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Distances Between Poisson k-Flats

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Abstract The distances between flats of a Poisson k-flat process in the d-dimensional Euclidean space with $k < d/2$ are discussed. Continuing an approach originally due to Rolf Schneider, the number of pairs of flats having distance less than a given threshold and midpoint in a fixed compact and convex set is considered. For a family of increasing convex subsets, the asymptotic variance is computed and a central limit theorem with an explicit rate of convergence is proven. Moreover, the asymptotic distribution of the m-th smallest distance between two flats is investigated and it is shown that the ordered distances form asymptotically after suitable rescaling an inhomogeneous Poisson point process on the positive real half-axis. A similar result with a homogeneous limiting process is derived for distances around a fixed, strictly positive value. Our proofs rely on recent findings based on the Wiener–Itô chaos decomposition and the Malliavin–Stein method.

Keywords Central limit theorem · Chaos decomposition · Extreme values · Limit theorems · Poisson flat process · Poisson point process · Poisson U-statistic · Stochastic geometry · Wiener–Itô integral

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1 Introduction

Point processes of k-dimensional flats in $\mathbb{R}^d$, especially Poisson point processes, are one of the most classical topics considered in stochastic geometry; cf. Mecke (1991), Mecke and Thomas (1986) and Weil (1987) for early works, Baumstark and Last (2009), Hug et al. (2003) and Spodarev (2001, 2003) for more recent papers and the book Schneider and Weil (2008) for an exhaustive reference. One of the problems considered in the theory of (Poisson) k-flat processes, the so-called **proximity problem**, is to describe the closeness or denseness of the arrangement of the flats in the case $k < d/2$, where the flats do not intersect each other (at least under suitable additional assumptions on their distribution). The notion of proximity generalizes the well-known second-order intersection density for k-flat processes in $\mathbb{R}^d$ with $k \geq d/2$ to the case $k < d/2$ and was originally introduced in Schneider (1999). There, only mean values of the proximity functional were considered, but no higher-order moments, limit theory or extreme values.

In this paper, we focus our attention to the Poisson case, for which we compute the asymptotic variance of the classical proximity as considered in Schneider (1999) and establish a Berry-Esseen-type central limit theorem. Moreover, we will not only deal with a cumulative proximity functional, but also investigate the order statistics induced by all distances between pairs of distinct flats, in particular the minimal distance, and the behaviour of the distances around a given positive value. This alternative approach to the proximity problem gives new insight into the geometry of Poisson k-flat processes.

The proofs of our limit theorems make use of a general central limit theorem from Schulte (2012) and a result about point process convergence and extreme value theory in Schulte and Thäle (2012). They are based on Berry-Esseen type inequalities in Peccati (2011) and Peccati et al. (2010) that were derived by combining the Malliavin calculus of variations on the Poisson space with Stein's method. The backbone of these methods is the fact that each square integrable Poisson functional can be represented as orthogonal sum of multiple Wiener–Itô integrals; see Last and Penrose (2011) and the references therein. It has recently turned out that this so-called Wiener–Itô chaos decomposition and related limit theorems can successfully be applied to problems in stochastic geometry. For example, in Reitzner and Schulte (2012) a general set-up was investigated as well as central limit theorems for Poisson hyperplanes, Last et al. (2012) deals with moment formulas and very general geometric functionals of intersection processes of Poisson k-flats, Lachièze-Rey and Peccati (2011, 2012) consider fine Gaussian fluctuations on the Poisson space and geometric random graphs. In all these works a crucial role is played by a special class of Poisson functionals, the so-called Poisson U-statistics.

The text is structured as follows: In Section 2, we introduce the proximity of a Poisson k-flat process and present our main results, Theorems 1–5. Their proofs rely on the Wiener–Itô chaos decomposition of Poisson functionals, whose background is briefly introduced in Section 3. The remaining three sections are devoted to the detailed proofs of our theorems.

2 Statement of the Main Results

2.1 Framework

A Poisson process of k-dimensional flats in $\mathbb{R}^d$ is a Poisson point process on the space $\mathbb{A}_k^d$ of k-dimensional affine subspaces of $\mathbb{R}^d$, where $k \in \{1, 2, \dots, d-1\}$ and $d \geq 1$. $\mathbb{A}_k^d$ can be equipped with its Borel σ-field as in Schneider and Weil (2008). We let η_t be such a **Poisson process of k-flats** having its **intensity measure** Θ_t given by

$$\int_{\mathbb{A}_k^d} f(E)\, \Theta_t(\mathrm{d}E) = t \int_{\mathbb{G}_k^d} \int_{L^\perp} f(L+x)\, \ell_{E^\perp}(\mathrm{d}x)\, \mathbb{Q}(\mathrm{d}E). \tag{1}$$

Here, $f : \mathbb{A}_k^d \to \mathbb{R}$ is a non-negative measurable function, $t > 0$, $\mathbb{G}_k^d$ is the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^d$, $\ell_{E^\perp}$ is the Lebesgue measure on $E^\perp$ and $\mathbb{Q}$ is a probability measure on $\mathbb{G}_k^d$. The Poisson k-flat process η_t is **stationary**, i.e., its distribution is invariant under all translations. In case that $\mathbb{Q}$ is the invariant probability measure (Haar measure) ν_k on $\mathbb{G}_k^d$, the distribution of η_t is also invariant under rotations and we call η_t **isotropic**. Two subspaces $L, M \in \mathbb{G}_k^d$ are said to be in general position if $\dim(L \cap M) = \max(0, 2k-d)$. Through the paper we make the following assumption on $\mathbb{Q}$.

(A1) Two independent random subspaces $M, L \in \mathbb{G}_k^d$ with distribution $\mathbb{Q}$ are in general position with probability one.

Assumption **(A1)** is for example fulfilled if $\mathbb{Q}$ is absolutely continuous with respect to ν_k, see Schneider and Weil (2008, Theorem 4.4.5). We note that under **(A1)** the flats of η_t are almost surely in general position (which means that the translates to the origin of any two flats are in general position). We also assume henceforth that

(A2) $1 \leq k < d/2$

holds, which ensures that the flats of η_t do not intersect each other with probability one (also notice that **(A2)** implies $d \geq 3$).

Before presenting our main findings in the following three subsections, we introduce some notions and notation used in the present paper. Let us write $\eta_{t,\neq}^2$ for the collection of pairs (E, F) of distinct k-flats of η_t, write $\mathrm{dist}(x, y)$ for the Euclidean distance of two points $x, y \in \mathbb{R}^d$ and let $\mathrm{dist}(E, F)$ be the **distance** of two k-flats $E, F \in \mathbb{A}_k^d$, i.e., $\mathrm{dist}(E, F) = \inf\{\mathrm{dist}(x, y) : x \in E,\ y \in F\}$. If E and F are in general position, this is the distance of two uniquely determined points $x_E \in E$ and $y_F \in F$ and we call $m(E, F) := (x_E + y_F)/2 \in \mathbb{R}^d$ the **midpoint** of E and F. For two linear subspaces $M, L \in \mathbb{G}_k^d$ we write $[M, L]$ for the **subspace determinant** of M and L, which is the $2k$-volume of a parallelepiped generated by two orthonormal bases of M and L; cf. Schneider and Weil (2008, Chapter 14.1). Furthermore, we denote in this paper by $V_k(K)$ the intrinsic volume of order $k \in \{0, \dots, d\}$ of a compact convex set $K \subset \mathbb{R}^d$; cf. Schneider and Weil (2008, Chapter 14.2). We also write κ_n for the volume of the unit ball in $\mathbb{R}^n$ ($n \geq 1$).

