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On Success Runs of Length Exceeded a Threshold

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Abstract Consider a sequence of n two state (success-failure) trials with outcomes arranged on a line or on a circle. The elements of the sequence are independent (identical or non identical distributed), exchangeable or first-order Markov dependent (homogeneous or non homogeneous) random variables. The statistic denoting the number of success runs of length at least equal to a specific length (a threshold) is considered. Exact formulae, lower/upper bounds and approximations are obtained for its probability distribution. The mean value and the variance of it are derived in an exact form. The distributions and the means of an associated waiting time and the length of the longest success run are provided. The reliability function of certain general consecutive systems is deduced using specific probabilities of the studied statistic. Detailed application case studies, covering a wide variety of fields, are combined with extensive numerical experimentation to illustrate further the theoretical results.

Keywords Success runs · Longest run length · Waiting time ·
Linear and circular binary sequences · Bernoulli trials · Poisson trials ·
Exchangeable trials · Markov chain · Reliability of consecutive systems ·
Exceedances · Polya-Eggenberger sampling scheme

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1 Introduction and Preliminaries

Runs are important in applied probability and statistical inference. They are used in many areas, such as hypothesis testing, quality control, meteorology, biology, learning models, radar astronomy and system reliability. A detailed and systematic exposition of past and recent developments on the theory and applications of runs is presented in Balakrishnan and Koutras (2002) as well as in Fu and Lou (2003). In particular, the study of the number of success runs and associated statistics, under various enumerative schemes, defined on binary sequences of several internal structures, have attracted the interest of many authors since de Moivre's era. Exact distributions of such statistics have been derived by combinatorial analysis, generating functions, recursive relations and Markov chain embedding technique. Some recent contributions on the subject, among others, are the works of Antzoulakos and Chadjikonstantinidis (2001), Fu et al. (2002, 2003), Sen et al. (2002, 2003), Antzoulakos et al. (2003), Koutras (2003), Eryilmaz (2005a, b, 2006), Antzoulakos and Boutsikas (2007), Eryilmaz and Demir (2007), Fu and Lou (2007), Makri et al. (2007a, b) and the references therein.

In this article we concentrate on a statistic of great importance; namely the number of success runs of length at least equal to a specific length (cf. Mood 1940 or Fu and Koutras 1994). It is defined on sequences of independent, exchangeable and dependent in a Markovian fashion binary random variables, ordered on a line or on a circle. An associated waiting time defined on linear sequences as well as the length of the longest success run defined on both linear and circular sequences are examined as derivative statistics. In the sequel we provide the necessary definitions and notations that are used throughout the article.

Consider a sequence $X_1, X_2,...,X_n$ $(n > 0)$ of binary trials with possible outcomes "success" ("S" or "1") or "failure" ("F" or "0"). The outcomes $x_i^\alpha \in \{0, 1\}, i \geq 1$ may be ordered on a line $(\alpha = \ell)$ or on a circle $(\alpha = c)$. In the latter case we assume that the first outcome is adjacent to (and follows) the n-th outcome. The elements of the sequence may be independent (identical or non-identical distributed), exchangeable or first-order Markov dependent (homogeneous or nonhomogeneous) binary random variables. A success run is defined to be a sequence of consecutive successes preceded and succeeded by failures or by nothing. The number of successes in a success run is referred to as its length.

Given a fixed length k (a threshold), $1 \leq k \leq n$, the random variable (RV) of interest i.e. the number of success runs of length at least k is denoted by $G_{n,k}^\alpha$. The support of $G_{n,k}^\alpha$ is the set $S_\alpha(n, k) = \left\{0, 1, \ldots, \left\lfloor \frac{n+1-\beta_\alpha}{k+1} \right\rfloor\right\}$, with $\beta_\alpha = 0$, if $\alpha = \ell$, $1 \leq k \leq n$ or $\alpha = c$, $k = n$; 1, if $\alpha = c$, $1 \leq k < n$. We denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x. We mention that the RVs $G_{n,1}^\ell$ and $G_{n,1}^c$ denote the number of success runs in a binary sequence ordered on a line and on a circle, respectively.

$G_{n,k}^\alpha$ may be formally established as a sum of the indicator RVs I_j^α with $j \in J_\alpha = \{k, k+1, \ldots, n\}$ if $\alpha = \ell$; $\{0, 1, \ldots, n\}$ if $\alpha = c, n > k$; and $\{0\}$, if $\alpha = c, n = k$, i.e.

$$G_{n,k}^\alpha = \sum_{j \in J_\alpha} I_j^\alpha. \tag{1}$$

The RVs I_j^α are defined as:

(a)

$$I_j^\ell = \begin{cases} 1, & \text{if } X_j = X_{j-1} = \ldots = X_{j-k+1} = 1, X_{j-k} = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

(Convention: $X_0 \equiv 0$).

(b)

$$I_0^c = \begin{cases} 1, & \text{if } X_1 = X_2 = \ldots = X_n = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

For $j = 1, 2, \ldots, k$,

$$I_j^c = \begin{cases} 1, & \text{if } X_j = X_{j-1} = \ldots = X_1 = 1,\ X_n = X_{n-1} = \ldots = X_{n-k+j+1} = 1,\ X_{n-k+j} = 0 \\ 0, & \text{otherwise.} \end{cases}$$

For $j = k+1, k+2, \ldots, n$,

$$I_j^c = \begin{cases} 1, & \text{if } X_j = X_{j-1} = \ldots = X_{j-k+1} = 1, X_{j-k} = 0 \\ 0, & \text{otherwise.} \end{cases}$$

The previous setup will be very helpful for deriving results referred to the mean value, variance and bounds of $G_{n,k}^\alpha$.

Two RVs closely related to $G_{n,k}^\alpha$ and extensively studied in the literature, are the length L_n^α, of the longest success run for linearly and circularly ordered trials and the waiting time $T_{r,k}$, until the r-th, $r \geq 1$, occurrence of a success run of length at least k for trials ordered on a line. They are defined as follows:

$$L_n^\alpha = \begin{cases} \max\left\{k \leq n : G_{n,k}^\alpha > 0\right\}, & \text{if } \left\{k \leq n : G_{n,k}^\alpha > 0\right\} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

and

$$T_{r,k} = \min\left\{n \geq r(k+1) - 1 : G_{n,k}^\ell = r\right\}. \tag{5}$$

For Bernoulli trials (i.i.d. binary trials) $T_{1,k}$ follows the geometric distribution of order k which was introduced by Philippou et al. (1983).

It is clear that for any n, k and r such that $1 \leq k \leq n$, $r \geq 1$ the dual relationships

$$L_n^\alpha < k \text{ iff } G_{n,k}^\alpha < 1; \quad T_{r,k} > n \text{ iff } G_{n,k}^\ell < r; \quad \text{and } L_n^\ell = k \text{ if } T_{1,k} = n \tag{6}$$

always hold. Relationships (6) offer alternative ways of obtaining the exact distributions of L_n^α and $T_{r,k}$ through formulae established for $G_{n,k}^\alpha$.

The foregoing definitions are illustrated using the following example. Let the first 10 binary trials be SSSFSFSSFS. Then, $S_\ell(10,2) = S_c(10,2) = \{0, 1, 2, 3\}$, $G_{10,2}^\ell = G_{10,2}^c = 2$; $G_{10,3}^\ell = G_{10,3}^c = 1$; $G_{10,4}^\ell = 0$, $G_{10,4}^c = 1$; $L_{10}^\ell = 3$, $L_{10}^c = 4$; and $T_{1,2} = L_2^\ell = 2$, $G_{2,2}^\ell = 1$, $T_{2,2} = 8$, $G_{8,2}^\ell = 2$, $T_{3,2} > 10$.

Throughout the article, for integers n, m, $\binom{n}{m}$ denotes the extended binomial coefficient; see Feller (1968, pp 50; 63). Further, in order to avoid repetitions we note here that: $\delta_{i,j}$ denotes the Kronecker delta function of the integer arguments i

and j; $I\{A\} = 1$ if A occurs, 0 otherwise; $I_A(x) = 1$ if $x \in A$, 0 otherwise; and by $\lceil x \rceil$ we denote the least integer greater than or equal to x. Also, we apply the conventions $\sum_{i=a}^{b} = 0$, $\prod_{i=a}^{b} = 1$, for $a > b$. Finally, we note that by independent, exchangeable and Markov dependent sequence, we mean that the elements of the sequence are independent, exchangeable and first-order Markov dependent RVs, respectively.

The rest of the article is organized as follows: In Section 2, exact formulae of the probability mass function of $G_{n,k}^{\alpha}$ are derived in closed form via combinatorial analysis and/or recursively. As a byproduct, they yield, via relationships (6), the exact distributions of L_n^{α} and $T_{r,k}$. The key results of the section are established in Lemma 2.1, Theorem 2.5 and Lemma 2.2 for independent, exchangeable and first-order Markov dependent trials, respectively. In Section 3, exact closed formulae for the mean value and the variance of $G_{n,k}^{\alpha}$ are given. Their expressions are determined by using the representation of $G_{n,k}^{\alpha}$ as a sum of the indicators I_j^{α} (cf. Eqs. 1 to 3). The same setup is latter used (Section 4) along with the expressions of the means and the variances for obtaining lower/upper bounds and approximations for the probability distribution of $G_{n,k}^{\alpha}$ which hold for all the types of the concerned sequences. As an engineering application the reliability function of certain general consecutive systems is determined in Section 5. Finally, in Section 6 some indicative, widely used in applied probability, linearly/circularly ordered sequences of independent (identical/non-identical), exchangeable and Markov dependent RVs are considered; e.g. the Polya-Eggenberger urn model, the fixed and random threshold models, the record indicator model and a communications model. In addition, two examples used in system reliability are discussed. They may represent telecommunication networking and vacuum system in accelerators. The concerned binary sequences combined with extensive numerical experimentation clarify further the theoretical results and their applicability in a variety of fields like: forecasting in finance and gambling, hypothesis testing, clustering in communications system and system reliability.

We end this section by noting that the vast majority of the presented formulae are new. In addition, the findings of the article generalize, unify and/or provide alternative formulae of recent results on RVs $G_{n,k}^{\alpha}$, L_n^{α}, $T_{r,k}$ and on the reliability of linear/circular consecutive systems. See Antzoulakos and Chadjikonstantinidis (2001), Eryilmaz and Tutuncu (2002), Sen et al. (2002, 2003), Eryilmaz (2005a, b, 2006), Agarwal et al. (2007), Eryilmaz and Demir (2007) and Makri et al. (2007b).

2 Exact Distributions

The exact probability mass function (PMF) of $G_{n,k}^{\alpha}$ depends on the internal structure considered for the binary sequence $\{X_i\}_{i=1}^{n}$. In this section we will examine cases in which the binary trials are considered as: (I) An independent (identical/non-identical) sequence, (II) an exchangeable sequence, and (III) a first-order Markov dependent sequence. For this reason and for compactness and convenience too in the presentation of the variety of the results, we use the next explicit formulation

$$h_{\alpha}^{d}(x; k, n; D) \equiv P\left(G_{n,k}^{\alpha} = x\right), \quad x \in S_{\alpha}(n, k) \text{ and } \alpha = \ell \text{ or } \alpha = c. \tag{7}$$

In the above formula the superscript d denotes the type of the (possible) dependence among the RVs of the assumed sequence, whereas D is the corresponding parameter set needed for the exact description of the internal structure of that sequence. Namely, we use:

$d \equiv I$, $D \equiv \mathbf{p} = (p_1, p_2, \ldots, p_n)$ for a sequence of independent (but not necessarily identically distributed) binary trials. These are called Poisson trials (see for instance, Mitzenmacher and Upfal 2005, p 63). For such trials it holds:

$$P\big(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\big) = \prod_{i=1}^{n} P\big(X_i = x_i\big),$$

with $p_i = P(X_i = 1) = 1 - q_i = 1 - P(X_i = 0)$, for $i = 1, 2, \ldots, n$.

$d \equiv II$, $D \equiv \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ for a sequence of exchangeable trials. It holds:

$$P\big(X_1 = x_{\pi_1}, X_2 = x_{\pi_2}, \ldots, X_n = x_{\pi_n}\big) = P\big(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\big)$$

for any permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of the index set $\{1, 2, \ldots, n\}$, with $\lambda_i = P(X_1 = X_2 = \ldots = X_i = 1)$, $i = 1, \ldots, n$ and $\lambda_0 \equiv 1$.

$d \equiv III$ and $D \equiv \big(\mathbf{P}^{(n)}, p_1^{(1)}\big)$ for a nonhomogeneous first-order two state Markov chain, with transition probability matrices $\mathbf{P}^{(n)}$ with elements $p_{ij}^{(n)}$, and initial probability vector

$$\mathbf{p}^{(1)} = \big(p_0^{(1)}, p_1^{(1)}\big) \quad \text{with} \quad p_0^{(1)} = P(X_1 = 0) = 1 - P(X_1 = 1) = 1 - p_1^{(1)}.$$

The one-step transition probabilities are:

$$p_{ij}^{(n)} = P\big(X_n = j \mid X_{n-1} = i\big), \quad \text{for} \quad n \geq 2 \quad \text{and} \quad i, j \in \{0, 1\}.$$

For homogeneous chains $p_{ij}^{(n)} = p_{ij}$, for any $n \geq 2$.

We spend special attention, and formulation too, for the case of Bernoulli trials (identical Poisson trials) with a common success probability p, $0 < p < 1$. This is so, because in addition to its own independent merit in studies of applied probability and statistics, it can be considered as a special case: of independent trials with common success probability $p = P(X_i = 1) = 1 - q = 1 - P(X_i = 0)$, $i = 1, \ldots, n$ or of an exchangeable sequence with $\lambda_i = \lambda_1^i$, $i = 1, 2, \ldots, n$ where $\lambda_1 = p$ or of a first-order homogeneous Markov chain with $p_{00} = p_{10} = q$, $p_{01} = p_{11} = p$ and initial probability vector $(p_0^{(1)}, p_1^{(1)}) = (q, p)$, with $p + q = 1$.

Hence, for Bernoulli trials ordered on a line ($\alpha = \ell$) or on a circle ($\ell = c$) we denote

$$h_\alpha(x; k, n; p) \equiv P\left(G_{n,k}^{\alpha} = x\right), \quad x \in S_\alpha(n, k), \quad 0 < p < 1 \ \text{ and } \ \alpha = \ell, \quad \alpha = c. \tag{8}$$

We mention that when no confusion is likely to arise, formulae referring to a type of the concerned sequences will be presented without the use of the superscript d or the explicit consideration of the internal parameter set D.

2.1 Independent Trials (Poisson and Bernoulli Trials)

Let $G(k; r, s)$ be a RV denoting the number of success runs of length at least k in the window X_r, X_{r+1},...,X_s of n Poisson trials, X_1, X_2,...,X_n, ordered on a line with

$n \geq s \geq r \geq 1$. The support of $G(k; r, s)$ is $S_{G(k;r,s)} = \left\{0, 1, \ldots, \left\lfloor \frac{s-r+2}{k+1} \right\rfloor\right\}$. By this setup we establish the following basic result.

