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Some Reward Paths in Semi-Markov Models with Stochastic Selection of the Transition Probabilities

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Abstract In the present paper, the reward paths in non homogeneous semi-Markov systems in discrete time are examined with stochastic selection of the transition probabilities. The mean entrance probabilities and the mean rewards in the course of time are evaluated. Then the rate of the total reward for the homogeneous case is examined and the mean total reward is evaluated by means of p.g.f's.

Keywords Stochastic selection · Semi-Markov process · Rewards

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1 Introduction

The definition of the non homogeneous semi-Markov process was provided in Iosifescu - Manu (1972) for the continuous time case, in Janssen and De Dominics (1984) for the discrete case, while the first definition of non homogeneous reward process was given in De Dominicis and Manca (1985). In Balcer and Sahin (1986) two extensions of a semi-Markov reward model of pension accumulation are examined and expressions for the mean expected benefits are derived. A multivariate reward process defined on a semi-Markov process is studied in Masuda and Sumita (1991) and transform results for the distributions of the multivariate reward processes are derived. In Masuda (1993) partially observable semi-Markov reward processes are examined and the conditional distribution of the vector of total rewards is studied.

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Later, in Janssen et al. (2004) a full treatment of non homogeneous semi-Markov reward process was presented. In Jianyong and Xiaobo (2004) average reward semi-Markov decision processes with multichain structure are examined. A general definition of rewards can be found in Limnios and Oprisan (2001) and the study of the asymptotic behaviour of semi-Markov reward process in Reza Soltani and Khorshidian (1998). The non homogeneous semi-Markov system in discrete time was examined in Vassiliou and Papadopoulou (1992), and the asymptotic behavior of the same model was studied in Papadopoulou and Vassiliou (1994). Important theoretical results and applications for semi-Markov models can be found in Cinlar (1969, 1975a,b); Howard (1971); Janssen (1986); Janssen and Limnios (1999); Keilson (1969, 1971); McClean (1980, 1986); Mclean and Neuts (1967); Pyke and Schaufele (1964) and in Teugels (1976). Continuing this effort in the present, we study the behaviour of the rewards paid during an interval of time along the reward paths. We consider rewards to be discrete random variables depending on the state, the corresponding holding time and the transition probabilities. In the present paper a new consideration is introduced by allowing the transition probability matrices to be selected from a pool of stochastic matrices. Under this aspect, this new model constitutes a generalization of the homogeneous semi Markov systems which are usually defined by means of a single transition matrix for every time point. In order to examine the characteristics of the reward paths for the new model established, we derive a formula expressing the distribution of the rewards per time unit for every state. The moments of the rewards for every time unit and every state can be easily evaluated by means of this formula which helps us to study the relative reward structure in the course of time.

In Section 2 we outline the semi-Markov reward model with stochastic selection of the transition matrices. In Section 3 the entrance probabilities, the moments of the rewards in one time unit, and the mean reward until time t, $t \epsilon N$, for the new model are evaluated. In Section 4 an asymptotic result for the rate of the total reward earned by time t for the homogeneous case of a semi-Markov process is given. Finally, in Section 5 the evaluation of the mean total reward is provided by means of p.g.f's, for the non homogeneous semi-Markov system with stochastic selection of the transition matrices.

2 The Semi-Markov Reward Model with Stochastic Selection of the Transition Matrix

We consider a population which is stratified into a set of states according to various characteristics and we denote by $\mathcal{S} = \{1, 2, \ldots, N\}$ the set of states assumed to be exclusive and exhaustive, so that each member of the system may be in one and only one state at any given time. Time t is considered to be a discrete parameter and the state of the system at any given time could be described by the vector $\mathbf{N}(t) = [N_1(t), N_2(t), \ldots, N_N(t)]'$, where $N_i(t)$ is the expected number of members of the system in the ith state at time t. The expected number of members of the system at time t is denoted by $T(t)$ and $N_{N+1}(t)$ is the expected number of leavers at time t. We assume that $T(t) = T$, i.e., the total number of leavers equals the total number of recruits for every t and that the individual transitions between the states occur according to a non homogeneous semi-Markov chain *(embedded non homogeneous* *semi-Markov chain)*. In this respect we denote by $\mathbf{F}(t)_{t=0}^{\infty}$ the sequence of matrices, whose (i, j)th element is the probability of a member of the system to make its next transition to state j, given that it entered state i at time t. Let also $\mathbf{p}_{N+1}(t)$ be the Nx1 vector whose ith element is the probability of leaving the system from i, given that the entrance in state i occured at time t and $\mathbf{p}_o(t)$ the Nx1 vector whose jth element is the probability of entering the system in state j as a replacement of a member who entered his last state at time t. A member entering the system holds a particular membership which moves within the states with the members (see also Bartholomew 1982; Vassiliou and Papadopoulou 1992; Vassiliou et al. 1990). Since the size of the system is constant, when a member decides to leave the system, the empty membership is taken by a new recruit who behaves like the former one. Denote by $\mathbf{P}(t)$ the matrix described by the relation

