Methodol Comput Appl Probab (2009) 11:145–157
DOI 10.1007/s11009-008-9085-3

Robust Optimal Portfolio Choice Under Markovian Regime-switching Model

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Received: 12 October 2007 / Revised: 14 May 2008 /
Accepted: 19 May 2008 / Published online: 7 June 2008

Abstract We investigate an optimal portfolio selection problem in a continuous-time Markov-modulated financial market when an economic agent faces model uncertainty and seeks a robust optimal portfolio strategy. The key market parameters are assumed to be modulated by a continuous-time, finite-state Markov chain whose states are interpreted as different states of an economy. The goal of the agent is to maximize the minimal expected utility of terminal wealth over a family of probability measures in a finite time horizon. The problem is then formulated as a Markovian regime-switching version of a two-player, zero-sum stochastic differential game between the agent and the market. We solve the problem by the Hamilton-Jacobi-Bellman approach.

Keywords Robust optimal portfolio · Utility maximization · Model uncertainty · Stochastic differential game · Change of measures

AMS 2000 Subject Classification 91A15 · 91A40 · 91B28 · 93E20

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1 Introduction and Summary

The optimal portfolio allocation problem is an important one in quantitative finance and financial economics. Its root can be traced back to the seminal work of Markowitz (1952), in which an elegant mathematical formulation to the problem is first provided. In particular, he considers a single-period model and uses variance (or standard deviation) as a measure of risk from the portfolio. The problem is formulated as a mean-variance optimization problem. The novelty of the Markowitz approach is that it simplifies the optimal portfolio selection problem to the one where only the mean and the variance of the rate of return are involved under the normality assumption. Despite its importance, the Markowitz paradigm is a static one-period model, which cannot incorporate the dynamic nature of optimal portfolio allocation in practice.

Merton (1969, 1971) pioneered the study of the optimal portfolio allocation problem in a continuous-time economy. He first explored the state of the art of the stochastic optimal control theory to develop an elegant (closed-form) solution to the problem. His work not only provides a theoretically sound and practical solution to the optimal portfolio allocation problem, but also stimulates the development of an important field in modern finance, namely, the continuous-time finance. Since the seminal works of Merton (1969, 1971), there have been numerous works on extending the basic paradigm of the Merton model. Recently, the spotlight seems to have turned to the optimal portfolio selection problem in the context of Markovian regime-switching models. These models allow the flexibility that one set of the model parameters switches to another set of the model parameters according to an underlying Markov chain. Empirically, Markovian regime-switching models provide better fit to many economic and financial time series compared with their non-regime-switching counterparts. From an economic perspective, the investment opportunity set varies stochastically over time in Markovian regime-switching models. Some novel and interesting economic insights into the optimal portfolio allocation problem can be gained when one moves from the constant opportunity set in a constant-coefficient model to the stochastic investment set described by a Markovian regime-switching model. Zhou and Yin (2003) and Yin and Zhou (2004) consider the Markowitz mean-variance portfolio selection problem under Markovian regime-switching models in a continuous-time setting and a discrete-time setting, respectively. They formulate the problem as a linear-quadratic optimization problem. Jang et al. (2007) investigate the optimal consumption/investment problem in a Markovian regime-switching model when transaction costs are present. They obtain some new insights into understanding the effect of transaction costs on liquidity premia under the assumption of the stochastic investment opportunity set described by the Markovian regime-switching model. Elliott and Siu (2007) consider the problem of minimizing portfolio risk under a Markovian regime-switching model. In particular, the portfolio risk is measured by a convex risk measure introduced by Föllmer and Schied (2002). The problem is formulated as a stochastic differential game and a verification of the Hamilton-Jacobi-Bellman (HJB) solution is provided.

In practice, there is no general agreement on which probability model or measure should be used for the optimal portfolio allocation problem. So, model uncertainty does exist and it is important to incorporate such uncertainty when considering the problem. In fact, model uncertainty is an important issue for any modelling exercises in finance. See Derman (1996) and Roma (2006) for excellent discussions for the impact of model uncertainty on financial modelling and derivatives pricing, respectively. Cvitanic and Karatzas (1999) adopt a family of real-world probability measures to incorporate model uncertainty in a continuous-time financial model and to generate a dynamic measure of risk from the family of probability measures, which resembles the representation of coherent risk measures introduced by Artzner et al. (1999). Model uncertainty may be related to a well-known paradox in economics, namely, the Ellsberg paradox, which is first proposed in the seminal work of Ellsberg (1961). It is noted in Ellsberg (1961) that economic agents behave differently under ambiguity and risk aversion and that the distinction between risk aversion and ambiguity aversion is important for economic analysis from a behavioral economics perspective. Different axiomatic frameworks of ambiguity aversion have been proposed in the literature (see, for example, Gilboa and Schmeidler (1989), Epstein and Schmeidler (2003), Hansen et al. (1999), Anderson et al. (2003), and others). These models for ambiguity aversion are based on various forms of max-min expected utility over a set of multiple distributions. See Leippold et al. (2007) for an excellent discussion. In the context of optimal portfolio selection, ambiguity aversion can be incorporated by considering the maximization of the minimal expected utility.