2.2 The Classical Proximity

After these preparations, we can now introduce the **proximity functional**

$$\pi_t(K,\delta) := \frac{1}{2} \sum_{(E,F)\in\eta_{t,\neq}^2} \mathbf{1}\{\mathrm{dist}(E,F) \leq \delta,\ m(E,F) \in K\},$$

where $\delta \in [0,\infty)$ is a fixed threshold, K is a compact and convex subset of $\mathbb{R}^d$ with $V_d(K) > 0$ (called **convex body** in this paper) and where $\mathbf{1}\{\cdot\}$ is the usual indicator function, which is one if the statement in brackets is fulfilled and zero otherwise. In other words, the functional $\pi_t(K,\delta)$ counts the number of pairs of flats in η_t with distance at most δ and midpoint in K. Schneider has calculated in Schneider (1999) the mean of $\pi_t(K,\delta)$ for K being the unit ball and $\delta = 1$; see also Schneider and Weil (2008, Theorem 4.4.10). More generally, we have the following result, which could also be directly deduced from Schneider (1999) or Schneider and Weil (2008, Theorem 4.4.10).

Theorem 1 *The expectation of $\pi_t(K,\delta)$ is given by*

$$\mathbb{E}\pi_t(K,\delta) = \frac{t^2}{2}\kappa_{d-2k}\delta^{d-2k}\, V_d(K) \int_{\mathbb{G}_k^d}\int_{\mathbb{G}_k^d} [M,L]\,\mathbb{Q}(\mathrm{d}L)\,\mathbb{Q}(\mathrm{d}M).$$

Remark 1 In the isotropic case $\mathbb{Q} = \nu_k$, we have

$$\psi_{d,k} := \int_{\mathbb{G}_k^d}\int_{\mathbb{G}_k^d} [M,L]\,\nu_k(\mathrm{d}L)\,\nu_k(\mathrm{d}M) = \frac{\kappa_k\kappa_{d-k}}{\binom{d}{k}\kappa_d}, \tag{2}$$

which is the content of Lemma 4.4 in Hug et al. (2008) or Corollary 4.5.5 in Materon (1975).

In what follows, we consider a family of increasing observation windows $(K_\varrho)_{\varrho\geq 1}$ with $K_\varrho = \varrho K$ and a convex body $K \subset \mathbb{R}^d$ and are interested in the asymptotic behaviour of $\pi_t(K_\varrho,\delta)$ as $\varrho \to \infty$. We first consider the asymptotic variance of $\pi_t(K_\varrho,\delta)$.

Theorem 2 *It holds that*

$$\lim_{\varrho\to\infty} \frac{\mathbb{V}\pi_t(K_\varrho,\delta)}{\varrho^{d+k}} = t^3\kappa_{d-2k}^2\delta^{2(d-2k)}\,\mathcal{I}(K),$$

where

$$\mathcal{I}(K) = \int_{\mathbb{G}_k^d}\int_{M^\perp} V_k\big(K\cap(M+y)\big)^2\,\ell_{M^\perp}(\mathrm{d}y)\left(\int_{\mathbb{G}_k^d}[M,L]\,\mathbb{Q}(\mathrm{d}L)\right)^2\mathbb{Q}(\mathrm{d}M).$$

Remark 2 In the case $\mathbb{Q} = \nu_k$, $\mathcal{I}(K)$ has an interpretation in terms of the order $k+1$ **chord-power integral** of K, which is defined as

$$\mathcal{J}_{k+1}(K) := \int_{\mathbb{A}_1^d} V_1(K \cap G)^{k+1} \, \mu_1(\mathrm{d}G),$$

where μ_1 is the Haar measure on $\mathbb{A}_1^d$ normalized as in Schneider and Weil (2008). Indeed, we first notice that the rotational average $\int_{\mathbb{G}_k^d} [M, L] \, \nu_k(\mathrm{d}L)$ does not depend on M; cf. Materon (1975, Corollary 4.5.5). Then identity (8.57) in Schneider and Weil (2008) implies that

$$\mathcal{I}(K) = \frac{\kappa_k}{k+1} \psi_{d,k}^2 \, \mathcal{J}_{k+1}(K)$$

with $\psi_{d,k}$ as in Eq. 2.

Remark 3 Theorem 2 shows that the variance of $\pi_t(K_\varrho, \delta)$ increases to infinity proportional to the $(d+k)$-th power of ϱ. This expresses long-range dependencies within the random set induced by the k-flat process η_t. A similar behaviour can also be observed for functionals of k-flat processes with $k \geq d/2$ for which we refer to Last et al. (2012) and the references cited therein.

Having investigated the expectation and the asymptotic variance of the proximity functional $\pi_t(K, \delta)$, we turn now to the central limit problem. Let the family $(K_\varrho)_{\varrho \geq 1}$ of convex bodies be as above.

Theorem 3 *Let $\mathcal{N}$ be a standard Gaussian random variable. Then there is a constant C depending on K, δ and t such that*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{\pi_t(K_\varrho, \delta) - \mathbb{E}\pi_t(K_\varrho, \delta)}{\sqrt{\mathbb{V}\pi_t(K_\varrho, \delta)}} \leq x \right) - \mathbb{P}(\mathcal{N} \leq x) \right| \leq C \, \varrho^{-\frac{d-k}{2}}$$

for $\varrho \geq 1$. In particular, we have the convergence in distribution

$$\frac{\pi_t(K_\varrho, \delta) - \mathbb{E}\pi_t(K_\varrho, \delta)}{\sqrt{\mathbb{V}\pi_t(K_\varrho, \delta)}} \xrightarrow{d} \mathcal{N} \quad \text{as} \quad \varrho \to \infty.$$

2.3 Small Distances

In the previous theorems, we have considered the number of midpoints of pairs of flats in a sequence of increasing observation windows, which have distance below a given threshold δ. A further natural question is to ask for the shortest or, more generally, the m-th shortest distance between two flats. To present the result, let $(K_\varrho)_{\varrho \geq 1}$ be a family of convex bodies as above. We denote by

$$\xi_\varrho^{(K,t)} = \{\mathrm{dist}(E, F) : (E, F) \in \eta_{t,\neq}^2 \text{ and } m(E, F) \in K_\varrho\} \tag{3}$$

the set of all distances between pairs of flats having a midpoint in K_ϱ (we count each value $\mathrm{dist}(E, F)$ only once, although (E, F) and (F, E) are both elements of $\eta_{t,\neq}^2$). Formally, $\xi_\varrho^{(K,t)}$ can be considered as a point process on the positive real half-line $\mathbb{R}_+$. By $D_m^{(K_\varrho,t)}$ we denote the m-th smallest element in $\xi_\varrho^{(K,t)}$ according to the natural ordering on $\mathbb{R}_+$. The following theorem describes the asymptotic distributions of $D_m^{(K_\varrho,t)}$ and $\xi_\varrho^{(K,t)}$ as the window size tends to infinity.

Theorem 4 *Define*

$$\beta = \frac{t^2}{2}\kappa_{d-2k} V_d(K) \int_{\mathbb{G}_k^d}\int_{\mathbb{G}_k^d} [M, L]\, \mathbb{Q}(\mathrm{d}L)\, \mathbb{Q}(\mathrm{d}M).$$

For every $u \geq 0$, there exists a constant C_u also depending on K and t such that

$$\left| \mathbb{P}\left(\varrho^{d/(d-2k)} D_m^{(K_\varrho,t)} > u\right) - e^{-\beta u^{(d-2k)}} \sum_{i=0}^{m-1} \frac{\left(\beta u^{(d-2k)}\right)^i}{i!} \right| \leq C_u\, \varrho^{-\frac{d-k}{2}}$$

for $m \in \{1, 2, 3, \ldots\}$ and $\varrho \in [1, \infty)$. Moreover, the family $\left(\varrho^{d/(d-2k)}\xi_\varrho^{(K,t)}\right)_{\varrho\geq 1}$ of rescaled point processes converges in distribution to a Poisson point process on $\mathbb{R}_+$ with the intensity measure

$$\nu(A) = \beta(d-2k) \int_A u^{d-2k-1}\, \mathrm{d}u, \qquad A \subset \mathbb{R}_+ \text{ a Borel set.}$$

Remark 4 We notice that

$$\beta = \frac{t^2}{2}\kappa_{d-2k}\, \psi_{d,k}\, V_d(K)$$

with $\psi_{d,k}$ given by Eq. 2 in the case where $\mathbb{Q} = \nu_k$ is the invariant probability measure on $\mathbb{G}_k^d$.

