{13}$$

In particular, from Eq. 13 it follows that interchanging x with $t-x$ and the distribution of $U^{(n)}$ with that of $D^{(n)}$, for all $n\ge1$, the expression of $\psi(x,t)$ remains unchanged.

Let us now come to the mean and the variance of occupation time.

Corollary 3.1 *For all* $t>0$ *we have*

$$\mathbb{E}[W(t)]=\frac{1}{2}\int_0^t\left[\overline{\Psi}_c(x,t)+\overline{\Psi}_{-v}(x,t)\right]\mathrm{d}x\tag{14}$$

and

$$\mathbb{V}\mathrm{ar}[W(t)] = \int_0^t x \left[\overline{\Psi}_c(x,t) + \overline{\Psi}_{-v}(x,t)\right] \mathrm{d}x - \{\mathbb{E}[W(t)]\}^2, \tag{15}$$

where, for $0 < x < t$*,*

$$\overline{\Psi}_c(x,t) := \mathbb{P}[W(t) > x \mid V(0) = c] = \sum_{n=0}^{+\infty} \left[F_U^{(n)}(x) - F_U^{(n+1)}(x)\right] F_D^{(n)}(t-x), \tag{16}$$

$$\overline{\Psi}_{-v}(x,t) := \mathbb{P}[W(t) > x \mid V(0) = -v] = \sum_{n=0}^{+\infty} \left[F_D^{(n)}(t-x) - F_D^{(n+1)}(t-x)\right] F_U^{(n)}(x). \tag{17}$$

Proof It follows from Theorem 3.1 by straightforward calculations. □

4 Distribution of $X(t)$

Before obtaining the distribution of process (1), we first express the probability law of the generalized telegraph process $Y(t)$ in terms of that of $W(t)$. We recall that (5) is the fraction of time that the motion moved with positive drift in $[0, t]$. Note that $-vt \le Y(t) \le ct$, where $Y(t)$ has a discrete component on the points $-vt$ and ct for any fixed $t > 0$. The absolutely continuous component over $(-vt, ct)$ is described by the following density:

$$g(x,t) := \frac{\partial}{\partial x}\mathbb{P}\{Y(t) \le x\}, \qquad -vt < x < ct, \quad t > 0. \tag{18}$$

Theorem 4.1 *For all* $t > 0$ *we have*

$$\mathbb{P}[Y(t) = -vt] = \frac{1}{2}\overline{F}_{D_1}(t), \qquad \mathbb{P}[Y(t) = ct] = \frac{1}{2}\overline{F}_{U_1}(t). \tag{19}$$

Moreover, the density (18) *is given by*

$$g(x,t) = \frac{1}{c+v}\,\psi\left(\frac{x+vt}{c+v}, t\right), \qquad -vt < x < ct, \tag{20}$$

where an expression of $\psi(x,t)$ *has been obtained in Theorem 3.1.*

Proof Due to Eqs. 3 and 5, the process $Y(t)$ is related to the occupation time by

$$Y(t) = (c+v)W(t) - vt, \qquad t \ge 0. \tag{21}$$

The proof immediately follows. □

Equation 20 allows to express the probability density of the generalized telegraph process $Y(t)$ in terms of the occupation time density obtained in Eq. 9.

We are now able to obtain the distribution of process (1). Let

$$p(x,t) = \frac{\partial}{\partial x}\mathbb{P}\{X(t) \le x\}, \qquad x \in \mathbb{R}, \quad t > 0 \tag{22}$$

be the probability density of $X(t)$ at time t. In the following, as usual, $\Phi(\cdot)$ denotes the standard normal distribution, and $\phi(\cdot)$ denotes the corresponding density.

Theorem 4.2 *For all $t > 0$ and $x \in \mathbb{R}$ we have*

$$p(x,t) = \frac{1}{\sqrt{t}}\left[\frac{1}{2}\,\overline{F}_{U_1}(t)\,\phi\left(\frac{x-ct}{\sqrt{t}}\right) + \frac{1}{2}\,\overline{F}_{D_1}(t)\,\phi\left(\frac{x+vt}{\sqrt{t}}\right) + \int_0^t \psi(w,t)\,\phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)\mathrm{d}w\right], \tag{23}$$

where $\psi(x,t)$ is given in Eq. 9.

Proof Since the Brownian motion is a process of independent increments (a Lévy process), and since $Y(t)$ is independent of $B(t)$, the conditional distribution of $X(t)$, given $W(t)$, due to Eqs. 1 and 21, is

$$[X(t) \mid W(t)] \sim (c+v)W(t) - vt + \sqrt{W(t)}\,Z_1 + \sqrt{t-W(t)}\,Z_2, \qquad t>0,$$

where Z_1 and Z_2 are independent standard normal random variables. Moreover,

$$\left[\sqrt{W(t)}\,Z_1 + \sqrt{t-W(t)}\,Z_2 \,\middle|\, W(t)\right] \sim \sqrt{t}\,Z, \qquad t>0,$$

where Z has a standard normal distribution and is independent of $W(t)$. Hence, we have

$$[X(t) \mid W(t)] \sim (c+v)\,W(t) - vt + \sqrt{t}\,Z, \qquad t>0, \tag{24}$$

so that the distribution function of $X(t)$ conditional on $W(t)$ is given by

$$\mathbb{P}\,[X(t) \le x \mid W(t)] = \Phi\left(\frac{x+vt-(c+v)\,W(t)}{\sqrt{t}}\right), \qquad t>0. \tag{25}$$

The distribution of $X(t)$ is thus given by the mean of Eq. 25 conditional on the distribution of $W(t)$. Hence, recalling Theorem 3.1, from Eq. 25 we get

$$\mathbb{P}[X(t) \le x] = \frac{1}{2}\,\overline{F}_{U_1}(t)\,\Phi\left(\frac{x-ct}{\sqrt{t}}\right) + \frac{1}{2}\,\overline{F}_{D_1}(t)\,\Phi\left(\frac{x+vt}{\sqrt{t}}\right) + \int_0^t \psi(w,t)\,\Phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)\mathrm{d}w, \qquad t>0.$$

Equation 23 thus follows by differentiation. □

Remark 4.1 From Remark 3.1 and Theorem 4.2 we have the following symmetry property:

$$p(x,t)\big|_{\{(U^{(n)},D^{(n)});\,n\ge1\}} = p((c-v)t-x,t)\big|_{\{(D^{(n)},U^{(n)});\,n\ge1\}} \qquad \text{for all } 0 \le x \le t. \tag{26}$$

The expected value and the variance of $X(t)$ can be found directly from Eq. 24. Indeed, the following result holds.