$$\mathbf{P}(t) = \mathbf{F}(t) + \mathbf{p}_{N+1}(t)\mathbf{p}'_o(t)$$

Obviously $\mathbf{P}(t)$ is a stochastic matrix whose (i, j)th element is the probability that a membership of the system which entered state i at time t makes its next transition to state j. For the present, we consider that the transition probability matrix $\mathbf{P}(t)$ is selected from a pool of matrices $\mathbf{L} = [\mathbf{P}_1(t), \mathbf{P}_2(t), \ldots, \mathbf{P}_v(t)]$ with corresponding probabilities $c_1(t), c_2(t), \ldots, c_v(t)$. Thus, whenever a membership enters state i at time t, it selects state j for its next transition according to the probabilities $p_{ij}(t)$. However before the entering state j, the membership 'holds' for a time in state i. Holding times for the memberships are described by the holding time mass functions $h_{ij}(m)$ which express the probabilities that a membership which entered state i at its last transition holds m time units in i before its next transition, given that state j has been selected.

Let also $y_{ij}(t)$ be the reward that a membership earns at time t after entering state i for occupying state i during the interval $[t, t+1)$ when its successor state is j, and $b_{ij}(m)$ be the bonus reward that the membership earns for making a transition from state i to j, after holding m time units in state i. Thus, if a membership enters state i at time s and decides to make a transition to j after m time units in i, then the total reward that it earns equals

$$\sum_{t=s}^{s+m-1} y_{ij}(t) + b_{ij}(m).$$

Now, denote

$$\begin{aligned} A_{ij;k}(t) = \; &\{\text{the reward that a membership earns during the time interval } [t, t+1) \\ &\text{given that the membership entered state } i \text{ at time } 0, \text{ possesses} \\ &\text{state } j \text{ at time } t, \text{ and will undertake its next transition to state } k\}. \end{aligned}$$

Moreover, entering some state j implies stay at j at least for one time unit. Also, denote

$$\begin{aligned} e_{ij}(t, n) = \text{prob}\{&\text{a membership which entered state } i \text{ at time } t \\ &\text{will enter state } j \text{ after } n \text{ time units}\}, \end{aligned}$$

with corresponding probability matrix $\mathbf{E}(t, n) = \{e_{ij}(t, n)\}$. It is apparent that $e_{ij}(t, n)$ depend on the transition probabilities $p_{ij}(t)$ (see also Papadopoulou 1997) or equivalently on the transition probability matrices $\mathbf{P}(t)$ which are selected from the pool $\mathbf{L} =$

$[\mathbf{P}_1(t), \mathbf{P}_2(t), \ldots, \mathbf{P}_v(t)]$ with probabilities $c_1(t), c_2(t), \ldots, c_v(t)$. Thus $e_{ij}(t, n)$ become (define) random variables and we are interested in the expected value of matrix $\mathbf{E}(t, n)$. From Papadopoulou (1997) we have that

$$\mathbf{E}(t,n) = \mathbf{P}(t)\diamond\mathbf{H}(n) + \sum_{j=2}^{n}\{\mathbf{P}(t)\diamond\mathbf{H}(j-1)\}\{\mathbf{P}(t+j-1)\diamond\mathbf{H}(n-j+1)\}$$
$$+\sum_{j=2}^{n}\sum_{k=1}^{j-2}\mathbf{S}_j(k,s,m_k)\{\mathbf{P}(t+j-1)\diamond\mathbf{H}(n-j+1)\},$$

for every $n \geq 1$ and $\mathbf{E}(t, 0) = \mathbf{I}$, where $\mathbf{P}(t)\diamond\mathbf{H}(n)$ is the Hadamard product of the matrices $\mathbf{P}(t)$ and $\mathbf{H}(n)$,