Remark 5 We notice that $e^{-\beta u^{(d-2k)}} \sum_{i=0}^{m-1} \frac{(\beta u^{(d-2k)})^i}{i!}$ is the tail of the distance from the origin to the m-th point of a Poisson point processes on $\mathbb{R}_+$ with intensity measure ν as in Theorem 4 above. A similar comment also applies to Theorem 5 below.

Remark 6 In Schulte and Thäle (2012) a similar problem was considered. Namely, for a pair (E, F) of flats of a stationary and isotropic Poisson k-flat process hitting a convex body K, the distance $\mathrm{dist}_K(E, F)$ was defined as

$$\mathrm{dist}_K(E, F) := \min_{x\in E\cap K, y\in F\cap K} \mathrm{dist}(x, y)$$

and it was shown that for increasing intensity the ordered distances converge to an inhomogeneous Poisson point process similar to that in Theorem 4. The fact that increasing the intensity is up to a factor the same as increasing the window size implies that that the normalization $\varrho^{d/(d-2k)}$ in Theorem 4 is the same as in Schulte and Thäle (2012). The constants β, however, are different in both settings since different pairs of flats and different approaches to measure the distance between two flats are considered.

2.4 Distances Around a Positive Value

The previous result describes the behaviour of very small distances and it is natural also to consider large distances. However, the maximal distance (and thus also the

m-th maximal distance for any $m \in \{1, 2, 3, \ldots\}$) of two flats having their midpoint in a test set K is not well defined since

$$\sup_{\substack{(E,F)\in\eta^2_{t,\neq}\\ m(E,F)\in K}} \mathrm{dist}(E, F) = \infty \quad \text{almost surely;} \tag{4}$$

see Section 6 for a proof.

To overcome this difficulty and in order to complete the picture, we fix some $\sigma > 0$ and consider the asymptotic behaviour of the point process $\xi_\varrho^{(K,t)}$ defined by Eq. 3 around σ. By $\overline{D}_m^{(K_\varrho,t,\sigma)}$ and $\underline{D}_m^{(K_\varrho,t,\sigma)}$, $m \in \{1, 2, 3, \ldots\}$, we denote the m-th element of $\xi_\varrho^{(K,t)}$ greater or less than σ, respectively.

Theorem 5 *Let $\sigma > 0$ and define*

$$\beta = \frac{t^2}{2}(d-2k)\kappa_{d-2k}\,\sigma^{d-2k-1}\, V_d(K)\int_{\mathbb{G}_k^d}\int_{\mathbb{G}_k^d}[M, L]\,\mathbb{Q}(\mathrm{d}L)\,\mathbb{Q}(\mathrm{d}M). \tag{5}$$

For every $u \geq 0$, there is a constant C_u also depending on K, t and σ such that

$$\left|\mathbb{P}\left(\varrho^d\left(\overline{D}_m^{(K_\varrho,t,\sigma)} - \sigma\right) > u\right) - e^{-\beta u}\sum_{i=0}^{m-1}\frac{(\beta u)^i}{i!}\right| \leq C_u\,\varrho^{-\frac{d-k}{2}}$$

and

$$\left|\mathbb{P}\left(-\varrho^d\left(\underline{D}_m^{(K_\varrho,t,\sigma)} - \sigma\right) > u\right) - e^{-\beta u}\sum_{i=0}^{m-1}\frac{(\beta u)^i}{i!}\right| \leq C_u\,\varrho^{-\frac{d-k}{2}}$$

for $m \in \{1, 2, 3, \ldots\}$ and $\varrho \in [1, \infty)$. Moreover, the family $\left(\varrho^d\,(\xi_\varrho^{(K,t)} - \sigma)\right)_{\varrho\geq 1}$ of rescaled and shifted point processes converges in distribution to a homogeneous Poisson point process on $\mathbb{R}$ with intensity β.

Remark 7 In the case where $\mathbb{Q}$ is the invariant probability measure ν_k on $\mathbb{G}_k^d$ we have that

$$\beta = \frac{t^2}{2}(d-2k)\kappa_{d-2k}\,\sigma^{d-2k-1}\,\psi_{d,k}\,V_d(K)$$

with $\psi_{d,k}$ given by Eq. 2.

Theorems 4 and 5 show the remarkable fact that very small distances near zero behave quite different compared with the distances around (i.e., above or below) every positive value σ. Indeed, in Theorem 4 an inhomogeneous Poisson point process on $\mathbb{R}_+$ appears after normalization with $\varrho^{d/(d-2k)}$, whereas in Theorem 5 a homogeneous Poisson point process on the whole real line shows up in the limit after rescaling with ϱ^d and the latter can be interpreted as the superposition of two independent homogeneous Poisson point process on $\mathbb{R}_+$ (for the distances greater than σ) and on $\mathbb{R}_-$ (for the distances less than σ).

3 Background Material on Chaos Decompositions

We let η_t be a Poisson point process on $\mathbb{A}_k^d$ with intensity measure Θ_t given by Eq. 1 and assume that **(A1)** and **(A2)** are satisfied. Given $n \in \mathbb{N}$ we write $L^2(\Theta_t^n)$ for the collection of measurable functions $f : (\mathbb{A}_k^d)^n \to \mathbb{R}$ such that

$$||f||_n := \left(\int_{(\mathbb{A}_k^d)^n} f^2 \, \mathrm{d}\Theta_t^n \right)^{1/2} < \infty$$

and $L^2_{\mathrm{sym}}(\Theta_t^n)$ for the subspace of $L^2(\Theta_t^n)$ consisting of functions that are invariant under permutation of their arguments, so called symmetric functions. We denote the inner product in $L^2(\Theta_t^n)$ by $\langle \cdot\,, \cdot \rangle_n$.

For $f \in L^2_{\mathrm{sym}}(\Theta_t^n)$ we let $I_n(f)$ be the (multiple) **Wiener–Itô integral** of f with respect to the **compensated Poisson process** $\hat{\eta}_t := \eta_t - \Theta_t$ (to make sense of the definition of $\hat{\eta}_t$, η_t has to be interpreted here as a random point measure so that the difference $\eta_t - \Theta_t$ is well defined). These stochastic integrals satisfy the following properties: for $n \in \mathbb{N}$ and $f \in L^2(\Theta_t^n)$,

$$\mathbb{E} I_n(f) = 0 \tag{6}$$

and for $n_1, n_2 \in \mathbb{N}$ and $f_1 \in L^2_{\mathrm{sym}}(\Theta_t^{n_1})$ and $f_2 \in L^2_{\mathrm{sym}}(\Theta_t^{n_2})$ it holds that

$$\mathbb{E}\left[I_{n_1}(f_1)\, I_{n_2}(f_2) \right] = n_1! \, \langle f_1, f_2 \rangle_{n_1} \, \mathbf{1}\{n_1 = n_2\}. \tag{7}$$

For further details on Wiener–Itô integrals we refer the reader to Last and Penrose (2011), Peccati and Taqqu (2008) and Peccati et al. (2010).

Let $g : (\mathbb{A}_k^d)^2 \to \mathbb{R}$ be integrable with respect to Θ_t^2 and be invariant under permutation of its two arguments. We define

$$U := \frac{1}{2} \sum_{(E,F) \in \eta^2_{t,\neq}} g(E, F)$$

and assume that U is square integrable with respect to the distribution of η_t. In this case, the random variable U is a so-called **Poisson U-statistic of order two**. It is a crucial fact that U can be written as

$$U = \mathbb{E}U + I_1(f_1) + I_2(f_2) \tag{8}$$

with

$$\mathbb{E}U = \frac{1}{2} \int_{(\mathbb{A}_k^d)^2} g(E, F) \, \Theta_t^2 \, (\mathrm{d}(E, F)) \tag{9}$$

by the classical Slivnyak–Mecke formula (Schneider and Weil 2008, Theorem 3.2.5) and with $f_1 \in L^2_{\mathrm{sym}}(\Theta_t)$ and $f_2 \in L^2_{\mathrm{sym}}(\Theta_t^2)$ given by

$$f_1(E) = \int_{\mathbb{A}_k^d} g(E, F) \, \Theta_t(\mathrm{d}F),$$

$$f_2(E, F) = \frac{1}{2} g(E, F);$$

cf. Lemma 3.5 in Reitzner and Schulte (2012), which is a consequence of the results in Last and Penrose (2011). The representation Eq. 8 is called the **Wiener–Itô chaos**

decomposition of U and we call f_1 and f_2 its **kernels**. This decomposition is a very powerful tool, which will be used extensively in our proofs below. In particular, squaring the expression in Eq. 8 and using the computation rules (Eqs. 6 and 7), we find the **variance formula**

$$\mathbb{V}U = \| f_1\|_1^2 + 2\| f_2\|_2^2 . \tag{10}$$

This will be essential in the proof of Theorem 2.