Lemma 2.1 *The PMF* $g(x; k, r, s) = P(G(k; r, s) = x)$, $x \in S_{G(k;r,s)}$, *satisfies the recursive scheme*

$$g(x; k, r, s) = \sum_{i=0}^{s-r} \beta_i g\big(x - I_A(i); k, r+i+1, s\big) + \beta \delta_{x,1}, \quad s - r + 1 \geq k;$$

where $\beta_i = q_{r+i}\big(\prod_{j=r}^{r+i-1} p_j\big)$, $i = 0, 1, \ldots, s-r$, $\beta = \prod_{j=r}^{s} p_j$, $A = \{k, k+1, \ldots, s-r\}$; *with initial conditions*

$$g(x; k, r, s) = 0, \quad \text{if} \quad x < 0 \ \text{ or } \ x > \left\lfloor \frac{s-r+2}{k+1} \right\rfloor,$$

$$g(x; k, r, s) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}, \quad \text{for } 0 \leq s - r + 1 < k.$$

Proof Obviously, for $x < 0$ or $x > \left\lfloor \frac{s-r+2}{k+1} \right\rfloor$ and for $0 \leq s - r + 1 < k$ the lemma holds. We define the events

$$A_i = \big\{X_r = X_{r+1} = \ldots = X_{r+i-1} = 1, X_{r+i} = 0\big\}, \quad i = 0, 1, \ldots, s-r$$

and

$$A_{s-r+1} = \big\{X_r = X_{r+1} = \ldots = X_s = 1\big\}.$$

Then, for $s - r + 1 \geq k$, $x = 0, 1, \ldots, \left\lfloor \frac{s-r+2}{k+1} \right\rfloor$ we have

$$\begin{aligned} g(x; k, r, s) &= P\left(\cup_{i=0}^{s-r+1}\big[(G(k; r, s) = x) \cap A_i\big]\right) \\ &= \sum_{i=0}^{k-1} P\big(G(k; r, s) = x \mid A_i\big) P\big(A_i\big) + \sum_{i=k}^{s-r} P\big(G(k; r, s) = x \mid A_i\big) P(A_i) \\ &\quad + P\big(G(k; r, s) = x \mid A_{s-r+1}\big) P\big(A_{s-r+1}\big). \end{aligned}$$

Also, we note that $P(G(k; r, s) = x \mid A_i) = g(x; k, r+i+1, s)$, for $i = 0, 1, \ldots, k-1$, $P(G(k; r, s) = x \mid A_i) = g(x-1; k, r+i+1, s)$, for $i = k, k+1, \ldots, s-r$, $P(G(k; r, s) = x \mid A_{s-r+1}) = \delta_{x,1}$ and $P(A_i) = p_r p_{r+1} \cdots p_{r+i-1} q_{r+i}$, $i = 0, 1, \ldots, s-r$, $P(A_{s-r+1}) = p_r p_{r+1} \cdots p_s$. The result follows. □

Setting $r = 1$ and $s = n$ we have the immediate consequence of Lemma 2.1.

Theorem 2.1 *The PMF of* $G_{n,k}^{\ell}$ *is given by*

$$h_{\ell}^{I}(x; k, n; \mathbf{p}) = g(x; k, 1, n), \quad x \in S_{\ell}(n, k). \tag{9}$$

For circularly ordered Poisson trials we have the next result.

Theorem 2.2 *The PMF of $G^c_{n,k}$, $x \in S_c(n,k)$, satisfies the recursive scheme:*

$$h^I_c(x;k,n;\mathbf{p}) = \sum_{i=0}^{n-2} \beta_i \sum_{j=0}^{n-2} \gamma_j g\big(x - I_A(i+j); k, i+2, n-j-1\big) + \beta\delta_{x,1}, \quad n \geq k+1; \tag{10}$$

where $\beta_i = q_{i+1}\big(\prod_{j=1}^{i} p_j\big)$, $\gamma_i = q_{n-i}\big(\prod_{j=0}^{i-1} p_{n-j}\big)$, $i = 0, 1, \ldots, n-2$, $\gamma = \prod_{j=1}^{n} p_j$, $\beta = \sum_{i=1}^{n}\big(\prod_{\substack{j=1\\ j\neq i}}^{n} p_j\big) - (n-1)\gamma$, $A = \{k, k+1, \ldots, n-2\}$*; with initial conditions*

$$h^I_c(x;k,n;\mathbf{p}) = 0, \quad \text{if } \; x < 0 \; \text{ or } \; \left(x > \left\lfloor \frac{n}{k+1} \right\rfloor, \quad n > k\right),$$

$$h^I_c(x;k,n;\mathbf{p}) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}, \quad \text{for } \; n < k,$$

$$h^I_c(x;k,n;\mathbf{p}) = \begin{cases} \gamma, & \text{if } x = 1 \\ 1-\gamma, & \text{if } x = 0 \end{cases}, \quad \text{for } \; n = k.$$

Proof Obviously for $x < 0$ or $(x > \lfloor \frac{n}{k+1} \rfloor, n > k)$, for $n < k$ and for $n = k$ the theorem holds. Next, we define the events

$$\begin{aligned} A_{ij} &= \big\{X_1 = \cdots X_i = 1, X_{i+1} = 0, X_n = X_{n-1} = \cdots = X_{n-j+1} = 1, X_{n-j} = 0\big\}, \\ &\qquad i, j = 0, 1, \ldots, n-2, \\ A_r &= \big\{X_1 = \cdots X_{r-1} = 1, X_r = 0, X_{r+1} = \cdots = X_n = 1\big\}, \; r = 1, 2, \ldots, n, \\ A_0 &= \big\{X_1 = \cdots = X_n = 1\big\}. \end{aligned}$$

Then, for $n \geq k+1, x = 0, 1, \ldots, \lfloor \frac{n}{k+1} \rfloor$,

$$\begin{aligned} P\left(G^c_{n,k} = x\right) &= P\big[\big(G^c_{n,k} = x\big) \cap \big[\big(\cup_{i,j} A_{i,j}\big) \cup (\cup_r A_r) \cup A_0\big]\big] \\ &= \sum_{i=0}^{n-2}\sum_{j=0}^{n-2} P\left(G^c_{n,k} = x \mid A_{ij}\right) P\left(A_{ij}\right) + \sum_{r=0}^{n} P\left(G^c_{n,k} = x \mid A_r\right) P(A_r). \end{aligned}$$

Also, we note that for $i, j = 0, 1, \ldots, n-2$, $P(G^c_{n,k} = x \mid A_{ij}) = g(x; k, i+2, n-j-1)$, if $i + j \leq k-1$, $P(G^c_{n,k} = x \mid A_{ij}) = g(x-1; k, i+2, n-j-1)$, if $i+j \geq k$, $P(A_{ij}) = p_1 \cdots p_i q_{i+1} p_n p_{n-1} \cdots p_{n-j+1} q_{n-j}$ and $P(G^c_{n,k} = x \mid A_r) = \delta_{x,1}$, $r = 0, 1, \ldots, n$, $P(A_r) = p_1 \cdots p_{r-1} q_r p_{r+1} \cdots p_n$, $r = 1, 2, \ldots, n$, $P(A_0) = p_1 \cdots p_n$. The result follows. □

Next, we consider Bernoulli trials. For such trials we provide two unified expressions for the PMF of $G^{\alpha}_{n,k}$, $\alpha = \ell, c$. The first is given by Corollary 2.1 recursively and the second by Theorem 2.4 via sums of binomial coefficients. The latter expression is obtained as a special case of the Polya-Eggenberger urn model trials with replacements studied by Makri et al. (2007b).

Corollary 2.1 *For Bernoulli trials $\{X_i\}_{i=1}^n$, with common success probability p, $0 < p < 1$, the PMF of $G_{n,k}^{\alpha}$, $x \in S_\alpha(n,k)$ is given by*

$$h_\alpha(x;k,n;p) = q\sum_{i=0}^{n-1-\beta_\alpha}\gamma_\alpha(i)p^i h_\ell(x - I_A(i);k,n-i-1-\beta_\alpha;p) + \left(p^n+\gamma_\alpha\right)\delta_{x,1},\quad n \geq k+1 \tag{11}$$

where

$$\beta_\alpha = \begin{cases}0, & \text{if } \alpha=\ell\\ 1, & \text{if } \alpha=c\end{cases},\quad \gamma_\alpha(i) = \begin{cases}1, & \text{if } \alpha=\ell\\ q(i+1), & \text{if } \alpha=c\end{cases},\quad \gamma_\alpha = \begin{cases}0, & \text{if } \alpha=\ell\\ nqp^{n-1}, & \text{if } \alpha=c\end{cases},$$

with initial conditions

$$h_\alpha(x;k,n;p) = 0,\quad \text{if}\ \ x<0\ \ \text{or}\ \ (x > \lfloor \tfrac{n+1-\beta_\alpha}{k+1}\rfloor,\ \ n>k),$$

$$h_\alpha(x;k,n;p) = \begin{cases}1, & \text{if } x=0\\ 0, & \text{if } x>0\end{cases},\quad \text{for}\ \ n<k,$$

$$h_\alpha(x;k,n;p) = \begin{cases}p^n, & \text{if } x=1\\ 1-p^n, & \text{if } x=0\end{cases},\quad \text{for}\ \ n=k.$$

Proof It is a direct consequence of Theorems 2.1 and 2.2. □

For Bernoulli trials ordered on a line an alternative recursive scheme of $P(G_{n,k}^{\ell} = x)$ is provided by Antzoulakos and Chadjikonstantinidis (2001).

If we assume that the number of successes and failures in a finite binary sequence of length n, are fixed quantities, then the probabilities of interest become conditional probabilities. To this end, let S_n denote the number of Ss in a sequence of n trials. Then combining Corollaries 3.1 and 3.2 of Makri et al. (2007b) we have the following unified result.

Theorem 2.3 *The conditional PMF $P\left(G_{n,k}^{\alpha} = x \mid S_n = n-y\right)$ is given by*

$$P\left(G_{n,k}^{c} = x \mid S_n = n\right) = \delta_{x,1},\quad \text{for}\ \ x \geq 0;$$

$$P\left(G_{n,k}^{\alpha} = x \mid S_n = n-y\right) = \binom{n-\beta_\alpha}{y-\beta_\alpha}^{-1} N_\alpha^{'}(x,y), \tag{12}$$

for $y = 0,1,\ldots,n$, if $\alpha=\ell$; $1,2,\ldots,n$, if $a=c$, and for $x = 0,1,\ldots,\left\lfloor\frac{n-y}{k}\right\rfloor$, with

$$N_\alpha^{'}(x,y) = \binom{y+1-\beta_\alpha}{x}\sum_{j=0}^{\lfloor\frac{n-y-kx}{k}\rfloor}(-1)^j\binom{y+1-x-\beta_\alpha}{j}\binom{n-k(x+j)-\beta_\alpha}{y-\beta_\alpha}$$

and β_α as in Corollary 2.1.

The conditional probabilities (12) may be used in certain nonparametric tests of randomness. See Koutras and Alexandrou (1997) and Makri et al. (2007b) for a recent use of the linear and the circular conditional distributions.

As a direct consequence of Theorem 2.3 and the law of total probability, we have

Theorem 2.4 *It is true that*

$$h_c(0; n, n; p) = 1 - p^n; \quad h_c(1; n, n; p) = p^n;$$

and

$$h_\alpha(x; k, n; p) = \beta_\alpha p^n \delta_{x,1} + \sum_{y=\beta_\alpha}^{n-kx} \gamma_\alpha N'_\alpha(x, y) p^{n-y} q^y, \tag{13}$$

for $x \in S_\ell(n, k)$, $1 \le k \le n$ or $x \in S_c(n, k)$, $1 \le k < n$ where $N'_\alpha(x, y)$ and β_α are as in Theorem 2.3 and $\gamma_\alpha = 1$ if $\alpha = \ell$; n/y if $\alpha = c$.

For Bernoulli trials a single summation formula for $P(G^\ell_{n,k} = x)$ has been given by Muselli (1996).

2.2 Exchangeable Trials

In this section we relax the strong assumption of independence of n binary trials with the weaker one of exchangeability (cf. Heath and Sudderth 1976). For such trials, we provide the PMF of $G^\alpha_{n,k}$ (Theorem 2.5), using the fact that the conditional distribution of it, given the number of failures in the sequence, in the exchangeable case is identical to the corresponding one in the i.i.d. case. This is so, because the exchangeability implies that all finite sequences with the same length and the same number of failures, and hence the same number of successes, are equally likely. See Lemma 2.1 of Schuster (1991), Lemma 2.2 of Eryilmaz and Demir (2007) and Remark 2.2 of Makri et al. (2007b).

Let Y_n denote the number of Fs in a sequence of exchangeable trials $X_1, X_2, \ldots, X_n$, $n > 0$ arranged on a line or on a circle. The success-failure composition of the sequence is assumed to be fixed. Then, because of the exchangeability any sequence with y Fs and $n - y$ Ss has probability

$$\begin{aligned} p_n(y) = P\big(X_1 = X_2 = \ldots = X_{n-y} = 1, \quad & X_{n-y+1} \\ = X_{n-y+2} = \ldots = X_n = 0\big), \quad & y = 0, 1, \ldots, n. \end{aligned}$$

i.e. all the sequences with the same number of failures have the same probability. By Theorem 2.1 of George and Bowman (1995)

$$p_n(y) = \sum_{i=0}^{y} (-1)^i \binom{y}{i} \lambda_{n-y+i}, \quad y = 0, 1, \ldots, n \tag{14}$$

where

$$\lambda_i = p_i(0) = P\big(X_1 = X_2 = \ldots = X_i = 1\big), \quad i = 1, 2, \ldots, n \text{ and } \lambda_0 = 1. \tag{15}$$

Then, according to the previous discussion on $p_n(y)$ and Theorem 2.4 we have the following unified result.

Theorem 2.5 *The PMF $h_\alpha^{II}(x; k, n; \lambda)$ of the RV $G_{n,k}^\alpha$ is*

$$h_c^{II}(0; n, n; \lambda) = 1 - \lambda_n; \quad h_c^{II}(1; n, n; \lambda) = \lambda_n$$

and

$$h_\alpha^{II}(x; k, n; \lambda) = \beta_\alpha \lambda_n \delta_{x,1} + \sum_{y=\beta_\alpha}^{n-kx} \gamma_\alpha N_\alpha^{'}(x, y) p_n(y), \tag{16}$$

for $x \in S_\ell(n, k)$, $1 \le k \le n$ or $x \in S_c(n, k)$, $1 \le k < n$. The numbers $N_\alpha^{'}(x, y)$, β_α and γ_α are as in Theorem 2.4 and the probability $p_n(y)$ is given by Eq. 14.

For $\alpha = \ell$ the result of Theorem 2.5 has also been provided in Eryilmaz and Demir (2007, Eq. 2.3). The Polya-Eggenberger urn model produces an exchangeable sequence with λ_i given by the forthcoming Eq. 42. Thus Eq. 16 for $\alpha = \ell$ and $\alpha = c$ extends, through a more efficiently computable formula, the results on $G_{n,k}^\alpha$ obtained by Sen et al. (2002, 2003), respectively.

