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On the First-Passage Area of a One-Dimensional Jump-Diffusion Process

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Abstract For a one-dimensional jump-diffusion process $X(t)$, starting from $x > 0$, it is studied the probability distribution of the area $A(x)$ swept out by $X(t)$ till its first-passage time below zero. In particular, it is shown that the Laplace transform and the moments of $A(x)$ are solutions to certain partial differential-difference equations with outer conditions. The distribution of the maximum displacement of $X(t)$ is also studied. Finally, some explicit examples are reported, regarding diffusions with and without jumps.

Keywords First-passage time · First-passage area · One-dimensional jump-diffusion

AMS 2000 Subject Classifications 60J60 · 60H05 · 60H10

1 Introduction

This paper is motivated by the aim to extend to one-dimensional jump-diffusion processes analogous results, present in literature, valid for drifted Brownian motion. In fact, several authors have dealt with first-passage area for Brownian motion with negative drift $-\mu$, that is $X(t) = x - \mu t + B_t,\ \ x > 0$, though sometimes they used a slightly different language, derived from Physics (Janson 2007; Kearney and Majumdar 2005; Kearney et al. 2007; Knight 2000; Perman and Wellner 1996); precisely, they focused on the first-passage time $\tau(x)$ of $X(t)$ through zero (which is indeed finite with probability one), and they studied the probability distribution of the random area swept out by $X(t)$ till the time $\tau(x)$, i.e. $A(x) = \int_0^{\tau(x)} X(t)dt$. Another variable that they studied was the maximum displacement $S(x) = \max\{X(t),\ \ t \in [0, \tau(x)]\}$. All these quantities have interesting applications, for instance, in Queueing

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Theory, if one identifies $X(t)$ with the length of a queue at time t, and $\tau(x)$ with the busy period, that is the time until the queue is first empty; then, $A(x)$ represents the cumulative waiting time experienced by all the "customers" during a busy period and $S(x)$ represents the maximum queue length experienced during a busy period.

As for an example from Economics, if the variable t represents the quantity of a commodity that producers have available for sale and $X(t)$, with $X(0) = x$, describes the price of the commodity as a function of the quantity in a supply-and-demand model, one can identify $\tau(x)$ with the amount of product at which the price falls to practically zero, while the area $A(x)$ under the demand curve provides a measure of the total value that consumers receive from consuming that amount of the product.

Other existing results in the literature concern the integral of the absolute value of Brownian motion over the interval [0, 1], i.e. $\int_0^1 |B_t| dt$, and the integral of Brownian excursion, i.e. $\int_0^1 B_t^+ dt$, where $B_t^+ = \max(B_t, 0)$ (see Janson 2007; Perman and Wellner 1996).

In the present article, we aim to extend results about random areas for Brownian motion and drifted Brownian motion to more general processes, such as one-dimensional diffusions with jumps; some results for the integral of certain jump-diffusion processes over a deterministic and fixed time interval were found in Abundo (2008).

We focus on one-dimensional dynamical systems whose time evolution is described by a so-called jump-diffusion equation. The state $X(t)$ of our system at time t can be considered as the size of a generalized population; at difference of a diffusion equation, the time trajectories are not continuous, but upward or downward jumps may occur at random instants. So, the process $X(t)$ turns out to be a piecewise-continuous process. We will suppose that $X(t)$ is the solution of a stochastic differential equation of the form:

$$dX(t) = b(X(t))dt + \sigma(X(t))dB_t + \int_{-\infty}^{+\infty} \gamma(X(t), u)\nu(dt, du) \tag{1.1}$$

with assigned initial condition $X(0) = x > 0$; here B_t is a standard Brownian motion, $\nu(\cdot, \cdot)$ is a temporally homogeneous Poisson random measure (see Section 2 for the definitions), and the functions $b(\cdot), \sigma(\cdot), \gamma(\cdot, \cdot)$ satisfy suitable conditions for the existence and uniqueness of the solution (see Section 2). The coefficients regulate the drift (b), the diffusion (σ), and the sizes of the jumps (γ) which occur at (random) exponentially distributed time intervals. The process $X(t)$ which is the solution of the Eq. 1.1 reduces to a simple diffusion (i.e. without jumps) if $\gamma(x, u) = 0$, and in particular to Brownian motion with negative drift if $b(x) = -\mu$ and $\sigma(x) = 1$.

Denote by $\tau(x)$ the first-passage time below zero of the process $X(t)$ starting from $x > 0$, and assume that $\tau(x)$ is finite with probability one. We will study the probability distributions of $\tau(x)$ and of the area swept out by $X(t)$ till its first-passage time, as well their moments; moreover, the distribution of the maximum displacement of $X(t)$ will be studied. In particular, we will show that the Laplace transforms of $A(x)$ and $\tau(x)$, their moments, as well the probability distribution of the maximum displacement of $X(t)$, are solutions to certain partial differential-difference equations (PDDEs) with outer conditions.

The paper is organized in the following way: Section 2 contains the statement of the problem and main results, in Section 3 some explicit examples are reported. Finally, Section 4 is devoted to conclusions and final remarks.

2 Notations, Formulation of the Problem and Main Results

Let $X(t) \in I := (0, a)$, $0 < a \leq +\infty$, be a time-homogeneous, one-dimensional jump-diffusion process which satisfies the stochastic differential equation (SDE) (1.1) with assigned initial condition $X(0) = x > 0$, where B_t is a standard Brownian motion and $\nu(\cdot, \cdot)$ is a Poisson random measure on $(0, \infty) \times \mathbb{R}$. Then, $X(t)$ can be represented as

$$X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB_s + \int_0^t \int_{-\infty}^{+\infty} \gamma(X(s), u)\nu(ds, du) \quad (2.1)$$

For the definitions of the integrals in the right hand side of Eq. 2.1 and the Poisson measure, see Gihman and Skorohod (1972). The coefficients b, σ and γ completely specify the law of $X(t)$. In particular, an atom (t, u) of the Poisson random measure ν causes a jump from x to $x + \gamma(x, u)$ at time t, if $X(t^-) = x$. We assume that ν is homogeneous with respect to time translation, that is, its intensity measure $E(\nu(dt, du))$ is of the form

$$E[\nu(dt, du)] = dt\pi(du) \quad (2.2)$$

for some positive measure π defined on $\mathcal{B}(\mathbb{R})$ ($\mathcal{B}(\mathbb{R})$ denotes the Borel $\sigma-$field of subsets of $\mathbb{R}$), and we suppose that the jump intensity

$$\Theta = \int_{-\infty}^{+\infty} \pi(du) \geq 0 \quad (2.3)$$

is finite.

We make the following assumptions on the coefficients:

A1 $b, \sigma : I \to \mathbb{R}$ are continuous functions and a constant $K > 0$ exists, such that, for every $x, y \in I$:

$$|b(x) - b(y)| \leq K|x - y|$$

$$b^2(x) + \sigma^2(x) \leq K(1 + x^2)$$

A2 σ is a non-negative, bounded function and it is differentiable for every x belonging to the interior of I. Moreover, there exists a strictly increasing function $G : \mathbb{R}^+ \to \mathbb{R}$ such that $G(0) = 0$, $\int_{0^+} G^{-2}(s)ds = +\infty$ and

$$|\sigma(x) - \sigma(y)| \leq G(|x - y|)$$

for every $x, y \in I$.