For more details on Poisson U-statistics (of arbitrary order) we refer to Lachièze-Rey and Peccati (2011, 2012), Last et al. (2012) and Reitzner and Schulte (2012). Poisson functionals that are a sum of a first and a second order Wiener–Itô integral, such as the functional U above, were also investigated in Peccati and Taqqu (2008).

4 Proof of Theorems 1 and 2

4.1 A Preparatory Lemma

In order to simplify our notation, we define for $E, F \in \mathbb{A}_k^d$,

$$h(E, F) = \mathbf{1}\{m(E, F) \in K, \operatorname{dist}(E, F) \le \delta\},$$

where K is a convex body and where $\delta > 0$.

Lemma 1 *Let $M, L \in \mathbb{G}_k^d$ be in general position and define $W = M + L$. Then*

$$\int_{L^\perp} h(M, L + x)\,\ell_{L^\perp}(\mathrm{d}x)$$
$$= [M, L] \int_{W^\perp} \mathbf{1}\{\|x\| \le \delta\}\, V_k((K - (x/2)) \cap M)\,\ell_{W^\perp}(\mathrm{d}x). \tag{11}$$

Proof By decomposing $x \in L^\perp$ in $x = x_1 + x_2$ with $x_1 \in L^\perp \cap W^\perp = W^\perp$ and $x_2 \in L^\perp \cap W$, we obtain

$$\int_{L^\perp} h(M, L + x)\,\ell_{L^\perp}(\mathrm{d}x)$$
$$= \int_{W^\perp} \int_{L^\perp \cap W} h(M, L + x_1 + x_2)\,\ell_{L^\perp \cap W}(\mathrm{d}x_2)\,\ell_{W^\perp}(\mathrm{d}x_1). \tag{12}$$

By the definition of x_1 and x_2, M and $L + x_2$ intersect in a unique point $z \in W$, $m(M, L + x_1 + x_2) = z + (x_1/2)$ and $\operatorname{dist}(M, L + x_1 + x_2) = \|x_1\|$.

Let B_M and $B_{L^\perp \cap W}$ be matrices whose columns form orthonormal bases of M and $L^\perp \cap W$, respectively. Rewrite x_2 as $x_2 = B_{L^\perp \cap W}\tilde{x}$ with $\tilde{x} \in \mathbb{R}^k$ and replace integration over $L^\perp \cap W$ in Eq. 12 by integration over $\mathbb{R}^k$. Moreover, we notice that the intersection point z of M and $L + x_2$ has the representation $z = B_M\tilde{z}$, where $\tilde{z} \in \mathbb{R}^k$ is the solution of

$$B_{L^\perp \cap W}^T B_M \tilde{z} = B_{L^\perp \cap W}^T x_2 = B_{L^\perp \cap W}^T B_{L^\perp \cap W}\tilde{x} = \tilde{x}.$$

This implies that

$$\tilde{z} = \left(B_{L^\perp \cap W}^T B_M\right)^{-1} \tilde{x}. \tag{13}$$

Using the representation of x_2, we now write the inner integral in Eq. 12 as

$$\int_{L^\perp \cap W} h\left(M, L + x_1 + x_2\right) \ell_{L^\perp \cap W}\left(\mathrm{d}x_2\right)$$
$$= \mathbf{1}\left\{\|x_1\| \le \delta\right\} \int_{\mathbb{R}^k} \mathbf{1}\left\{m\left(M, L + x_1 + B_M \tilde{x}\right) \in K\right\} \mathrm{d}\tilde{x}. \tag{14}$$

Continuing by using Eq. 13, we find

$$\int_{\mathbb{R}^k} \mathbf{1}\left\{m\left(M, L + x_1 + B_M \tilde{x}\right) \in K\right\} \mathrm{d}\tilde{x}$$
$$= \int_{\mathbb{R}^k} \mathbf{1}\left\{B_M \left(B_{L^\perp \cap W}^T B_M\right)^{-1} \tilde{x} \in \left(K - (x_1/2)\right) \cap M\right\} \mathrm{d}\tilde{x}$$
$$= \int_{\mathbb{R}^k} \mathbf{1}\left\{\tilde{x} \in B_{L^\perp \cap W}^T B_M B_M^{-1} \left(K - (x_1/2)\right) \cap M\right\} \mathrm{d}\tilde{x}$$
$$= V_k\left(B_{L^\perp \cap W}^T B_M B_M^{-1} \left(K - (x_1/2)\right) \cap M\right). \tag{15}$$

Combining Eq. 14 with Eq. 15 and using the fact that $\det(B_{L^\perp \cap W}^T B_M) = [M, L]$, we arrive at

$$\int_{L^\perp \cap W} h\left(M, L + x_1 + x_2\right) \ell_{L^\perp \cap W}\left(\mathrm{d}x_2\right)$$
$$= [M, L]\, \mathbf{1}\{\|x_1\| \le \delta\} V_k\left(\left(K - (x_1/2)\right) \cap M\right).$$

Integration with respect to $W^\perp$ finally yields Eq. 11. □

4.2 Proof of Theorem 1

By Eq. 9, the expectation of $\pi_t(K, \delta)$ is given by

$$\mathbb{E}\pi_t(K, \delta) = \frac{1}{2} \int_{\mathbb{A}_k^d} \int_{\mathbb{A}_k^d} h(E, F)\, \Theta_t(\mathrm{d}F)\, \Theta_t(\mathrm{d}E).$$

A glance at Eq. 1 shows that this equals

$$\frac{t^2}{2} \int_{\mathbb{G}_k^d} \int_{\mathbb{G}_k^d} \int_{M^\perp} \int_{L^\perp} h(M + y, L + x)\, \ell_{L^\perp}(\mathrm{d}x)\, \ell_{M^\perp}(\mathrm{d}y)\, \mathbb{Q}(\mathrm{d}L)\, \mathbb{Q}(\mathrm{d}M). \tag{16}$$

We evaluate now the inner double integral in Eq. 16. Translating M and L by $-y$, substituting $x - y$ and using Lemma 1 thereafter, we obtain

$$\int_{M^\perp} \int_{L^\perp} h(M + y, L + x)\, \ell_{L^\perp}(\mathrm{d}x)\, \ell_{M^\perp}(\mathrm{d}y)$$
$$= \int_{M^\perp} \int_{L^\perp} \mathbf{1}\{m(M, L + x) \in K - y,\ \mathrm{dist}(M, L + x) \le \delta\}\, \ell_{L^\perp}(\mathrm{d}x)\, \ell_{M^\perp}(\mathrm{d}y)$$
$$= [M, L] \int_{M^\perp} \int_{W^\perp} V_k\big((K - y - (x/2)) \cap M\big)\, \mathbf{1}\{\|x\| \le \delta\}\, \ell_{W^\perp}(\mathrm{d}x)\, \ell_{M^\perp}(\mathrm{d}y),$$

where $W = L + M$. Fubini's theorem further implies that

$$
\begin{aligned}
&[M, L] \int_{M^\perp} \int_{W^\perp} V_k\big((K - y - (x/2)) \cap M\big)\, \mathbf{1}\{\|x\| \le \delta\}\, \ell_{W^\perp}(\mathrm{d}x)\, \ell_{M^\perp}(\mathrm{d}y) \\
&\quad = [M, L] \int_{W^\perp} \int_{M^\perp} V_k\big((K - y - (x/2)) \cap M\big)\, \mathbf{1}\{\|x\| \le \delta\}\, \ell_{M^\perp}(\mathrm{d}y)\, \ell_{W^\perp}(\mathrm{d}x) \\
&\quad = [M, L]\, V_d(K)\, \kappa_{d-2k} \delta^{d-2k},
\end{aligned}
$$

which in view of Eq. 16 completes the proof. □

Remark 8 The above proofs of Lemma 1 and Theorem 1 are very similar to that of the main result in Schneider (1999) and use the same ideas, generalized to a slightly more general setting. We decided to state the first part as lemma since Eq. 11 is applied several times below.