$$\mathbf{S}_j(k,s,m_k)$$
$$= \sum_{m_k=2}^{j-k}\;\sum_{m_{k-1}=1+m_k}^{j-k+1}\cdots\sum_{m_1=1+m_2}^{j-1}\;\prod_{r=-1}^{k-1}\{\mathbf{P}(t+m_{k-r}-1)\diamond\mathbf{H}(m_{k-r-1}-m_{k-r})\},$$

and the (i, r)th element of $\mathbf{S}_j(k, s, m_k)$ is the probability that a membership which entered state i at time s makes a transition to state r after $j-1$ time units and k intermediate transitions during the interval $(s, s+j-1)$.

3 Evaluation of the Expected Entrance Probabilities, the Moments of the Rewards in One Time Unit, and the Mean Reward until Time t

In the present section the expected values of the entrance probabilities, the moments of the rewards in one time unit, and the mean reward until time t are evaluated for the non homogeneous case.

◇ *Evaluation of the expected entrance probabilities*
Taking into account the form of the entrance probabilities given by the entries of $\mathbf{E}(t, n)$, and the stochastic selection of the transition probabilities, we easily get

$$E(\mathbf{E}(t,n)) = E(\mathbf{P}(t))\diamond\mathbf{H}(n)$$
$$+\sum_{j=2}^{n}\{E(\mathbf{P}(t))\diamond\mathbf{H}(j-1)\}\{E(\mathbf{P}(t+j-1))\diamond\mathbf{H}(n-j+1)\}$$
$$+\sum_{j=2}^{n}\sum_{k=1}^{j-2}\tilde{\mathbf{S}}_j(k,s,m_k)\{E(\mathbf{P}(t+j-1))\diamond\mathbf{H}(n-j+1)\},$$

where:

$$E(\mathbf{P}(t)) = \sum_{x=1}^{v} c_x(t)\mathbf{P}_x(t),$$

$$\tilde{\mathbf{S}}_j(k,s,m_k,\beta) = \sum_{m_k=2}^{j-k}\;\sum_{m_{k-1}=1+m_k}^{j-k+1}\cdots\sum_{m_1=1+m_2}^{j-1}\;\prod_{r=-1}^{k-1} E(\mathbf{P}(t+m_{k-r}-1))\diamond\mathbf{H}(m_{k-r-1}-m_{k-r}).$$

◇ *Evaluation of the moments of the rewards in one time unit*
There are three different ways for a membership starting from state i (at time $t = 0$) to occupy state j at time t. The three different ways are exclusive and exhaustive, and are illustrated below:

Time t	Time $t+1$
New entrance in j	New entrance in k
Age in j equal to $m_p \geq 0$	Residual life in j equal to $m_f > 0$
Age in j equal to $m_p > 0$	New entrance in k

Thus $A_{ij;k}(t)$ is a random variable taking the values: $y_{jk}(t) + b_{jk}(1)$, $y_{jk}(t)$, $y_{jk}(t) + b_{jk}(m_p + 1)$. The corresponding probabilities can be easily evaluated, for example:

$$\begin{aligned} P\{A_{ij;k}(t) = \; & y_{jk}(t) + b_{jk}(1)\} = \text{prob}\{\text{a membership which entered state } i \text{ at} \\ & \text{time 0 will enter state } j \text{ at time } t\} \cdot \text{prob}\{\text{a membership which} \\ & \text{entered state } j \text{ at time } t \text{ will take its next transition to state } k\} \\ & \cdot \text{prob}\{\text{a membership which entered state } j \text{ at its last transition} \\ & \text{holds one time unit in } j \text{ before its next transition given that} \\ & \text{state } k \text{ has been selected}\} \\ = \; & e_{ij}(0, t) p_{jk}(t) h_{jk}(1). \end{aligned}$$

Similarly we have that

$$P\{A_{ij;k}(t) = y_{jk}(t)\} = \\ = \sum_{m_p} \sum_{m_f} e_{ij}(0, t - m_p) p_{jk}(t - m_p) h_{jk}(m_p + m_f + 1)$$

where $m_f \neq 0$,

$$P\{A_{ij;k}(t) = y_{jk}(t) + b_{jk}(m_p + 1)\} = \\ = e_{ij}(0, t - m_p) p_{jk}(t - m_p) h_{jk}(m_p + 1).$$