B1 For every u and $x \in I$:

$$\int_{-\infty}^{+\infty} \gamma^2(x,u)\pi(du) \le K(1+x^2)$$

B2 For every u and $x, y \in I$:

$$\int_{-\infty}^{+\infty} |\gamma(x,u) - \gamma(y,u)|\pi(du) \le K|x-y|$$

Remark 2.1 The conditions A1, A2, and B1, B2 ensure that there exists a unique non-explosive solution of Eq. 1.1 which is a temporally homogeneous Markov process (see Gihman and Skorohod 1972; Ikeda and Watanabe 1981); A2 holds, for instance, if $\sigma(\cdot)$ is Lipschitz-continuous ($G(x) = const \cdot x$), or Hölder-continuous of order $\alpha \in [1/2, 1)$ ($G(x) = const \cdot x^\alpha$).

Notice that, if $\gamma = 0$, or $\nu = 0$, then the SDE (1.1) becomes the usual Itô's stochastic differential equation for a simple-diffusion (i.e. without jumps).

In the special case when the measure π is concentrated over the set $\{u_1, u_2\} = \{-1, 1\}$ with $\pi(u_i) = \theta_i$ and $\gamma(u_i) = \epsilon_i$, we can rewrite the SDE (1.1) as

$$dX(t) = b(X(t))dt + \sigma(X(t))dB_t + \epsilon_2 dN_2(t) + \epsilon_1 dN_1(t) \tag{2.4}$$

where $\epsilon_1 < 0$, $\epsilon_2 > 0$ and $N_i(t), t \ge 0$ are independent homogeneous Poisson processes of amplitude 1 and rates θ_1 and θ_2, respectively governing downward (N_1) and upward (N_2) jumps.

Let D be the class of function $f(t,x)$ defined in $\mathbb{R}^+ \times \mathrm{I}$, differentiable with respect to t and twice differentiable with respect to x, for which the function $f(t, x+\gamma(x,u)) - f(t,x)$ is $\pi-$integrable for any (t,x). We recall the generalized Itô's formula for jump-diffusion processes giving the differential of a function $f \in D$ (see Gihman and Skorohod 1972):

$$\begin{aligned} df(t,X(t)) = &\left[\frac{\partial f}{\partial t}(t,X(t)) + b(X(t))\frac{\partial f}{\partial x}(t,X(t)) + \frac{1}{2}\sigma^2(X(t))\frac{\partial^2 f}{\partial x^2}(t,X(t))\right]dt \\ &+ \frac{\partial f}{\partial x}(t,X(t))\sigma(X(t))dB_t \\ &+ \int_{-\infty}^{+\infty} \left[f(t, X(t)+\gamma(X(t),u)) - f(t,X(t))\right]\nu(dt,du). \end{aligned} \tag{2.5}$$

The differential operator associated to the process $X(t)$ which is the solution of Eq. 1.1, is defined for any function $f \in D$ by:

$$Lf(t,x) = L_d f(t,x) + L_j f(t,x) \tag{2.6}$$

where the "diffusion part" is

$$L_d f(t,x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2}(t,x) + b(x)\frac{\partial f}{\partial x}(t,x)$$

and the "jump part" is

$$L_j f(t,x) = \int_{-\infty}^{+\infty} [f(t,x+\gamma(x,u)) - f(t,x)]\pi(du).$$

Then, from Eq. 2.5, taking expectation, one obtains

$$E[f(t,X(t))] = f(0,X(0)) + E\left(\int_0^t \left[\frac{\partial f}{\partial s}(s,X(s)) + Lf(s,X(s))\right] ds\right). \tag{2.7}$$

Let us define, for $x > 0$:

$$\tau(x) = \inf\{t > 0 : X(t) \le 0 | X(0) = x\} \tag{2.8}$$

that is the first-passage time below zero of $X^x(t)$ (i.e. the process $X(t)$ starting from x) and suppose that $\tau(x)$ is finite with probability one. Really, it is possible to show (see Abundo 2000; Tuckwell 1976) that the probability $p_0(x)$ that $X^x(t)$ ever leaves the interval $(0,+\infty)$ satisfies the partial differential–difference equation (PDDE):

$$Lp_0 = 0 \tag{2.9}$$

with outer condition:

$$p_0(x) = 1 \text{ if } x \le 0.$$

The equality $p_0(x) = 1$ is equivalent to say that $\tau(x)$ is finite with probability one. For diffusion processes without jumps (i.e. $\gamma = 0$) sufficient conditions are also available which ensure that $\tau(x)$ is finite w.p. 1, and they concern the convergence of certain integral associated to the coefficients of Eq. 1.1 (see Section 3.1 and also Gihman and Skorohod 1972; Has' minskij 1980).

Remark 2.2 Let us consider the special case when $X(t) = \widetilde{X}(t) + J(t)$, where $\widetilde{X}$ is a simple-diffusion (i.e. without jumps) and J is a pure-jump process, set $\widetilde{m}(t) = E(\widetilde{X}(t))$, and suppose that

$$\exists\, \bar{t} > 0 : \widetilde{m}(t) + E(J(t)) < 0, \text{ for any } t \ge \bar{t}; \tag{2.10}$$

then, $P(\tau(x) < \infty) = 1$. Indeed, $P(\tau(x) = \infty) > 0$ implies that, for a set of trajectories having positive probability, it results $X(t) > 0$ for any $t \ge 0$. Thus, by taking expectation one obtains $E(X(t)) = \widetilde{m}(t) + E(J(t)) > 0$ which contradicts Eq. 2.10.

If $J(t) = \epsilon_1 N_1(t) + \epsilon_2 N_2(t)$ (cf. Eq. 2.4), condition (2.10) becomes $\widetilde{m}(t) + (\epsilon_1\theta_1 + \epsilon_2\theta_2)t < 0$; in particular, for $\epsilon_2 = \epsilon = -\epsilon_1$ it writes $\widetilde{m}(t) < \epsilon(\theta_1 - \theta_2)t$.

Let U be a functional of the process X; assume that $\tau(x)$ is finite with probability one, and for $\lambda > 0$ denote by

$$M_{U,\lambda}(x) = E\left[e^{-\lambda \int_0^{\tau(x)} U(X(s))ds}\right] \tag{2.11}$$

the Laplace transform of the integral $\int_0^{\tau(x)} U(X(s))ds$. Then, it holds:

Theorem 2.3 *Let $X(t)$ be the solution of the SDE (1.1), starting from $X(0) = x > 0$; then, under the above assumptions, $M_{U,\lambda}(x)$ is the solution of the problem with outer conditions:*

$$\begin{cases} LM_{U,\lambda}(x) = \lambda U(x) M_{U,\lambda}(x) \\ M_{U,\lambda}(y) = 1, \text{ for } y \leq 0 \\ \lim_{x\to+\infty} M_{U,\lambda}(x) = 0 \end{cases} \tag{2.12}$$

where L is the generator of X, which is defined by Eq. 2.6.