Corollary 4.1 *For all $t > 0$ we have*

$$\mathbb{E}[X(t)] = (c+v)\,\mathbb{E}[W(t)] - vt, \tag{27}$$

and

$$\mathbb{V}\mathrm{ar}[X(t)] = (c+v)^2\,\mathbb{V}\mathrm{ar}[W(t)] + t, \tag{28}$$

where $\mathbb{E}[W(t)]$ and $\mathbb{V}\mathrm{ar}[W(t)]$ can be evaluated by using Eqs. 14 *and* 15.

We conclude this section by introducing the *flow function* of $X(t)$, which is defined as

$$j(x,t) := f(x,t) - b(x,t), \qquad x \in \mathbb{R}, \quad t>0. \tag{29}$$

Here $f(x,t)$ and $b(x,t)$ denote the probability densities of the motion with *forward* and *backward* drift, respectively, and are given by

$$f(x,t) = \frac{\partial}{\partial x}\mathbb{P}\{X(t) \le x, V(t) = c\}, \qquad b(x,t) = \frac{\partial}{\partial x}\mathbb{P}\{X(t) \le x, V(t) = -v\}, \quad (30)$$

for $x \in \mathbb{R}$ and $t > 0$. Similarly as for the telegraph process (see Orsingher (1990)), in a large ensamble of particles moving as $X(t)$, function (29) measures, at each time t, the excess of particles moving with drift c with respect to the ones moving with drift $-v$ near point x. Let us now obtain the expressions of densities (30).

Corollary 4.2 *For all $t > 0$ and $x \in \mathbb{R}$ we have*

$$\begin{aligned} f(x,t) &= \frac{1}{\sqrt{t}}\left[\frac{1}{2}\overline{F}_{U_1}(t)\,\phi\left(\frac{x-ct}{\sqrt{t}}\right) + \int_0^t \psi(w,t;c)\,\phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)\mathrm{d}w\right], \\ b(x,t) &= \frac{1}{\sqrt{t}}\left[\frac{1}{2}\overline{F}_{D_1}(t)\,\phi\left(\frac{x+vt}{\sqrt{t}}\right) + \int_0^t \psi(w,t;-v)\,\phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)\mathrm{d}w\right], \end{aligned} \quad (31)$$

where densities $\psi(x,t;c)$ and $\psi(x,t;-v)$ are defined in Eq. 7*, and can be evaluated by means of* Eqs. 10 *and* 11.

Proof Expressions (31) easily follow from a reasoning similar to the proof of Theorem 4.2. □

5 Special Cases

In this section we analyze some special cases, in which the distributions of the random times U_i and D_i are specified explicitly. Note that in Sections 5.1 and 5.2 the random variables U_i (D_i), $i = 1, 2, \ldots$, are i.i.d., whereas in Section 5.3 they are not identically distributed.

5.1 Exponential Times with Constant Rates

Some experimental studies, as those shown in Berg and Brown (1972) on Escherichia *coli* bacteria, show that the motion of certain micro-organisms can be approximated by random trajectories which change directions at exponentially distributed random times. This suggests to investigate the Brownian motion governed by telegraph process in the special case when the random alternating times are exponentially distributed.

In this section we thus assume that U_i and D_i, for all $i = 1, 2, \ldots$, are exponentially distributed with parameter $\lambda > 0$ and $\mu > 0$, respectively. Under these assumptions we first show that the probability densities $f(x,t)$ and $b(x,t)$ are solution of a differential system.

Theorem 5.1 *For all $i = 1, 2, \ldots$, let U_i and D_i be exponentially distributed with parameter λ and μ, respectively. Then, for $x \in \mathbb{R}$ and $t > 0$ the densities defined in* Eq. 30 *satisfy the differential system*

$$\begin{cases} \dfrac{\partial f}{\partial t} = -c\,\dfrac{\partial f}{\partial x} + \dfrac{1}{2}\,\dfrac{\partial^2 f}{\partial x^2} - \lambda\, f + \mu\, b, \\ \dfrac{\partial b}{\partial t} = v\,\dfrac{\partial b}{\partial x} + \dfrac{1}{2}\,\dfrac{\partial^2 b}{\partial x^2} - \mu\, b + \lambda\, f. \end{cases} \quad (32)$$

The proof of Theorem 5.1 is omitted, since it follows by typical arguments, by which $X(t)$ is viewed as the suitable limit of a random walk with drift-dependent step probabilities. For instance, we refer to Orsingher (1985) and Ratanov (1999), where a limit procedure is employed to obtain differential equations of finite-velocity random motions. Moreover, it is interesting to note the analogies between system (32) and system (2.4) of Ratanov (2010), which is satisfied by the (generalized) probability densities of a jump telegraph-diffusion process.

Corollary 5.1 *Under the assumptions of Theorem 5.1, for $x \in \mathbb{R}$ and $t > 0$ the following differential system holds:*

$$\begin{cases} \dfrac{\partial p}{\partial t} = -\dfrac{c-v}{2}\dfrac{\partial p}{\partial x} - \dfrac{c+v}{2}\dfrac{\partial j}{\partial x} + \dfrac{1}{2}\dfrac{\partial^2 p}{\partial x^2}, \\ \dfrac{\partial j}{\partial t} = -\dfrac{c+v}{2}\dfrac{\partial p}{\partial x} - \dfrac{c-v}{2}\dfrac{\partial j}{\partial x} - (\lambda-\mu)\,p - (\lambda+\mu)\,j, \end{cases} \tag{33}$$

where $p(x,t)$ and $j(x,t)$ are defined in Eqs. 22 *and* 29, *respectively.*

Proof Making use of Eq. 29 and noting that density (22) is related to functions (30) by $p(x,t) = f(x,t) + b(x,t)$, $x \in \mathbb{R}$, $t > 0$, system (33) follows from Eq. 32. □

It is worth noting that if $c = v = 0$ then system (33) yields the classical heat equation of the standard Brownian motion

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2},$$

whereas if the diffusive term $\frac{1}{2}\frac{\partial^2 p}{\partial x^2}$ is neglected then (33) becomes the differential system that governs the telegraph process with drift (see Eq. 3.2 of Beghin et al. (2001)).