2.3 Markov Dependent Trials

Let $G(k; r, n)$ be a RV denoting the number of success runs of length at least k in the window $X_r, X_{r+1}, \ldots, X_n$ of a non-homogenous two state Markov dependent sequence $X_1, X_2, \ldots, X_n$, $n > r \ge 1$. We first present a recursive formula for the conditional probability $g(x; k, r, n) = P(G(k; r+1, n) = x \mid X_r = 0)$.

Lemma 2.2 *The conditional probabilities $g(x; k, r, n)$, $r = 1, 2, \ldots,$ satisfy the relations, for $x < 0$ or $x > \lfloor \frac{n-r+1}{k+1} \rfloor$, $g(x; k, r, n) = 0$; for $0 \le n - r < k$, $g(0; k, r, n) = 1$ and $g(x; k, r, n) = 0$, $x \ne 0$; and for $n - r \ge k$, $x = 0, 1, \ldots, \lfloor \frac{n-r+1}{k+1} \rfloor$,*

$$g(x; k, r, n) = p_{00}^{(r+1)} g(x; k, r+1, n) + p_{01}^{(r+1)} \sum_{i=1}^{n-r-1} \beta_i g(x - I_A(i); k, r+i+1, n)$$
$$+ p_{01}^{(r+1)} \beta \delta_{x,1},$$

where $\beta_i = p_{10}^{(r+i+1)} \prod_{j=2}^{i} p_{11}^{(r+j)}$, $\beta = \prod_{j=2}^{n-r} p_{11}^{(r+j)}$ and $A = \{k, \ldots, n-r-1\}$.

Proof For $x < 0$ or $x > \lfloor \frac{n-r+1}{k+1} \rfloor$ and for $0 \leq n-r < k$, the theorem obviously holds. For $n-r \geq k$, $x=0, 1, \ldots, \lfloor \frac{n-r+1}{k+1} \rfloor$, let $A_{r,i} = \{X_{r+1} = X_{r+2} = \cdots = X_{r+i} = 1,\ X_{r+i+1} = 0\}$, $i = 0, 1, \ldots, n-r-1$ and $A_{r,n-r} = \{X_{r+1} = \cdots = X_n = 1\}$. Then,

$$
\begin{aligned}
g(x; k, r, n) &= \sum_{i=0}^{n-r} P\big[(G(k; r+1, n) = x) \cap A_{r,i} \mid X_r = 0\big] \\
&= \sum_{i=0}^{n-r-1} P\big(G(k; r+i+2, n) = x - I_A(i) \mid X_{r+i+1} = 0\big) \\
&\quad \times P\big(X_{r+1} = \cdots X_{r+i} = 1, X_{r+i+1} = 0 \mid X_r = 0\big) \\
&\quad + \delta_{x,1} P(X_{r+1} = \cdots = X_n = 1 \mid X_r = 0) \\
&= P\big(G(k; r+2, n) = x \mid X_{r+1} = 0\big) p_{00}^{(r+1)} \\
&\quad + \sum_{i=1}^{n-r-1} P\big(G(k; r+i+2, n) \\
&\qquad\qquad = x - I_A(i) \mid X_{r+i+1} = 0\big) p_{01}^{(r+1)} \left(\prod_{j=2}^{i} p_{11}^{(r+j)} \right) p_{10}^{(r+i+1)} \\
&\quad + \delta_{x,1} p_{01}^{(r+1)} \prod_{j=2}^{n-r} p_{11}^{(r+j)}.
\end{aligned}
$$

The result follows. □

Theorem 2.6 *The PMF of the RV $G_{n,k}^{\ell}$ for $x = 0, 1, \ldots, \lfloor \frac{n+1}{k+1} \rfloor$, is given by*

$$
\begin{aligned}
h_{\ell}^{III}\left(x; k, n; \mathbf{P}^{(n)}, p_1^{(1)}\right) &= p_0^{(1)} g(x; k, 1, n) + p_1^{(1)} \sum_{i=1}^{n-1} \gamma_i g(x - I_A(i); k, i+1, n) \\
&\quad + \delta_{x,1} p_1^{(1)} \gamma, \qquad (17)
\end{aligned}
$$

where $\gamma_i = p_{10}^{(i+1)} \prod_{j=2}^{i} p_{11}^{(j)}$, $i = 1, 2, \ldots, n-1$, $\gamma = \prod_{j=2}^{n} p_{11}^{(j)}$, $A = \{k, \ldots, n-1\}$.

Proof Given the nonhomogeneous two state Markov dependent trials, $X_1, X_2, \ldots, X_n$, $n \geq k \geq 1$, we define the events $A_0 = \{X_1 = 0\}$, $A_i = \{X_1 = \ldots = X_i = 1, X_{i+1} = 0\}$, $i = 1, 2, \ldots, n-1$ and $A_n = \{X_1 = \ldots = X_n = 1\}$. Then, for $x = 0, 1, \ldots, \lfloor \frac{n+1}{k+1} \rfloor$,

$$
\begin{aligned}
h_{\ell}^{III}\left(x; k, n; \mathbf{P}^{(n)}, p_1(1)\right) &= P\left(\cup_{i=0}^{n} \left[\left(G_{n,k}^{\ell} = x\right) \cap A_i\right]\right) \\
&= \sum_{i=0}^{n} P\left(G_{n,k}^{\ell} = x \mid A_i\right) P(A_i)
\end{aligned}
$$

$$
\begin{aligned}
&= p_0^{(1)} g(x; k, 1, n) \\
&\quad + \sum_{i=1}^{k-1} g\big(x; k, i+1, n\big) p_1^{(1)} \left(\prod_{j=2}^{i} p_{11}^{(j)} \right) p_{10}^{(i+1)} \\
&\quad + \sum_{i=k}^{n-1} g\big(x-1; k, i+1, n\big) p_1^{(1)} \left(\prod_{j=2}^{i} p_{11}^{(j)} \right) p_{10}^{(i+1)} \\
&\quad + \delta_{x,1} p_1^{(1)} \left(\prod_{j=2}^{n} p_{11}^{(j)} \right).
\end{aligned}
$$

The result follows. □

For homogeneous Markov chains recursive schemes of the PMF of $G^{\ell}_{n,k}$ are also provided by Hirano and Aki (1993) and Antzoulakos and Chadjikonstantinidis (2001).

2.4 Exact Distributions of the Longest Success Run and the Waiting Time

The exact PMF of the derivative RVs L^{α}_n and $T_{r,k}$ are readily obtained using relations (6). Concretely, we have

$$
P(L^{\alpha}_n = k) = \begin{cases} P\big(G^{\alpha}_{n,1} = 0\big), & \text{if } k = 0 \\ P\big(G^{\alpha}_{n,k+1} = 0\big) - P\big(G^{\alpha}_{n,k} = 0\big), & \text{if } k = 1, 2, \ldots, n-1 \\ 1 - P\left(G^{\alpha}_{n,k} = 0\right), & \text{if } k = n \end{cases} \tag{18}
$$

and

$$
P(T_{r,k} = t) = \begin{cases} 1 - P\big(G^{\ell}_{t,k} = 0\big), & \text{if } t = k \\ \sum_{x=0}^{r-1} \left\{ P\left(G^{\ell}_{t-1,k} = x\right) - P\left(G^{\ell}_{t,k} = x\right)\right\}, & \text{if } t > k. \end{cases} \tag{19}
$$

where the probabilities $P(G^{\alpha}_{n,k} = x) = h^{d}_{\alpha}(x; k, n; D)$ for $d = I, II, III$ are given by the proper Eqs. 9 to 17.

Since $S_{\alpha}(n, k) = \{0, 1\}$ if $n \leq 2k$ and $\alpha = \ell$ or $n \leq 2k + 1$ and $\alpha = c$, it is evident that

$$
P(T_{1,k} > n) = P\left(L^{\ell}_n < k\right) = P\left(G^{\ell}_{n,k} = 0\right) = 1 - E\left(G^{\ell}_{n,k}\right), \quad \text{if} \quad n \leq 2k
$$

and

$$
P\left(L^{c}_n < k\right) = P\left(G^{c}_{n,k} = 0\right) = 1 - E\left(G^{c}_{n,k}\right), \quad \text{if} \quad n \leq 2k + 1, \tag{20}
$$

where $E(G^{\alpha}_{n,k})$, $\alpha = \ell, c$, are given by the forthcoming Eqs. 22 to 28 for Poisson, Bernoulli, exchangeable and Markov dependent trials.

3 Mean Values and Variances

In this section we obtain exact closed formulae for the mean values and the variances of the RVs $G^{\alpha}_{n,k}$, $\alpha = l$ or $\alpha = c$. The expressions are derived using the representation of $G^{\alpha}_{n,k}$ as a sum of the indicator RVs I^{α}_{j} defined in Section 1. Hence,

$$E\left(G^{\alpha}_{n,k}\right) = \sum_{j\in J_\alpha} E\left(I^{\alpha}_{j}\right), \quad \text{and} \quad V\left(G^{\alpha}_{n,k}\right) = \sum_{j\in J_\alpha} V\left(I^{\alpha}_{j}\right) + 2 \sum_{\substack{j_1<j_2 \\ j_1,j_2\in J_\alpha}} Cov\left(I^{\alpha}_{j_1}, I^{\alpha}_{j_2}\right) \tag{21}$$

with

$$E\left(I^{\alpha}_{j}\right) = P\left(I^{\alpha}_{j} = 1\right), \quad V\left(I^{\alpha}_{j}\right) = P\left(I^{\alpha}_{j} = 1\right) - \left\{P\left(I^{\alpha}_{j} = 1\right)\right\}^2 \quad \text{and}$$

$$Cov\left(I^{\alpha}_{j_1}, I^{\alpha}_{j_2}\right) = P\left(I^{\alpha}_{j_1} = 1, I^{\alpha}_{j_2} = 1\right) - P\left(I^{\alpha}_{j_1} = 1\right)P\left(I^{\alpha}_{j_2} = 1\right).$$

The foregoing relations offer a framework to establish $E\left(G^{\alpha}_{n,k}\right)$ and $V\left(G^{\alpha}_{n,k}\right)$ for all the under study sequences. This is done in Propositions 3.1–3.2, 3.3–3.4, and 3.5, for independent, exchangeable and Markov dependent sequences, respectively. Corollaries 3.1 and 3.2 give the results for Bernoulli sequences.

3.1 Independent Trials

Proposition 3.1 *For $n \geq k$, the mean value $E\left(G^{\ell}_{n,k}\right)$, and the variance $V\left(G^{\ell}_{n,k}\right)$, of $G^{\ell}_{n,k}$ are given by*

$$E\left(G^{\ell}_{n,k}\right) = \sum_{j=k}^{n} \mu^{\ell}_{j}$$

and

$$V(G^{\ell}_{n,k}) = \begin{cases} \sum_{j=k}^{n} \mu^{\ell}_{j}\left(1-\mu^{\ell}_{j}\right) - 2\sum_{j=k}^{n-1} \mu^{\ell}_{j} \sum_{i=1}^{n-j} \mu^{\ell}_{j+i}, & \text{for } n < 2k \\ \sum_{j=k}^{n} \mu^{\ell}_{j}\left(1-\mu^{\ell}_{j}\right) - 2\sum_{j=k}^{n-k} \mu^{\ell}_{j} \sum_{i=1}^{k} \mu^{\ell}_{j+i} & \\ \quad - 2\sum_{j=n-k+1}^{n-1} \mu^{\ell}_{j} \sum_{i=1}^{n-j} \mu^{\ell}_{j+i}, & \text{for } n \geq 2k; \end{cases} \tag{22}$$

where $\mu^{\ell}_{j} = q_{j-k} \prod_{i=j-k+1}^{j} p_i$, $j = k, k+1, \ldots, n$, with $q_0 \equiv 1$.

Proof It is clear that

$$P\left(I^{\ell}_{k} = 1\right) = P\left(X_1 = X_2 = \ldots = X_k = 1\right) = p_1 p_2 \cdots p_k = \mu^{\ell}_{k}$$

and for $j = k+1, \ldots, n$,

$$P(I^{\ell}_{j} = 1) = P\left(X_{j-k} = 0, X_{j-k+1} = X_{j-k+2} = \ldots = X_j = 1\right)$$
$$= q_{j-k} p_{j-k+1} p_{j-k+2} \cdots p_j = \mu^{\ell}_{j}.$$

Next, because of the internal structure of the RVs $I^{\ell}_{j_1}$, $I^{\ell}_{j_2}$ and the independence of the RVs X_i's we observe that $P\big(I^{\ell}_{j_1}=1, I^{\ell}_{j_2}=1\big)=0$, if $0 < j_2 - j_1 \le k$, which implies that $Cov\big(I^{\ell}_{j_1}, I^{\ell}_{j_2}\big) = -\mu^{\ell}_{j_1}\mu^{\ell}_{j_2}$, whereas $I^{\ell}_{j_1}$ and $I^{\ell}_{j_2}$ are independent RVs for $j_2 - j_1 \ge k+1$, so that $Cov\big(I^{\ell}_{j_1}, I^{\ell}_{j_2}\big) = 0$. The results follow. □

Proposition 3.2 *The mean value $E\big(G^c_{n,k}\big)$ and the variance $V\big(G^c_{n,k}\big)$, of $G^c_{n,k}$ are given by*

(a) *for $n = k$,*

$$E\left(G^c_{n,k}\right) = \mu^c_0 \quad \text{and} \quad V\left(G^c_{n,k}\right) = \mu^c_0\left(1-\mu^c_0\right)$$

(b) *for $n \ge k+1$*

$$E\left(G^c_{n,k}\right) = \sum_{j=0}^{n} \mu^c_j \tag{23}$$

and

$$V\left(G^c_{n,k}\right) = \sum_{j=0}^{n} \mu^c_j\left(1-\mu^c_j\right) - 2\mu^c_0\sum_{j=1}^{n}\mu^c_j - 2\begin{cases} \displaystyle\sum_{j=1}^{n-1}\mu^c_j\sum_{i=1}^{n-j}\mu^c_{j+i}, & \text{for } n \le 2k+1 \\ \displaystyle\sum_{j=1}^{n}\mu^c_j\sum_{i=1}^{k}\mu^c_{j+i}, & \text{for } n > 2k+1; \end{cases}$$

where $\mu^c_0 = \prod_{i=1}^{n} p_i$, $\mu^c_j = q_{n-k+j}\big(\prod_{i=1}^{j} p_i\big)\big(\prod_{i=0}^{k-j-1} p_{n-i}\big)$, for $j = 1, 2\ldots, k$ and $\mu^c_j = q_{j-k}\big(\prod_{i=0}^{k-1} p_{j-i}\big)$, for $j{=}k+1, k+2, \ldots, n$, with the convention $\mu^c_{n+i} \equiv \mu^c_i$, $i = 1, 2, \ldots, k$.