In this article we investigate an optimal portfolio selection problem in a continuous-time Markov-modulated financial market when an economic agent faces model uncertainty and seeks a robust optimal portfolio strategy. We consider a market with two primitive assets, namely, a bank account and a risky stock. The key market parameters, including the market interest rate, the appreciation rate and the volatility of the stock are modulated by a continuous-time, finite-state Markov chain. We suppose that the agent is to maximize the minimal expected utility of terminal wealth over a family of (real-world) probability measures in a finite time horizon. The problem is then formulated as a Markovian regime-switching version of a two-player, zero-sum stochastic differential game between the agent and the market. We solve the problem using the HJB approach.

The article is structured as follows. The next section presents the model dynamics in the Markovian regime-switching financial market. We describe the robust portfolio selection problem and formulate it as the stochastic differential game in Section 3. In Section 4, we provide a verification theorem for the HJB solution to the game corresponding to the robust optimal portfolio selection problem and derive its solution in terms of a system of first-order nonlinear ordinary differential equations.

2 The Model Dynamics

We consider a continuous-time financial market with two primitive assets, namely, a bank account and a risky stock, that are tradable continuously over time on a fixed horizon $\mathcal{T} := [0, T]$, where $T \in (0, \infty)$. These are the available investment assets of an economic agent. We suppose that the market is frictionless; that is, there are no transaction costs and taxes. We also assume that assets are divisible so that fractional units of assets can be traded. Fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Here, $\mathcal{P}$ represents a reference probability measure from which a family of absolutely continuous real-world probability measures are generated. In the sequel, we describe an observable continuous-time, finite-state Markov chain whose states represent states of an economy. Throughout the paper, we use bold-face letters to denote matrices (or vectors).

Let $\mathbf{X} := \{\mathbf{X}(t) | t \in \mathcal{T}\}$ denote a continuous-time, finite-state Markov chain on $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\mathcal{X} := \{\mathbf{x}_1, \mathbf{x}_2 \ldots, \mathbf{x}_N\}$. Without loss of generality, we identify the state space $\mathcal{X}$ with a finite set of unit vectors $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where $\mathbf{e}_i \in \Re^N$ and the j^{th} component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij}, for each $i, j = 1, 2, \ldots, N$. We call $\mathcal{E}$ the canonical state space of $\mathbf{X}$. With this, a representation to the chain, which describes its evolution over time, can be derived. To specify the statistical properties of the chain $\mathbf{X}$, we define a family of generators, or rate matrices, $\mathbf{A}(t) = [a_{ij}(t)]_{i,j=1,2,\ldots,N}$, $t \in \mathcal{T}$, of $\mathbf{X}$ under $\mathcal{P}$, where, for $i \neq j$, $a_{ij}(t)$ is the instantaneous intensity of the transition of $\mathbf{X}$ from state j to state i at time t. Here, for each $t \in \mathcal{T}$, $a_{ij}(t) \geq 0$, for $i \neq j$ and $\sum_{i=1}^{N} a_{ij}(t) = 0$, so $a_{ii}(t) \leq 0$. We suppose that $a_{ij}(t) > 0$, for each $i, j = 1, 2, \ldots, N$ and each $t \in \mathcal{T}$. With the canonical state space, Elliott et al. (1994) provide the following semimartingale representation for $\mathbf{X}$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}(u)\mathbf{X}(u)du + \mathbf{M}(t) , \tag{1}$$

where $\{\mathbf{M}(t) | t \in \mathcal{T}\}$ is an $\Re^N$-valued martingale with respect to the filtration generated by $\mathbf{X}$ under $\mathcal{P}$.

Write $\mathbf{y}'$ for the transpose of a matrix (or a vector) $\mathbf{y}$ and $\langle \cdot, \cdot \rangle$ for the scalar product in $\Re^N$. The instantaneous market interest rate $r(t)$ of the bank account is determined by the chain as:

$$r(t) = \langle \mathbf{r}, \mathbf{X}(t) \rangle \ ,$$

where $\mathbf{r} := (r_1, r_2, \ldots, r_N)' \in \Re^N$ with $r_i > 0$, for each $i = 1, 2, \ldots, N$. We assume that the r_i's are all distinct.

Then, the evolution of the balance of the bank account over time is:

$$B(t) = \exp\left(\int_0^t r(u)du \right) , \quad B(0) = 1 .$$

We further suppose that the appreciation rate $\mu(t)$ and the volatility $\sigma(t)$ of the stock are, respectively, determined by the chain as:

$$\mu(t) = \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle \ , \quad \sigma(t) = \langle \boldsymbol{\sigma}, \mathbf{X}(t) \rangle \ ,$$

where $\boldsymbol{\mu} := (\mu_1, \mu_2, \ldots, \mu_N)' \in \Re^N$ and $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \ldots, \sigma_N)' \in \Re^N$ with $\mu_i > r_i$ and $\sigma_i > 0$, for each $i = 1, 2, \ldots, N$. We further assume that the components of $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are all distinct.

Let $W := \{W(t) | t \in \mathcal{T}\}$ denote a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to the $\mathcal{P}$-augmentation of its natural filtration. We suppose that W and $\mathbf{X}$ are stochastically independent under the measure $\mathcal{P}$. The evolution of the price process of the stock over time follows a Markovian regime-switching GBM:

$$\begin{aligned} dS(t) &= \mu(t)S(t)dt + \sigma(t)S(t)dW(t) , \\ S(0) &= s > 0 . \end{aligned} \tag{2}$$

Under the assumption that the σ_i's are all distinct, there is no loss of generality to assume that the chain $\mathbf{X}$ is observable since $\mathbf{X}(t)$ is identified exactly by the (local) quadratic variation of the price process of the stock in any small time duration to the left of t, (see Guo (2001)).