In the following we resort to Theorem 4.2 to obtain the distribution of $W(t)$, the density $p(x,t)$ and the flow function $j(x,t)$ in the present case. We recall that since U_i and D_i, $i = 1, 2, \ldots$, are exponentially distributed with parameter λ and μ, respectively, then the sums (4) have Erlang distribution with parameters (λ, n) and (μ, n). Hence,

$$f_U^{(n)}(x) = \frac{\lambda^n x^{n-1}}{(n-1)!}\,\mathrm{e}^{-\lambda x}, \qquad F_U^{(n)}(x) = 1 - \mathrm{e}^{-\lambda x}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}, \qquad x \ge 0, \tag{34}$$

while $f_D^{(n)}(x)$ and $F_D^{(n)}(x)$ are defined analogously, with parameter μ.

Proposition 5.1 *If U_i and D_i, $i = 1, 2, \ldots$, are exponentially distributed with parameter λ and μ, respectively, then*

$$\mathbb{P}[W(t) = 0] = \frac{1}{2}\,\mathrm{e}^{-\mu t}, \qquad \mathbb{P}[W(t) = t] = \frac{1}{2}\,\mathrm{e}^{-\lambda t}, \qquad t > 0, \tag{35}$$

and

$$\psi(x,t) = \frac{\mathrm{e}^{-\lambda x - \mu(t-x)}}{2}\left\{(\lambda+\mu)\,I_0\Big[2\sqrt{\lambda\mu x(t-x)}\Big] + \frac{\sqrt{\lambda\mu}\,t}{\sqrt{x(t-x)}}\,I_1\Big[2\sqrt{\lambda\mu x(t-x)}\Big]\right\}, \tag{36}$$

for $0 < x < t$, where

$$I_n(z) = \sum_{k=0}^{+\infty}\frac{(z/2)^{2k+n}}{k!\,(k+n)!}, \qquad n = 0, 1, \ldots,$$

is the modified Bessel function. Moreover, for all $t > 0$ *and* $x \in \mathbb{R}$ *we have*

$$p(x,t) = \frac{1}{\sqrt{t}}\Bigg[\frac{1}{2}\,\mathrm{e}^{-\lambda t}\,\phi\left(\frac{x-ct}{\sqrt{t}}\right) + \frac{1}{2}\,\mathrm{e}^{-\mu t}\,\phi\left(\frac{x+vt}{\sqrt{t}}\right) + \int_0^t \psi(w,t)\,\phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)\mathrm{d}w\Bigg], \tag{37}$$

with $\psi(x,t)$ *given in* Eq. 36. *The flow function* (29) *is given by*

$$j(x,t) = \frac{1}{\sqrt{t}}\Bigg[\frac{1}{2}\,\mathrm{e}^{-\lambda t}\,\phi\left(\frac{x-ct}{\sqrt{t}}\right) - \frac{1}{2}\,\mathrm{e}^{-\mu t}\,\phi\left(\frac{x+vt}{\sqrt{t}}\right) + \int_0^t [\psi(w,t;c)-\psi(w,t;-v)]\,\phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)\mathrm{d}w\Bigg], \tag{38}$$

where

$$\psi(w,t;c)-\psi(w,t;-v) = \frac{\mathrm{e}^{-\lambda w-\mu(t-w)}}{2}\Bigg\{(\mu-\lambda)\,I_0\Big[2\sqrt{\lambda\mu w(t-w)}\Big] + \frac{\sqrt{\lambda\mu}\,(2w-t)}{\sqrt{w(t-w)}}\,I_1\Big[2\sqrt{\lambda\mu w(t-w)}\Big]\Bigg\}. \tag{39}$$

Proof Equation 35 immediately follows from Eq. 8. Under the given assumptions, from Eqs. 10 and 34 after some calculations we obtain

$$\psi_c(x,t;c) = \mathrm{e}^{-\lambda x-\mu(t-x)}\,\frac{\sqrt{\lambda\mu x}}{\sqrt{t-x}}\,I_1\Big[2\sqrt{\lambda\mu x(t-x)}\Big], \qquad 0<x<t,$$

$$\psi_c(x,t;-v) = \lambda\mathrm{e}^{-\lambda x-\mu(t-x)}\,I_0\Big[2\sqrt{\lambda\mu x(t-x)}\Big], \qquad 0<x<t.$$

Recalling Theorem 3.1, density $\psi_{-v}(x,t;c)$ can be obtained from $\psi_c(x,t;-v)$ by interchanging λ with μ, and x with $t-x$. Clearly, $\psi_{-v}(x,t;-v)$ follows from $\psi_c(x,t;c)$ in the same way. Density (36) thus can be obtained by making use of Eq. 9. Moreover, Eq. 37 holds by virtue of Eq. 23. Finally, the right-hand-side of Eq. 38 is due to Eqs. 31 and 29, whereas Eq. 39 follows from the densities evaluated in the first part of the proof. □

We remark the analogies between the results given in Proposition 5.1 and those provided in Section 2 of Ratanov (2010) for a jump telegraph-diffusion process. Even though closed-form expressions of density (37) and flow function (38) seem not feasible, some plots of $p(x,t)$ and $j(x,t)$ have been obtained via numerical computation (see Figs. 3 and 4, respectively).

Remark 5.1 An alternative formula to density (39) can be established as follows. In the present case, according to the gamma-Poisson relationship, the functions given in Eq. 34 can be expressed as

$$f_U^{(n)}(x) = \lambda\,p(n-1;\lambda x), \qquad F_U^{(n)}(x) = 1-P(n-1;\lambda x), \qquad x\ge 0, \tag{40}$$

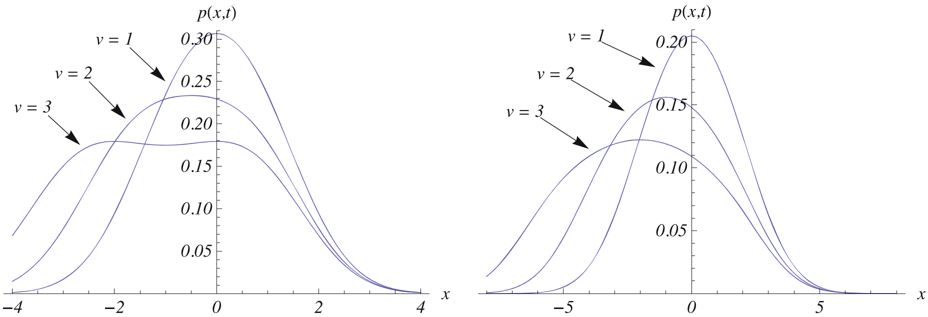


Fig. 3 Density (37) for $\lambda = \mu = 1$, $c = 1$ and $v = 1, 2, 3$, for $t = 1$ (*left*) and $t = 2$ (*right*)

where $p(j;\eta)$ and $P(j;\eta)$ denote respectively the Poisson p.d.f. and c.d.f. with mean η evaluated at j. Hence, from Theorem 3.1 after few calculations we obtain:

$$\psi(x,t) = \frac{1}{2}\sum_{n=0}^{+\infty}\Big\{\lambda\, p(n;\lambda x)[p(n;\mu(t-x)) + p(n+1;\mu(t-x))] \\ + \mu\, p(n;\mu(t-x))[p(n;\lambda x) + p(n+1;\lambda x)]\Big\}, \qquad x \geq 0.$$

Let us conclude this section by evaluating the mean of $W(t)$.