Proof Obviously, $P\big(I^c_0{=}1\big){=}\mu^c_0$, and for $j{=}1, \ldots, n$, $P\big(I^c_j{=}1\big){=}\mu^c_j$. Also, for $n < 2k{+}2$ and $j{=}0, 1, \ldots, n{-}1$, $i{=}1, 2, \ldots, n{-}j$, we have that $P\big(I^c_j = 1, I^c_{j+i} = 1\big) = 0$, so that $Cov\big(I^c_j, I^c_{j+i}\big) = -\mu^c_j\mu^c_{j+i}$ and

$$\sum_{0\le j_1<j_2\le n} Cov\left(I^c_{j_1}, I^c_{j_2}\right) = -\sum_{j=0}^{n-1}\sum_{i=1}^{n-j}\mu^c_j\mu^c_{j+i}.$$

Further, for $n \ge 2k+2$, considering $I^c_{n+m} \equiv I^c_m$, $m = 1, 2, \ldots, k$, we have again that $P\big(I^c_j = 1, I^c_{j+i} = 1\big) = 0$, for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, k$ which implies that $Cov\big(I^c_j, I^c_{j+i}\big) = -\mu^c_j\mu^c_{j+i}$, whereas $Cov\big(I^c_0, I^c_i\big) = -\mu^c_0\mu^c_i$, $i = 1, 2, \ldots, n$. Noting that, $Cov\big(I^c_{j_1}, I^c_{j_2}\big) = 0$, for every 2-combination $\{j_1, j_2\}$ of the n numbers of the set $\{1, 2, \ldots, n\}$, displayed on a circle possessing the property that between them there are at least k numbers (e.g. j_1 and j_2 are k numbers apart) we have

$$\sum_{0\le j_1<j_2\le n} Cov\left(I^c_{j_1}, I^c_{j_2}\right) = -\sum_{i=1}^{n}\mu^c_0\mu^c_i - \sum_{j=1}^{n}\sum_{i=1}^{k}\mu^c_j\mu^c_{j+i}.$$

The Proposition follows. □

For Bernoulli trials Propositions 3.1 and 3.2 imply the following corollaries.

Corollary 3.1 *For Bernoulli trials the mean value and the variance of* $G^{\ell}_{n,k}$ *are given by*

$$E\left(G^{\ell}_{n,k}\right) = p^k + (n-k)qp^k, \quad n \geq k$$

and

$$V\big(G^{\ell}_{n,k}\big) = \begin{cases} p^k\big(1-p^k\big) + (n-k)qp^k - 2(n-k)qp^{2k} \\ \quad -\big\{(n-k)qp^k\big\}^2, & \text{for } k \leq n < 2k \\ p^k\big(1-p^k\big) + (n-k)qp^k - 2kqp^{2k} \\ \quad -\big\{n+2(n-1)k-3k^2\big\}q^2p^{2k}, & \text{for } n \geq 2k. \end{cases} \tag{24}$$

Corollary 3.2 *For Bernoulli trials the mean value and the variance of* $G^{c}_{n,k}$ *are given by*

(a) *for* $n = k$,

$$E\left(G^{c}_{n,k}\right) = p^n \quad \text{and} \quad V\left(G^{c}_{n,k}\right) = p^n(1-p^n)$$

(b) *for* $n \geq k+1$

$$E\left(G^{c}_{n,k}\right) = p^n + nqp^k \tag{25}$$

and

$$V\left(G^{c}_{n,k}\right) = \begin{cases} p^n\big(1-p^n\big) + nqp^k - 2nqp^{n+k} - n^2q^2p^{2k}, & \text{for } n \leq 2k+1 \\ p^n\big(1-p^n\big) + nqp^k - 2nqp^{n+k} \\ \quad -n(2k+1)q^2p^{2k}, & \text{for } n > 2k+1. \end{cases}$$

For Bernoulli trials, alternative derivations of $E\big(G^{\ell}_{n,k}\big)$ are in Mood (1940), Goldstein (1990) and Hirano and Aki (1993). The last authors and Mood (1940) provided expressions for $V\big(G^{\ell}_{n,k}\big)$ too. Further, Makri et al. (2007b) established the same formulae for $E\big(G^{\alpha}_{n,k}\big)$, $\alpha = \ell, c$ using a different approach.

3.2 Exchangeable Trials

Proposition 3.3 *For* $n \geq k$, *the mean value* $E\big(G^{\ell}_{n,k}\big)$ *and the variance* $V\big(G^{\ell}_{n,k}\big)$, *of* $G^{\ell}_{n,k}$ *are given by*

(a)

$$E\left(G^{\ell}_{n,k}\right) = (n-k+1)\lambda_k - (n-k)\lambda_{k+1}$$

(b)

$$V\left(G^{\ell}_{n,k}\right)=\begin{cases}\lambda_k\left(1-\lambda_k\right)+(n-k)\left(\lambda_k-\lambda_{k+1}\right)\left(1-\lambda_k+\lambda_{k+1}\right)\\ \quad -2(n-k)\lambda_k\left(\lambda_k-\lambda_{k+1}\right)\\ \quad -(n-k)(n-k-1)\left(\lambda_k-\lambda_{k+1}\right)^2, & \text{for } n<2k\\ \lambda_k(1-\lambda_k)+(n-k)\left(\lambda_k-\lambda_{k+1}\right)\left(1-\lambda_k+\lambda_{k+1}\right)\\ \quad -2k\lambda_k\left(\lambda_k-\lambda_{k+1}\right)\\ \quad +2(n-2k)\left\{\lambda_{2k}-\lambda_{2k+1}-\lambda_k\left(\lambda_k-\lambda_{k+1}\right)\right\}\\ \quad -(n-k)(n-k-1)(\lambda_k-\lambda_{k+1})^2\\ \quad +(n-2k)(n-2k-1)\left(\lambda_{2k}-2\lambda_{2k+1}+\lambda_{2k+2}\right), & \text{for } n\geq 2k.\end{cases} \tag{26}$$

Proof First by using Theorem 2.1 of George and Bowman (1995), we observe that $P\left(I^{\ell}_k=1\right)=P\left(X_1=\ldots=X_k=1\right)=p_k(0)=\lambda_k$, $P\left(I^{\ell}_j=1\right)=P\left(X_1=\ldots=X_k=1, X_{k+1}=0\right)=p_{k+1}(1)=\lambda_k-\lambda_{k+1}$, $j=k+1,\ldots,n$. Further, $P\left(I^{\ell}_{j_1}=1, I^{\ell}_{j_2}=1\right)=0$, if $0<j_2-j_1\leq k$, whereas $P\left(I^{\ell}_k=1, I^{\ell}_{j_2}=1\right)=P\left(X_1=\ldots=X_{2k}=1, X_{2k+1}=0\right)=p_{2k+1}(1)=\lambda_{2k}-\lambda_{2k+1}$, if $j_2-k\geq k+1$, and $P\left(I^{\ell}_{j_1}=1, I^{\ell}_{j_2}=1\right)=P\left(X_1=\ldots=X_{2k}=1, X_{2k+1}=0, X_{2k+2}=0\right)=p_{2k+2}(2)=\lambda_{2k}-2\lambda_{2k+1}+\lambda_{2k+2}$, if $j_2-j_1\geq k+1$. Next, noting that the number of 2-combinations $\{j_1, j_2\}$ of the $n-k$ numbers $\{k+1,\ldots,n\}$ for which it holds $j_2-j_1\geq k+1$ equals $\binom{n-k-(2-1)k}{2}=\binom{n-2k}{2}$ (see, Charalambides 2002, p 99), and the number of them for which it holds $j_2-j_1\leq k$ equals $\binom{n-k}{2}-\binom{n-2k}{2}$ the results follow after some algebraic manipulations. □

Proposition 3.4 *The mean value, $E\left(G^c_{n,k}\right)$, and the variance, $V\left(G^c_{n,k}\right)$, of $G^c_{n,k}$ are given by*

(a) *for $n=k$,*

$$E\left(G^c_{n,k}\right)=\lambda_n, \quad \text{and} \quad V\left(G^c_{n,k}\right)=\lambda_n\left(1-\lambda_n\right)$$

(b) *for $n\geq k+1$,*

$$E\left(G^c_{n,k}\right)=\lambda_n+n\left(\lambda_k-\lambda_{k+1}\right) \tag{27}$$

and

$$V\left(G^c_{n,k}\right)=\begin{cases}\lambda_n\left(1-\lambda_n\right)+n\left(\lambda_k-\lambda_{k+1}\right)\left(1-\lambda_k+\lambda_{k+1}\right)\\ \quad -2n\lambda_n\left(\lambda_k-\lambda_{k+1}\right)-n(n-1)\left(\lambda_k-\lambda_{k+1}\right)^2, & \text{for } n\leq 2k+1\\ \lambda_n(1-\lambda_n)+n\left(\lambda_k-\lambda_{k+1}\right)\left(1-\lambda_k+\lambda_{k+1}\right)\\ \quad -2n\lambda_n\left(\lambda_k-\lambda_{k+1}\right)-n(n-1)\left(\lambda_k-\lambda_{k+1}\right)^2\\ \quad +n\left(n-2k-1\right)\left(\lambda_{2k}-2\lambda_{2k+1}+\lambda_{2k+2}\right), & \text{for } n>2k+1.\end{cases}$$

Proof It is clear that $P\big(I_0^c=1\big)=p_n(0)=\lambda_n$ and for $n \geq k+1$ $P\big(I_j^c=1\big)=p_{k+1}(1)=\lambda_{k+1}-\lambda_k$, $j=1,2,\ldots,n$. Also, for any of the $\frac{n}{n-2k}\binom{n-2k}{2}=\frac{n(n-2k-1)}{2}$ 2-combinations $\{j_1, j_2\}$ of the set of the n numbers $\{1,2,\ldots,n\}$ displayed on a circle, for which there are at least k points between j_1 and j_2, (see, Charalambides 2002, p. 99) it holds that

$$P\left(I_{j_1}^c=1, I_{j_2}^c=1\right)=p_{2k+2}(2)=\lambda_{2k}-2\lambda_{2k+1}+\lambda_{2k+2},$$

whereas

$$P\left(I_{j_1}^c=1, I_{j_2}^c=1\right)=0$$

for any 2-combination $\{j_1, j_2\}$ of the $\binom{n}{2}-\frac{n}{n-2k}\binom{n-2k}{2}=\frac{n(n-1)-n(n-2k-1)}{2}$ ones for which there are less than k points between j_1 and j_2. The results follow after some algebraic manipulations. □

An alternative formula for $E\big(G_{n,k}^{\ell}\big)$ was given by Eryilmaz and Demir (2007) which, however, is more complicated than ours as it is expressed as a sum of $n-k$ algebraic expressions of λ_i's.

3.3 Markov Dependent Trials

Proposition 3.5 *For $n \geq k$, the mean value $E\big(G_{n,k}^{\ell}\big)$, and the variance $V\big(G_{n,k}^{\ell}\big)$, of $G_{n,k}^{\ell}$ are given by*

$$E\left(G_{n,k}^{\ell}\right)=\sum_{j=k}^{n}\mu_j$$

and

$$V\left(G_{n,k}^{\ell}\right)=\begin{cases}\displaystyle\sum_{j=k}^{n}\mu_j^{\ell}\left(1-\mu_j^{\ell}\right)-2\sum_{j=k}^{n-1}\mu_j^{\ell}\sum_{i=1}^{n-j}\mu_{j+i}^{\ell}, & \text{for } n<2k\\ \displaystyle\sum_{j=k}^{n}\mu_j^{\ell}\left(1-\mu_j^{\ell}\right)-2\sum_{j=k}^{n-k}\mu_j^{\ell}\sum_{i=1}^{k}\mu_{j+i}^{\ell}-2\sum_{j=n-k+1}^{n-1}\mu_j^{\ell}\sum_{i=1}^{n-j}\mu_{j+i}^{\ell} & \\ \displaystyle\quad+2\sum_{j=k}^{n-k-1}\sum_{i=k+1}^{n-j}\left(\mu_{j,j+i}-\mu_j\mu_{j+i}\right), & \text{for } n\geq 2k;\end{cases} \tag{28}$$

where

$$\mu_j=\begin{cases}p_1^{(1)}\displaystyle\prod_{r=2}^{k}p_{11}^{(r)}, & \text{if } j=k\\ \phi(j-k)p_{01}^{(j-k+1)}\displaystyle\prod_{r=2}^{k}p_{11}^{(j-k+r)}, & \text{if } j=k+1,\ldots,n,\end{cases}$$

$$\mu_{k,j} = \begin{cases} p_1^{(1)} p_{10}^{(k+1)} p_{01}^{(k+2)} \prod_{r=2}^{k} p_{11}^{(r)} p_{11}^{(k+1+r)}, & \text{if } j = 2k+1 \\ p_1^{(1)} p_{01}^{(j-k+1)} \theta(k, j) \prod_{r=2}^{k} p_{11}^{(r)} p_{11}^{(j+2-r)}, & \text{if } j \geq 2k+2. \end{cases}$$

For $i \geq k+1$

$$\mu_{i,j} = \begin{cases} \phi(i-k) p_{01}^{(i-k+1)} p_{10}^{(i+1)} p_{01}^{(i+2)} \prod_{r=0}^{k-2} p_{11}^{(i-r)} p_{11}^{(i+k+1-r)}, & \text{if } j = i+k+1 \\ \phi(i-k) p_{01}^{(i-k+1)} p_{01}^{(j-k+1)} \theta(i, j) \prod_{r=2}^{k} p_{11}^{(i-k+r)} p_{11}^{(j-k+r)}, & \text{if } j \geq i+k+2, \end{cases}$$

with $\phi(j) = P(X_j = 0) = p_0^{(j)} = \left(p_{00}^{(j)} - p_{10}^{(j)}\right)\phi(j-1) + p_{10}^{(j)}$, $j = 2, \ldots, n$, $\phi(1) = p_0^{(1)}$, $\theta(i, j) = P(X_{j-k} = 0 \mid X_i = 1) = \sum_{s_{i+1} \in S} \cdots \sum_{s_{j-k-1} \in S} p_{1,s_{i+1}}^{(i+1)} \left(\prod_{r=i+1}^{j-k-2} p_{s_r, s_{r+1}}^{(r+1)} \right) p_{s_{j-k-1},0}^{(j-k)}$, $j \geq i+k+2$ *and* $S = \{0, 1\}$.