We then describe the information structure of the model. Let $F^{\mathbf{X}}$ and F^S denote the right-continuous, complete filtration generated by the values of the chain and the stock price process, respectively. Write, for each $t \in \mathcal{T}$, $\mathcal{G}(t) := \mathcal{F}^{\mathbf{X}}(t) \vee \mathcal{F}^S(t)$, the enlarged σ-field generated by $\mathcal{F}^{\mathbf{X}}(t)$ and $\mathcal{F}^S(t)$. Note that at each time $t \in \mathcal{T}$, $\mathcal{G}(t)$ is known to the economic agent. Write $G := \{\mathcal{G}(t) | t \in \mathcal{T}\}$.

Now, we describe the wealth of the agent. Suppose $\pi(t)$ represents the proportion of the total wealth invested in the stock at time $t \in \mathcal{T}$, so $1 - \pi(t)$ is the proportion invested in the bank account. Here, we assume that the portfolio process $\pi := \{\pi(t) | t \in \mathcal{T}\}$ is G-progressively measurable and cadlag (i.e. right continuous with left limit). We further impose two standard assumptions for the portfolio process π.

1. π is self-financing; that is, there is no income and consumption;
2. $$\int_0^T \pi^2(t) dt < \infty \, , \quad \mathcal{P}\text{-a.s.}$$

We denote the space of all such processes as $\mathcal{A}$. A portfolio process $\pi \in \mathcal{A}$ is said to be admissible.

Suppose

$$V^{\pi}(t) := \pi(t) S(t) + (1 - \pi(t)) B(t)$$

denotes the total wealth of the portfolio process π at time t. Then, it can be shown that the evolution of the wealth process $V := \{V(t) | t \in \mathcal{T}\}$ over time is governed by:

$$\begin{aligned} dV^{\pi}(t) &= V^{\pi}(t) \{[r(t) + \pi(t)(\mu(t) - r(t))] dt + \pi(t) \sigma(t) dW(t)\} \, , \\ V^{\pi}(0) &= v > 0 \, . \end{aligned} \tag{3}$$

3 The Portfolio Selection Problem

In this section we present the robust optimal portfolio selection problem of the economic agent who selects the portfolio process to maximize the minimal expected utility of terminal wealth over a family of real-world probability measures. We suppose that the agent has a power utility function:

$$U(v) = \frac{v^{1-\gamma}}{1-\gamma} \, , \quad v \in [0, \infty) \, . \tag{4}$$

Here, γ is the risk aversion parameter. The relative risk aversion of the agent is γ.

In the sequel, we generate a family P_a of real-world probability measures by a product of two density processes, one for the Brownian motion W and one for the Markov chain $\mathbf{X}$. This allows us to incorporate model uncertainty in both the stock price process and the Markov chain describing the states of the economy. A product of two density processes is also considered in Elliott and Siu (2007). However, we simplify the parametrisation of the product of two density processes of Elliott and Siu (2007) and introduce a new parametrisation for instantaneous intensities of transitions of the chain under the family of real-world probability measures. In particular, we suppose that all instantaneous intensities of transitions of the chain are scaled or rotated by the same amount when changing the probability measures. This provides a more parsimonious result.

Firstly, we define a family of density processes for the Brownian motion. Let $\{\theta(t)|t \in \mathcal{T}\}$ denote a Markovian regime-switching process such that for each $t \in \mathcal{T}$,

$$\theta(t) = \langle \boldsymbol{\theta}(t), \mathbf{X}(t) \rangle \ ,$$

where $\boldsymbol{\theta}(t) := (\theta_1(t), \theta_2(t), \ldots, \theta_N(t))' \in \Re^N$, for some functions $\theta_i(t)$ $(i = 1, 2, \ldots, N)$ with $\theta_i(t) \geq 0$ and $\theta_{(N)}(t) := \max_{1 \leq i \leq N} \theta_i(t) < \infty$, $\forall t \in \mathcal{T}$. Write Θ for the space of all such processes $\{\theta(t)|t \in \mathcal{T}\}$.

Then, for each $\theta \in \Theta$, the density process for the Brownian motion associated with θ is defined as a G-adapted process $\Lambda_1^\theta := \{\Lambda_1^\theta(t)|t \in \mathcal{T}\}$ given as below:

$$\Lambda_1^\theta(t) := \exp\left(- \int_0^t \theta(s) dW(s) - \frac{1}{2} \int_0^t \theta^2(s) ds \right) . \tag{5}$$

Applying Itô's differentiation rule to Eq. 5 gives

$$d\Lambda_1^\theta(t) = -\Lambda_1^\theta(t)\theta(t)dW(t) \ ,$$
$$\Lambda^\theta(0) = 1 \ . \tag{6}$$

So, Λ_1^θ is a $(G, \mathcal{P})$-local-martingale.