Proposition 5.2 *Under the assumptions of Proposition 5.1, we have*

$$E[W(t)] = \frac{\mu}{\lambda+\mu}t + \frac{\lambda-\mu}{2(\lambda+\mu)^2}\left(1 - \mathrm{e}^{-(\lambda+\mu)t}\right), \qquad t \geq 0. \tag{41}$$

Proof Consider the second identity of Corollary 1 of Ratanov (2007) for $s = 0$, and for $(c_-, \lambda_-) = (-v, \mu)$ and $(c_+, \lambda_+) = (c, \lambda)$. By combining the two cases concerning the initial velocity with equal probability, and recalling relation (21) we obtain Eq. 41. □

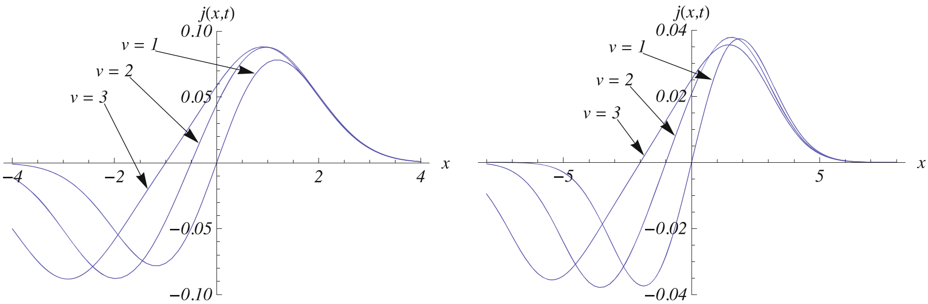


Fig. 4 Flow function (38) for the same cases of Fig. 3

5.2 Erlang Distributed Random Times

Let the random times U_i and D_i, $i = 1, 2, \ldots$, have Erlang distribution with parameters (λ, k) and (μ, r), respectively. Hence, the sums (4) are Erlang-distributed with parameters (λ, nk) and (μ, nr), so that

$$f_U^{(n)}(x) = \frac{\lambda^{nk} x^{nk-1}}{(nk-1)!} \mathrm{e}^{-\lambda x}, \qquad F_U^{(n)}(x) = 1 - \mathrm{e}^{-\lambda x} \sum_{j=0}^{nk-1} \frac{(\lambda x)^j}{j!}, \qquad x \geq 0, \tag{42}$$

with $f_D^{(n)}(x)$ and $F_D^{(n)}(x)$ having similar expressions. In the forthcoming proposition we express the occupation time density in terms of the following two-index pseudo-Bessel function (see Di Crescenzo (2001) for its use in the description of finite-velocity random motions with Erlang-distributed random times):

$$S_{i,j}^{(k,r)}(x, y) = \sum_{\ell=0}^{+\infty} \frac{x^{k\ell+i}\, y^{r\ell+j}}{(k\ell+i)!\,(r\ell+j)!}, \tag{43}$$

defined for integers $k, r \geq 1$ and $i, j \geq 0$.

Proposition 5.3 *If U_i and D_i have Erlang distribution with parameters (λ, k) and (μ, r), respectively, then for all $t > 0$*

$$\mathbb{P}[W(t) = 0] = \frac{1}{2}\mathrm{e}^{-\mu t} \sum_{j=0}^{r-1} \frac{(\mu t)^j}{j!}, \qquad \mathbb{P}[W(t) = t] = \frac{1}{2}\mathrm{e}^{-\lambda t} \sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!},$$

and, for $0 < x < t$,

$$\psi(x, t) = \frac{\mathrm{e}^{-\lambda x - \mu(t-x)}}{2} \left[\mu \sum_{j=0}^{2k-1} S_{j,r-1}^{(k,r)}(\lambda x, \mu(t-x)) + \lambda \sum_{j=0}^{2r-1} S_{k-1,j}^{(k,r)}(\lambda x, \mu(t-x)) \right]. \tag{44}$$

Moreover, for all $t > 0$ and $x \in \mathbb{R}$ we have

$$\begin{aligned} p(x, t) = \frac{1}{\sqrt{t}} \Bigg[& \frac{1}{2}\, \mathrm{e}^{-\lambda t} \sum_{j=0}^{k-1} \frac{(\lambda t)^j}{j!}\, \phi\left(\frac{x - ct}{\sqrt{t}}\right) + \frac{1}{2}\, \mathrm{e}^{-\mu t} \sum_{j=0}^{r-1} \frac{(\mu t)^j}{j!}\, \phi\left(\frac{x + vt}{\sqrt{t}}\right) \\ & + \int_0^t \psi(w, t)\, \phi\left(\frac{x + vt - (c+v)w}{\sqrt{t}}\right) \mathrm{d}w \Bigg], \end{aligned} \tag{45}$$

with $\psi(x, t)$ given in Eq. 44.