Proof First we note that $\mu_k = P(I_k^\ell = 1) = P(X_1 = 1) P(X_2 = 1 \mid X_1 = 1) \cdots P(X_k = 1 \mid X_{k-1} = 1) = p_1^{(1)} p_{11}^{(2)} \cdots p_{11}^{(k)}$ and for $j = k+1, \ldots, n$, $\mu_j = P(I_j^\ell = 1) = P(X_{j-k} = 0) P(X_{j-k+1} = 1 \mid X_{j-k} = 0) P(X_{j-k+2} = 1 \mid X_{j-k+1} = 1) \cdots P(X_j = 1 \mid X_{j-1} = 1) = \phi(j-k) p_{01}^{(j-k+1)} p_{11}^{(j-k+2)} \cdots p_{11}^{(j)}$, because of the Markovian property. Also, for $i \geq k+1$,

$$\begin{aligned} \mu_{i,i+k+1} &= P(I_i = 1, I_{i+k+1} = 1) \\ &= P(X_{i-k} = 0, X_{i-k+1} = \ldots = X_i = 1, X_{i+1} = 0, X_{i+2} = \cdots = X_{i+k+1} = 1) \\ &= P(X_{i-k} = 0) p_{01}^{(i-k+1)} p_{11}^{(i-k+2)} \cdots p_{11}^{(i)} p_{10}^{(i+1)} p_{01}^{(i+2)} p_{11}^{(i+3)} \cdots p_{11}^{(i+k+1)} \\ &= \phi(i-k) p_{01}^{(i-k+1)} \left(\prod_{r=0}^{k-2} p_{11}^{(i-r)} \right) p_{10}^{(i+1)} p_{01}^{(i+2)} \left(\prod_{r=0}^{k-2} p_{11}^{(i+k+1-r)} \right), \end{aligned}$$

and for $j \geq i+k+2$,

$$\begin{aligned} \mu_{i,j} &= P(I_i = 1, I_j = 1) \\ &= P(X_{i-k} = 0, X_{i-k+1} = \ldots = X_i = 1, X_{j-k} = 0, X_{j-k+1} = \cdots = X_j = 1) \\ &= \sum_{s_{i+1} \in S} \cdots \sum_{s_{j-k-1} \in S} P(X_{i-k} = 0, X_{i-k+1} = \cdots = X_i = 1, \\ &\qquad X_{i+1} = s_{i+1}, \cdots, X_{j-k-1} = s_{j-k-1}, X_{j-k} = 0, X_{j-k+1} = \cdots = X_j = 1) \\ &= \phi(i-k) p_{01}^{(i-k+1)} \left(\prod_{r=2}^{k} p_{11}^{(i-k+r)} \right) \theta(i, j) p_{01}^{(j-k+1)} \left(\prod_{r=2}^{k} p_{11}^{(j-k+r)} \right). \end{aligned}$$

Next, by expressing in an similar way,

$$\mu_{k,k+(k+1)} = P(X_1 = \cdots = X_k = 1, X_{k+1} = 0, X_{k+2} = \cdots = X_{2k+1} = 1)$$

and for $j \geq 2k+2$

$$\mu_{k,j} = P\big(X_1 = \cdots = X_k = 1, X_{j-k} = 0, X_{j-k+1} = \cdots = X_j = 1\big)$$

and noting that

$$\phi(j) = \sum_{s \in S} P\big(X_j = 0 \mid X_{j-1} = s\big) P(X_{j-1} = s), \quad j = 2, 3, \ldots, n$$

and for $j \geq i+k+2$

$$\begin{aligned}\theta(i, j) &= \sum_{s_{i+1} \in S} \cdots \sum_{s_{j-k-1} \in S} P\big(X_{j-k} = 0, X_{j-k-1} = s_{j-k-1}, \ldots, X_{i+1} = s_{i+1} \mid X_i = 1\big) \\ &= \sum_{s_{i+1} \in S} \cdots \sum_{s_{j-k-1} \in S} P(X_{i+1} = s_{i+1} \mid X_i = 1) P(X_{i+2} = s_{i+2} \mid X_{i+1} = s_{i+1}) \times \\ &\qquad \times \cdots P(X_{j-k} = 0 \mid X_{j-k-1} = s_{j-k-1}),\end{aligned}$$

we get the result. □

In Eryilmaz (2005b) an alternative expression of $E\big(G_{n,k}^{\ell}\big)$ is given. However, it is more complicated than ours as it eventually contains two successive sums of μ_j terms instead of one sum of ours. The same is also true for homogeneous chains. For the latter case, Antzoulakos and Chadjikonstantinidis (2001) provided a simpler expression for the mean.

Remark 3.1 Using first principles, the following alternative expressions for the probabilities $\phi(j)$ and $\theta(i, j)$ are derived

$$\phi(j) = \mathbf{p}^{(1)} \left(\prod_{t=2}^{j} \mathbf{P}^{(t)} \right) \mathbf{e}_0^{'}, \quad j = 2, 3, \ldots, n, \quad \text{with} \quad \phi(1) = 1 - p_1^{(1)};$$

and

$$\theta(i, j) = \mathbf{e}_1 \left(\prod_{t=i+1}^{j-k} \mathbf{P}^{(t)} \right) \mathbf{e}_0^{'}, \quad j-k-i \geq 2, \quad i \geq 1 \tag{29}$$

where $\mathbf{e}_0 = (1, 0)$, $\mathbf{e}_1 = (0, 1)$ and $\mathbf{e}_0^{'}$ the transpose of $\mathbf{e}_0$. In Eq. 29 the probabilities $\phi(j)$ and $\theta(i, j)$ are evaluated iteratively using matrices multiplication. Hence, in some computers the required computational time may be reduced enough. For homogeneous Markov chains with $1 - p_{00} + p_{10} \neq 0$ the expressions for $\phi(j)$ and $\theta(i, j)$ may be simplified even more. Concretely, we have

$$\phi(j) = (1 - p_1^{(1)})(p_{00} - p_{10})^{j-1} + \frac{p_{10}}{1 - p_{00} + p_{10}}[1 - (p_{00} - p_{10})^{j-1}], \quad j = 1, 2 \ldots, n$$

and

$$\theta(i, j) = \frac{p_{10}}{1 - p_{00} + p_{10}}[1 - (p_{00} - p_{10})^{j-k-i}], \quad j-k-i \geq 2 \quad \text{and} \quad i \geq 1. \tag{30}$$

3.4 Expected Longest Run Length and Waiting Time

In this section we give the mean values of RVs L_n^α and $T_{r,k}$ as a direct application of the PMF of $G_{n,k}^\alpha$. Using elementary probability and relationships (6) we get

$$E\left(L_n^\alpha\right) = n - \sum_{k=1}^{n} P\left(G_{n,k}^\alpha = 0\right) \tag{31}$$

and

$$\begin{aligned} E(T_{r,k}) &= r(k+1) - 1 + \sum_{t=r(k+1)-1}^{\infty} P(G_{t,k}^\ell < r) \\ &\simeq r(k+1) - 1 + \sum_{t=r(k+1)-1}^{t_\infty} \sum_{x=0}^{r-1} P(G_{t,k}^\ell = x). \end{aligned} \tag{32}$$

$t_\infty = t_\infty(\varepsilon; r, k)$ is a stopping time such that

$$\sum_{x=0}^{r-1} P(G_{t_\infty,k}^\ell = x) \le \varepsilon \sum_{t=r(k+1)-1}^{t_\infty} \sum_{x=0}^{r-1} P(G_{t,k}^\ell = x) \tag{33}$$

where ε is a pre-specified small positive number, defined by the demanded accuracy of the results. In relations (31) to (33) the required probabilities are determined via Eqs. 9 to 17.

4 Bounds and Approximations

For large n, determining the exact probability $h_\alpha^d(x; k, n; D)$ is often a hard task, because of the computational effort needed for calculating the required recursions or the binomial coefficients involved. Therefore, the need for easily computed sharp bounds and approximations is apparent. In this section, computational tractable lower and upper bounds and approximations, combined with error analysis, for the distribution of $G_{n,k}^\alpha$, are established. The bounds are based on the well known Markov, Chebychev and conditional expectation inequalities and offer an additional application of the material presented in Section 3. The concerned approximations are derived using the obtained lower and upper bounds. Both bounds and approximations imply that similar results hold true for the RVs L_n^α and $T_{r,k}$ as well. We note in ending that the presented bounds hold for all the types of the concerned internal structure of the under study sequences, i.e. Poisson, Bernoulli, exchangeable and Markov dependent ones. Furthermore, extensive numerical investigations, showed that the bounds and the approximations are well behaved even for moderate values of n.

In the sequel, let $m_\alpha = E(G_{n,k}^\alpha)$, $\sigma_\alpha^2 = V(G_{n,k}^\alpha)$ and $F_\alpha(x) = P(G_{n,k}^\alpha < x) = \sum_{y=0}^{x-1} h_\alpha^d(y; k, n; D)$, for $x \in S_\alpha(n, k)$ and $d = I, II, III$. First, we derive a lower and an upper bound of $F_\alpha(x)$, using Markov's and one-sided Chebychev's inequalities.

(Markov's bound) For any $x \in S_\alpha(n, k) - \{0\}$, it holds

$$F_\alpha(x) \ge l_M^\alpha(x) = 1 - m_\alpha / x. \tag{34}$$

Since, $P(L_n^\alpha < k) = P(G_{n,k}^\alpha < 1)$, Eq. 34 implies $P(L_n^\alpha < k) \geq l_M^\alpha(1) = 1 - m_\alpha$. The lower bound $l_M^\alpha(1)$ is also derived by considering the Worsley's (1982) variant of a Bonferroni-type inequality applied for the sets $A_i^\alpha = \{I_i^\alpha = 1\}$, $i \in J_\alpha$. Specifically, we have $P(\cup_{i\in J_\alpha} A_i^\alpha) \leq \sum_{i\in J_\alpha} P(A_i^\alpha) - \sum_{i=\min J_\alpha}^{\max J_\alpha - 1} P(A_i^\alpha A_{i+1}^\alpha)$. But $P(A_i^\alpha A_{i+1}^\alpha) = P(I_i^\alpha = 1, I_{i+1}^\alpha = 1) = P(\emptyset) = 0$. Thus, $F_\alpha(1) = P(L_n^\alpha < k) = 1 - P(L_n^\alpha \geq k) \geq 1 - m_\alpha$, since $m_\alpha = \sum_{i\in J_\alpha} P(A_i^\alpha)$ and $P(L_n^\alpha \geq k) = P(\cup_{i\in J_\alpha} A_i^\alpha)$. Eryilmaz (2005b, 2006) and Eryilmaz and Demir (2007) have used the Worsley's inequality applied for different sets to obtain the same lower bound of $P(L_n^\ell < k)$ for Markov dependent and exchangeable trials, respectively.

Further, for $x = 1$ Eq. 34 combined with Eq. 20 provide an upper bound of the probability of having no success run of length at least k, i.e. $P(G_{n,k}^\alpha = 0) = P(L_n^\alpha < k)$, $1 \leq k \leq n$. This is true for any specific type of the considered sequences of a fixed length n. Concretely, the relation $\{L_n^\alpha < k_1\} \subseteq \{L_n^\alpha < k_2\}$, $1 \leq k_1 \leq k_2 \leq n$ implies that $P(L_n^\alpha < k) \leq P(L_n^\alpha < k_0)$ for $1 \leq k \leq k_0 \leq n$ with $2k_0 \geq n$ if $\alpha = \ell$ and $2k_0 + 1 \geq n$ if $\alpha = c$. This means that the exact value of $P(L_n^\alpha < k_0)$ $[P(L_n^\alpha \geq k_0)]$ is always an upper [lower] bound of $P(L_n^\alpha < k)$ $[P(L_n^\alpha \geq k)]$ for $k \leq k_0$. In other words, $\max\{0, 1 - E(G_{n,k}^\alpha)\} \leq P(L_n^\alpha < k) < 1 - E(G_{n,k_0}^\alpha)$ for $k < k_0$, and readily $P(L_n^\alpha < k) = 1 - E(G_{n,k}^\alpha)$ for $k \geq k_0$. Similar ideas are used in Eryilmaz (2008).

As an example, we consider n trials with outcomes ordered on a line ($\alpha = \ell$). Let $k_0 = \lceil n/2 \rceil$ and the trials be exchangeable ($d = II$). Then, Eqs. 26(a) and 34 imply $l_M^\ell(1) = 1 - E(G_{n,k}^\ell) = 1 - (n-k+1)\lambda_k + (n-k)\lambda_{k+1}$. Therefore, $\max\{0, 1 - (n-k+1)\lambda_k + (n-k)\lambda_{k+1}\} \leq P(L_n^\ell < k) < 1 - (n-k_0+1)\lambda_{k_0} + (n-k_0)\lambda_{k_0+1}$, for $1 \leq k < k_0$ and $P(L_n^\ell < k) = 1 - (n-k+1)\lambda_k + (n-k)\lambda_{k+1}$, for $k_0 \leq k \leq n$.

(Chebychev's bounds). For $x \in S_\alpha(n,k)$ it holds

$$F_\alpha(x) \geq l_C^\alpha(x) = 1 - \sigma_\alpha^2/\{\sigma_\alpha^2 + (x - m_\alpha)^2\}, \quad \text{if} \quad x > m_\alpha$$

$$F_\alpha(x) \leq U_C^\alpha(x) = \sigma_\alpha^2/\{\sigma_\alpha^2 + (1 + m_\alpha - x)^2\}, \quad \text{if} \quad x < m_\alpha + 1. \tag{35}$$

It is evident that $l_M^\alpha(x) \geq 0$ if $x \geq m_\alpha$, $l_C^\alpha(x) > 0$ for $x > m_\alpha$ and $l_C^\alpha(x) > l_M^\alpha(x)$ if $x > m_\alpha + \sigma_\alpha^2/m_\alpha$. Thus we have:

For any $x \in S_\alpha(n,k) \cap \{y \in N : y \geq m_\alpha\}$ it holds

$$F_\alpha(x) \geq L_{MC}^\alpha(x), \tag{36}$$

where

$$L_{MC}^\alpha(x) = \begin{cases} 0, & \text{if } x = m_\alpha \\ l_M^\alpha(x), & \text{if } m_\alpha < x \leq m_\alpha + \sigma_\alpha^2/m_\alpha \\ l_C^\alpha(x), & x > m_\alpha + \sigma_\alpha^2/m_\alpha. \end{cases}$$

Equations 35 and 36 along with the dual relationships

$$P(T_{r,k} > n) = F_\ell(r) \quad \text{and} \quad P(L_n^\alpha < k) = F_\alpha(1)$$

establish lower/upper bounds for the tail probability of the waiting time $T_{r,k}$ and the probability distribution of the length of the longest success run L_n^α. Next, new upper bounds for the latter distribution as well as for the tail probability of the waiting time $T_{1,k}$ are given.