Moments about 0 can be easily found taking into account the previously mentioned p.d.f. of $A_{ij;k}(t)$:

$$\begin{aligned} E((A_{ij;k}(t))^x) \; &= E(e_{ij}(0, t)) E(p_{jk}(t)) h_{jk}(1) [y_{jk}(t) + b_{jk}(1)]^x + \\ & + \sum_{m_p} \sum_{m_f} E(e_{ij}(0, t - m_p)) E(p_{jk}(t - m_p)) h_{jk}(m_p + m_f + 1) (y_{jk}(t))^x + \\ & + \; E(e_{ij}(0, t - m_p)) E(p_{jk}(t - m_p)) h_{jk}(m_p + 1) [y_{jk}(t) + b_{jk}(m_p + 1)]^x, \\ x \; &= 1, 2, \ldots . \end{aligned}$$

Similar relations can also be derived for the central moments.

◇ *Evaluation of the mean reward in* $[0, t + 1)$
Let us define the r.v.

$$A_i(t) = \{\text{the reward that a membership earns during the time interval } [0, t + 1) \\ \text{given that the membership entered state i at time 0}\}.$$

Then a recursive relation can be found for the mean reward in the interval $[0, t+1)$ as follows:

$$E(A_i(t)) = E(A_i(t-1)) + \sum_k \sum_j E(A_{ij;k}(t)) \cdot$$
$$\cdot \text{prob}\{\text{a membership possesses state } j \text{ at time } t \text{ and will move to state } k \text{ at its next transition}\}$$

Thus,

$$E(A_i(t)) = E(A_i(t-1)) +$$
$$+ \sum_k \sum_j E(A_{ij;k}(t)) \sum_{x=0}^{t} E(e_{ij}(0, t-x)) E(p_{jk}(t-x)) \sum_{m>x} h_{jk}(m),$$

with initial condition

$$E(A_i(0)) = \sum_{m=1}^{M} \sum_k E(p_{ik}(0)) h_{ik}(m) y_{ik}(0) + \sum_{k=0} E(p_{ik}(0)) h_{ik}(1) b_{ik}(1).$$

4 The Rate of the Total Reward Earned by a Membership by Time t for the Homogeneous Case

In the non homogeneous case the transition matrices are of the form $\mathbf{P}_i(t)$ while in the homogeneous case transition probabilities do not depend on time t. As a consequence the matrices of the pool are time independent, i.e., $\mathbf{L} = [\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_v]$ with corresponding probabilities $c_1, c_2, \ldots, c_v$. Let us define the r.v.

$A_{i;k} = \{$the reward that a membership earns at its kth transition given that the membership entered state i at time 0$\}$.

We are interested in evaluating the evolution of the ratio $\frac{A_i'(t)}{t}$, as $t \to \infty$, where $A_i'(t) = \sum_{k=1}^{N(t)} A_{i;k}$ and $N(t)$ denotes the number of transitions by time t.

For a renewal reward process it is known that (Ross 1996):

Proposition *If $E(A_{i;k}) < \infty$ and $E(X) < \infty$, then*

$$\frac{A_i'(t)}{t} \to \frac{E(A_{i;k})}{E(X)} \quad \textit{and} \quad \frac{E(A_i'(t))}{t} \to \frac{E(A_{i;k})}{E(X)}, \quad \textit{as} \quad t \to \infty, \tag{1}$$

where $E(X)$ stands for the expected value of the interarrival times (between successive renewals-transitions).

Note that for an ordinary renewal process the interarrival times are considered to be independent and identically distributed (Ross 1996). Thus, the latter proposition is not directly applicable for the semi-Markov process. In order to apply the proposition for our case we focus especially at the renewals-transitions where (say) state i is visited. Then the ordinary semi-Markov process can be considered as consisting of i-cycles, i.e., cycles which begin whenever state i is visited.