Proof The first result is due to Eq. 8. From Eq. 42 we get

$$\left[F_U^{(n)}(x) - F_U^{(n+1)}(x) \right] f_D^{(n)}(t - x) = \mu \mathrm{e}^{-\lambda x - \mu(t-x)} \sum_{j=nk}^{nk+k-1} \frac{(\lambda x)^j}{j!} \frac{(\mu(t-x))^{nr-1}}{(nr-1)!}.$$

The first density of Eq. 10 thus becomes

$$\psi_c(x, t; c) = \mu \mathrm{e}^{-\lambda x - \mu(t-x)} \sum_{j=0}^{k-1} S_{k+j,r-1}^{(k,r)}(\lambda x, \mu(t - x)), \qquad 0 < x < t,$$

where $S_{i,j}^{(k,r)}(x, y)$ is defined in Eq. 43. Analogously, we have

$$\psi_c(x, t; -v) = \lambda e^{-\lambda x - \mu(t-x)} \sum_{j=0}^{r-1} S_{k-1,j}^{(k,r)}(\lambda x, \mu(t-x)), \qquad 0 < x < t.$$

Due to Theorem 3.1, density $\psi_{-v}(x, t; c)$ can be obtained from $\psi_c(x, t; -v)$ by interchanging λ with μ, k with r, and x with $t - x$. Similarly, $\psi_{-v}(x, t; -v)$ follows from $\psi_c(x, t; c)$. Hence, recalling (9), and making use of the symmetry property $S_{i,j}^{(k,r)}(x, y) = S_{j,i}^{(r,k)}(y, x)$, we obtain the density (44). In conclusion, from Eq. 23 we have Eq. 45. □

As for the case of Section 5.1, it seems that density (45) cannot be expressed in closed form. Nevertheless in Fig. 5 we give various plots of $p(x, t)$, obtained by means of numerical computations.

5.3 Exponential Times with Linear Rates

We now assume that the random times U_i and D_i, $i = 1, 2, \ldots$, have exponential distribution with linear parameters λi and μi, respectively, so that they are not identically distributed. A telegraph process characterized by such random times and its geometric version have been studied by Di Crescenzo and Martinucci (2010) and Di Crescenzo et al. (2012), respectively. In this case the sums (4) have generalized exponential distribution, with

$$f_U^{(n)}(x) = n \left(1 - e^{-\lambda x}\right)^{n-1} \lambda e^{-\lambda x}, \qquad F_U^{(n)}(x) = \left(1 - e^{-\lambda x}\right)^n, \qquad x \geq 0. \tag{46}$$

Functions $f_D^{(n)}(x)$ and $F_D^{(n)}(x)$ have similar expressions, with parameter μ. We stress the damped nature of the random times, since in the present case U_i and D_i are stochastically decreasing in $i \geq 1$. Hence, under the above assumptions X_t changes drift faster and faster as time proceeds.

Proposition 5.4 *If U_i and D_i are exponentially distributed with parameter λi and μi, $i = 1, 2, \ldots$, respectively, then*

$$\mathbb{P}[W(t) = 0] = \frac{1}{2} e^{-\mu t}, \qquad \mathbb{P}[W(t) = t] = \frac{1}{2} e^{-\lambda t}, \qquad t > 0,$$

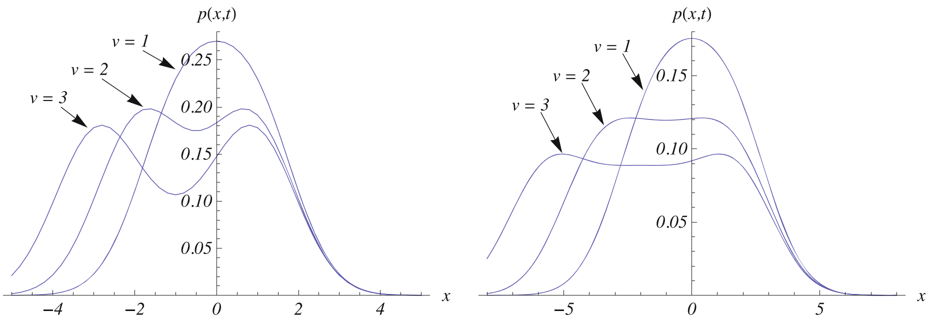


Fig. 5 Density (45) for $\lambda = \mu = 1$, $k = r = 2$, $c = 1$ and $v = 1, 2, 3$, for $t = 1$ (*left*) and $t = 2$ (*right*)

and

$$\psi(x,t) = \mathrm{e}^{-\lambda x-\mu(t-x)}\,\frac{\big(2(\lambda+\mu)-\mu\mathrm{e}^{-\lambda x}-\lambda\mathrm{e}^{-\mu(t-x)}\big)}{2[1-(1-\mathrm{e}^{-\lambda x})(1-\mathrm{e}^{-\mu(t-x)})]^2}, \qquad 0<x<t. \tag{47}$$

Furthermore, for all $t>0$ and $x\in\mathbb{R}$ we have

$$\begin{aligned} p(x,t) = \frac{1}{\sqrt{t}}\bigg[&\frac{1}{2}\mathrm{e}^{-\lambda t}\,\phi\left(\frac{x-ct}{\sqrt{t}}\right)+\frac{1}{2}\mathrm{e}^{-\mu t}\,\phi\left(\frac{x+vt}{\sqrt{t}}\right)\\ &+\int_0^t \psi(w,t)\,\phi\left(\frac{x+vt-(c+v)w}{\sqrt{t}}\right)\mathrm{d}w\bigg], \end{aligned} \tag{48}$$

where $\psi(x,t)$ is given in Eq. 47.

Proof The discrete component of the probability law of $W(t)$ follows from Eq. 8. From Eq. 10 we have

$$\psi_c(x,t;c) = \mu\mathrm{e}^{-\lambda x-\mu(t-x)}\,\frac{1-\mathrm{e}^{-\lambda x}}{[1-(1-\mathrm{e}^{-\lambda x})(1-\mathrm{e}^{-\mu(t-x)})]^2}, \qquad 0<x<t,$$

$$\psi_c(x,t;-v) = \lambda\mathrm{e}^{-\lambda x-\mu(t-x)}\,\frac{1}{[1-(1-\mathrm{e}^{-\lambda x})(1-\mathrm{e}^{-\mu(t-x)})]^2}, \qquad 0<x<t.$$

Due to Theorem 3.1, density $\psi_{-v}(x,t;c)$ can be obtained from $\psi_c(x,t;-v)$ by interchanging λ with μ, k with r, and x with $t-x$. In a similar way $\psi_{-v}(x,t;-v)$ follows from $\psi_c(x,t;c)$. Recalling (9), we get the density (47). Finally, from Eq. 23 we have Eq. 48. □

Figure 6 shows some plots of density (48) obtained via suitable numerical computations.