Proof It easily follows from the next lemma, by taking $r(x) = \lambda U(x)$, $f(x) \equiv 1$, $T = \min(t, \tau(x))$, and letting t go to infinity. For a diffusion process, the result of Lemma 2.4 is a special case of the well-known Feynman–Kac formula (see Klebaner 1999). Although it is rather straightforward to show that it holds also for jump-diffusion equations, we report the proof for the completeness of the exposition. □

Lemma 2.4 *Let $X(t)$ be the solution of the jump-diffusion equation:*

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dB_t + \int_{-\infty}^{+\infty} \gamma(X(t), u)\nu(dt, du) \\ X(0) = x \end{cases} \tag{2.13}$$

where the coefficients b, σ and γ satisfy the assumptions A1, A2 and B1, B2 for the existence and uniqueness of the solution, and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a twice differentiable function of x and $r(x)$ a continuous function. Let $v(x)$ be the solution of the equation

$$Lv(x) = r(x)v(x) \tag{2.14}$$

with the final condition

$$v(X(T)) = f(X(T)). \tag{2.15}$$

Then it holds

$$v(x) = E^x[f(X(T))e^{-\int_0^T r(X(t))dt}], \; 0 \leq t \leq T. \tag{2.16}$$

Proof Set $V(t) = v(X(t))$, $R(t) = r(X(t))$. By the generalized Itô's formula (see Eq. 2.5) we obtain

$$dV(t) = \left(Lv - \int \gamma_0(t)\pi(du)\right)dt + \beta_0(t)dB_t + \int \gamma_0(t)\nu(dt, du)$$

where

$$\gamma_0(t) = v\big(X(t) + \gamma(X(t), u)\big) - v(X(t));\ \ \beta_0(t) = v'(X(t))\sigma(X(t)).$$

By using Eq. 2.14, we get

$$dV(t) = [\alpha_1(t)v(X(t)) + \alpha_0(t)]dt + \beta_0(t)dB_t + \int \gamma_0(t)\nu(dt, du) \tag{2.17}$$

where

$$\alpha_0(t) = -\int \gamma_0(t)\pi(du),\ \ \alpha_1(t) = R(t).$$

Equation 2.17 is a linear SDE whose solution with initial condition $V(0) = v(X(0)) = v(x)$ is explicitly given by (see Gihman and Skorohod 1972, p. 271):

$$\begin{aligned} V(t) &= e^{\int_0^t R(s)ds} \Big\{ v(x) + \int_0^t \beta_0(s)\xi_0(s)dB_s + \int \xi_0(s)\alpha_0(s)ds \\ &\qquad + \int_0^t \int \gamma_0(s)\xi_0(s)\nu(ds, du) \Big\} \\ &= e^{\int_0^t R(s)ds} \Big\{ v(x) + \int_0^t \beta_0(s)\xi_0(s)dB_s + \int_0^t \int \xi_0(s)[v(X(s) + \gamma(X(s), u)) \\ &\qquad - v(X(s))](\nu(ds, du) - \pi(du)ds) \Big\} \end{aligned}$$

where $\xi_0(t) = e^{-\int_0^t R(s)ds}$.

Now, setting $t = T$, multiplying both members by $e^{-\int_0^T R(s)ds}$, taking expectation in the above equation and using Eq. 2.15, we easily obtain Eq. 2.16. □

Remark 2.5 Theorem 2.3 can be also proved directly by using an approach based on Kolmogorov's equation. For the sake of simplicity, we outline this kind of proof for a diffusion without jumps, i.e. $\gamma = 0$. For $\Delta t > 0$, we split the integral $\int_0^{\tau(x)} U(X(s))ds$ into two parts, obtaining

$$M_{U,\lambda}(x) = E\left[e^{-\lambda \int_0^{\Delta t} U(X(s))ds} e^{-\lambda \int_{\Delta t}^{\tau(x)} U(X(s))ds}\right]. \tag{2.18}$$

Setting $u = s - \Delta t$, and by considering the process Y such that $Y(u) = X(u + \Delta t)$, $Y(0) = X(\Delta t) = x + \Delta x$, that is the solution to Eq. 1.1 (with $\gamma = 0$) starting from $x + \Delta x$ at time 0, the second integral in Eq. 2.18 becomes

$$\int_0^{\tau_1(x)} U(Y(u))du$$

where $\tau_1(x) = \tau(x) - \Delta t = \tau(x + \Delta x)$. Then the second exponential in Eq. 2.18 becomes $M_{U,\lambda}(x + \Delta x)$.

The first integral is equal to $U(x)\Delta t + o(\Delta t)$, so the first exponential in Eq. 2.18 becomes $1 - \lambda U(x)\Delta t + o(\Delta t)$. Thus, from Eq. 2.18, we finally obtain at the first order in Δt:

$$M_{U,\lambda}(x) = (1 - \lambda U(x)\Delta t) E[M_{U,\lambda}(x + \Delta x)].$$

By Itô's formula, we get $E[M_{U,\lambda}(x + \Delta x)] = M_{U,\lambda}(x) + LM_{U,\lambda}(x)\Delta t$, so at the first order in Δt :

$$\begin{aligned} M_{U,\lambda}(x) &= (1 - \lambda U(x)\Delta t)(M_{U,\lambda}(x) + LM_{U,\lambda}(x)\Delta t) \\ &= M_{U,\lambda}(x) + (LM_{U,\lambda}(x) - \lambda U(x)M_{U,\lambda}(x))\Delta t \end{aligned}$$

from which it follows $LM_{U,\lambda}(x) - \lambda U(x)M_{U,\lambda}(x) = 0$, that is the equation in Eq. 2.12. The first condition in Eq. 2.12 (i.e. $M_{U,\lambda}(0) = 1$ in the present case) immediately follows since, if the process starts from 0, then it takes zero time to reach the position zero, while the second condition follows by the fact that the process starting from $+\infty$ takes an infinite time to reach the origin.

We recall that the nth order moment of $\int_0^{\tau(x)} U(X(s))ds$, if it exists finite, is given by $(n = 1, 2, \dots)$:

$$T_n(x) = E\left[\left(\int_0^{\tau(x)} U(X(s))ds\right)^n\right] = (-1)^n \left[\frac{\partial^n}{\partial \lambda^n} M_{U,\lambda}(x)\right]_{\lambda=0}.$$

Then, taking the nth derivative with respect to λ in both members of the Eq. 2.12, and calculating it for $\lambda = 0$, one easily obtains that the nth order moment $T_n(x)$ $(n = 1, 2, \dots)$ of $\int_0^{\tau(x)} U(X(s))ds$, whenever it exists finite, is the solution of the PDDE:

$$LT_n(x) = -nU(x)T_{n-1}(x), \quad x > 0 \tag{2.19}$$

which satisfies

$$T_n(x) = 0, \ \text{for } x \le 0 \tag{2.20}$$

and an appropriate additional condition.