4.3 Proof of Theorem 2

First, Eq. 8 implies that the proximity functional $\pi_t(K, \delta)$ has chaos decomposition

$$\pi_t(K, \delta) = \mathbb{E}\pi_t(K, \delta) + I_1(f_1^{(K,\delta,t)}) + I_2(f_2^{(K,\delta,t)}).$$

Here, the kernels $f_n^{(K\delta,t)}$ $(n = 1, 2)$ are given by

$$f_1^{(K,\delta,t)}(M + y) = \int_{\mathbb{A}_k^d} h(M + y, F)\, \Theta_t(\mathrm{d}F) \tag{17}$$

for $M \in \mathbb{G}_k^d$ and $y \in M^\perp$ and

$$f_2^{(K,\delta,t)}(E, F) = \frac{1}{2} \mathbf{1}\{m(E, F) \in K,\ \mathrm{dist}(E, F) \le \delta\} \tag{18}$$

for $E, F \in \mathbb{A}_k^d$, respectively. Now, the variance formula in Eq. 10 implies that

$$\mathbb{V}\pi_t(K, \delta) = \| f_1^{(K,\delta,t)} \|_1^2 + 2\| f_2^{(K,\delta,t)} \|_2^2. \tag{19}$$

We determine the asymptotic behaviour of the right hand side in Eq. 19. For the second term we find

$$
\begin{aligned}
\| f_2^{(K,\delta,t)} \|_2^2 &= \frac{1}{4} \int_{\mathbb{A}_k^d} \int_{\mathbb{A}_k^d} \mathbf{1}\{m(E, F) \in K,\ \mathrm{dist}(E, F) \le \delta\}^2\, \Theta_t(\mathrm{d}E)\, \Theta_t(\mathrm{d}F) \\
&= \frac{t^2}{4} \kappa_{d-2k}\, \delta^{d-2k}\, V_d(K) \int_{\mathbb{G}_k^d} \int_{\mathbb{G}_k^d} [L, M]\, \mathbb{Q}(\mathrm{d}L)\, \mathbb{Q}(\mathrm{d}M)
\end{aligned}
$$

by using the formula for $\mathbb{E}\pi_t(K, \delta)$ in Theorem 1. Thus,

$$
\begin{aligned}
\lim_{\varrho \to \infty} \frac{\| f_2^{(K_\varrho,\delta,t)} \|_2^2}{\varrho^{d+k}} &= \lim_{\varrho \to \infty} \frac{t^2}{4} \kappa_{d-2k}\, \delta^{d-2k} V_d(K) \varrho^{-k} \\
&\quad \times \int_{\mathbb{G}_k^d} \int_{\mathbb{G}_k^d} [M, L]\, \mathbb{Q}(\mathrm{d}L)\, \mathbb{Q}(\mathrm{d}M) = 0.
\end{aligned} \tag{20}
$$

We continue with the first term in Eq. 19 and observe that Eq. 1 and Lemma 1 imply that

$$\begin{aligned}
&f_1^{(K,\delta,t)}(M+y)\\
&\quad= \int_{\mathbb{A}_k^d} h(M+y,\,F)\,\Theta_t(\mathrm{d}F)\\
&\quad= t\int_{\mathbb{G}_k^d}\int_{L^\perp} h(M+y,\,L+x)\,\ell_{L^\perp}(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}L)\\
&\quad= t\int_{\mathbb{G}_k^d}\int_{W^\perp} V_k\big((K-y-(x/2))\cap M\big)\,\mathbf{1}\{\|x\|\le\delta\}\,\ell_{W^\perp}(\mathrm{d}x)\,[M,L]\,\mathbb{Q}(\mathrm{d}L),
\end{aligned}$$

where, as before, $W=L+M$. We now observe that the scaling relation

$$f_1^{(K_\varrho,\delta,t)}(M+y) = \varrho^{d-k}\, f_1^{(K,\delta/\varrho,t)}(M+(y/\varrho)) \tag{21}$$

holds. Indeed, from a simple change of variables and from the fact that $\dim W^\perp = d-2k$ it follows that

$$\begin{aligned}
&\int_{W^\perp} V_k\big((K_\varrho-y-(x/2))\cap M\big)\,\mathbf{1}\{\|x\|\le\delta\}\,\ell_{W^\perp}(\mathrm{d}x)\\
&\quad= \varrho^k\int_{W^\perp} V_k\big((K-(y/\varrho)-(x/2\varrho))\cap M\big)\,\mathbf{1}\{\|x\|\le\delta\}\,\ell_{W^\perp}(\mathrm{d}x)\\
&\quad= \varrho^{d-k}\int_{W^\perp} V_k\big((K-(y/\varrho)-(x/2))\cap M\big)\,\mathbf{1}\{\|x\|\le\delta/\varrho\}\,\ell_{W^\perp}(\mathrm{d}x),
\end{aligned}$$

which shows Eq. 21. As a consequence, we have

$$\begin{aligned}
\|f_1^{(K_\varrho,\delta,t)}\|_1^2 &= \varrho^{2(d-k)}\,t\int_{\mathbb{G}_k^d}\int_{M^\perp} f_1^{(K,\delta/\varrho,t)}(M+(y/\varrho))^2\,\ell_{M^\perp}(\mathrm{d}y)\,\mathbb{Q}(\mathrm{d}M)\\
&= \varrho^{3(d-k)}\,t\int_{\mathbb{G}_k^d}\int_{M^\perp} f_1^{(K,\delta/\varrho,t)}(M+y)^2\,\ell_{M^\perp}(\mathrm{d}y)\,\mathbb{Q}(\mathrm{d}M).
\end{aligned}\tag{22}$$

Moreover, the dominated convergence theorem implies that

$$\begin{aligned}
&\lim_{\varrho\to\infty}\varrho^{d-2k}\,f_1^{(K,\delta/\varrho,t)}(M+y)\\
&\quad= t\int_{\mathbb{G}_k^d}\lim_{\varrho\to\infty}\varrho^{d-2k}\int_{W^\perp} V_k\big((K-y-(x/2))\cap M\big)\\
&\qquad\times\mathbf{1}\{\|x\|\le\delta/\varrho\}\,\ell_{W^\perp}(\mathrm{d}x)\,[M,L]\,\mathbb{Q}(\mathrm{d}L)\\
&\quad= t\,\kappa_{d-2k}\,\delta^{d-2k}\,V_k\big((K-y)\cap M\big)\int_{\mathbb{G}_k^d}[M,L]\,\mathbb{Q}(\mathrm{d}L)\\
&\quad= t\,\kappa_{d-2k}\,\delta^{d-2k}\,V_k\big(K\cap(M+y)\big)\int_{\mathbb{G}_k^d}[M,L]\,\mathbb{Q}(\mathrm{d}L),
\end{aligned}$$

where we used that $V_k\big((K-y-(x/2)\cap M)\big)$ behaves like $V_k\big((K-y)\cap M\big)$ whenever $\|x\|$ is small. Combining this with Eq. 22, writing $\varrho^{3(d-k)}$ as $\varrho^{2(d-2k)}\varrho^{d+k}$ and applying the dominated convergence Theorem once again, yields

$$\lim_{\varrho\to\infty}\frac{\| f_1^{(K_\varrho,\delta,t)}\|_1^2}{\varrho^{d+k}} = t^3\,\kappa_{d-2k}^2\,\delta^{2(d-2k)}\,\mathcal{I}(K)$$

with $\mathcal{I}(K)$ as in the statement of the theorem. This together with the asymptotic behaviour (Eq. 20) of the second term in the variance expansion (Eq. 19) of the proximity functional proves the claim. □