Since $\theta^{(N)}(t) < \infty$, the Novikov condition

$$E\left[\exp\left(\frac{1}{2} \int_0^T \theta^2(t) dt \right) \right] < \infty \ ,$$

is satisfied. So, Λ_1^θ is a $(G, \mathcal{P})$-martingale, and

$$E[\Lambda_1^\theta(T)] = 1 \ .$$

Now, we define a family of density processes for the Markov chain $\mathbf{X}$. To begin with, we define some notation adopted in Dufour and Elliott (1999). For any rate matrix $\boldsymbol{\Pi}(t)$, let $\boldsymbol{\pi}(t) := (\pi_{11}(t), \ldots, \pi_{ii}(t), \ldots, \pi_{NN}(t))^*$ and $\boldsymbol{\Pi}_0(t) := \boldsymbol{\Pi}(t) - \mathbf{diag}(\boldsymbol{\pi}(t))$, where $\mathbf{diag}(\mathbf{y})$ is a diagonal matrix with the diagonal elements given by the vector $\mathbf{y}$.

For each $\theta \in \Theta$, let $\mathbf{A}^\theta(t) := [a_{ij}^\theta(t)]_{i,j=1,2,\ldots,N}$ denote a second family of generators, or rate matrices, of the chain $\mathbf{X}$ such that for each $i, j = 1, 2, \ldots, N$,

$$a_{ij}^\theta(t) = \theta(t) a_{ij}(t) \ , \quad t \in \mathcal{T} \ .$$

We then define a real-world probability measure under which $\mathbf{A}^\theta$ is a family of generators of the chain $\mathbf{X}$. Define, for each $t \in \mathcal{T}$, $\mathbf{D}^\theta(t) := \mathbf{A}^\theta(t)/\mathbf{A}(t)$ as the matrix $\mathbf{D}^\theta(t) = [a_{ij}^\theta(t)/a_{ij}(t)]_{i,j=1,2,\ldots,N}$. Recall that $a_{ij}(t) > 0$, for each $t \in \mathcal{T}$, so $\mathbf{D}^\theta(t)$ is well-defined. Let $\mathbf{1} := (1, 1, \ldots, 1)' \in \Re^N$ and $\mathbf{I}$ denote the $(N \times N)$-identity matrix.

Consider a vector of counting processes $\mathbf{N} := \{\mathbf{N}(t)|t \in \mathcal{T}\}$ defined by:

$$\mathbf{N}(t) := \int_0^t (\mathbf{I} - \mathbf{diag}(\mathbf{X}(u-))) d\mathbf{X}(u) \ , \quad t \in \mathcal{T} \ , \tag{7}$$

where its component $N_i(t)$ counts the number of times the chain $\mathbf{X}$ jumps to state $\mathbf{e}_i$ in the time interval $[0, t]$, for each $i = 1, 2, \ldots, N$.

Then, the compensator of $\mathbf{N}$ is identified with the following result from Dufour and Elliott (1999). We cite the result here without giving the proof.

Lemma 1 *Let* $\mathbf{A}_0(t) := \mathbf{A}(t) - \mathbf{diag}(\mathbf{a}(t))$, *where* $\mathbf{a}(t) := (a_{11}(t), a_{22}(t), \ldots, a_{NN}(t))'$, *for each* $t \in \mathcal{T}$. *Suppose*

$$\tilde{\mathbf{N}}(t) := \mathbf{N}(t) - \int_0^t \mathbf{A}_0(u)\mathbf{X}(u)du , \quad t \in \mathcal{T} . \tag{8}$$

Then, $\tilde{\mathbf{N}} := \{\tilde{\mathbf{N}}(t)|t \in \mathcal{T}\}$ *is an* $(F^{\mathbf{X}}, \mathcal{P})$*-martingale.*

Consider another density process $\Lambda_2^\theta := \{\Lambda_2^\theta(t)|t \in \mathcal{T}\}, \theta \in \Theta$, defined by:

$$\Lambda_2^\theta(t) = 1 + \int_0^t \Lambda_2^\theta(u-)[\mathbf{D}_0(u)\mathbf{X}(u-) - \mathbf{1}]'(d\mathbf{N}(u) - \mathbf{A}_0(u)\mathbf{X}(u)du) . \tag{9}$$

From Lemma 1, Λ_2^θ is an $(F^{\mathbf{X}}, \mathcal{P})$-martingale.

Now, for each $\theta \in \Theta$, we define a G-adapted process $\Lambda^\theta := \{\Lambda^\theta(t)|t \in \mathcal{T}\}$ as the product of the two density processes Λ_1^θ and Λ_2^θ:

$$\Lambda^\theta(t) := \Lambda_1^\theta(t) \cdot \Lambda_2^\theta(t) .$$

Lemma 2 *For each* $\theta \in \Theta$, Λ^θ *is a* $(G, \mathcal{P})$*-martingale.*

Proof Let $\bar{\mathcal{G}}(t) := \mathcal{F}^{\mathbf{X}}(T) \vee \mathcal{F}^W(t)$, for each $t \in \mathcal{T}$. Write $\bar{G} := \{\bar{\mathcal{G}}(t)|t \in \mathcal{T}\}$. Then, the result follows by noting that Λ_1^θ is a $(\bar{G}, \mathcal{P})$-martingale and Λ_2^θ is a $(F^{\mathbf{X}}, \mathcal{P})$-martingale. □