Indeed, the only condition $T_n(x) = 0$ for $x \le 0$ is not sufficient to determinate uniquely the desired solution of the PDDE (2.19), because it is a second order equation. We will return to this problem when we will consider some explicit examples. Note that for a diffusion without jumps ($\gamma = 0$) and for $U(x) \equiv 1$, Eq. 2.19 is nothing but the celebrated Darling and Siegert's equation (Darling and Siegert 1953) for the moments of the first-passage time, and Eq. 2.20 becomes simply the boundary condition $T_n(0) = 0$.

2.1 Distribution of the Maximum Displacement

In this subsection, we will study the probability distribution of the maximum displacement of the jump-diffusion $X(t)$ starting from $x > 0$, till its first passage below zero, that is $S(x) = \max_{t\in[0,\tau(x)]}\{X(t)|X(0) = x\}$. For any $z > 0$, the event $\{S(x) \leq z\}$ is nothing but the event "$X(t)$ first exit the interval $[0, z]$ through the left end 0"; so, by the well-known result about the exit probability of a jump-diffusion from the left end of an interval (see Abundo 2000), we obtain:

Proposition 2.6 *The probability distribution of $S(x)$, i.e. $P(S(x) \leq z)$, is the solution of the problem with outer conditions:*

$$\begin{cases} Lw(x) = 0, \ x \in (0, z) \\ w(y) = 1, \ y \leq 0 \\ w(y) = 0, \ y \geq z \end{cases} \tag{2.21}$$

3 A Few Examples

In this section we will compute explicitly the moments of $\tau(x)$ and those of the first-passage area $A(x)$ for certain jump-diffusion processes. We start with considering diffusions without jumps.

3.1 Simple Diffusions (i.e with no Jump)

Let $X(t)$ be the solution of Eq. 1.1, with $\gamma \equiv 0$, that is:

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dB_t \\ X(0) = x > 0 \end{cases} \tag{3.1}$$

In this case $\tau(x) \equiv \inf\{t > 0 : X(t) = 0|X(0) = x\}$, that is the first-passage time of $X^x(t)$ through zero.

Let us consider the functions (c is a constant):

$$\phi(x) = \exp\left(-\int_c^x \frac{2b(s)}{\sigma^2(s)}ds\right) \tag{3.2}$$

$$\xi(x) = \phi(x)\int_c^x \frac{2}{\sigma^2(s)\phi(s)}ds. \tag{3.3}$$

As it is well-known (see e.g. Gihman and Skorohod 1972), a sufficient condition in order that $\tau(x)$ is finite with probability one, namely the boundary 0 is attainable, is that the function $\xi(x)$ is integrable in a neighbor of 0.

Since the generator L coincides with its diffusion part L_d, by Theorem 2.3 we obtain that $M_{U,\lambda}(x) = E\left[e^{-\lambda\int_0^{\tau(x)} U(X(s))ds}\right]$ is the solution of the problem with boundary conditions (M' and M'' denote first and second derivative with respect to x):

$$\begin{cases} \frac{1}{2}\sigma^2(x)M''_{U,\lambda}(x) + b(x)M'_{U,\lambda}(x) = \lambda U(x)M_{U,\lambda}(x) \\ M_{U,\lambda}(0) = 1 \\ \lim_{x\to+\infty} M_{U,\lambda}(x) = 0 \end{cases} . \tag{3.4}$$

Moreover, by Eqs. 2.19 and 2.20 the nth order moments $T_n(x)$ of $\int_0^{\tau(x)} U(X(s))ds$, if they exists, satisfy the recursive ODEs:

$$\frac{1}{2}\sigma^2(x)T_n''(x) + b(x)T_n'(x) = -nU(x)T_{n-1}(x), \text{ for } x > 0 \tag{3.5}$$

with the condition $T_n(0) = 0$, plus an appropriate additional condition.

Finally, as for the maximum displacement $S(x)$, its distribution $F(z) = P(S(x) \le z)$ is the solution of the problem with boundary conditions:

$$\begin{cases} Lw(x) = 0, \ x \in (0, z) \\ w(0) = 1 \\ w(z) = 0 \end{cases} . \tag{3.6}$$

Example 1 (Brownian Motion with Drift $-\mu < 0$) Let be $X(t) = x - \mu t + \sigma B_t$, with $\mu, \sigma > 0$. Without loss of generality, we can assume $\sigma = 1$ (otherwise dividing by σ one reduces to this case). Note that, since the drift is negative, $\tau(x) = \tau^\mu(x) = \inf\{t > 0 | X(t) = 0\}$ is finite with probability one, for any $x > 0$. Taking $b(x) = -\mu,\ \sigma(x) = 1$, the equation in Eq. 3.4 for $M_{U,\lambda}(x) = E(e^{-\lambda \int_0^{\tau^\mu(x)} U(X(s)ds})$ becomes

$$\frac{1}{2}M_{U,\lambda}''(x) - \mu M_{U,\lambda}'(x) - \lambda U(x)M_{U,\lambda}(x) = 0. \tag{3.7}$$

(i) The moment generating function of $\tau^\mu(x)$
By solving Eq. 3.7 with $U(x) = 1$, we explicitly obtain:

$$M_{U,\lambda}(x) = c_1 e^{\rho_1 x} + c_2 e^{\rho_2 x}$$

where $\rho_1 = \mu - \sqrt{\mu^2 + 2\lambda} < 0,\ \rho_2 = \mu + \sqrt{\mu^2 + 2\lambda} > 0$; the constants c_1 and c_2 must be determined by the boundary conditions. Indeed, $M_{U,\lambda}(0) = 1$ implies $c_1 + c_2 = 1$, while $M_{U,\lambda}(+\infty) = 0$ gives $c_2 = 0$. Thus, we get:

$$M_{U,\lambda}(x) = E(\exp(-\lambda\tau^\mu(x)) = \exp[(\mu - \sqrt{\mu^2 + 2\lambda})x]. \tag{3.8}$$

This Laplace transform can be explicitly inverted (see Karlin and Taylor 1975), so obtaining the well-known expression of the density of $\tau^\mu(x)$:

$$f_{\tau^\mu(x)}(t) = \frac{x}{\sqrt{2\pi}t^{3/2}} e^{-(x-\mu t)^2/2t}. \tag{3.9}$$

For $\mu > 0$ the moments $T_n(x) = E(\tau^\mu(x))^n$ of any order n, are finite and they can be easily obtained by calculating $(-1)^n[\partial^n M_{U,\lambda}(x)/\partial\lambda^n]_{\lambda=0}$. We obtain, for instance, $E(\tau^\mu(x)) = \frac{x}{\mu}$ and $E((\tau^\mu(x))^2) = \frac{x}{\mu^3} + \frac{x^2}{\mu^2}$. Note that $E(\tau^\mu(x)) \to +\infty$, as $\mu \to 0$. As easily seen, for any $x > 0$ it results $M_{U,\lambda}(x) \to 1$, as $\mu \to +\infty$, or, equivalently, $\tau^\mu(x)$ converges to 0 in distribution, and so $E((\tau^\mu(x))^n) \to 0$, as $\mu \to +\infty$, for any n and $x > 0$.