Remark 5.2 According to Remark 3.1, it is easy to check the validity of the symmetry property

$$\psi(x,t)\big|_{(\lambda,\mu)} = \psi(t-x,t)\big|_{(\mu,\lambda)} \qquad \text{for all } 0\le x\le t$$

for each of the three cases considered in this section, i.e. for densities Eqs. 36, 44 and 47. Similarly, due to Remark 4.1, the following symmetry property holds true:

$$p(x,t)\big|_{(\lambda,\mu)} = p((c-v)t-x,t)\big|_{(\mu,\lambda)} \qquad \text{for all } 0\le x\le t,$$

and is satisfied by densities Eqs. 37, 45 and 48.

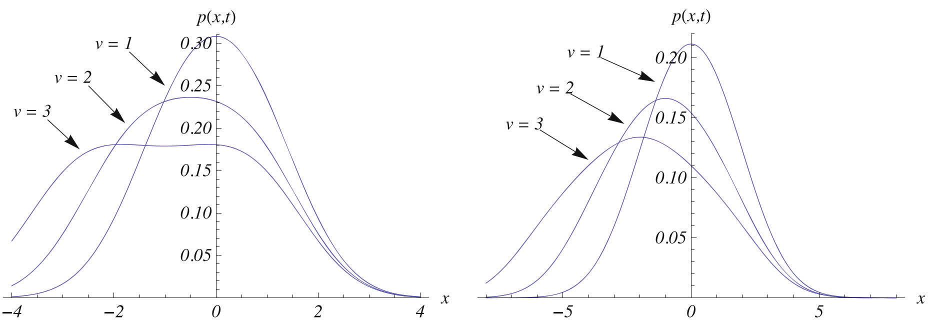


Fig. 6 Density (48) for $\lambda=\mu=1$, $c=1$ and $v=1,2,3$, for $t=1$ (*left*) and $t=2$ (*right*)

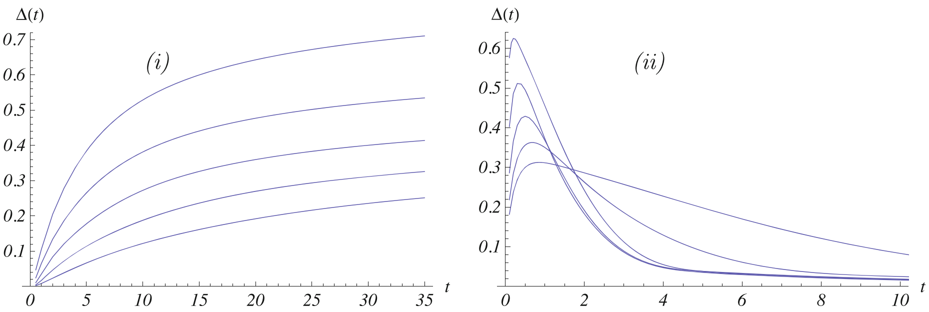


Fig. 7 Plots of $\Delta(t)$, given in Eq. 49, for the 2 cases described in the text; in both cases $\lambda = 1$ and $\mu = 0.25$, 0.5, 1, 2, 4 (from *bottom* to *top* near the origin)

5.4 Some Comparisons

The similarity exhibited by the plots of Figs. 3, 5 and 6 suggests the existence of certain analogies among the three cases. However, density $p(x, t)$ is shown only for few instances of the involved parameters, and thus the figures are uneffective to find out the closeness among the three considered models. Hence, looking for a more appropriate comparison we consider the following measure of closeness:

$$\Delta(t) = d_{(2)}(\psi_1(t), \psi_2(t)) = \left\{ \int_0^t |\psi_1(x) - \psi_2(x)|^2 \, dx \right\}^{1/2}, \qquad t > 0. \tag{49}$$

The right-hand-side of Eq. 49 is the L^2-distance of $\psi_1(x)$ and $\psi_2(x)$, defined on $(0, t)$. Figure 7 shows some plots of $\Delta(t)$ obtained via numerical evaluations, in 2 cases where ψ_1 and ψ_2 are respectively the densities

(*i*) Equations 36 and 47, with random times having exponential distribution with constant rates, and linear rates;
(*ii*) Equations 36 and 44, when the random times have exponential distribution with constant rates, and Erlang distribution.

In case (*i*) the measure $\Delta(t)$ is increasing in t, see Fig. 7 (*i*), so that the sojourn times of the two models become remarkable different when t grows, and when μ grows. This is a consequence of the significant discrepancy between the two schemes with random times having exponential distribution with constant and linear rates, the latter case yielding a damped behavior.

On the contrary, in case (*ii*) the measure $\Delta(t)$ is definitely decreasing and vanishing as t grows, see Fig. 7 (*ii*). This shows that when t is large the sojourn times of the two models are very similar. This is intuitively justified by the fact that the Erlang distribution is the sum of i.i.d. exponential distributions, and thus the distribution of the sojourn times in both models is a suitable combination of exponential distributions in which the constant parameters play a similar role.

6 Application in Environmental Sciences

In this section we shall indicate an application of the foregoing results in environmental sciences.

Freidlin and Pavlopoulos (1997) proposed a stochastic model aimed to describe the temporal evolution of the moisture content in a given oceanic atmospheric column. In order to take into account the intermittence of rainfall and the dynamics of moisture processes, the authors considered the alternation of two Wiener processes with drift of opposite signs and unequal diffusion coefficients, the alternation being governed by the attainment of certain thresholds. Such model was suitably approximated in Di Crescenzo et al. (2005) by a Wiener process with alternating drift and infinitesimal variance. We point out that the above results on process (1) can be suitably extended to such a case. In particular, similarly to Theorem 4.2 it can be shown that the probability density of a Wiener process with alternating infinitesimal moments $\{c, \sigma_1^2\}$ and $\{-v, \sigma_2^2\}$, for all $x \in \mathbb{R}$ and $t > 0$, is given by

$$p(x,t) = \frac{1}{2\sqrt{t}}\left[\overline{F}_{U_1}(t)\,\frac{1}{\sigma_1}\,\phi\left(\frac{x-ct}{\sigma_1\sqrt{t}}\right) + \overline{F}_{D_1}(t)\,\frac{1}{\sigma_2}\,\phi\left(\frac{x+vt}{\sigma_2\sqrt{t}}\right)\right] + \int_0^t \psi(w,t)\,\frac{1}{\sqrt{\sigma_1^2 w + \sigma_2^2(t-w)}}\,\phi\left(\frac{x-cw+v(t-w)}{\sqrt{\sigma_1^2 w + \sigma_2^2(t-w)}}\right)\mathrm{d}w, \quad (50)$$

where $\psi(x,t)$ is the density of the occupation time given in Eq. 9, and $\phi(\cdot)$ is the standard normal density.