Define, for each $\theta \in \Theta$, a real-world probability measure $\mathcal{Q}^\theta \sim \mathcal{P}$ on $\mathcal{G}(T)$ as:

$$\frac{d\mathcal{Q}^\theta}{d\mathcal{P}} := \Lambda^\theta(T) . \tag{10}$$

So, a family of real-world probability measures is generated as follows:

$$P_a := P_a(\Theta) = \{\mathcal{Q}^\theta | \theta \in \Theta\} .$$

Let

$$\tilde{\mathbf{N}}^\theta(t) := \mathbf{N}(t) - \int_0^t \mathbf{A}_0^\theta(u)\mathbf{X}(u)du , \quad t \in \mathcal{T} , \quad \theta \in \Theta . \tag{11}$$

Suppose $\mathcal{Q}^\theta$, $\theta \in \Theta$, is defined by Eq. 10. Then, Dufour and Elliott (1999) show that $\tilde{\mathbf{N}}^\theta := \{\tilde{\mathbf{N}}^\theta(t)|t \in \mathcal{T}\}$ is an $(F^{\mathbf{X}}, \mathcal{Q}^\theta)$-martingale. So, we have the following result similar to Lemma 2.3 in Dufour and Elliott (1999).

Proposition 1 *Suppose, for each* $\theta \in \Theta$, W *and* $\mathbf{X}$ *are stochastically independent under* $\mathcal{Q}^\theta$. *Then,* $\mathbf{X}$ *is a Markov chain with a family of generators* $\mathbf{A}^\theta(t)$, $t \in \mathcal{T}$, *under* $\mathcal{Q}^\theta$.

So, from Proposition 1 and the semimartingale decomposition of the chain, $\mathbf{X}$ has the following representation under $\mathcal{Q}^\theta$:

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{A}^\theta(u)\mathbf{X}(u)du + \mathbf{M}^\theta(t) , \tag{12}$$

where $\mathbf{M}^\theta := \{\mathbf{M}^\theta(t)|t \in \mathcal{T}\}$ is an $(F^{\mathbf{X}}, \mathcal{Q}^\theta)$-martingale.

Define a process $W^{\theta} := \{W^{\theta}(t) | t \in \mathcal{T}\}$ by:

$$W^{\theta}(t) = W(t) + \int_0^t \theta(s) ds \, . \tag{13}$$

By Girsanov's theorem, W^{θ} is a standard Brownian motion under $\mathcal{Q}^{\theta}$.

Let $V(t) = V^{\pi,\theta}(t)$ denote the total wealth at time t of the portfolio $\pi(t)$ when the market selects the strategy $\theta(t)$. Then, we can rewrite the wealth process under $\mathcal{Q}^{\theta}$ as follows:

$$dV(t) = V(t)\{[r(t) + \pi(t)(\mu(t) - r(t)) - \theta(t)\pi(t)\sigma(t)]dt + \pi(t)\sigma(t)dW^{\theta}(t)\} \, . \tag{14}$$

As in Mataramvura and Øksendal (2007), we define a vector-valued process $\mathbf{Z} := \{\mathbf{Z}(t) | t \in \mathcal{T}\}$ by

$$\begin{aligned} d\mathbf{Z}(t) &= (dZ_0(t), dZ_1(t), d\mathbf{Z}_2(t))' \\ &= (dZ_0(t), dZ_1^{\pi,\theta}(t), d\mathbf{Z}_2^{\theta}(t))' \\ &= (dt, dV^{\pi,\theta}(t), d\mathbf{X}(t))' \, , \\ \mathbf{Z}(0) &= \mathbf{z} = (s, z_1, \mathbf{z}_2)' \, , \end{aligned}$$

where under $\mathcal{P}$,

$$\begin{aligned} dZ_0(t) &= dt \, , \quad Z_0(0) = s \in \mathcal{T} \, , \\ dZ_1(t) &= Z_1(t)\{[r(t) + \pi(t)(\mu(t) - r(t)) - \theta(t)\pi(t)\sigma(t)]dt + \pi(t)\sigma(t)dW^{\theta}(t)\} \, , \\ Z_1(0) &= z_1 > 0 \, , \\ d\mathbf{Z}_2(t) &= \mathbf{A}^{\theta}(t)\mathbf{Z}_2(t)dt + d\mathbf{M}^{\theta}(t) \, , \\ \mathbf{Z}_2(0) &= \mathbf{z}_2 \, . \end{aligned}$$

To simplify the notation, we suppress the notation π and θ in $Z_1(t)$ and $\mathbf{Z}_2(t)$ and write $Z_1(t) := Z_1^{\pi,\theta}(t)$ and $\mathbf{Z}_2(t) := \mathbf{Z}_2^{\theta}(t)$.

Let $E_{\theta}^{\mathbf{z}}[\cdot]$ denote expectation under $\mathcal{Q}^{\theta}$ given that $\mathbf{Z}(0) = \mathbf{z}$. Then, conditional on $\mathbf{Z}(0) = \mathbf{z}$, the robust utility maximization problem is to find the portfolio process $\pi \in \mathcal{A}$ so as to maximize:

$$\inf_{\theta \in \Theta} \left\{ E_{\theta}^{\mathbf{z}}[U(V^{\pi,\theta}(T))] \right\} \, .$$

Here, the minimal utility can be thought of as the expected utility in the "worst-case" scenario and is called a robust utility of the terminal wealth.