Then relations (1) become

$$\frac{A_i'(t)}{t} \rightarrow \frac{E(A_{ii})}{\mu_{ii}} \quad \text{and} \quad \frac{\mathrm{E}(\mathrm{A}_i'(\mathrm{t}))}{\mathrm{t}} \rightarrow \frac{\mathrm{E}(\mathrm{A}_{ii})}{\mu_{ii}}, \quad \text{as} \ \ \mathrm{t} \rightarrow \infty, \tag{2}$$

where A_{ii} denotes the reward earned by a membership during an i-cycle and μ_{ii} is the expected time of an i-cycle (providing that this expected time is finite). Now we proceed as follows:

◇ *Evaluation of μ_{ii} (for the homogeneous semi-Markov case)*
Let us assume that the transition probabilities are time homogeneous and the expected interarrival time is finite. Denote by

$$p_i = \text{prob}\{\text{the semi-Markov process is in state } i, \text{ as } t \rightarrow \infty\}.$$

Then

$$p_i = \frac{\mu_i}{\mu_{ii}}, \qquad i = 1, 2, ..., N, \tag{3}$$

where μ_i represents the mean of the time the process is in state i during an i-cycle, and μ_{ii} represents the mean of the interarrival times between successive visits of state i (see also Ross 1996).
Using probabilistic arguments it can be derived that

$$\begin{aligned} \mu_i &= \sum_{x=1}^{v} c_x \sum_{k \neq i} p_{ik}(x) \sum_{n=1}^{\infty} n h_{ik}(n) \\ &= \sum_{x=1}^{v} c_x E[\text{holding time in i / next state selected is different from } i]. \end{aligned}$$

It is also known that if the embedded Markov chain with stochastic selection of the transition probabilities is irreducible and positive recurrent, then

$$p_i = \frac{\pi_i \mu_i}{\sum\limits_{j=1} \pi_j \mu_j}, \qquad i = 1, 2, ..., N\ , \tag{4}$$

where π_i, $i = 1, 2, ..., N$, stand for the stationary probabilities of the embedded Markov chain with stochastic selection of the transition probabilities (Ross 1996). The quantities π_i, $i = 1, 2, ..., N$, can be evaluated by solving the system

$$\pi^{\top} \cdot E[\mathbf{P}] = \pi^{\top},$$

where $\pi = (\pi_i)$, $i = 1, 2, ..., N$.
Now from Eqs. 3 and 4 we get that

$$\mu_{ii} = \frac{\sum_{j=1} \pi_j \mu_j}{\pi_i}, \qquad i = 1, 2, ..., N. \tag{5}$$

◇ *Evaluation of the expected reward of the i-cycle*
Let us define the r.v.

$$A_{ki}(n) = \{\text{the reward earned by a membership which entered state k, until its first visit to state i after n time units}\}.$$

Then the expected reward earned by a membership during an i-cycle is given by

$$E(A_{ii}) = \sum_{n=1}^{\infty}\left[\sum_{k\neq i} p_{ik}\sum_{m=1}^{n-1} h_{ik}(m)\left[\sum_{x=0}^{m-1} y_{ik}(x) + b_{ik}(m) + E(A_{ki}(n-m)\right]\right] + p_{ii}h_{ii}(n)\left[\sum_{x=0}^{n-1} y_{ii}(x) + b_{ii}(n)\right], \tag{6}$$

where

$$E(A_{ki}(n)) = \sum_{m=1}^{n-1}\sum_{l\neq i} p_{kl}h_{kl}(m)\left[\sum_{x=0}^{m-1} y_{kl}(x) + b_{kl}(m) + E(A_{li}(n-m))\right] + p_{ki}h_{ki}(n)\left[\sum_{x=0}^{n-1} y_{ki}(x) + b_{ki}(n)\right], \tag{7}$$

with $E(A_{ki}(0)) = 0$ (initial condition). The term

$$\sum_{m=1}^{n-1}\sum_{k\neq i} p_{ik}h_{ik}(m)\left[\sum_{x=0}^{m-1} y_{ik}(x) + b_{ik}(m)\right],$$

on the right-hand side of Eq. 6 interprets the expected reward until the first transition out of state i in case at least one transition occurs during the time interval $(0, n)$. Also the term

$$\sum_{m=1}^{n-1}\sum_{k\neq i} p_{ik}h_{ik}(m)[E(A_{ki}(n-m)],$$

stands for the expected reward after the first transition out of state i in case at least one transition occurs during the time interval $(0, n)$, and the term

$$p_{ii}h_{ii}(n)\left[\sum_{x=0}^{n-1} y_{ii}(x) + b_{ii}(n)\right], \tag{8}$$

is the contribution to the expected value of the reward earned when the first transition occurs after n time units. Now let us define

$$r_{ki}(n) = \sum_{m=1}^{n-1}\sum_{l\neq i} p_{kl}h_{kl}(m)\left[\sum_{x=0}^{m-1} y_{kl}(x) + b_{kl}(m)\right],$$

which represents the expected reward until the first transition out of state k, and

$$d_{ki}(n) = p_{ki}h_{ki}(n)\left[\sum_{x=0}^{n-1} y_{ki}(x) + b_{ki}(n)\right].$$