(Conditional Expectation bound) It is true that

$$F_\alpha(1) \le U_E^\alpha = 1 - \sum_{i \in J_\alpha} \frac{P(I_i^\alpha = 1)}{E(G_{n,k}^\alpha \mid I_i^\alpha = 1)}, \quad \alpha = \ell, c. \tag{37}$$

Explicitly we have,

$$P(T_{1,k} > n) = P(L_n^\ell < k) = P(G_{n,k}^\ell < 1) \le U_E^\ell$$

where

$$U_E^\ell = 1 - \sum_{i=k}^{n} \frac{P(I_i^\ell = 1)}{1 + \sum_{j=k}^{i-k-1} P(I_j^\ell = 1 \mid I_i^\ell = 1) + \sum_{j=i+k+1}^{n} P(I_j^\ell = 1 \mid I_i^\ell = 1)}, \quad n \ge 2k+1$$

and

$$P(L_n^c < k) = P(G_{n,k}^c < 1) \le U_E^c$$

with

$$\begin{aligned} U_E^c = 1 - P(I_0^c = 1) &- \sum_{i=1}^{k+1} \frac{P(I_i^c = 1)}{1 + \sum_{j=i+k+1}^{n-k+i-1} P(I_j^c = 1 \mid I_i^c = 1)} \\ &- \sum_{i=k+2}^{n-k-1} \frac{P(I_i^c = 1)}{1 + \sum_{j=1}^{i-k-1} P(I_j^c = 1 \mid I_i^c = 1) + \sum_{j=i+k+1}^{n} P(I_j^c = 1 \mid I_i^c = 1)} \\ &- \sum_{i=n-k}^{n} \frac{P(I_i^c = 1)}{1 + \sum_{j=k+1-n+i}^{i-k-1} P(I_j^c = 1 \mid I_i^c = 1)}, \quad n \ge 2k+2. \end{aligned}$$

We can utilize the probabilities $P(I_i^\alpha = 1)$, $P(I_j^\alpha = 1 \mid I_i^\alpha = 1) = \frac{P(I_j^\alpha = 1, I_i^\alpha = 1)}{P(I_i^\alpha = 1)}$ given in the proofs of Propositions 3.1 to 3.5 to obtain explicit formulae of U_E^α for Poisson, exchangeable and Markov dependent trials, e.g. we see that for Poisson trials

$$U_E^\ell = 1 - \sum_{i=k}^{n} \frac{\mu_i^\ell}{1 + \sum_{j=k}^{i-k-1} \mu_j^\ell + \sum_{j=i+k+1}^{n} \mu_j^\ell}, \quad n \ge 2k+1$$

and

$$\begin{aligned} U_E^c = 1 - \mu_0^c - \sum_{i=1}^{k+1} \frac{\mu_i^c}{1 + \sum_{j=i+k+1}^{n-k+i-1} \mu_j^c} - \sum_{i=k+2}^{n-k-1} \frac{\mu_i^c}{1 + \sum_{j=1}^{i-k-1} \mu_j^c + \sum_{j=i+k+1}^{n} \mu_j^c} \\ - \sum_{i=n-k}^{n} \frac{\mu_i^c}{1 + \sum_{j=k+1-n+i}^{i-k-1} \mu_j^c}, \quad n \ge 2k+2. \end{aligned}$$

The following relationships establish general arguments for a commonly used approximation of a certain nonnegative quantity and an upper bound of the associated error committed by this approximation (cf. Corollary 1 of Makri and Psillakis 1997). We emphasize that the latter error estimate does not assume the knowledge of the exact value of the studied quantity. It depends only on the lower and upper bounds. Therefore, it gives an advantage in cases for which the exact value is difficult to be computed.

Let LB and UB be a lower and an upper bound of F (≥ 0). Then, F can be estimated by

$$\hat{F} = (LB + UB)/2,$$

with relative error

$$B = \mid F - \hat{F} \mid / F, \quad F > 0. \tag{38}$$

An upper bound of B, for $LB > 0$, is

$$\hat{B} = (UB - LB)/(2LB).$$

For instance, if $\lceil m_\alpha \rceil > 1$, we set $LB = L^\alpha_{MC}(\lceil m_\alpha \rceil)$ and $UB = U^\alpha_C(\lceil m_\alpha \rceil)$ in Eq. 38 to obtain an approximation $\hat{F}_\alpha(\lceil m_\alpha \rceil)$ of $F_\alpha(\lceil m_\alpha \rceil)$ and an upper bound $\hat{B}_\alpha(\lceil m_\alpha \rceil)$ of the accompanied error $B_\alpha(\lceil m_\alpha \rceil)$ of this approximation. Furthermore, for $\lceil m_\alpha \rceil = 1$, Eq. 38 with $LB = L^\alpha_{MC}(1)$ and $UB = \min\{U^\alpha_E, U^\alpha_c(1)\}$ gives analogous results for the probabilities $F_\ell(1) = P(L^\ell_n < k) = P(T_{1,k} > n)$ and $F_c(1) = P(L^c_n < k)$. Numerical investigations indicated that U^α_E is at most equal to $U^\alpha_C(1)$.

Next, we consider a lower bound ℓ_α, an upper bound u_α, an approximation $\hat{e}_\alpha$, and an upper bound $\hat{b}_\alpha$ of the associated error of $\hat{e}_\alpha$, of the expected length of the longest success run $e_\alpha = E(L^\alpha_n)$, for $\alpha = \ell, c$ and $d = I, II, III$.

Let $k_\alpha = \lceil n/2 \rceil$ for $\alpha = \ell$, $\lceil (n-1)/2 \rceil$ for $\alpha = c$; then it is true by Eq. 20 that $\sum_{k=k_\alpha}^n F_\alpha(1) = \sum_{k=k_\alpha}^n (1 - E(G^\alpha_{n,k}))$. Since, $E(L^\alpha_n) = n - \left\{ \sum_{k=1}^{k_\alpha - 1} F_\alpha(1) + \sum_{k=k_\alpha}^n (1 - E(G^\alpha_{n,k})) \right\}$, using the bounds $L^\alpha_{MC}(1)$ and $U^\alpha_{EC}(1)$ of $F_\alpha(1)$, for $1 \leq k \leq k_\alpha - 1$ and Eq. 38 we have:

$$\ell_\alpha \leq e_\alpha \simeq \hat{e}_\alpha = (\ell_\alpha + u_\alpha)/2 \leq u_\alpha$$

and for $e_\alpha > 0$,

$$b_\alpha = \mid e_\alpha - \hat{e}_\alpha \mid / e_\alpha \leq \hat{b}_\alpha = (u_\alpha - \ell_\alpha)/(2\ell_\alpha), \quad \text{for} \quad \ell_\alpha > 0,$$

where

$$\ell_\alpha = C_\alpha - \sum_{k=1}^{k_\alpha - 1} U^\alpha_{EC}(1), \quad u_\alpha = C_\alpha - \sum_{k=1}^{k_\alpha - 1} L^\alpha_{MC}(1),$$

with

$$C_\alpha = k_\alpha - 1 + \sum_{k=k_\alpha}^{n} E(G^\alpha_{n,k}), \;\; \text{and} \;\; U^\alpha_{EC}(1) = \min\{U^\alpha_E, U^\alpha_C(1)\}.$$

5 Reliability of Consecutive Systems

The statistic $G^\alpha_{n,k}$ is useful to study the reliability of a general class of consecutive systems; namely the linear/circular r-consecutive-at-least-k-out-of-n:F systems denoted by $C^+_\alpha(r; k, n : F)$. Such a system consists of n ($n \geq 1$) components. Each component and the system itself is either good (working, "0") or not-good (failed, "1"). The system's components are ordered linearly ($\alpha = \ell$) or circularly ($\alpha = c$). The system fails if and only if there are at least r ($1 \leq r \leq r_\alpha = \lfloor \frac{n+1-\beta_\alpha}{k+1} \rfloor$) runs of at least k consecutive failed components. We mention that the system $C^+_\alpha(r; k, n : F)$ generalizes the well known consecutive-k-out-n:F system denoted by $C_\alpha(k, n; F)$; i.e. $C^+_\alpha(1; k, n : F) \equiv C_\alpha(k, n : F)$. The $C^+_\ell(r; k, n : F)$ was introduced by Agarwal et al. (2007) under the assumption that the states of the components are Bernoulli random variables. Since Kontoleon (1980) and Derman et al. (1982) introduced $C^+_\ell(1; k, n : F)$ and $C^+_c(1; k, n : F)$, respectively, and studied them for the Bernoulli case, a great deal of research has been produced on this subject. We refer to Kuo and Zuo (2003) as well as to Balakrishnan and Koutras (2002), for a review on the literature on such systems whenever the components are independent (identical-nonidentical), exchangeable or dependent in a Markovian fashion.

Assume that the component states are binary RVs, $\{X_i\}_{i=1}^n$, with $P(X_i = 0)$ and $P(X_i = 1) = 1 - P(X_i = 0)$ considered as the reliability and the unreliability of the i-th component, $i = 1, 2, \ldots, n$; then $G^\alpha_{n,k}$ counts the number of runs of at least k consecutive failed components. Therefore, the reliability $R^d_\alpha(r; k, n; D)$ of a $C^+_\alpha(r; k, n : F)$ consisting of independent (Poisson or Bernoulli), exchangeable or Markov dependent components, is defined as

$$R^d_\alpha(r; k, n; D) \equiv P(G^\alpha_{n,k} < r) - P^d_n = 1 - P(G^\alpha_{n,k} \geq r) - P^d_n = 1 - \tilde{R}^d_\alpha(r; k, n; D) \quad (39)$$

where $P^d_n = P\big(\prod_{i=1}^n X_i = 1\big)$, if $r > 1$; 0 if $r = 1$, and

$$\tilde{R}^d_\alpha(r; k, n; D) = \sum_{x=r}^{r_\alpha} h^d_\alpha(x; k, n; D) + P^d_n$$

is the failure probability (unreliability) of a $C^+_\alpha(r; k, n : F)$. The reliability and the unreliability of a $C^+_\alpha(r; k, n : F)$ with i.i.d. components with common component unreliability $p = P(X_i = 1) = 1 - P(X_i = 0) = 1 - q$, are denoted by $R_\alpha(r; k, n; p)$ and $\tilde{R}_\alpha(r; k, n; p)$, respectively.

Alternatively, R^d_α is written

$$R^d_\alpha(r; k, n; D) = \sum_{x=0}^{r-1} h^d_\alpha(x; k, n; D) - P^d_n.$$

Hence, depending on the position of r in the sequence $1, 2, \ldots, r_\alpha$ we can use the next formula for efficient computation of $R^d_\alpha(r; k, n; D)$,

$$R^d_\alpha(r; k, n; D) = \begin{cases} \sum_{x=0}^{r-1} h^d_\alpha(x; k, n; D) - P^d_n, & \text{if } 2r \leq r_\alpha + 1 \\ 1 - \sum_{x=r}^{r_\alpha} h^d_\alpha(x; k, n; D) - P^d_n, & \text{otherwise.} \end{cases} \quad (40)$$

In many applications, exact system reliability is not necessary. Reasonably good bounds or approximations that can be easily computed are usually sufficient. Next, bounds and approximations of $R^d_\alpha(r; k, n; D)$ are derived directly using the material of Section 4. More specifically, we have:

For given α, n, k let $R_\alpha(r) = R^d_\alpha(r; k, n; D)$, $1 \leq r \leq r_\alpha$ and $m_\alpha = E(G^\alpha_{n,k})$. Then it is true that

$$R_\alpha(r) \geq LR_\alpha(r) = L^\alpha_{MC}(r) - P^d_n, \quad \text{for} \quad r \geq \max\{1, m_\alpha\} \quad (41)$$

$$R_\alpha(r) \leq UR_\alpha(r) = \begin{cases} U^\alpha_C(r) - P^d_n, & \text{for } 1 < r < m_\alpha + 1 \\ \min\{U^\alpha_C(1), U^\alpha_E\}, & \text{for } r = 1. \end{cases}$$

Specifically, for $r = \lceil m_\alpha \rceil$ with $m_\alpha > 0$,

$$LR_\alpha(r) \leq R_\alpha(r) \simeq \hat{R}_\alpha(r) = (LR_\alpha(r) + UR_\alpha(r))/2 \leq UR_\alpha(r)$$

and if $R_\alpha(r) > 0$, $LR_\alpha(r) > 0$ we have

$$BR_\alpha(r) = \mid R_\alpha(r) - \hat{R}_\alpha(r) \mid / R_\alpha(r) \leq \hat{BR}_\alpha(r) = (UR_\alpha(r) - LR_\alpha(r))/(2LR_\alpha(r)).$$

The bounds $L^\alpha_{MC}(r)$, $U^\alpha_C(r)$ and U^α_E are given by Eqs. 35 to 37.

We mention that the error bound $\hat{BR}_\alpha(r)$, is a very conservative one. Although, it is several times larger than $BR_\alpha(r)$, its use in cases in which the exact reliability is difficult to evaluate -due to computer limitations, is inestimable. Further, if $\hat{BR}_\alpha(r) \leq 0.5 \times 10^{-m}$, then $\hat{R}_\alpha(r)$ and the (unknown) $R_\alpha(r)$ have to agree in at least m significant decimal digits.

For systems with i.i.d. components Agarwal et al. (2007) obtained an alternative formula of $R_\ell(r; k, n; p)$ using a graphical evaluation and review technique. Hwang (1986), Muselli (2000a, b) and Fu et al. (2003) provided formulae of $R_\ell(1; k, n; p)$ along with lower/upper bounds, whereas Charalambides (1994), Makri and Philippou (1994) gave expressions of $R_c(1; k, n; p)$. Lambiris and Papastavridis (1985) derived $R_\alpha(1; k, n; p)$ for both $\alpha = \ell$ and $\alpha = c$.

6 Applications and Numerics

In this section some examples of binary sequences (independent, exchangeable and Markov dependent ones) are considered. They are indicative of real situations and they have appeared in numerous fields of application of run related statistics. For these sequences we present a summary of the involved concepts and the necessary notation. For details the interested reader may consult the quoted references. After that, we select some parametric configurations which we use in indicative case studies. These give to someone a sense of the involved numerics and a gain of insight into the formulae presented in Sections 2 to 5. Further, the use of the concerned sequences as models on which the RVs $G^\alpha_{n,k}$, L^α_n and $T_{r,k}$ are defined, imply some attractive applications of these RVs.

6.1 Application Binary Sequences

6.1.1 Polya-Eggenberger urn Model

We start with an example of an exchangeable sequence of special interest in applied probability; namely, the Poya-Eggenberger sampling scheme (cf. Johnson and Kotz 1977, pp 176–178). In this scheme a ball is drawn at random from an urn initially containing w white balls and b black balls, its color is observed, and it is then returned to the urn along with s additional balls of the same color as the ball drawn. Drawing a white ball is considered as a success and drawing a black ball is considered as a failure. We denote this scheme as $PE(w, b, s)$. A finite number of n repetitions of the scheme derives an exchangeable binary sequence with

$$\lambda_i = \prod_{j=0}^{i-1} \frac{w + js}{w + b + js}, \quad i = 0, 1, \ldots, n. \tag{42}$$

For $w = b = s = 1$, Eq. 42 reduces to $\lambda_i = \frac{1}{1+i}$, $i = 0, 1, \ldots, n$.

For Bernoulli trials which correspond to a Polya-Eggenberger sampling scheme with replacements, i.e. $s = 0$, Eq. 42 gives

$$\lambda_i = \lambda_1^i, \quad i = 1, 2, \ldots, n \;\text{ and }\; \lambda_0 = 1, \tag{43}$$

with

$$\lambda_1 = w/(w + b) = p$$

where p, $0 < p < 1$, is the common success probability of the trials. For such trials $p_n(y)$ reduces to

$$p_n(y) = p^{n-y} q^y, \quad p + q = 1.$$

A Bernoulli sequence, that may be used as an example of threshold exceedances in a diverse field of applications, is the fixed truncation or fixed threshold model (see, Sen 1991; Boutsikas and Koutras 2002; Eryilmaz 2005a). In this model let $\{Y_i\}_{i=1}^n$ be i.i.d. RVs with $P(Y_i \le y) = F_Y(y)$, carrying some crucial information about a studied quantity. F_Y is a continuous distribution function. If t is a given threshold value then the Bernoulli sequence associated with the fixed threshold model (FTM) is defined as

$$X_i = \begin{cases} 1, \text{ if } Y_i > t \\ 0, \text{ otherwise,} \end{cases} \quad i = 1, 2 \ldots, n, \tag{44}$$

with a common exceedance (success) probability $p = E(X_i) = P(X_i = 1) = P(Y_i > t)$.