(ii) The moments of $A^\mu(x) = \int_0^{\tau^\mu(x)} (x - \mu t + B_t)dt$
For $U(x) = x$ the Eq. 3.7 becomes

$$\frac{1}{2}M_{U,\lambda}'' - \mu M_{U,\lambda}' = \lambda x M_{U,\lambda} \tag{3.10}$$

where now $M_{U,\lambda}(x) = E(e^{-\lambda A^\mu(x)})$. It is the Schrodinger equation for a quantum particle moving in a uniform field (see e.g. Kearney and Majumdar 2005; Kearney et al. 2007). Unfortunately, its explicit solution cannot be found in terms of elementary functions, but it can be written in terms of the Airy function (see Kearney and Majumdar 2005; Grandshteyn and Ryzhik 1980) though, for $\mu \neq 0$, it is impossible to invert the Laplace transform $M_{U,\lambda}$ to obtain the probability density of $A^\mu(x)$.

In the special case $\mu = 0$, it can be shown (Kearney and Majumdar 2005) that the solution of Eq. 3.10 is:

$$M_{U,\lambda}(x) = 3^{2/3}\Gamma\left(\frac{2}{3}\right) \mathrm{Ai}(2^{1/3}\lambda^{1/3}x) \tag{3.11}$$

where Ai(x) denotes the Airy function (see Grandshteyn and Ryzhik 1980); then, by inverting this Laplace transform one finds that the first-passage area density is (Kearney and Majumdar 2005):

$$f_{A^0(x)}(a) = \frac{2^{1/3}}{3^{2/3}\Gamma(\frac{1}{3})} \frac{x}{a^{4/3}} e^{-2x^3/9a} . \tag{3.12}$$

Thus, the distribution of $A^0(x)$ has an algebraic tail of order $\frac{4}{3}$ and so the moments of all orders are infinite.

For $\mu > 0$, we will find closed form expression for the first two moments of $A^\mu(x)$, by solving Eq. 3.5 with $U(x) = x$. For $n = 1$, we get that $T_1(x) = E(A^\mu(x))$ must satisfy the equation:

$$\begin{cases} \frac{1}{2}T_1''(x) - \mu T_1'(x) = -x \\ T_1(0) = 0 \end{cases} \tag{3.13}$$

Of course, the only condition $T_1(0) = 0$ is not sufficient to uniquely determinate the solution. The general solution of Eq. 3.13 involves an arbitrary constant c and, as easily seen, it is given by

$$T_1(x) = c(1 - e^{2\mu x}) + \frac{x^2}{2\mu} + \frac{x}{2\mu^2} .$$

By imposing that $T_1(x) \to +\infty$, as $x \to +\infty$, we find that c must be nonpositive. Moreover, by the condition that $T_1(x) \to 0$, as $\mu \to +\infty$, we finally obtain $c = 0$, thus the mean first-passage area is

$$E(A^\mu(x)) = \frac{x^2}{2\mu} + \frac{x}{2\mu^2} . \tag{3.14}$$

As far as the second moment of $A^\mu(x)$ is concerned, we have to solve Eq. 3.5 with $U(x) = x$ and $n = 2$, obtaining the equation for $T_2(x) = E[(A^\mu(x))^2]$:

$$\begin{cases} \frac{1}{2}T_2''(x) - \mu T_2'(x) = -2x\left(\frac{x^2}{2\mu} + \frac{x}{2\mu^2}\right) \\ T_2(0) = 0 \end{cases} \tag{3.15}$$

As before, the only condition $T_2(0) = 0$ is not sufficient to uniquely determinate the solution, The general solution of Eq. 3.15, which involves an arbitrary constant c, is given by

$$T_2(x) = c(1 - e^{2\mu x}) + \frac{x^4}{4\mu^2} + \frac{5x^3}{6\mu^3} + \frac{5x^2}{4\mu^4} + \frac{5x}{4\mu^5} .$$

By imposing that $T_2(x) \to +\infty$, as $x \to +\infty$, we find that c must be nonpositive. Moreover, by the condition that $T_2(x) \to 0$, as $\mu \to +\infty$, we finally obtain $c = 0$; thus the second order moment of the first-passage area is

$$E[(A^\mu(x))^2] = \frac{x^4}{4\mu^2} + \frac{5x^3}{6\mu^3} + \frac{5x^2}{4\mu^4} + \frac{5x}{4\mu^5} . \tag{3.16}$$

Finally, by Eqs. 3.14 and 3.16, we obtain the variance of $A^\mu(x)$:

$$Var(A^\mu(x)) = E(A^\mu(x)^2) - E^2(A^\mu(x)) = \frac{x^3}{3\mu^3} + \frac{x^2}{\mu^4} + \frac{5x}{4\mu^5} . \tag{3.17}$$

As for the density of the first-passage area $A^\mu(x)$, since a closed form expression cannot be found for $\mu > 0$, we estimate it numerically by simulating a large number of trajectories of Brownian motion with drift $-\mu$, starting from the initial state $x > 0$ (see Abundo and Abundo 2011); in Fig. 1 the estimated density of the first-passage area is reported, for several values of the parameter μ.

(iii) The distribution of the maximum displacement $S^\mu(x)$

The distribution function $P(S^\mu(x) \leq z)$ is the solution of the problem with boundary conditions:

$$\begin{cases} \frac{1}{2} w''(x) - \mu w'(x) = 0 \\ w(0) = 1, \ \ w(z) = 0 \end{cases} \tag{3.18}$$

By solving the above equation, we obtain, for $z \geq x$:

$$P(S^\mu(x) \leq z) = w(x) = \frac{e^{2\mu x} - e^{2\mu z}}{1 - e^{2\mu z}}.$$

Then, calculating the derivative with respect to z, we get the probability density of $S^\mu(x)$:

$$f_{S^\mu(x)}(z) = \frac{2\mu e^{2\mu z}\left(e^{2\mu x} - 1\right)}{\left(1 - e^{2\mu z}\right)^2} = \mu e^{\mu x} \frac{\sinh(\mu x)}{\sinh^2(\mu z)}, \ \ z \geq x > 0.$$

For $\mu = 0$, taking the limit in the above expression, we get $f_{S^0(x)}(z) = \frac{x}{z^2}$, and so the moments of $S^0(x)$ of all orders $n \geq 1$ are infinite.