5 Proof of Theorem 3

5.1 A General Bound

For two random variables Y and Z define the Kolmogorov distance $d_K(Y, Z)$ by

$$d_K(Y,Z) = \sup_{x\in\mathbb{R}} |\mathbb{P}(Y\le x) - \mathbb{P}(Z\le x)|.$$

This is to say, $d_K(Y, Z)$ is the supremum norm of the difference between the distribution functions of Y and Z. We consider a second-order Poisson U-statistic

$$U = \frac{1}{2}\sum_{(E,F)\in\eta_{t,\neq}^2} g(E,F),$$

where we assume that g is bounded, symmetric and satisfies

$$\Theta_t^2\left(\left\{(E,F)\in\mathbb{A}_k^d\times\mathbb{A}_k^d : g(E,F)\neq 0\right\}\right) < \infty, \qquad t>0.$$

We denote the kernels of the chaos decomposition of U given in Eq. 8 by f_1 and f_2 and define M_{11} by

$$M_{11} = \int_{\mathbb{A}_k^d} f_1(E)^4\,\Theta_t(\mathrm{d}E).$$

We also define M_{12} by

$$\begin{aligned} M_{12} = {} & 8\int_{\left(\mathbb{A}_k^d\right)^3} f_1(E_1)\,f_2(E_1,E_2)\,f_1(E_3)\,f_2(E_2,E_3)\,\Theta_t^3(\mathrm{d}(E_1,E_2,E_3)) \\ & + 4\int_{\left(\mathbb{A}_k^d\right)^2} f_1(E_1)\,f_2(E_1,E_2)\,f_1(E_2)\,f_2(E_1,E_2)\,\Theta_t^2(\mathrm{d}(E_1,E_2)) \end{aligned}$$

and finally M_{22} by

$$\begin{aligned} M_{22} = {} & 48\int_{\left(\mathbb{A}_k^d\right)^4} f_2(E_1,E_2)\,f_2(E_2,E_3)\,f_2(E_3,E_4) \\ & \times f_2(E_4,E_1)\,\Theta_t^4(\mathrm{d}(E_1,E_2,E_3,E_4)) \\ & + 96\int_{\left(\mathbb{A}_k^d\right)^3} f_2(E_1,E_2)\,f_2(E_1,E_3)\,f_2(E_1,E_3) \\ & \times f_2(E_2,E_3)\,\Theta_t^3(\mathrm{d}(E_1,E_2,E_3)) \\ & + 8\int_{\left(\mathbb{A}_k^d\right)^2} f_2(E_1,E_2)^4\,\Theta_t^2(\mathrm{d}(E_1,E_2)). \end{aligned}$$

We can now rephrase a special situation of Theorem 4.2 in Schulte (2012).

Proposition 1 *Let $\mathcal{N}$ be a standard Gaussian random variable. Then*

$$d_K\left(\frac{U-\mathbb{E}U}{\sqrt{\mathbb{V}U}},\mathcal{N}\right)\le 1088\,\frac{\sqrt{M_{11}}+\sqrt{M_{12}}+\sqrt{M_{22}}}{\mathbb{V}U}. \tag{23}$$

5.2 Proof of Theorem 3

Let us introduce the abbreviation

$$\pi_\varrho := \pi_t(K_\varrho,\delta)=\frac{1}{2}\sum_{(E,F)\in\eta^2_{t,\neq}}\mathbf{1}\{m(E,F)\in K_\varrho,\ \mathrm{dist}(E,F)\le\delta\}.$$

Since t, K and δ are fixed in the following, we suppress this dependency in our notation. We further let $f_1^{(\varrho)}$ and $f_2^{(\varrho)}$ be the kernels of the Wiener–Itô chaos decomposition of π_ϱ given by Eqs. 17 and 18, respectively. In the following, we prove Theorem 3 by bounding the right hand side of Eq. 23 for the Poisson U-statistic $U=\pi_\varrho$.

Step 1: An inequality for $f_1^{(\varrho)}$. We show that

$$f_1^{(\varrho)}(E)=\Theta_t\big(\{F\in\mathbb{A}_k^d:\ f_2^{(\varrho)}(E,F)\neq 0\}\big)\le C\varrho^k \tag{24}$$

for all $E\in\mathbb{A}_k^d$, where $C=t\kappa_k\kappa_{d-2k}\delta^{d-2k}\,(\mathrm{diam}(K)/2)^k$. To see the equality we notice that

$$\begin{aligned}
f_1^{(\varrho)}(E)&=\int_{\mathbb{A}_k^d}\mathbf{1}\{m(E,F)\in K_\varrho,\ \mathrm{dist}(E,F)\le\delta\}\,\Theta_t(\mathrm{d}F)\\
&=\Theta_t\big(\{F\in\mathbb{A}_k^d:\ m(E,F)\in K_\varrho,\ \mathrm{dist}(E,F)\le\delta\}\big)\\
&=\Theta_t\big(\{F\in\mathbb{A}_k^d:\ f_2^{(\varrho)}(E,F)\neq 0\}\big).
\end{aligned}$$

The estimate in Eq. 24 is a consequence of Eq. 17, Lemma 1 and the inequality $V_k(\tilde{K})\le\kappa_k(\mathrm{diam}(\tilde{K})/2)^k$ from Bonnesen and Fenchel (1934, page 76) for a convex body $\tilde{K}\subset\mathbb{R}^k$. Indeed, writing $E=M+x$ and $W=M+L$, it holds that

$$\begin{aligned}
f_1^{(\varrho)}(E)&=f_1^{(\varrho)}(M+x)\\
&=t\int_{\mathbb{G}_k^d}[M,L]\int_{W^\perp}\mathbf{1}\{\|y\|\le\delta\}\,V_k\big((K_\varrho-x-(y/2))\cap M\big)\,\ell_{W^\perp}(\mathrm{d}y)\,\mathbb{Q}(\mathrm{d}L)\\
&\le t\int_{\mathbb{G}_k^d}[M,L]\int_{W^\perp}\mathbf{1}\{\|y\|\le\delta\}\,\kappa_k\,(\mathrm{diam}(K_\varrho)/2)^k\,\ell_{W^\perp}(\mathrm{d}y)\,\mathbb{Q}(\mathrm{d}L)\\
&\le t\kappa_k\kappa_{d-2k}\delta^{d-2k}\,(\mathrm{diam}(K)/2)^k\,\varrho^k,
\end{aligned}$$

where we have used additionally the fact that $[M,L]\le 1$.

Step 2: Completing the proof. Let B_δ^d be the d-dimensional centred ball with radius δ and denote by $+$ the usual Minkowski sum.

All integrands occurring in M_{11}, M_{12} and M_{22} have the structure that after choosing the first k-flat E hitting $K_\varrho+B_\delta^d$, the second flat must be in the set $\{F\in\mathbb{A}_k^d:\ f_2^{(\varrho)}(E,F)\neq 0\}$ or the integrand is zero otherwise. For the remaining flats there are similar conditions so that, by Step 1, the measure of the support of each integrand is at most

$$\Theta_t\big([K_\varrho + B^d_\delta]\big)\,(C\varrho^k)^{m-1} \le \Theta_t\big([K + B^d_\delta]\big)\,C^{m-1}\varrho^{d+(m-2)k}$$

for $\varrho \ge 1$. Here, $m \in \{1, 2, 3, 4\}$ is the number of k-flats the integration runs over and for a set $A \subset \mathbb{R}^d$, $[A]$ stands for the collection of k-flats that have non-empty intersection with A. Combining this with the fact that $f_1^{(\varrho)} \le C\varrho^k$, recall Eq. 24, and $f_2^{(\varrho)} \le \frac{1}{2}$, we obtain

$$\begin{aligned}
M_{11} &\le C^4\,\Theta_t\big([K + B^d_\delta]\big)\,\varrho^{d+3k},\\
M_{12} &\le 2C^4\,\Theta_t\big([K + B^d_\delta]\big)\,\varrho^{d+3k} + C^3\,\Theta_t\big([K + B^d_\delta]\big)\,\varrho^{d+2k},\\
M_{22} &\le 3C^3\,\Theta_t\big([K + B^d_\delta]\big)\,\varrho^{d+2k} + 6C^2\,\Theta_t\big([K + B^d_\delta]\big)\,\varrho^{d+k}\\
&\quad + \frac{C}{2}\,\Theta_t\big([K + B^d_\delta]\big)\,\varrho^{d}.
\end{aligned}$$