Another type of truncation which may be viewed as a random truncation or random threshold model is defined as follows (see, Sarkadi 1957; Eryilmaz 2005a). Suppose that we have $m + n$ independent observations regarded as two samples of sizes m (≥ 1) and n (≥ 1), respectively, from a population with a continuous distribution function, say F; specifically the samples $Y_1, Y_2, \ldots, Y_m$ and $Y_{m+1}, Y_{m+2}, \ldots, Y_{m+n}$. The j-th smallest element with $1 \le j \le m$, i.e. the j-th order statistic $Y_{j:m}$ of the first sample is chosen as a random threshold. Then, the sequence associated with the random threshold model (RTM)

$$X_i = \begin{cases} 1, \text{ if } Y_{m+i} > Y_{j:m} \\ 0, \text{ otherwise,} \end{cases} \quad i = 1, 2 \ldots, n, \tag{45}$$

is exchangeable. This is so, since the random threshold model corresponds to n repetitions of a Polya-Eggenberger sampling scheme with $s = 1$, $w = m + 1 - j$ and $b = j$, and links the latter scheme with the study of the order statistics (cf. Johnson and Kotz 1977, p 181 or David and Nagaraja 2003). In this model

$$\lambda_1 = P(Y_{m+i} > Y_{j:m}) = 1 - \frac{j}{m+1} = \frac{w}{w+b} \tag{46}$$

represents the exceedance (success) probability.

We note that both FTM and RTM are derived by a $PE(w, b, s)$ urn model with initial composition w and b such that $\lambda_1 = \frac{w}{w+b}$. However, FTM corresponds to $s = 0$ (drawings with replacements such that the urn composition remains always constant) whereas RTM corresponds to $s = 1$ (drawings such that the urn composition increases). Hence, the sequence $\{X_i\}_{i=1}^n$ is an exchangeable one in both cases. Thus, the behavior of RVs $G^{\alpha}_{n,k}$, $T_{r,k}$ and L^{α}_{n} is reasonable to be different whenever they are defined on a sequence derived by using a FTM and a RTM with the same λ_1. It clearly explains why the findings of Sen (1991) and Eryilmaz (2005a) with $\lambda_1 = 0.5$, concerning the expected length of the longest success run in a linear sequence are different.

6.1.2 Record Indicator Model

A known example of independent but not identically distributed RVs is the record indicator model (RIM). Let $\{Y_i\}_{i\geq 1}$ be a sequence of RVs with continuous distribution function F and we consider the nondecreasing sequence

$$M_j = \max\{Y_1, Y_2, \ldots, Y_j\}, \quad j = 1, 2, \ldots.$$

Then, we define the sequence (cf. Nevzorov 2001, pp 57–58 or Eryilmaz and Tutuncu 2002, p 76)

$$X_1 = 1 \quad \text{and} \quad X_j = I\{M_j > M_{j-1}\}, \quad j = 2, 3, \ldots. \tag{47}$$

That is $X_j = 1$, if Y_j is a record and $X_j = 0$, otherwise. X_j's are called record indicators. For sequences of independent and identically distributed RVs Y_j the record indicators X_j have two important properties (cf. Renyi 1962 or Lemma 13.1 of Nevzorov 2001). First, they are independent RVs and second

$$P(X_j = 1) = 1/j, \quad j = 1, 2, \ldots. \tag{48}$$

Similar properties also hold for the record indicators defined on a sequence of exchangeable (symmetrically dependent) RVs. See, Theorem 28.2 of Nevzorov (2001) and the discussion on the subject in Eryilmaz and Tutuncu (2002, pp 76, 80).

6.1.3 Communications Model

An useful example of a homogeneous Markov chain is the following (cf. Ross 2002, p 104). Consider a communication system that transmits the digits 0 and 1. Each digit transmitted must pass through several stages, at each of which there is a probability p $(0 < p < 1)$ that the digit entered will be unchanged when it leaves. Letting X_n denote the digit entering the n-th stage, then X_n, $n \geq 1$ is a two-state Markov chain having a one-step transition probability matrix

$$\mathbf{P} = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}. \tag{49}$$

6.2 Case Studies and Numerics

Before we proceed in applied case studies, let us consider an example dealing with the formulae presented in the Sections 2 to 4.

6.2.1 Numerical Comparisons

In this example we give some numerics concerning several model sequences of various length, which belong to the different kinds of the binary sequences considered in the article. They show a variety of possible configurations and also shed some light to the similarities and discrepancies among the corresponding probabilities, means and variances of the RV $G^{\alpha}_{n,k}$, defined on a linear ($\alpha = \ell$) or on a circular ($\alpha = c$) sequence. The used sequences are:

Case I: A Bernoulli sequence with a common success probability $p = 1/2$.
Case II: A sequence of Poisson trials with $p_i = 1/(1+i)$ (Table 1) or $p_i = 1/i$ (Table 2), $i = 1, 2, \ldots, n$.
Case III: An exchangeable sequence with $\lambda_i = 1/(1+i), i = 1, 2, \ldots, n$.
Case IV: An homogeneous Markov chain with $p_{00} = 3/4$ and $p_{10} = 1/4$.
Case V: An non-homogeneous Markov chain with $p^{(t)}_{00} = 1/t^2$ and $p^{(t)}_{10} = 1 - 1/t$, $t \geq 2$.

The initial probability vector used in cases IV and V is $\mathbf{p}^{(1)} = (1/2, 1/2)$.

Table 1 gives exact probabilities, means and variances of $G^{\alpha}_{5,2}$ defined on the sequences quoted in cases I to V. The value $n = 5$ was chosen small so that the required computations can also be carried out by hand, and thus it is possible to gain insight in to formulae (9) to (11), (13) to (17) and (22) to (30) presented in the text.

The results of Table 1 can be used further to mine some information—via the given means and variances, about bounds and approximations of $F_{\alpha}(x) = P(G^{\alpha}_{5,2} < x)$. As an indicative example we consider the case II, $\alpha = \ell$ and $x = 1$. We assume that the exact probability $F_{\ell}(1)$ is difficult to evaluate but we know in advance $E(G^{\ell}_{5,2})$ and $V(G^{\ell}_{5,2})$. Since, $x = \lceil E(G^{\ell}_{5,2}) \rceil = 1$, the remarks after Eq. 38 along with Eqs. 34 to 36 imply that $L^{\ell}_{MC}(1) = l^{\ell}_{M}(1) = 0.7333$, $U^{\ell}_{C}(1) = 0.7414$, $\hat{F}_{\ell}(1) = 0.7374$ and $\hat{B}_{\ell}(1) = 0.0055 \leq 0.5 \times 10^{-1}$. Therefore, $\hat{B}_{\ell}(1)$ suggests that $\hat{F}_{\ell}(1)$ has to agree with respect to the (unknown) exact value $F_{\ell}(1)$, in at least 1 significant decimal digits. In fact, they agree in 3 digits since $B_{\ell}(1) = 0.00014 \leq 0.5 \times 10^{-3}$. Similar analysis may be useful in cases in which $P(G^{\alpha}_{n,k} < x)$ is really difficult to evaluate but the mean and the variance of $G^{\alpha}_{n,k}$ are efficiently computed.

In Table 2 we present means and variances of $G^{\alpha}_{n,k}$ for $\alpha = \ell, c$, for various values of n and for the sequences I to V. The chosen values of n range from small ($n = 10$) to large ($n = 1000$). The selected values of k are such that to have the same percentages, k/n, of the success threshold length k in different values of n. They are $k/n = 0.5\%$,

Table 1 Exact probabilities, means and variances of $G^{\alpha}_{5,2}$

Case	α	$E(G^{\alpha}_{5,2})$	$V(G^{\alpha}_{5,2})$	x	$P(G^{\alpha}_{5,2} < x)$
I	ℓ	0.62500	0.29688	1	0.4063
				2	0.9688
	c	0.65625	0.22559	1	0.3438
II	ℓ	0.26667	0.20389	1	0.7375
				2	0.9958
	c	0.30694	0.21273	1	0.6931
III	ℓ	0.58333	0.30972	1	0.4500
				2	0.9667
	c	0.58333	0.24306	1	0.4167
IV	ℓ	0.65625	0.26074	1	0.3613
				2	0.9824
V	ℓ	0.54427	0.31054	1	0.4870
				2	0.9688

Table 2 Means and variances of $G^{\alpha}_{n,k}$, $\alpha = \ell, c$

n	k	k/n	case	$E(G^{\ell}_{n,k})$	$V(G^{\ell}_{n,k})$	$E(G^{c}_{n,k})$	$V(G^{c}_{n,k})$
10	1	10%	I	2.750	0.688	2.501	0.621
			II	2.029	0.649	1.929	0.579
			III	2.000	1.200	1.758	1.002
			IV	1.625	0.547		
			V	4.363	0.366		
100	1	1%	I	25.250	6.313	25.000	6.250
			II	4.197	2.582	4.187	2.572
			III	17.000	61.200	16.677	61.902
			IV	12.875	4.766		
			V	48.754	0.665		
	5	5%	I	1.516	1.262	1.563	1.294
			II	0.009	0.009	0.009	0.009
			III	2.429	7.230	2.391	7.048
			IV	3.916	1.952		
			V	0.006	0.006		
	10	10%	I	0.045	0.044	0.049	0.048
			II	2.79×10^{-7}	2.79×10^{-7}	3.06×10^{-7}	3.06×10^{-7}
			III	0.773	1.711	0.767	1.665
			IV	0.882	0.703		
			V	1.63×10^{-7}	1.63×10^{-7}		
1000	5	0.5%	I	15.578	12.907	15.625	12.939
			II	0.009	0.009	0.009	0.009
			III	23.857	610.848	23.811	609.550
			IV	39.511	19.246		
			V	0.006	0.006		
	10	1%	I	0.484	0.479	0.488	0.483
			II	2.79×10^{-7}	2.79×10^{-7}	2.81×10^{-7}	2.81×10^{-7}
			III	7.591	134.791	7.577	134.435
			IV	9.329	7.326		
			V	1.63×10^{-7}	1.63×10^{-7}		

1%, 5% and 10%. The depicted values give a sense of feeling of the variation of the means and the variances of $G^{\alpha}_{n,k}$, defined on different in structure and in length sequences and of how these values variate with a pre-specified threshold k. To derive the results of the Table Eqs. 22 to 30 have been used.

In the sequel, we consider some case studies which show potential uses of $G^{\alpha}_{n,k}$ and its associated statistics L^{α}_{n} and $T_{r,k}$ in applied research.

6.2.2 Forecasting in Gambling and Finance

A. (*Avoid ruin in a roulette*) Consider a player who bets in a Las Vegas roulette wheel. The Vegas wheel is divided in 38 congruent (and supposedly equal likely) sectors numbered 1 through 36, 0, and 00. The numbers 1 through 36 consist of 18 red numbers and 18 black (0 and 00 are green). We prefer to consider a Las Vegas wheel instead of a European roulette wheel used by Binswanger and Embrechts (1994) because in the latter roulette a wager referring to an even money bet (for instance, red) is a little more complicated than the corresponding wager in a Vegas roulette (cf. Packell 1981, pp 17, 25, 78–79, 84). Let the player bets only on red with win probability $p = P(win) = P(red) = P(\text{“}success\text{”}) =$

18/38. Although the latter authors recommend that a player is better not to play, the answering to some reasonable questions of the kind that are listed below, allows the player to estimate certain risk probabilities. Further, the results may be applied in the famous Martingale or "double when you loose" strategy to help a gambler to estimate the financial reserves needed in order to avoid ruin.

(I) What is the probability to appear 2 streaks of at least 3 consecutive reds in a sequence of 20 games? How this probability is changed if the player has the willing to bet 80 additional times?
(II) What is the expected number of streaks of at least 3 consecutive wins each in a sequence of $n = 20, 100$ games? How much on the average the number of streaks variate around its mean value?
(III) What is the longest streak of consecutive wins that the player expects to see in $n = 20$ and 100 games?
(IV) How long on the average does the player has to wait until 2 win streaks, each of length at least 3, occur?
(V) Let the player decide to abandon the game after a specific number, say n, of spins of the wheel. If we suppose that n is determined by the probability $P(T_{r,k} > n) \leq 0.5$, how long the player has to wait until abandons for $k = 3$ and $r = 1, 2, 3, 4$?

Denoting by $G^{\ell}_{n,k}$, L^{ℓ}_n and $T_{r,k}$ the number of streaks, each of at least k consecutive wins, the length of the longest winning run in n games, and the waiting time until r streaks each of at least k consecutive wins occur one gets using formulae (11) or (13), (24) and (31) to (33), with $p = 18/38$, the following answers:

(I) $P(G^{\ell}_{20,3} = 2) = 0.2368$ and $P(G^{\ell}_{100,3} = 2) = 0.0341$.
(II) $E(G^{\ell}_{20,3}) = 1.0572$, $V(G^{\ell}_{20,3}) = 0.6755$ and $E(G^{\ell}_{100,3}) = 5.5323$, $V(G^{\ell}_{100,3}) = 3.3982$.
(III) $E(L^{\ell}_{20}) = 3.4923$ and $E(L^{\ell}_{100}) = 5.5907$.
(IV) $E(T_{2,3}) = 33.85$.
(V) For $r = 1, 2, 3, 4$ we have $n = 12, 29, 47, 65$ games, respectively.

B. (*Predicting in capital markets*) Consider a trader at a certain capital market (a stock market, a commodity market, an exchange market, etc). Suppose that the trader makes transactions on an individual security or on an index defined on that market at time units i (days, weeks, months, etc), $i = 1, 2, \ldots, m + n$. Let the respective prices of the security be Y_i. We assume that the samples $\{Y_i\}_{i=1}^{m}$ and $\{Y_{m+i}\}_{i=1}^{n}$ are independent from each other and represent the past and the future prices of the security. We consider as a "success" ("1") the attribute that a certain future value exceeds $Y_{j:m}$, i.e. the j-th smallest value of the sample of the past values $Y_1, Y_2, \ldots, Y_m$. The occurrence $X_i = 1$, or the not occurrence $X_i = 0$, of this attribute is defined by the exchangeable sequence given by Eq. 45. Therefore, the probabilistic analysis of the RVs $G^{\ell}_{n,k}$, L^{ℓ}_n and $T_{r,k}$ defined on the sequence $\{X_i\}_{i=1}^{n}$ may be helpful in predicting recurrences of successes within a certain time horizon of n units. It helps the trader, mainly, to avoid ruin or even hopefully to evaluate risk probabilities in order to design profitable strategies.
As a practical example we note that in stock markets samples of size $m = 9, 21$ and 39 weeks are of common use in representing the "short-term", the

"mid-term" and the "long-term" past behavior of a stock or a stock index. Next, some indicative questions of the same sense of the ones quoted in part A are presented.