So, conditional on $\mathbf{Z}(0) = \mathbf{z}$, the robust utility maximization problem can be formulated as the following two-player, zero-sum, Markovian regime-switching stochastic differential game between the investor and the market.

$$\begin{aligned} \Phi(\mathbf{z}) &= \sup_{\pi \in \mathcal{A}} \left(\inf_{\theta \in \Theta} E_{\theta}^{\mathbf{z}}[U(V^{\pi,\theta}(T))] \right) \\ &= E_{\hat{\theta}}^{\mathbf{z}}[U(V^{\hat{\pi},\hat{\theta}}(T))] \, . \end{aligned} \tag{15}$$

Here, we need to find the value function $\Phi(\mathbf{z})$ of the game and the optimal strategies $\hat{\pi} \in \mathcal{A}$ and $\hat{\theta} \in \Theta$ of the investor and the market, respectively.

4 Solution to the Game

In this section, we first provide a verification theorem for the Markovian regime-switching HJB solution to the stochastic differential game. Then, we derive the optimal strategies of the investor and the market and the value function of the game.

Firstly, we discuss the relationship between the information structure and the control processes π and θ. Let $F^{\theta,W} := \{\mathcal{F}^{\theta,W}(t) | t \in \mathcal{T}\}$ denote the right-continuous, complete filtration generated by the process W^θ. Write, for each $t \in \mathcal{T}$, $\mathcal{G}^\theta(t) := \mathcal{F}^{\theta,W}(t) \vee \mathcal{F}^{\mathbf{X}}(t)$, the enlarged σ-field generated by information about the values of the processes W^θ and $\mathbf{X}$ up to and including time t. Then, it is obvious that $\mathcal{G}^\theta(t) = \mathcal{G}(t)$, for each $t \in \mathcal{T}$. Since there are only two driving processes for the vector-valued process $\mathbf{Z}$, namely, W^θ and $\mathbf{X}$, $\mathbf{Z}$ is adapted to the filtration $G^\theta := \{\mathcal{G}^\theta(t) | t \in \mathcal{T}\}$, and so, it is adapted to the filtration G as well. The vector-valued process $\mathbf{Z}$ is also a Markovian process with respect to G. Under mild technical conditions, the performance of Markovian controls may be as good as the more general adapted controls in the "classical" stochastic optimal control theory (see Øksendal (2003) and Øksendal and Sulem (2004)). It is also noted in Elliott (1982) that if the state processes are Markovian, the optimal controls can be Markovian. So, as in Mataramvura and Øksendal (2007), we impose some restrictions and consider only Markovian controls for the robust utility maximization problem. Let $\mathcal{O} := (0, T) \times (0, \infty)$, which represents our solvency region. Write K_1 and K_2 for the sets such that $\pi(t) \in K_1$ and $\theta(t) \in K_2$, respectively. To restrict ourselves to Markovian controls, we suppose that

$$\pi(t) := \bar{\pi}(\mathbf{Z}(t)) \ , \quad \theta(t) := \bar{\theta}(\mathbf{Z}(t)) \ ,$$

for some functions $\bar{\pi} : \mathcal{O} \times \mathcal{E} \to K_1$ and $\bar{\theta} : \mathcal{O} \times \mathcal{E} \to K_2$.

With slight abuse of notation, we do not distinguish notationally between π and $\bar{\pi}$, and between θ and $\bar{\theta}$. So, we can simply identify the control processes with deterministic functions $\pi(\mathbf{z})$ and $\theta(\mathbf{z})$, for each $\mathbf{z} \in \mathcal{O} \times \mathcal{E}$. These are referred to as feedback controls.

Let $\mathcal{H}$ denote the space of functions $h(\cdot, \cdot, \cdot) : \mathcal{T} \times \Re^+ \times \mathcal{E} \to \Re$ such that for each $\mathbf{x} \in \mathcal{E}$, $h(\cdot, \cdot, \mathbf{x})$ is $\mathcal{C}^{1,2}(\mathcal{T} \times \Re^+)$. Write

$$\mathbf{H}(s, z_1) := (h(s, z_1, \mathbf{e}_1), h(s, z_1, \mathbf{e}_2), \ldots, h(s, z_1, \mathbf{e}_N))' \in \Re^N \ .$$

Define the Markovian regime-switching generator $\mathcal{L}^{\theta,\pi}$ acting on a function $h \in \mathcal{H}$ for the Markov process $\{\mathbf{Z}(t) | t \in \mathcal{T}\}$ as:

$$\begin{aligned}
&\mathcal{L}^{\theta,\pi}[h(s, z_1, \mathbf{x})] \\
&\quad = \frac{\partial h}{\partial s} + z_1[r(s) + (\mu(s) - r(s))\pi(\mathbf{z}) - \theta(\mathbf{z})\pi(\mathbf{z})\sigma(s)]\frac{\partial h}{\partial z_1} + \frac{1}{2} z_1^2 \pi^2(\mathbf{z})\sigma^2(s)\frac{\partial^2 h}{\partial z_1^2} \\
&\quad\quad + \theta(\mathbf{z}) \langle \mathbf{H}(s, z_1), \mathbf{A}(s)\mathbf{x} \rangle \ . \qquad (16)
\end{aligned}$$

Then, we have the following lemma.