On the other hand, Theorem 2 tells us that $\mathbb{V}\pi_\varrho$ is asymptotically of order ϱ^{d+k}, so that $\sqrt{M_{11}}/\mathbb{V}\pi_\varrho$, $\sqrt{M_{12}}/\mathbb{V}\pi_\varrho$ and $\sqrt{M_{22}}/\mathbb{V}\pi_\varrho$ are of order $\varrho^{-(d-k)/2}$ or less and Proposition 1 implies Theorem 3. □

6 Proofs of Theorems 4, 5 and Eq. 4

6.1 An Auxiliary Limit Theorem

We consider the following general setting. Let $(g_\varrho)_{\varrho\ge1}$ be a family of symmetric functions $g_\varrho : (\mathbb{A}^d_k)^2 \to \mathbb{R}$ satisfying $\Theta_t^2\big(g_\varrho^{-1}([-u, u])\big) < \infty$ for all $u \ge 0$ (this will always be the case in our applications below). Next, we define a point process

$$\xi_\varrho = \big\{g_\varrho(E, F) : (E, F) \in \eta^2_{t,\ne},\ m(E, F) \in K_\varrho\big\}$$

on $\mathbb{R}$, where we count the point $g_\varrho(E, F) = g_\varrho(F, E)$ only once and where the family $(K_\varrho)_{\varrho\ge1}$ is a family of convex bodies as in Section 2 (that ξ_ϱ is indeed a point processes follows from our assumption on g_ϱ). By $\overline{D}_m^{(\varrho)}$ we denote the m-th smallest point of ξ_ϱ greater than zero and $\underline{D}_m^{(\varrho)}$ stands for the m-th largest point of ξ_ϱ less than zero (with respect to the natural ordering). To neatly formulate a result about the asymptotic distributions of ξ_ϱ, $\overline{D}_m^{(\varrho)}$ and $\underline{D}_m^{(\varrho)}$, we use the following notation. For $\gamma > 0$ and $a, b \in \mathbb{R}$ with $a < b$ let us define

$$\alpha_\varrho(a, b) = \frac{1}{2}\,\mathbb{E}\sum_{(E,F)\in\eta^2_{t,\ne}} \mathbf{1}\big\{m(E, F) \in K_\varrho,\ \varrho^{-\gamma}a < g_\varrho(E, F) \le \varrho^{-\gamma}b\big\},$$

which is the expected number of pairs of flats with midpoint in K_ϱ such that $\varrho^{-\gamma}a < g_\varrho(E, F) \le \varrho^{-\gamma}b$. We further define

$$r_\varrho(u) = \sup_{E\in\mathbb{A}^d_k} \Theta_t\left(\big\{F \in \mathbb{A}^d_k : m(E, F) \in K_\varrho,\ -\varrho^{-\gamma}u \le g_\varrho(E, F) \le \varrho^{-\gamma}u\big\}\right)$$

for any $u > 0$. We are now in the position to formulate a two-sided version of Theorem 1.1 in Schulte and Thäle (2012).

Proposition 2 *Let $\gamma > 0$ and let ν be a σ-finite non-atomic Borel measure on $\mathbb{R}$ such that*

$$\lim_{\varrho\to\infty} \alpha_\varrho(a,b) = \nu\big((a,b]\big) \quad \textit{and} \quad \lim_{\varrho\to\infty} r_\varrho(u) = 0 \tag{25}$$

for any $-\infty < a < b < \infty$ and $u > 0$. Then there is a constant C_u for every $u \ge 0$ such that

$$\left|\mathbb{P}\big(\varrho^\gamma\, \overline{D}_m^{(\varrho)} > u\big) - e^{-\nu((0,u])}\sum_{i=0}^{m-1}\frac{\nu((0,u])^i}{i!}\right| \le \big|\nu((0,u]) - \alpha_\varrho(0,u)\big| + C_u\sqrt{r_\varrho(u)}$$

and

$$\left|\mathbb{P}\big(\varrho^\gamma\, \underline{D}_m^{(\varrho)} < -u\big) - e^{-\nu((-u,0])}\sum_{i=0}^{m-1}\frac{\nu((-u,0])^i}{i!}\right| \le \big|\nu((-u,0]) - \alpha_\varrho(-u,0)\big| + C_u\sqrt{r\varrho(u)}$$

for all $m \in \{1,2,3,\ldots\}$ and $\varrho \in [1,\infty)$. Furthermore, the rescaled point processes $\big(\varrho^\gamma \xi_\varrho\big)_{\varrho\ge 1}$ converge in distribution to a Poisson point process on $\mathbb{R}$ with intensity measure ν.

Remark 9 In Schulte and Thäle (2012), it is assumed that functions $(g_\varrho)_{\varrho\ge1}$ are non-negative and that the measure ν satisfies $\nu(\mathrm{d}u) = \beta\,\tau u^{\tau-1}\,\mathbf{1}\{u>0\}\,\mathrm{d}u$ for some constants $\beta, \tau > 0$. However, these assumptions—tailored to the applications in that paper—can be relaxed so that Proposition 2 can be shown by repeating literally the proof of Theorem 1.1 in Schulte and Thäle (2012).

The assumptions on ν in the statement of Proposition 2 ensure that a Poisson point process on $\mathbb{R}$ with intensity measure ν exists; cf. Chapter 12 in Kallenberg (2002).

In contrast to $\alpha_\varrho(a,b)$, it is not necessary to consider r_ϱ for arbitrary intervals $(a,b]$ because $(a,b] \subset [-u,u]$ for an appropriate choice of u and $\lim_{\varrho\to\infty} r_\varrho(u) = 0$ for all $u > 0$ already implies the same behaviour for all $(a,b]$ with $-\infty < a < b < \infty$.

6.2 Proof of Theorem 4

We apply Proposition 2 to the functions $(g_\varrho)_{\varrho\ge1}$ given by

$$g_\varrho(E,F) = \mathrm{dist}(E,F).$$

It remains to determine γ and the measure ν as well as to check the condition of Eq. 25. As a consequence of Theorem 1, we find that

$$\begin{aligned}\alpha_\varrho(a,b) &= \mathbb{E}\pi_t(K_\varrho, (\varrho^{-\gamma}b)_+) - \mathbb{E}\pi_t(K_\varrho, (\varrho^{-\gamma}a)_+)\\ &= \frac{t^2}{2}\kappa_{d-2k}V_d(K)\,\varrho^d\big((\varrho^{-\gamma}b)_+^{d-2k} - (\varrho^{-\gamma}a)_+^{d-2k}\big)\\ &\quad\times\int_{\mathbb{G}_k^d}\int_{\mathbb{G}_k^d}[M,L]\,\mathbb{Q}(\mathrm{d}L)\,\mathbb{Q}(\mathrm{d}M)\end{aligned}$$

for any reals $a < b$ (here $x_+ = \max\{x,0\}$ for $x \in \mathbb{R}$). Thus, choosing $\gamma = d/(d-2k)$ and putting ν as in the statement of the Theorem, we obtain

$$\alpha_\varrho(a,b) = \nu\big((-\infty,b]\big) - \nu\big((-\infty,a]\big) = \nu\big((a,b]\big)$$

for all $-\infty < a < b < \infty$. Moreover, from Eq. 24 in Step 1 of the proof of Theorem 3 it follows that

$$r_\varrho(u) = \sup_{E\in\mathbb{A}_k^d} \Theta_t\left(\left\{F \in \mathbb{A}_k^d : m(E,F) \in K_\varrho,\ 0 \le \operatorname{dist}(E,F) \le \varrho^{-d/(d-2k)}u\right\}\right)$$
$$\le t\kappa_k\kappa_{d-2k}(\operatorname{diam}(K)/2)^k u^{d-2k}\varrho^{-(d-k)},$$

which because of $k < d$ tends to zero, as $\varrho \to \infty$ for any $u \ge 0$. So, Theorem 4 is a direct consequence of Proposition 2. □