Then Eq. 7 becomes

$$E(A_{ki}(n)) = r_{ki}(n) + \sum_{m=1}^{n-1}\sum_{l\neq i} p_{kl}h_{kl}(m)E(A_{li}(n-m)) + d_{ki}(n). \tag{9}$$

In matrix notation (9) can be written as

$$E(\mathbf{A}(n)) = \mathbf{R}(n) + \sum_{m=1}^{n-1} \mathbf{P} \diamond \mathbf{H}(m)[E(\mathbf{A}(n-m)) \diamond (\mathbf{U} - \mathbf{I})] + \mathbf{D}(n), \tag{10}$$

with $E(\mathbf{A}(0)) = \mathbf{0}$ (initial condition), where $\mathbf{A}(n) = \{A_{ij}(n)\}$, $\mathbf{R}(n) = \{r_{ij}(n)\}$, $\mathbf{1}' = \{1, 1, ..., 1\}$, $\mathbf{D}(n) = \{d_{ij}(n)\}$ and $\mathbf{U} = \{u_{ij}\}$ where $u_{ij} = 1$ for every i, j.
In the same way, if we denote $\mathbf{A} = \{A_{ii}\}$, then relation (6) becomes in matrix notation

$$E(\mathbf{A}) = \sum_{n=1}^{\infty} \Bigg[[[\mathbf{D}(n) + \mathbf{R}(n)] \diamond \mathbf{I}]\mathbf{1} + \sum_{m=1}^{n-1} [[(\mathbf{P} \diamond \mathbf{H}(m))[E(\mathbf{A}(n-m)) \diamond [\mathbf{U} - \mathbf{I}]]] \diamond \mathbf{I}]\mathbf{1} \Bigg]. \tag{11}$$

Using similar reasoning we can derive the analogous relation to Eq. 10 for the case of stochastic selection of the transition probabilities, as follows:

$$E(\mathbf{A}) = \sum_{n=1}^{\infty} \Bigg[[[E(\mathbf{D}(n) + \mathbf{R}(n))] \diamond \mathbf{I}]\mathbf{1} + \sum_{m=1}^{n-1} [[(E(\mathbf{P}) \diamond \mathbf{H}(m))[E(\mathbf{A}(n-m)) \diamond [\mathbf{U} - \mathbf{I}]]] \diamond \mathbf{I}]\mathbf{1} \Bigg] \tag{12}$$

5 Evaluation of the Mean total Reward by Means of P.G.F's

Now, let us number by $1, 2, ..., N_i(0)$ the memberships of a non homogeneous semi-Markov system having started their motion from state i, and denote by $A_i^{(r)}(t)$ the reward of the rth membership paid in the time interval $[t, t+1)$. Let us assume that the rth membership having started its motion from state i, possesses at time t state j, having hold for m_p time units in j and will attain the next state k after m_f time units. We assume without loss of generality that the holding times are bounded by M, $M \epsilon N$ (M is considered to be big enough), where M is the maximum of all bounds of the age and life residuals (m_p, m_f, respectively) for all i, j, k. Then we correspond to the rth membership an $s-$dimensional vector $\mathbf{v}_r(t; i, j, m_p, m_f)$, where $s = N \cdot N \cdot M \cdot M$, which expresses its status concerning the reward attained at time t as a function of the parameters i, j, m_p and m_f. In this matter, the element of the above mentioned vector in the position $(i-1)NM^2 + (j-1)M^2 + m_p M + m_f$ equals the reward $A_{ij;k}^{(r)}(t; m_p, m_f)$, where

$$A_{ij;k}^{(r)}(t; m_p, m_f) = y_{jk}(t) + \delta_{(m_f - 1)} b_{jk}(m_p + m_f),$$

while $\delta_{(x)}$=1 when x=0 and $\delta_{(x)}$=0 elsewhere. All other entries of the vector $\mathbf{v}_r(t; i, j, m_p, m_f)$ are zero. The total reward paid in the interval $[t, t+1)$ for the memberships having started their motion from state i, is the r.v.