According to Di Crescenzo et al. (2005) we assume that the random times between consecutive infinitesimal moments' changes follow the inverse Gaussian distribution, this being the distribution of the first-passage time of a Wiener process through a constant boundary. We recall that a random variable having inverse Gaussian distribution $IG(\mu, \lambda)$, with mean $\mu > 0$ and shape parameter $\lambda > 0$, has density

$$f_{IG}(x;\mu,\lambda) := \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \qquad x > 0,$$

and distribution function

$$F_{IG}(x;\mu,\lambda) := \Phi\left(\left(\frac{\lambda}{x}\right)^{1/2}\left(\frac{x}{\mu}-1\right)\right) + \exp\left\{\frac{2\lambda}{\mu}\right\}\Phi\left(-\left(\frac{\lambda}{x}\right)^{1/2}\left(\frac{x}{\mu}+1\right)\right), \qquad x > 0,$$

where $\Phi(\cdot)$ is the standard normal distribution function. We now aim to investigate process (1) when the random times U_i and D_i, $i = 1, 2, \ldots$, have inverse Gaussian distribution $IG(m_1, l_1)$ and $IG(m_2, l_2)$, respectively. Since the sequences $\{U_i\}$ and $\{D_i\}$ are constituted by independent random variables, from a well-known property of inverse Gaussian distributions we have that the random variables (4) are distributed as $U^{(n)} \sim IG(nm_1, n^2 l_1)$ and $D^{(n)} \sim IG(nm_2, n^2 l_2)$, respectively, so that:

$$\begin{aligned} f_U^{(n)}(x) &= f_{IG}(x; nm_1, n^2 l_1), & f_D^{(n)}(x) &= f_{IG}(x; nm_2, n^2 l_2), \\ F_U^{(n)}(x) &= F_{IG}(x; nm_1, n^2 l_1), & F_D^{(n)}(x) &= F_{IG}(x; nm_2, n^2 l_2). \end{aligned} \quad (51)$$

We consider the following choices of the involved parameters, as suggested by the estimates obtained in Freidlin and Pavlopoulos (1997) via the method of moments:

$$\begin{aligned} &l_1 = 0.7518, \quad m_1 = 1.4215, \quad l_2 = 0.5073, \quad m_2 = 1.0476, \\ &c = 0.2313, \quad \sigma_1 = 0.3792, \quad v = 0.3139, \quad \sigma_2 = 0.4616. \end{aligned} \quad (52)$$

For the considered model, the density of the occupation time seems not obtainable in closed form. However, it can be numerically evaluated via Theorem 3.1, due to the relations given in Eq. 51. As example, Fig. 8 shows four instances of $\psi(x,t)$ for different times t. The

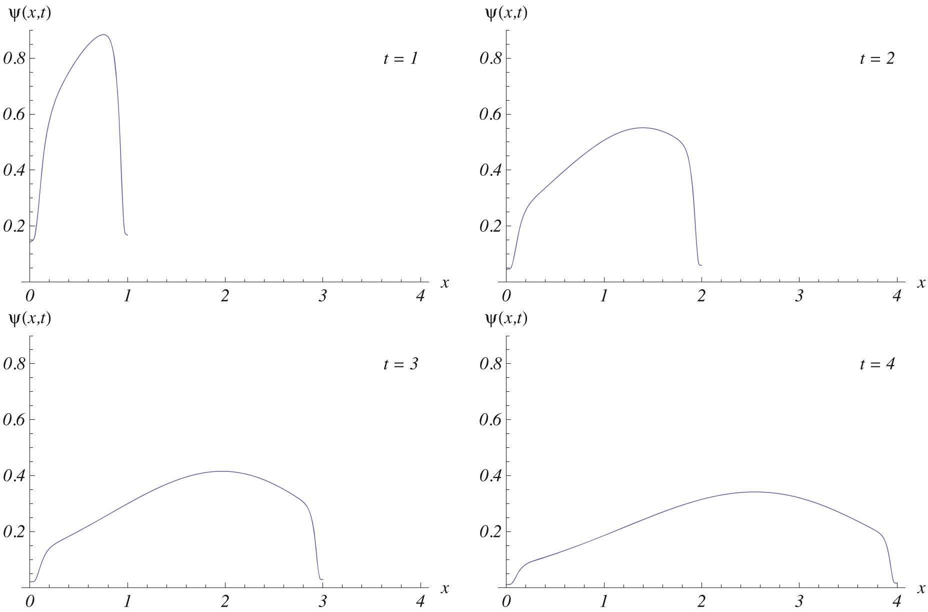


Fig. 8 Plots of density $\psi(x, t)$ for the model with random intertimes having inverse Gaussian distribution with parameters given in Eq. 52, for $0 < x < t$, with $t = 1, 2, 3, 4$. The probability mass of the density is 0.6513, 0.8324, 0.9044, 0.9407, respectively

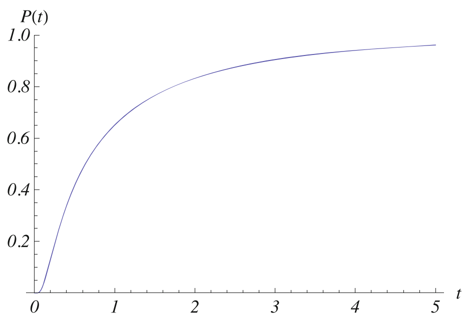
displayed asymmetry is due to the asymmetric values of parameters (m_1, l_1) and (m_2, l_2) in Eq. 52. It is clear that the density spreads out within the support $(0, t)$ as t increases, and its probability mass increases at the same time. Indeed, recalling Eq. 12 the probability mass is given by the following function, which is plotted in Fig. 9 for $0 \leq t \leq 5$:

$$P(t) = \frac{1}{2}\left[F_{IG}(t; m_1, l_1) + F_{IG}(t; m_2, l_2)\right], t \geq 0. \tag{53}$$

Some plots of the density $p(x, t)$ are finally shown in Fig. 10, the parameters being chosen as in Eq. 52.