On the contrary, for $\mu > 0$ the maximum displacement $S^\mu(x)$ possesses finite moments of all orders; for instance its mean is

$$E(S^\mu(x)) = \int_x^{+\infty} z f_S^\mu(z) dz = x + \int_x^{+\infty} P(S^\mu(x) > z) dz$$

$$= x e^{2\mu x} + \frac{e^{2\mu x} - 1}{2\mu} \ln\left(\frac{1}{1 - e^{-2\mu x}}\right).$$

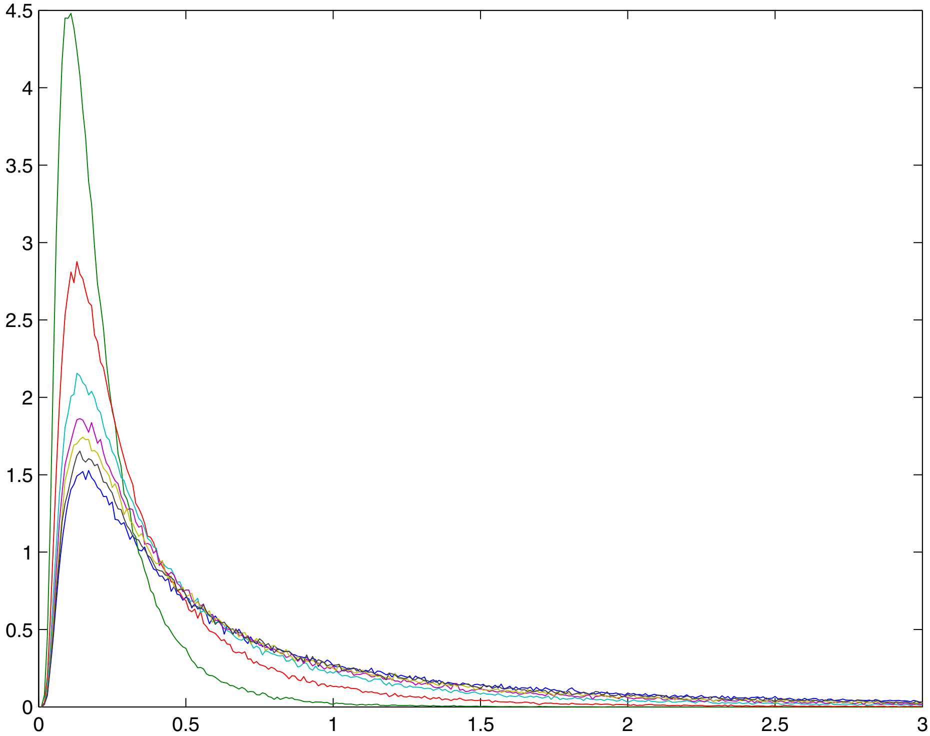

Fig. 1 Estimated density of the first-passage area $A^{\mu}(x)$ for $x = 1$ and several values of μ. From top to bottom, with respect to the peak of the curve: $\mu = 3$; $\mu = 2$; $\mu = 1.5$; $\mu = 1.3$; $\mu = 1.2$; $\mu = 1.1$; $\mu = 1$

Definition 3.1 We say that a one-dimensional diffusion $X(t)$ with $X(0) = x$, is conjugated to BM if there exists an increasing differentiable function $u(x)$, with $u(0) = 0$, such that $X(t) = u^{-1}\left(B_t + u(x)\right)$.

Remark 3.2 If $X(t)$ is conjugated to Brownian motion via the function u, then

$$\begin{aligned}\tau(x) &= \inf\{t > 0 : X(t) = 0 | X(0) = x\} = \inf\{t > 0 : u(X(t)) = 0 | u(X(0)) = u(x)\} \\ &= \inf\{t > 0 : B_t + u(x) = 0\} = \tau^0(u(x))\end{aligned}$$

where $\tau^0(y)$ is the first hitting time to zero of $y + B_t$ (i.e. BM without drift starting from y). Thus, the first-passage time through zero of the process $X(t)$ starting from x, is nothing but the first hitting time to zero of BM starting from $u(x)$, and

$$A(x) = \int_0^{\tau(x)} X(t)dt = \int_0^{\tau^0(u(x))} u^{-1}(B_t + u(x))dt.$$

Note that, though $\tau(x)$ turns out to be finite with probability one, it results $E(\tau(x)) = +\infty$. Moreover

$$M_{U,\lambda}(x) = E\left[\exp\left(-\lambda \int_0^{\tau(x)} U(X(t))dt\right)\right] = E\left[\exp\left(-\lambda \int_0^{\tau^0(u(x))} V(B_t)dt\right)\right),$$

where $V(s) = U\left(u^{-1}(s + u(x))\right)$.

Therefore, the Laplace transform $M_{U,\lambda}(x)$ of $\int_0^{\tau(x)} U(X(t))dt$, associated to the functional U of the process X, is nothing but the Laplace transform $M^0_{V,\lambda}(y)$ of $\int_0^{\tau^0(y)} V(y)dt$, associated to the functional V of BM, where $y = u(x)$.

This means that the Eq. 3.4 is easily reduced to the analogous equation for BM starting from $u(x)$, with U replaced by V.

Example 2 A class of diffusions conjugated to BM is given by processes $X(t)$ which are solutions of SDEs such as

$$dX(t) = \frac{1}{2}\sigma(X(t))\sigma'(X(t))dt + \sigma(X(t))dB_t,\ X(0) = x_0 \tag{3.19}$$

with $\sigma(\cdot) \geq 0$. Indeed, if the integral $v(x) \doteq \int_{x_0}^{x} \frac{1}{\sigma(r)}dr$ is convergent for every x, by Itô's formula, we obtain that $X(t) = v^{-1}(B_t + v(x_0))$.

(i) (Feller Process)
For $a, b \geq 0$, let $X(t)$ be the solution of the SDE

$$dX(t) = (a + b\,X(t))dt + \sqrt{X(t) \vee 0}\,dB_t\,,\ X(0) = x_0 \geq 0 \tag{3.20}$$

(note that, although $\sqrt{x}$ is not Lipschitz-continuous, the solution is unique because $\sqrt{x}$ is Hölder-continuous of order $\frac{1}{2}$ (see condition A2)). The process $X(t)$ turns out to be non-negative for all $t \geq 0$ (see Abundo 1997). If $b = 0$ and $a = \frac{1}{4}$, the SDE (3.20) becomes:

$$dX(t) = \frac{1}{4}dt + \sqrt{X(t) \vee 0}\,dB_t\,,\ X(0) = x_0$$

and $X(t)$ turns out to be conjugate to BM via the function $v(x) = 2\sqrt{x}$ i.e. $X(t) = \frac{1}{4}(B_t + 2\sqrt{x_0})^2$; the SDE is obtained by taking $\sigma(x) = \sqrt{x \vee 0}$ in Eq. 3.19.

(ii) (Wright & Fisher-like Process)
The diffusion described by the SDE:

$$dX(t) = (a + b\,X(t))dt + \sqrt{X(t)(1 - X(t)) \vee 0}\,dB_t,\ \ X(0) = x_0 \in [0, 1] \tag{3.21}$$

with $a \geq 0$ and $a + b \leq 0$ does not exit from the interval $[0, 1]$ for any time (see Abundo 1997). This equation is used for instance in the Wright-Fisher model for population genetics and in certain diffusion models for neural activity

(Lanska et al. 1994). For $a = \frac{1}{4}$ and $b = -\frac{1}{2}$, $X(t)$ turns out to be conjugated to BM via the function $v(x) = 2\arcsin\sqrt{x}$, i.e. $X(t) = \sin^2(B_t/2 + \arcsin\sqrt{x_0})$. For these special values of parameters, the SDE (3.21) becomes:

$$dX(t) = \left(\frac{1}{4} - \frac{1}{2}X(t)\right)dt + \sqrt{X(t)(1-X(t))}\,dB_t\ ,\ X(0) = x_0 \in [0,1]$$

and it is obtained by Eq. 3.19 by taking $\sigma(x) = \sqrt{x(1-x) \vee 0}$. Note that, since $0 \le X(t) \le 1$, it results $A(x) = \int_0^{\tau(x)} X(t)dt \le \tau(x)$.