(I) What is the probability that in the next 9 weeks will occur 2 clumps each of which contains at least 2 consecutive exceedances of the median price that a stock attained during the past 21 weeks?
(II) How the previous probability changes if the predicting time horizon increases by 6 additional weeks?
(III) How much the probability (II) is affected if our predictions are based on a larger set of past data, for instance 39 weeks?
(IV) What is the probability that in the next 21 weeks will occur 2 clumps each of which contains at least 2 consecutive exceedances of the third smallest price that a stock attained during the past 39 weeks?
(V) Which is the respective probability to the one quoted in (IV), if the trader uses FTM with the same exceedance probability?
(VI) How long on the average the trader has to wait until two consecutive exceedances of the smallest value of a stock of the past 39 weeks occur? Which is the respective predicting time that the trader has to wait if he uses FTM?
(VII) Suppose that a trader wants to design a buy-sell strategic based on the expected length of the longest success run. A success is defined as an exceedance over the third smallest price of a data set consisting of 39 past prices. Which are the predictions of a RTM for the next $n = 10$ and 20 weeks? Which are the analogous predictions if the trader prefers to use FTM, with the same exceedance probability, instead of RTM?

Using formulae (11) or (13), (16), (31) to (33) and (44) to (46) the respective answers are:
A RTM with $m = 21$ and $j = 11$ gives:

(I) $P(G^{\ell}_{9,2} = 2) = 0.2709$ and
(II) $P(G^{\ell}_{15,2} = 2) = 0.4029$.
(III) A RTM with $m = 39$ and $j = 20$ implies: $P(G^{\ell}_{15,2} = 2) = 0.4168$.
(IV)–(V) A RTM with $m = 39$, $j = 3$ gives $P(G^{\ell}_{21,2} = 2) = 0.3789$ whereas a FTM with the same exceedance probability $\lambda_1 = p = 0.925$ gives $P(G^{\ell}_{21,2} = 2) = 0.4325$, respectively.
(VI) A RTM with $m = 39$, $j = 1$ and $\lambda_1 = 0.975$ gives $E(T_{1,2}) = 2.0804$ whereas a corresponding FTM implies $E(T_{1,2}) = 2.0776$.
(VII) A RTM with $m = 39$ and $j = 3$ gives $E(L^{\ell}_{10}) = 8.1057$, $E(L^{\ell}_{20}) = 14.1280$ and a FTM with the same exceedance probability 0.925 implies $E(L^{\ell}_{10}) = 8.0391$, $E(L^{\ell}_{20}) = 13.7842$, respectively.

We mention that the expected discrepancies among the corresponding results derived by FTM and RTM have been already explained at the end of Section 6.1.

C. (*Waiting for consecutive upper or lower record stock prices*) We consider a sequence of Poisson trials X_i with different success probabilities $p_i, i = 1, 2, \ldots, n$. As a possible application we assume that the binary RVs $\{X_i\}_{i=1}^n$ are derived by applying RIM given by Eqs. 47–48, on the prices $\{Y_i\}_{i=1}^n$ of a certain stock. Then, $X_i = 1$ means the achievement of a new price maximum and we identify it as a new upper record. The RVs L_n^ℓ and $T_{r,k}$ defined on the sequence X_i admit the following interpretation: L_n^ℓ represents the maximum number of successive increases of the stock price (consecutive achievements of new upper record price) in n transactions, whereas $T_{r,k}$ represents the waiting time until the r-th (≥ 1) occurrence of a group with at least k (≥ 1) successive increases of the stock price.

Therefore, the expected values of those RVs may be helpful to a trader. As an example we consider that the prices of a certain stock are recorded on a monthly basis. Let a trader wonder about: the expected maximum number of successive increases of the stock prices in the next $n = 20$ months and how many months on the average has to wait until two not successive increases of the stock price will occur?

Using RIM, Eqs. 31 to 33 give $E(L_{20}^\ell) = 1.8602$ and $E(T_{2,1}) = 14.24$.

By symmetry, price minimums (lower records) of $\{Y_i\}_{i=1}^n$ are obtained by considering the upper records of the sequence $\{-Y_i\}_{i=1}^n$. Namely, they are the Y_i's with $X_i = 1$.

6.2.3 Non-parametric Test of Randomness

We consider two samples S1 and S2 of m and n elements, respectively. First, we test the null hypothesis that the two samples have been randomly selected from the same population. Second, we consider them as a past and a future sample, respectively. After that, we are interested in testing for randomness the two sequences that are derived according to: (I) a RTM and (II) a RIM. These tasks can be accomplished by using the conditional distribution of $G_{n,k}^\ell$, given the number of failures U_n.

To become even clear we consider the following two samples of observations (for instance, they may represent prices of a security in USD) of size $m = 11$ and $n = 10$, respectively.

Sample S1: 26.3, 28.6, 25.4, 29.3, 27.6, 25.6, 26.4, 27.7, 28.2, 29.0, 28.9
Sample S2: 25.3, 26.5, 27.2, 27.5, 26.2, 29.2, 28.5, 30.0, 28.8, 28.4.

Using the run test for randomness of two related samples (cf. Kanji 2001, p 107) we conclude that we do not reject at significant level of $\alpha = 0.05$ the null hypothesis that the samples S1, S2 have been randomly selected from the same population. This is so, because the applied test statistic is found to be (using the number of runs of the combined samples) $Z = 0.91$ whereas the Z-test gives $Z_{0.05} = 1.96$.

After that, we proceed to test for randomness the sequences $\{X_i\}_{i=1}^n$ that are derived using:

(I) RTM with $m = 11$, $j = 6$ and as $\{Y_i\}_{i=1}^m$ and $\{Y_{i+m}\}_{i=1}^n$, the elements of the samples S1 and S2, respectively.
(II) RIM on the sequence $\{Y_i\}_{i=1}^n$ of the elements of sample S2.

Accordingly, we have (I) Since, $Y_{6:11} = 27.7$, X^I : 0000011111 and (II) X^{II} : 1111010100.

Let $G_{10,k} \equiv G_{10,k}^\ell$ and $\hat{G}_{10,k}$ be the observed value of the statistic $G_{10,k}$ in any of the sequences X^I and X^{II}. We choose a significant level α, say $\alpha = 5\%$, and we denote by $G_{10,k}^*$ the critical values of $G_{10,k}$ (if there are any) such that the p-value

$\alpha^* = P(G_{10,k} \geq G^*_{10,k} \mid U_{10} = y) \leq \alpha$ and $P(G_{10,k} \geq g \mid U_{10} = y) > \alpha$ for every $g < G^*_{10,k}$. The number of failures in the sequences X^I and X^{II} are $y^I = 5$ and $y^{II} = 4$, respectively. By Eq. 12 the pairs of critical values $(k, G^*_{10,k})$ and the corresponding probabilities α^* are: (I) (1, 5) and (5, 1) both with $\alpha^* = 0.0238$ and (II) (1, 5), (2, 3), (3, 2) and (6, 1) with $\alpha^* = 0.0238, 0.0476, 0.0476$ and 0.0238, respectively. Therefore:

(I) we reject the null hypothesis of randomness of the sequence X^I since $\hat{G}_{10,5} = G^*_{10,5} = 1$ with $P(G_{10,5} \geq G^*_{10,5} \mid U_{10} = 5) = 0.0238$ and

(II) we do not reject the same hypothesis for the sequence X^{II} since $\hat{G}_{10,3} = 1 < G^*_{10,3} = 2$ with $P(G_{10,3} \geq G^*_{10,3} \mid U_{10} = 4) = 0.0476$ or since $\hat{G}_{10,2} = 1 < G^*_{10,2} = 3$ with $P(G_{10,2} \geq G^*_{10,2} \mid U_{10} = 4) = 0.0476$.

We note that in the second sequence for $k = 3$ we followed the empirical rule suggested by Agin and Godbole (1992), i.e. $k - 1$ is taken equal to the expected length of the longest success run in a random sequence of 10 Bernoulli trias ($p = 0.5$). The latter mean value is $E(L^\ell_{10}) = 2.79883$ as it was derived by Eq. 31. Alternatively, for $k = 2$ we followed the rule proposed by Koutras and Alexandrou (1997), i.e. we choose k so that $P(L^\ell_{10} = k)$ is maximized. By Eq. 18 we find that for $k = 2$ we have the maximum probability which is 0.351563. Further, in the first sequence we selected $k = 5$, that is equal to the observed longest run length, and we did not follow the pre-mentioned rules since for $k = 2, 3$ there is no $G^*_{10,k}$ such that $P(G_{10,k} \geq G^*_{10,k} \mid U_{10} = 5) \leq 0.05$.

6.2.4 Clustering in a Communications System

We suppose that an engineer wants to study the clustering of the binary digits transmitted through the communications system discussed in 6.1.3. Readily, the clustering depends on the stage probability p. The use of the RVs $G^\ell_{n,k}$ and L^ℓ_n, defined on the Markov chain with transition probability matrix given by Eq. 49 and initial probability vector $(1 - p_1^{(1)}, p_1^{(1)})$, may be proven helpful. Among the characteristic numbers of consecutive 1's in a binary sequence could be (a) a k that maximizes $P(L^\ell_n = k)$ and (b) the nearest integer to $E(L^\ell_n)$.

Let us consider a detector that counts the number of "long" runs of 1's; that is, these with length at least equal to a threshold k. If it observes r or more such runs, an order is given to the system- for instance, it sends an alarm signal. The detector's tolerance (the false alarm probability) is taken to be at most equal to γ $(0 < \gamma < 1)$. For a given γ the upper-tailed critical value of r (if there is such a value for the selected γ and the system parameters n, k, p and $p_1^{(1)}$), r^* is: $r^* = \min\{r \geq 1 : P(G^\ell_{n,k} \geq r) \leq \gamma\}$.

Table 3 Communications system: $n = 20$, $p_1^{(1)} = 0.5$, $\gamma = 0.05$

p	$E(L^\ell_{20})$	k	$\max_{1\leq k\leq n} P(L^\ell_{20} = k)$	$E(G^\ell_{20,k})$	$V(G^\ell_{20,k})$	r^*	$P(G^\ell_{20,k} \geq r^*)$
0.75	6.03531	4	0.15198	1.05469	0.49851	3	0.01400
		6		0.53394	0.33685	2	0.04395
0.90	8.53534	6	0.07129	0.70859	0.30997	3	0.00001
		9		0.45199	0.25307	2	0.00269

Table 4 Reliabilities of $C^+_\alpha(r; 3, n : F)$ with unequal component reliabilities $q_i = 0.85$ if i is odd, 0.95 otherwise for $i = 1, 2, \ldots, n$

n	α	r	$LR_\alpha(r)$	$R_\alpha(r)$	$\hat{R}_\alpha(r)$	$UR_\alpha(r)$	$BR_\alpha(r)$	$\hat{BR}_\alpha(r)$
20	ℓ	1	0.98745625	0.98750824	0.98750789	0.98755952	0.00000036	0.000052
		2	0.99684779	0.99994810				
	c	1	0.98612500	0.98618886	0.98618839	0.98625178	0.00000048	0.000064
		2	0.99651120	0.99993625				
100	ℓ	1	0.93195625		0.93398662	0.93601699		0.002179
		4	0.99563806					
	c	1	0.93062500		0.93273356	0.93484211		0.002266
		4	0.99555044					

As a numerical example let us consider that: $\gamma = 5\%$, $n = 20$, $p_1^{(1)} = 0.5$ and $p = 0.75, 0.90$. In Table 3 we present the critical values r^* and the corresponding probabilities $P(G^\ell_{n,k} \geq r^*)$ for the chosen values of k according to the empirical rules (a) and (b). The last two entries of its second row suggest that the detector sends an alarm signal if it finds at least 2 runs of at least 6 consecutive 1's in a system with stage probability 0.75; whereas the last two entries of its third row suggest that the detector sends an alarm signal if it finds at least 3 runs of at least 6 consecutive 1's in a system with stage probability 0.90. The respective false alarm probabilities are smaller than or equal to 4.4% and 0.001%, respectively. Further, as we see an increasing by 1.2 times of the stage probability p, implies an increasing of the corresponding value r^* by 1.5 times and a decreasing of the false alarm probability $P(G^\ell_{20,6} \geq r^*)$ by 4395 times. The latter observations are in accordance with the ones that someone expects to find in well designed detectors of systems with varying stage probability.

In Table 3, $E(G^\ell_{20,k})$ and $V(G^\ell_{20,k})$ for the depicted values of k are also presented. To get the results formulae (17), (18) and (28) to (31) have been used.

6.2.5 System Reliability

In applied reliability studies we need to work with specific systems. To this end, we consider two possible representative examples of a $C^+_\alpha(r; k, n : F)$ system, $\alpha = \ell$ or $\alpha = c$. Concretely in Example 1, a linear and a circular system both with independent components with unequal reliabilities are presented and in Example 2 a circular system with i.i.d. components is provided. The original versions of the examples have been used by several researchers in connection with the reliability of a $C^+_\alpha(1; k, n : F)$ system. Kuo and Zuo (2003, pp 326–327) present some details of the initial version of them.

Table 5 Reliabilities of $C^+_c(r; 3, n : F)$ with common component reliability $q = 0.95$

n	r	$LR_c(r)$	$R_c(r)$	$\hat{R}_c(r)$	$UR_c(r)$	$BR_c(r)$	$\hat{BR}_c(r)$
100	1	0.98812500	0.98819035	0.98818986	0.98825471	0.000001	0.000066
200	1	0.97625000	0.97652017	0.97651606	0.97678212	0.000004	0.000273
	2	0.99396068	0.99973182				
1000	1	0.88125000		0.88751291	0.89377582		0.007107
	4	0.99218513					
100000	16	0.58916831					
	32	0.97153831					

Example 1 (Telecom networks) A sequence of n (≥ 2) microwave stations relay signals from place A to B. Stations are equally spaced between places A and B with the 1st and nth stations identifying places A and B, respectively. Each microwave station is able to transmit signals to a distance including k (≥ 1) other microwave stations. We consider that such a system fails if and only if there are at least r (≥ 1) clusters of stations each of which has at least k consecutive stations failed. The reliabilities of the stations may be different because of differences in environmental conditions and operational procedures among the individual microwave stations and station failures are likely to be independent. This is an example of a $C_\ell^+(r; k, n : F)$ system. If we suppose that the stations form a ring such that the first station is adjacent to (and follows) the n-th station then a $C_c^+(r; k, n : F)$ system is derived.

Example 2 (Vacuum system in an electronic accelerator) In the vacuum system of an electronic accelerator, the core consists of a large number of n (≥ 100) identical components (vacuum bulbs). The vacuum system fails if a pre-specified number of r (≥ 1) non-overlapping component blocks, each containing at least a certain number of k (≥ 1) failed components that are adjacent to one another occurs. The components are placed sequentially along a ring. This is an example of a $C_c^+(r; k, n : F)$ system.

Next, in order to evaluate the reliability of the systems discussed in the previous examples we consider some specific configurations of them. Tables 4 and 5 present exact and approximate values as well as bounds for the reliabilities of the systems considered in Examples 1 and 2, respectively. To obtain the results we have used Eqs. 40 and 41.

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