6.3 Proof of Theorem 5

Let us apply Proposition 2 to the family of functions $(g_\varrho)_{\varrho\ge1}$ given by

$$g_\varrho(E,F) = \operatorname{dist}(E,F) - \sigma$$

so that the point process $\xi_\varrho^{(K,t)}$ in Theorem 5 and the point process ξ_ϱ in Proposition 2 are related by $\xi_\varrho^{(K,t)} = \xi_\varrho + \sigma$. As a consequence of Theorem 1, we find that in this case

$$\alpha_\varrho(a,b) = \mathbb{E}\pi_t\left(K_\varrho, \left(\sigma + \varrho^{-\gamma}b\right)_+\right) - \mathbb{E}\pi_t\left(K_\varrho, \left(\sigma + \varrho^{-\gamma}a\right)_+\right)$$
$$= \frac{t^2}{2}\kappa_{d-2k}V_d(K)\varrho^d\left(\left(\sigma + \varrho^{-\gamma}b\right)_+^{d-2k} - \left(\sigma + \varrho^{-\gamma}a\right)_+^{d-2k}\right)$$
$$\times \int_{\mathbb{G}_k^d}\int_{\mathbb{G}_k^d}[M,L]\,\mathbb{Q}(\mathrm{d}L)\,\mathbb{Q}(\mathrm{d}M)$$

for any reals $a < b$ (again, $x_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$). Since $\sigma + \varrho^{-\gamma}a \to \sigma$ and $\sigma + \varrho^{-\gamma}b \to \sigma$ as $\varrho \to \infty$ for all reals $a < b$, the measure ν is this time supported on the whole real axis. Together with $\gamma = d$ in the equation for $\alpha_\varrho(a,b)$ above, we obtain that

$$\lim_{\varrho\to\infty}\alpha_\varrho(a,b) = \beta\,(b-a) \quad \text{for all reals} \quad a < b,$$

where β is given by Eq. 5. Moreover, there is a finite constant $C_{a,b}^{(1)} > 0$ for any $a < b$ also depending on K, t and σ such that $\left|\alpha_\varrho(a,b) - \beta(b-a)\right| \le C_{a,b}^{(1)}\varrho^{-d}$ for $\varrho \ge 1$. Using Lemma 1 and $[M,L] \le 1$ in a similar way as in the proof of Eq. 24, we have for any $u > 0$,

$$r_\varrho(u) = \sup_{E\in\mathbb{A}_k^d} \Theta_t\left(\left\{F \in \mathbb{A}_k^d : m(E,F) \in K_\varrho,\ -\varrho^{-d}u \le \operatorname{dist}(E,F) - \sigma \le \varrho^{-d}u\right\}\right)$$
$$= t \sup_{M\in\mathbb{G}_k^d,\, y\in M^\perp} \int_{\mathbb{G}_k^d}\int_{(M+L)^\perp} V_k\left((K_\varrho - (x/2) - y) \cap M\right)$$
$$\times \mathbf{1}\left\{\sigma - \varrho^{-d}u \le \|x\| \le \sigma + \varrho^{-d}u\right\}\ell_{(M+L)^\perp}(\mathrm{d}x)[M,L]\,\mathbb{Q}(\mathrm{d}L)$$
$$\le t\kappa_k(\operatorname{diam}(K)/2)^k\varrho^k\kappa_{d-2k}\left(\left(\sigma + \varrho^{-d}u\right)^{d-2k} - \left(\sigma - \varrho^{-d}u\right)^{d-2k}\right)$$
$$\le C_u^{(2)}\varrho^{-(d-k)}$$

for all $\varrho \geq 1$ with a finite constant $C_u^{(2)} > 0$ depending on K, t, σ and u. Thus, the conditions in Eq. 25 are satisfied with $\gamma = d$ there and with ν equal to β times the Lebesgue measure on $\mathbb{R}$, where β is given by Eq. 5. Whence Theorem 5 is again a consequence of Proposition 2. □

6.4 Proof of Eq. 4

Since each convex body includes a ball with positive radius, it is sufficient to assume that $K = B_r^d$, where $B_r^d \subset \mathbb{R}^d$ is a ball with fixed radius $r > 0$ around the origin. For $n = 1, 2, 3, \dots$ we define Poisson U-statistics

$$S_n = \frac{1}{2} \sum_{(E,F) \in \eta_{t,\neq}^2} \mathbf{1}\{m(E,F) \in B_r^d,\ a_n < \mathrm{dist}(E,F) \leq b_n\}$$

with $a_n = 2(3n-1)r$ and $b_n = 6nr$. From Theorem 1, it follows that

$$\mathbb{E}S_n = \mathbb{E}\pi_t\left(B_r^d, b_n\right) - \mathbb{E}\pi_t\left(B_r^d, a_n\right) = c_1\left(b_n^{d-2k} - a_n^{d-2k}\right)$$

with $c_1 = \mathbb{E}\pi_t(B_r^d, 1)$. S_n has a Wiener–Itô chaos decomposition

$$S_n = \mathbb{E}S_n + I_1(f_1^{(n)}) + I_2(f_2^{(n)})$$

with kernels

$$f_1^{(n)}(E) = \int_{\mathbb{A}_k^d} \mathbf{1}\left\{m(E,F) \in B_r^d,\ a_n < \mathrm{dist}(E,F) \leq b_n\right\} \Theta_t(\mathrm{d}F)$$

and

$$f_2^{(n)}(E,F) = \frac{1}{2}\mathbf{1}\left\{m(E,F) \in B_r^d,\ a_n < \mathrm{dist}(E,F) \leq b_n\right\}.$$

As a consequence of Lemma 1, we have

$$f_1^{(n)}(E) \leq c_2\left(b_n^{d-2k} - a_n^{d-2k}\right) \quad \text{for} \quad E \in \mathbb{A}_k^d,$$

where $c_2 = t\kappa_k\kappa_{d-2k}\, r^k \int_{\mathbb{G}_k^d} [M, L]\, \mathbb{Q}(\mathrm{d}L)$ and where $M \in \mathbb{G}_k^d$ is E shifted to the origin. Hence, we obtain

$$\mathbb{V}S_n = \| f_1^{(n)}\|_1^2 + 2\| f_2^{(n)}\|_2^2 \leq 2c_2\, \left(b_n^{d-2k} - a_n^{d-2k}\right)\mathbb{E}S_n + \mathbb{E}S_n.$$

In order to belong to a pair (E, F) with $a_n < \mathrm{dist}(E,F) \leq b_n$ and $m(E,F) \in B_r^d$, a flat $E \in \mathbb{A}_k^d$ must satisfy $\frac{a_n}{2} - r < \mathrm{dist}(E, 0) \leq \frac{b_n}{2} + r$. Since $\frac{a_n}{2} - r = (3n-2)r$ and $\frac{b_n}{2} + r = (3n+1)r$, the random variables $(S_n)_{n\in\mathbb{N}}$ are determined by disjoint sets of k-flats and are independent by the Poisson assumption on η_t. As a consequence, the normalized random variables $\widetilde{S}_n = S_n/\mathbb{E}S_n$ with $\mathbb{E}\widetilde{S}_n = 1$ for any $n \geq 1$ are independent, too. Together with the fact that $b_n^{d-2k} - a_n^{d-2k} \geq (b_n - a_n)^{d-2k}$, we obtain

$$\mathbb{V}\widetilde{S}_n = (\mathbb{E}S_n)^{-2}\, \mathbb{V}S_n \leq \frac{2c_2}{c_1} + \frac{1}{c_1\left(b_n^{d-2k} - a_n^{d-2k}\right)} \leq \frac{2c_2}{c_1} + \frac{1}{c_1(2r)^{d-2k}} < \infty.$$

Now, a version of the strong law of large numbers for independent, but not identically distributed random variables yields that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \widetilde{S}_n = 1 \qquad \text{with probability one;}$$

see Kallenberg (2002, Corollary 4.22). Since each S_n is almost surely bounded, this means that there is almost surely a sequence $(n_k)_{k\in\mathbb{N}}$ with $S_{n_k} > 0$ for all k. This implies Eq. 4. □

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