Lemma 3 *Let τ be a stopping time, where $\tau < \infty$, $\mathcal{P}$-almost surely. Assume further that $h(\mathbf{Z}(t))$ and $\mathcal{L}^{\theta,\pi}[h(\mathbf{Z}(t))]$ are bounded on $t \in [0, \tau]$.*

Then,

$$E_\theta^{\mathbf{z}}[h(\mathbf{Z}(\tau))] = h(\mathbf{z}) + E_\theta^{\mathbf{z}}\left(\int_0^\tau \mathcal{L}^{\theta,\pi}[h(\mathbf{Z}(t))]dt \right). \tag{17}$$

Proof The result can be established by applying Itô's differentiation rule to $h(\mathbf{Z}(s))$ and conditioning on $\mathbf{Z}(0) = \mathbf{z}$ under $\mathcal{Q}^\theta$. □

The following proposition describes a Markovian regime-switching HJB solution to the stochastic differential game, and so provides a verification theorem for an optimal Markov control.

Proposition 2 *Let $\bar{\mathcal{O}}$ denote the closure of $\mathcal{O}$. Suppose there exists a function ϕ such that for each $\mathbf{x} \in \mathcal{E}$, $\phi(\cdot, \cdot, \mathbf{x}) \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}(\bar{\mathcal{O}})$ and a Markovian control $(\hat{\theta}(t), \hat{\pi}(t)) \in \Theta \times \mathcal{A}$, such that:*

1. $\mathcal{L}^{\theta,\hat{\pi}}[\phi(s, z_1, \mathbf{x})] \geq 0$, *for all* $\theta \in \Theta$ *and* $(s, z_1, \mathbf{x}) \in \mathcal{O} \times \mathcal{E}$,
2. $\mathcal{L}^{\hat{\theta},\pi}[\phi(s, z_1, \mathbf{x})] \leq 0$, *for all* $\pi \in \mathcal{A}$ *and* $(s, z_1, \mathbf{x}) \in \mathcal{O} \times \mathcal{E}$,
3. $\mathcal{L}^{\hat{\theta},\hat{\pi}}[\phi(s, z_1, \mathbf{x})] = 0$, *for all* $(s, z_1, \mathbf{x}) \in \mathcal{O} \times \mathcal{E}$,
4. *for all* $(\theta, \pi) \in \Theta \times \mathcal{A}$,

$$\lim_{t \to T^-} \phi(t, Z_1^{\theta,\pi}(t), \mathbf{X}(t)) = U\left(Z_1^{\theta,\pi}(T)\right),$$

5. *let $\mathcal{K}$ denote the set of stopping times $\tau \leq T$. The family $\{\phi(\mathbf{Z}^{\theta,\pi}(\tau)) | \tau \in \mathcal{K}\}$ is uniformly integrable.*

Write, for each $\mathbf{z} \in \mathcal{O} \times \mathcal{E}$ and $(\theta, \pi) \in \Theta \times \mathcal{A}$,

$$J^{\theta,\pi}(\mathbf{z}) := E_\theta^{\mathbf{z}}\left\{U\left(Z_1^{\theta,\pi}(T)\right)\right\}.$$

Then,

$$\begin{aligned}
\phi(\mathbf{z}) &= \Phi(\mathbf{z}) \\
&= \inf_{\theta \in \Theta}\left(\sup_{\pi \in \mathcal{A}} J^{\theta,\pi}(\mathbf{z})\right) \\
&= \sup_{\pi \in \mathcal{A}}\left(\inf_{\theta \in \Theta} J^{\theta,\pi}(\mathbf{z})\right) \\
&= \sup_{\pi \in \mathcal{A}} J^{\hat{\theta},\pi}(\mathbf{z}) = \inf_{\theta \in \Theta} J^{\theta,\hat{\pi}}(\mathbf{z}) \\
&= J^{\hat{\theta},\hat{\pi}}(\mathbf{z}), \quad \mathbf{z} \in \mathcal{O} \times \mathcal{E},
\end{aligned} \tag{18}$$

and $(\hat{\theta}, \hat{\pi})$ is an optimal Markovian control.

The proof of Proposition 2 is adapted from the proof of Theorem 3.2 in Mataramvura and Øksendal (2007) to the Markovian regime-switching case and uses Lemma 3 here.

In the sequel, we derive explicit formulae for the optimal strategies of the investor and the market and the value function of the game. Firstly, from the form of the utility function in Eq. 4, we guess that the function ϕ has the following form:

$$\phi(\mathbf{z}) = \frac{z_1^{1-\gamma}(g(s,\mathbf{x}))^{1-\gamma}}{1-\gamma}, \quad \forall \mathbf{z} \in \mathcal{O} \times \mathcal{E}, \tag{19}$$

where, for each $(s, \mathbf{x}) \in (0, T) \times \mathcal{E}$, $g(s, \mathbf{x})$ does not vanish and $g(T, \mathbf{x}) = 1$.

Write

$$\mathbf{G}(s,\gamma) = \left(g^{1-\gamma}(s,\mathbf{e}_1), g^{1-\gamma}(s,\mathbf{e}_2), \ldots, g^{1-\gamma}(s,\mathbf{e}_N)\right)'.$$

From Conditions 1–3 of Proposition 2, we get

$$\begin{aligned} &\inf_{\theta \in \Theta} \mathcal{L}^{\theta,\hat{\pi}}[\phi(\mathbf{z})] = \mathcal{L}^{\hat{\theta},\hat{\pi}}[\phi(\mathbf{z})] = 0, \\ &\sup_{\pi \in \mathcal{A}} \mathcal{L}^{\hat{\theta},\pi}[\phi(\mathbf{z})] = \mathcal{L}^{\hat{\theta},\hat{\pi}}[\phi(\mathbf{z})] = 0. \end{aligned} \tag{20}$$