Let us now discuss some features of the performed computations. A crucial role is played by the density of the occupation time. The three cases investigated in Section 5 deal with exact expressions for the right-hand-sides of Eqs.10 and 11. On the contrary, for the application treated in this Section we have been forced to resort to suitable approximations by

Fig. 9 Plots of probability mass (53)



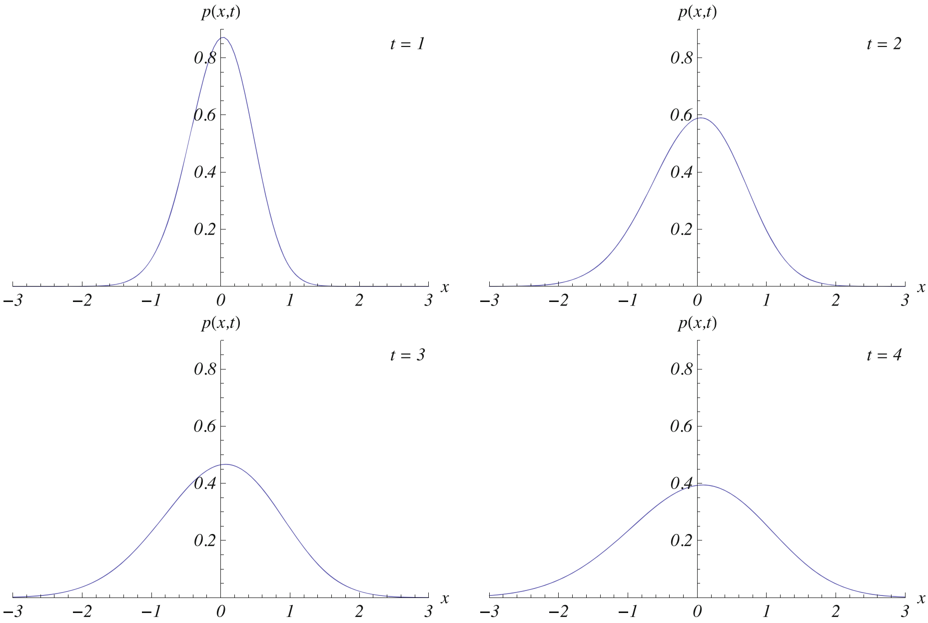


Fig. 10 Plots of density (50) for the same cases of Fig. 8

appropriate truncation of the related series, and by numerical integration of the integral in the right-hand-side of Eq. 50. All the performed evaluations have been implemented in MATHEMATICA on a desktop PC, the truncation errors being controlled in order to attain a precision of at least 6 digits.

Acknowledgments The authors thank the anonymous referee for helpful comments that improved the paper.

References

Beghin L, Nieddu L, Orsingher E (2001) Probabilistic analysis of the telegrapher's process with drift by means of relativistic transformations. J Appl Math Stoch Anal 14:11–25

Berg HC, Brown DA (1972) Chemotaxis in Escherichia coli analysed by three-dimensional tracking. Nature 239:500–504

Bshouty D, Di Crescenzo A, Martinucci B, Zacks S (2012) Generalized telegraph process with random delays. J Appl Prob 49:850–865

Buonocore A, Di Crescenzo A, Di Nardo E (2002) Input-output behaviour of a model neuron with alternating drift. Biosystems 67:27–34

Chernoff H, Zacks S (1964) Estimating the current mean of a normal distribution which is subjected to changes in time. Ann Math Statist 35:999–1018

Dayanik S (2010) Wiener disorder problem with observations at fixed discrete time epochs. Math Oper Res 35:756–785

De Gregorio A, Iacus SM (2011) Least-squares change-point estimation for the telegraph process observed at discrete times. Statistics 45:349–359

Di Crescenzo A (2001) On random motions with velocities alternating at Erlang-distributed random times. Adv Appl Prob 33:690–701

Di Crescenzo A, Di Nardo E, Ricciardi LM (2005) Simulation of first-passage times for alternating Brownian motions. Method Comput Appl Prob 7:161–181

Di Crescenzo A, Martinucci B (2010) A damped telegraph random process with logistic stationary distribution. J Appl Prob 47:84–96
Di Crescenzo A, Martinucci B, Zacks S (2012) On the damped geometric telegrapher's process. In: Perna C, Sibillo M (eds) Mathematical and statistical methods for actuarial sciences and finance. Springer, Dordrecht, pp 175–182. ISBN 978-88-470-2341-3
Elliott RJ, Siu TK, Chan L, Lau JW (2007) Pricing options under a generalized Markov-modulated jump-diffusion model. Stoch Anal Appl 25:821–843
Esser A, Mönch B (2007) The navigation of an iceberg: the optimal use of hidden orders. Financ Res Lett 4:68–81
Freidlin M, Pavlopoulos H (1997) On a stochastic model for moisture budget in an eulerian atmospheric column. Environmetrics 8:425–440
Gapeev PV, Peskir G (2006) The Wiener disorder problem with finite horizon. Stoch Proc Appl 116:1770–1791
Guo X (2001) Information and option pricings. Quant Financ 1:38–44
Jobert A, Rogers LCG (2006) Option pricing with Markov-modulated dynamics. SIAM J Control Optim 44:2063–2078
Keilson J, Wellner JA (1978) Oscillating Brownian motion. J Appl Prob 15:300–310
Lagache T, Holcman D (2008) Effective motion of a virus trafficking inside a biological cell. SIAM J Appl Math 68:1146–1167
Orsingher E (1985) Hyperbolic equations arising in random models. Stoch Process Appl 21:93–106
Orsingher E (1986) Brownian fluctuations in space-time with applications to vibrations of rods. Stoch Proc Appl 23:221–234
Orsingher E (1990) Probability law, flow function, maximum distribution of wave-governed random motions and their connections with Kirchoff's laws. Stoch Proc Appl 34:49–66
Perry D (1997) A double band control policy of a Brownian perishable inventory system. Prob Engin Inf Sci 11:361–373
Ratanov NE (1999) Telegraph evolutions in inhomogeneous media. Markov Process Relat Fields 5:53–68
Ratanov N (2007) A jump telegraph model for option pricing. Quant Finance 7:575–583
Ratanov N (2010) Option pricing model based on a Markov-modulated diffusion with jumps. Braz J Probab Stat 24:413–431
Sezer SO (2010) On the Wiener disorder problem. Ann Appl Probab 20:1537–1566
Zacks S (2004) Generalized integrated telegraph processes and the distribution of related stopping times. J Appl Prob 41:497–507
Zacks S (2012) Distribution of the total time in a mode of an alternating renewal process with applications. Sequential Anal 31:397–408
Zhang C, Wu R (2002) Total duration of negative surplus for the compound Poisson process that is perturbed by diffusion. J Appl Prob 39:517–532