$$\sum_{r=1}^{N_i(0)} A_i^{(r)}(t).$$

Symbolize by $f_i^{(r)}(t)$ the probability generating function (p.g.f.) of $A_i^{(r)}(t)$ and by $F_i(t)$ the p.g.f. of $\sum_{r=1}^{N_i(0)} A_i^{(r)}(t)$. Since the r.v's $A_i^{(r)}(t)$ are independent with common p.g.f. $f_i^{(r)}(t) = f_i(t)$, $r = 1, 2, .., N_i(0)$, then

$$F_i(t) = \prod_{r=1}^{N_i(0)} f_i^{(r)}(t) = (f_i(t))^{N_i(0)}.$$

Now let us define

$N_{i,j;k}(t; m_p, m_f)$ = {the number of the memberships having started their motion from state i and which possess at time t the "cell" respecting to the values of j, k, m_p, m_f}.

Then the probability function of $N_{i,j;k}(t; m_p, m_f)$ is multinomial with probabilities

$$e_{ij}(0, t-m_p)p_{jk}(t-m_p)h_{jk}(m_p+m_f) \tag{13}$$

and thus the p.g.f. of $N_{i,j;k}(t; m_p, m_f)$ is

$$f_i(t, \mathbf{z}) = (\mathbf{e}_i{}^\top \Phi(0, t)\mathbf{z})^{N_i(0)},$$

where

- $\mathbf{e}_i$ is an Nx1 vector with 1 in the ith position and 0 elsewhere
- $\mathbf{z} = (z_i)$ is an sx1 vector
- $\Phi(0, t)$ is the Nxs matrix of the probabilities given in Eq. 13, i.e., the element given in Eq. 13 possesses the entry in the ith row and in the $(i-1)NM^2+$ $(j-1)M^2+m_pM+m_f$ column of $\Phi(0, t)$.

The mean number of the memberships having started their motion from state i at $t = 0$, and possessing some of the s states at time t is

$$\bar{\mathbf{m}}_i = \left(\frac{\partial}{\partial z_r} f_i(t; \mathbf{z})|_{\mathbf{z}=\mathbf{1}}\right)_{r=1,2,\ldots,s},$$

thus the rth entry of $\bar{\mathbf{m}}_i$ is given by

$$(\bar{\mathbf{m}}_i)_r = N_i(0)\mathbf{e}_i{}^\top \Phi(0, t)\mathbf{e}_r \ \ , \ r = 1, 2, ..., s. \tag{14}$$

The matrix $\Phi(0, t)$ can be expressed by means of the matrices of the entrance probabilities, the transition probabilities and the holding times mass functions, as follows:

Let us denote

- $\mathbf{e}_{ij}{}^\top(0, \preceq t) = \mathbf{1}^\top \bigotimes \{e_{ij}(0, t-0), ..., e_{ij}(0, t-(M-1))\}$, $j = 1, 2, ..., N$ where $\bigotimes$ stands for the Kronecker product and $\mathbf{1}^\top$ is the $1\times(N \cdot M)$ vector of ones

– $\mathbf{p}_{jk}{}^{\top}(t) = \{p_{jk}(t-0), ..., p_{jk}(t-(M-1))\}$, $j, k = 1, 2, ..., N$
– $\mathbf{h}_{jk}{}^{\top} = \{\mathbf{h}_{jk}{}^{\top}(0), \mathbf{h}_{jk}{}^{\top}(1), ..., \mathbf{h}_{jk}{}^{\top}(M-1)\}$, $j, k = 1, 2, ..., N$
with $\mathbf{h}_{jk}{}^{\top}(r) = \{h_{jk}(r+1), h_{jk}(r+2), ..., h_{jk}(r+M)\}$, $r = 1, 2, ..., M$, and $h_{jk}(u) = 0$, for $u > M$.