Example 3 (Ornstein–Uhlenbeck Process)
Let $X(t)$ be the solution of the SDE:

$$dX(t) = -\mu X(t)dt + \sigma dB_t,\ \ X(0) = x \tag{3.22}$$

where μ and σ are positive constants. By calculating the functions ϕ and ξ in Eqs. 3.2 and 3.3, we obtain:

$$\phi(x) = const \cdot \exp\left(\frac{\mu x^2}{\sigma^2}\right)$$

and

$$\xi(x) = const \cdot \frac{2}{\sigma}\sqrt{\frac{\pi}{\mu}}\exp\left(\frac{\mu x^2}{\sigma^2}\right)\left[\Phi\left(\frac{x\sqrt{2\mu}}{\sigma}\right) - \frac{1}{2}\right]$$

where Φ denotes the distribution function of the standard Gaussian variable. Since $\xi(x)$ is integrable in a neighbor of $x = 0$, the boundary 0 is attainable.

The explicit solution of Eq. 3.22 is $X(t) = e^{-\mu t}\left(x + \int_0^t \sigma e^{\mu s} dB_s\right)$ (see Abundo 2009). By using a time–change, we can write $\int_0^t \sigma e^{\mu s} dB_s = B_{\rho(t)}$, where $\rho(t) = \frac{\sigma^2}{2\mu}\left(e^{2\mu t} - 1\right)$. Then, one gets $X(t) = e^{-\mu t}(x + B(\rho(t))$ and

$$\tau(x) = \inf\{t > 0 : X(t) = 0 | X(0) = x\} = \inf\{t > 0 : B(\rho(t)) + x = 0\}$$

and so

$$\rho(\tau(x)) = \inf\{s > 0 : x + B_s = 0\} = \tau^0(x),$$

that is $\tau(x) = \rho^{-1}(\tau^0(x))$, where $\tau^0(x)$ is the first-passage time through zero of Brownian motion starting from x. Thus, $\tau(x)$ has density

$$f_{\tau(x)}(t) = f_{\tau^0(x)}(\rho(t))\rho'(t)$$

where $f_{\tau^0(x)}$ denotes the density of $\tau^0(x)$, which is given by Eq. 3.9 with $\mu = 0$. Note that even the mean of $\tau(x)$ is impractical to be directly calculated by using $f_{\tau(x)}(t)$. On the other hand, also the problems (3.4) and (3.5) for the Laplace transform and the moments of $\tau(x)$, cannot be solved explicitly. Thus, to obtain $E(\tau(x))$, we first calculate the first-exit time $\tau_{(0,\beta)}(x)$ of $X(t)$ from the interval $(0, \beta)$ with the condition

$X(0) = x > 0$, and then we let β go to $+\infty$. Indeed, as it is well-known $\tau_{(0,\beta)}(x)$ is the solution of the problem:

$$\begin{cases} \frac{1}{2}\sigma^2 z''(x) - \mu x z'(x) = -1 \\ z(0) = z(\beta) = 0 \end{cases}$$

whose solution can be explicitly found in terms of functions ϕ, ξ (see Abundo 1997), thus obtaining:

$$E(\tau_{(0,\beta)}(x)) = \left(\int_0^\beta \exp\left(\frac{\mu s^2}{\sigma^2}\right) ds \right)^{-1} \int_0^\beta \frac{2}{\sigma}\sqrt{\frac{\pi}{\mu}} \exp\left(\frac{\mu s^2}{\sigma^2}\right) \left[\Phi\left(\frac{s\sqrt{2\mu}}{\sigma}\right) - \frac{1}{2} \right] ds$$
$$\times \int_0^x \exp\left(\frac{\mu s^2}{\sigma^2}\right) ds - \int_0^x \frac{2}{\sigma}\sqrt{\frac{\pi}{\mu}} \exp\left(\frac{\mu s^2}{\sigma^2}\right) \left[\Phi\left(\frac{s\sqrt{2\mu}}{\sigma}\right) - \frac{1}{2} \right] ds.$$

Taking the limit as $\beta \to +\infty$ and using the Hospital's rule, we finally get:

$$E(\tau(x)) = \frac{2}{\sigma}\sqrt{\frac{\pi}{\mu}} \int_0^x \exp\left(\frac{\mu s^2}{\sigma^2}\right) \left[1 - \Phi\left(\frac{s\sqrt{2\mu}}{\sigma}\right) \right] ds.$$

Note that, for $\mu \to 0$, $E(\tau(x))$ tends to infinity, since the OU process reduces to BM; for $\mu \to +\infty$, as easily verified by the Hospital's rule, $E(\tau(x))$ tends to zero, which is indeed natural, since the process points to zero with infinite drift.

As regards the maximum displacement $S(x)$, by Proposition 2.6 its distribution $w(x) = P(S(x) \leq z)$ satisfies the problem:

$$\begin{cases} \frac{1}{2}\sigma^2 w''(x) - \mu x w'(x) = 0, \ z \geq x \\ w(0) = 1, \ w(z) = 0 \end{cases}$$

whose solution is:

$$P(S(x) \leq z) = 1 - \left(\int_0^x e^{\frac{\mu}{\sigma^2} t^2} dt \right) \left(\int_0^z e^{\frac{\mu}{\sigma^2} t^2} dt \right)^{-1}, \ z \geq x.$$

Thus, the density of $S(x)$ is:

$$\frac{d}{dz} P(S(x) \leq z) = e^{\frac{\mu}{\sigma^2} z^2} \int_0^x e^{\frac{\mu}{\sigma^2} t^2} dt \left(\int_0^z e^{\frac{\mu}{\sigma^2} t^2} dt \right)^{-2}, \ z \geq x$$

while it is zero for $z < x$.

3.2 Diffusions with Jumps

Example 4 (Poisson Process) For $x > 0$, let us consider the jump-process $X(t) = x - N_t$, where N_t is a homogeneous Poisson process with intensity $\theta > 0$, namely

$N_0 = 0$ and its jumps, of amplitude 1, occur at independent instants, exponentially distributed with parameter θ. This means that

$$P(N_t = k) = \frac{e^{-\theta t}(\theta t)^k}{k!}, \text{ for } k = 0, 1, \dots$$

The infinitesimal generator of the process is defined by:

$$Lg(x) = \theta[g(x-1) - g(x)], \; g \in C^0(\mathbb{R})$$

and $\tau(x) = \inf\{t > 0 : x - N_t \leq 0\}$.