The first-order condition for a maximum point $\hat{\pi}$ by maximizing $\mathcal{L}^{\hat{\theta},\pi}[\phi(\mathbf{z})]$ over $\pi \in \mathcal{A}$ is given by:

$$z_1^{1-\gamma}[g(s,\mathbf{x})]^{1-\gamma}(\mu(s) - r(s) - \hat{\theta}(\mathbf{z})\sigma(s) - \gamma\hat{\pi}(\mathbf{z})\sigma^2(s)) = 0, \quad \forall \mathbf{z} \in \mathcal{O} \times \mathcal{E}. \tag{21}$$

The first-order condition for a minimum point $\hat{\theta}$ by minimizing $\mathcal{L}^{\theta,\hat{\pi}}[\phi(\mathbf{z})]$ over all $\theta \in \Theta$ is:

$$z_1^{1-\gamma}\left(-\hat{\pi}(\mathbf{z})\sigma(s)g^{1-\gamma}(s,\mathbf{x}) + \frac{1}{1-\gamma}\langle \mathbf{G}(s,\gamma), \mathbf{A}(s)\mathbf{x}\rangle\right) = 0. \tag{22}$$

So, we have

$$\hat{\pi}(\mathbf{z}) = \frac{\langle \mathbf{G}(s,\gamma), \mathbf{A}(s)\mathbf{x}\rangle}{(1-\gamma)\sigma(s)g^{1-\gamma}(s,\mathbf{x})} = \hat{\pi}^{\dagger}(s,\mathbf{x}), \tag{23}$$

and

$$\hat{\theta}(\mathbf{z}) = \frac{\mu(s) - r(s) - \gamma\hat{\pi}^{\dagger}(s,\mathbf{x})\sigma^2(s)}{\sigma(s)} = \hat{\theta}^{\dagger}(s,\mathbf{x}). \tag{24}$$

From Condition 3 of Proposition 2,

$$\begin{aligned} &\frac{dg(s,\mathbf{x})}{ds} + \left(r(s) + (\mu(s) - r(s))\hat{\pi}^{\dagger}(s,\mathbf{x}) - \hat{\theta}^{\dagger}(s,\mathbf{x})\hat{\pi}^{\dagger}(s,\mathbf{x})\sigma(s) - \frac{\gamma}{2}(\hat{\pi}^{\dagger}(s,\mathbf{x}))^2\sigma^2(s)\right) \\ &\times g(s,\mathbf{x}) + \frac{\hat{\theta}^{\dagger}(s,\mathbf{x})}{(1-\gamma)g^{1-\gamma}(s,\mathbf{x})}\langle \mathbf{G}(s,\gamma), \mathbf{A}(s)\mathbf{x}\rangle = 0. \end{aligned} \tag{25}$$

From Eqs. 23, 24 and 25, it can be shown that $g(s, \mathbf{x})$ satisfies the following Markovian regime-switching first-order nonlinear ordinary differential equation (O.D.E.):

$$\frac{dg(s,\mathbf{x})}{ds} + \left(r(s) + \frac{\gamma}{2}(\hat{\pi}^{\dagger}(s,\mathbf{x}))^2\sigma^2(s)\right)g(s,\mathbf{x}) + \hat{\theta}^{\dagger}(s,\mathbf{x})\hat{\pi}^{\dagger}(s,\mathbf{x})\sigma(s) = 0, \tag{26}$$

where $\hat{\pi}^{\dagger}(s, \mathbf{x})$ and $\hat{\theta}^{\dagger}(s, \mathbf{x})$ satisfy Eqs. 23 and 24, respectively.

Let $g_i(s) := g(s, \mathbf{e}_i)$, for each $i = 1, 2, \ldots, N$. Write $\mathbf{g}(s) := (g_1(s), g_2(s), \ldots, g_N(s))'$. So, if $\mathbf{x} = \mathbf{e}_i$ $(i = 1, 2, \ldots, N)$, then

$$\mu(s) = \mu_i \,, \quad r(s) = r_i \,.$$

Hence, $\mathbf{g}$ satisfies the following system of first-order nonlinear O.D.E.s:

$$\frac{dg_i(s)}{ds} + \left(r_i + \frac{\gamma}{2}(\hat{\pi}^{\dagger}(s, \mathbf{e}_i))^2 \sigma_i^2\right) g_i(s) + \hat{\theta}^{\dagger}(s, \mathbf{e}_i)\hat{\pi}^{\dagger}(s, \mathbf{e}_i)\sigma_i = 0 \,,$$

where

$$\hat{\pi}^{\dagger}(s, \mathbf{e}_i) = \frac{\langle \mathbf{G}(s, \gamma), \mathbf{A}(s)\mathbf{e}_i \rangle}{(1-\gamma)\sigma_i g_i^{1-\gamma}(s)} \,,$$

and

$$\hat{\theta}^{\dagger}(s, \mathbf{e}_i) = \frac{\mu_i - r_i - \gamma \hat{\pi}^{\dagger}(s, \mathbf{e}_i)\sigma_i^2}{\sigma_i} \,, \quad i = 1, 2, \ldots, N \,.$$

Acknowledgements We would like to thank the referee for many valuable and helpful comments and suggestions. Robert J. Elliott would like to thank SSHRC for its continued support.

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