Then, the ith row $\Phi_i{}^{\top}(0, t)$ of the matrix $\Phi(0, t)$ can be written as a product of three matrices, i.e.,

$$\begin{aligned}\Phi_i^{\top}(0, t) = \{&\mathbf{e}_{i1}{}^{\top}(0, \preceq t), \mathbf{e}_{i2}{}^{\top}(0, \preceq t), ..., \mathbf{e}_{iN}{}^{\top}(0, \preceq t)\} \\ &\times diag\{\mathbf{p}_{11}{}^{\top}(t), ..., \mathbf{p}_{1N}{}^{\top}(t), \mathbf{p}_{21}{}^{\top}(t), ..., \mathbf{p}_{NN}{}^{\top}(t)\} \\ &\times diag\{\mathbf{h}_{11}{}^{\top}(t), ..., \mathbf{h}_{1N}{}^{\top}(t), \mathbf{h}_{21}{}^{\top}(t), ..., \mathbf{h}_{NN}{}^{\top}(t)\}.\end{aligned}$$

Let

$$\begin{aligned}\mathbf{e}_i{}^{\top}(0, t) &= \{\mathbf{e}_{i1}{}^{\top}(0, \preceq t), ..., \mathbf{e}_{iN}{}^{\top}(0, \preceq t)\}, \\ \mathbf{p}_i{}^{\top}(t) &= \{\mathbf{p}_{i1}{}^{\top}(t), ..., \mathbf{p}_{iN}{}^{\top}(t)\}, \\ \mathbf{h}_i{}^{\top} &= \{\mathbf{h}_{i1}{}^{\top}, ..., \mathbf{h}_{iN}{}^{\top}\}\end{aligned}$$

and denote

$$\begin{aligned}\tilde{\mathbf{E}}^{\top}(0, t) &= \{\mathbf{e}_1(0, t), ..., \mathbf{e}_N(0, t)\}, \\ \tilde{\mathbf{P}}^{\top}(t) &= \{\mathbf{p}_1(t), ..., \mathbf{p}_N(t)\}, \\ \tilde{\mathbf{H}}^{\top} &= \{\mathbf{h}_1, ..., \mathbf{h}_N\}.\end{aligned}$$

Then $\Phi(0, t)$ can be written as

$$\Phi(0, t) = \tilde{\mathbf{E}}(0, t)\tilde{\mathbf{P}}(t)\tilde{\mathbf{H}},$$

and substituting for $\Phi(0, t)$ in Eq. 14 we get

$$(\bar{\mathbf{m}}_i)_r = N_i(0)\mathbf{e}_i{}^{\top}\tilde{\mathbf{E}}(0, t)\tilde{\mathbf{P}}(t)\tilde{\mathbf{H}}\mathbf{e}_r \ \ , \ r = 1, 2, ..., s.$$

Thus in the *stochastic selection case* of the transition probabilities the mean number $(\bar{\mathbf{m}}_i)_r$ equals to

$$(\bar{\mathbf{m}}_i)_r = N_i(0)\mathbf{e}_i{}^{\top} E(\tilde{\mathbf{E}}(0, t)\tilde{\mathbf{P}}(t))\tilde{\mathbf{H}}\mathbf{e}_r \ \ , \ r = 1, 2, ..., s. \tag{15}$$

Since the transition matrices $\mathbf{P}(t)$ are selected independently over disjoint time intervals, then Eq. 15 leads to

$$(\bar{\mathbf{m}}_i)_r = N_i(0)\mathbf{e}_i{}^{\top} E(\tilde{\mathbf{E}}(0, t)) E(\tilde{\mathbf{P}}(t))\tilde{\mathbf{H}}\mathbf{e}_r \ \ , \ r = 1, 2, ..., s.$$

Now let $\mathbf{r}(t)$ denote the $s \times 1$ reward vector the components of which correspond to the rewards of the (j, k, m_p, m_f)-cells at time t. Then, the expected reward paid in $[t, t+1)$ for the memberships having started their motion from state i at time 0, is given by

$$\bar{\mathbf{m}}_i{}^{\top}\mathbf{r}(t) = N_i(0)\mathbf{e}_i{}^{\top} E(\tilde{\mathbf{E}}(0, t)) E(\tilde{\mathbf{P}}(t))\tilde{\mathbf{H}}\mathbf{r}(t) \ \ , \ i = 1, 2, ..., N. \tag{16}$$

Then the mean total reward paid in $[t, t+1)$ for all the memberships of the system can be evaluated by summing over all i in Eq. 16.

The mean reward paid in some time interval for the memberships having started their motion from state *i* at time 0, can be evaluated by adding the mean rewards for the respective time units by using Eq. 15. In the same way the mean total reward paid for all the memberships in some time interval can be evaluated by using Eq. 16.

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