By Theorem 2.3 with $U(x) = 1$, it follows that the Laplace transform $M_{U,\lambda}(x)$ of $\tau(x)$ is the solution of the equation $LM_{U,\lambda}(x) = \lambda M_{U,\lambda}(x)$, with outer condition $M_{U,\lambda}(y) = 1$ for $y \leq 0$. By solving this equation, we get:

$$M_{U,\lambda}(x) = \begin{cases} \left(\frac{\theta}{\theta+\lambda}\right)^x & \text{if } x \in \mathbb{N} \\ \left(\frac{\theta}{\theta+\lambda}\right)^{[x]+1} & \text{if } x \notin \mathbb{N} \end{cases}$$

where $[x]$ denotes the integer part of x. Note that the condition $M_{U,\lambda}(+\infty) = 0$ also holds. Thus, recalling the expression of the Laplace transform of the Gamma density, we find that $\tau(x)$ has Gamma distribution with parameters (x, θ) if x is a positive integer, while it has Gamma distribution with parameters $([x] + 1, \theta)$ if x is not an integer. This fact also follows directly, if one considers the nature of the Poisson process. The moments $T_n(x) = E[\tau^n(x)]$ are soon obtained from the density or also by the formula $T_n(x) = (-1)^n \left[\frac{\partial^n}{\partial\lambda^n} M_\lambda(x)\right]_{\lambda=0}$. We have:

$$E(\tau(x)) = \begin{cases} \frac{x}{\theta} & \text{if } x \in \mathbb{N} \\ \frac{[x]+1}{\theta} & \text{if } x \notin \mathbb{N} \end{cases}$$

and

$$E(\tau^2(x)) = \begin{cases} \frac{x^2}{\theta^2} + \frac{x}{\theta^2} & \text{if } x \in \mathbb{N} \\ \frac{([x]+1)^2}{\theta^2} + \frac{[x]+1}{\theta^2} & \text{if } x \notin \mathbb{N} \end{cases}$$

Thus:

$$Var(\tau(x)) = \begin{cases} \frac{x}{\theta^2} & \text{if } x \in \mathbb{N} \\ \frac{[x]+1}{\theta^2} & \text{if } x \notin \mathbb{N} \end{cases}$$

By Theorem 2.3 with $U(x) = x$, we get the Laplace transform $M_{U,\lambda}(x)$ of $A(x)$ as the solution of the equation $LM_{U,\lambda}(x) = \lambda x M_{U,\lambda}(x)$, with outer condition $M_{U,\lambda}(y) = 1$ for $y \leq 0$. By solving this equation, we get:

$$M_{U,\lambda}(x) = \begin{cases} \theta^x \cdot \{(\theta+\lambda)(\theta+2\lambda)\cdots(\theta+x\lambda)\}^{-1} & \text{if } x \in \mathbb{N} \\ \theta^{[x]+1} \cdot \{(\theta+\lambda x)(\theta+\lambda(x-1))\cdots(\theta+\lambda(x-[x]))\}^{-1} & \text{if } x \notin \mathbb{N} \end{cases}$$

that can be written in the unique form, valid for any $x > 0$:

$$M_{U,\lambda}(x) = \frac{\theta^{[x]+1}}{(\theta+\lambda x)(\theta+\lambda(x-1))\cdots(\theta+\lambda(x-[x]))}.$$

Note that the condition $M_{U,\lambda}(+\infty) = 0$ is fulfilled.

We observe that $M_{U,\lambda}(x)$ turns out to be the Laplace transform of a linear combination of $[x]+1$ independent exponential random variables with parameter θ, with coefficients $x, x-1, \ldots, x-[x]$. The $n-$th order moment of $A(x)$ is given by $(-1)^n \left[\frac{\partial^n}{\partial \lambda^n} M_{U,\lambda}(x)\right]_{\lambda=0}$; calculating the first and second derivative, after some tedious computations, we obtain

$$E(A(x)) = \frac{(2x-[x])([x]+1)}{2\theta}$$

and

$$E(A^2(x)) = \frac{[x]+1}{12\theta^2}\Big(12x(x-[x])([x]+2) + [x](3[x]^2 + 7[x] + 2)\Big).$$

The expression found for $E(A(x))$ also follows directly, by considering the nature of the Poisson process.

Example 5 (A Levy Process) Let us consider the process $X(t) = x + \beta t + B_t - N_t$, where N_t is a homogeneous Poisson Process with intensity $\theta > 0$ and $\beta < \theta$. We have $\tau(x) = \inf\{t > 0 : X(t) \le 0\} = \inf\{t > 0 : N_t \ge x + \beta t + B_t\}$; since $-B_t$ is also Brownian motion, we obtain $\tau(x) = \inf\{t > 0 : -\beta t + B_t \ge x - N_t\}$ and the condition $\beta < \theta$ assures that $\tau(x)$ is finite with probability one (see Abundo 2010). Now, the infinitesimal generator of the process is $Lf(x) = \frac{1}{2} f''(x) + \beta f'(x) + f(x-1) - f(x)$ and the differential-difference equations involved to find the Laplace transforms of $\tau(x)$ and $A(x)$, as well those for the moments of $\tau(x)$ and $A(x)$, cannot be solved explicitly; these quantities have to be found by a numerical procedure (see also Abundo 2010 as regards the density of $\tau(x)$).

4 Conclusions and Final Remarks

We have considered a one-dimensional jump-diffusion process $X(t)$ starting from $x > 0$, that is, a diffusion to which jumps at Poisson-distributed instants are superimposed; then, we have studied the probability distribution of the (random) area $A(x)$ swept out by $X(t)$ till its first-passage time below zero, i.e. the (random) time $\tau(x) = \inf\{t > 0 : X(t) \le 0 | X(0) = x\}$. In this way, we have generalized analogous results, already present in literature (see Janson 2007; Kearney and Majumdar 2005; Kearney et al. 2007; Knight 2000; Perman and Wellner 1996), which hold for Brownian motion with negative drift, that is a very special case of diffusion without jumps. In particular, we have shown that the Laplace transforms of $A(x)$ and $\tau(x)$, their moments, as well the probability distribution of the maximum displacement of $X(t)$, are solutions to certain partial differential-difference equations (PDDEs) with outer conditions. Note that, in the absence of jumps, these PDDEs with outer conditions become simply PDEs with boundary conditions. The quantities here investigated have interesting applications in Queueing Theory and in Economics (see the Introduction for a brief discussion).

After stating and proving theoretical results for diffusions $X(t)$ with and without jumps, in the final part of the paper we reported some examples for which we have carried out explicit calculations. Notice that, in general, it is not possible to solve explicitly the equations involved, in order to find closed formulae for the Laplace transform of the first-passage area, and for its moments; moreover, even if one is able to find explicitly the Laplace transform, it is not always possible to invert it, to get the distribution of $A(x)$.

The aim of this paper was to illustrate, also by means of practical examples, some techniques and results which can be useful to study the first-passage area of not only jump-diffusions, but also of more complex processes which are well approximated by jump-diffusions, including non-diffusion processes such as the piecewise-deterministic Markov processes. When the analytical solution is not available, due to the complexity of calculations, one can resort to numerical solution of the PDDEs involved; alternatively, one can carry out computer simulation of a large enough number of trajectories of the process $X(t)$, in order to obtain statistical estimations of the quantities of interest.

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