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Improving the Performance of the Chi-square Control Chart via Runs Rules

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Abstract The most popular multivariate process monitoring and control procedure used in the industry is the chi-square control chart. As with most Shewhart-type control charts, the major disadvantage of the chi-square control chart, is that it only uses the information contained in the most recently inspected sample; as a consequence, it is not very efficient in detecting gradual or small shifts in the process mean vector. During the last decades, the performance improvement of the chi-square control chart has attracted continuous research interest. In this paper we introduce a simple modification of the chi-square control chart which makes use of the notion of runs to improve the sensitivity of the chart in the case of small and moderate process mean vector shifts.

Keywords Multivariate statistical quality control · Chi-square control chart · Average run length · Runs rules

AMS 2000 Subject Classification 62N10

1 Introduction

Process-monitoring pertaining to the simultaneous control of two or more dependent variables (quality characteristics) is usually referred in the literature as *multivariate*

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quality control. The area of multivariate quality control was initiated by the pioneering work of Hotelling (1947) who applied several novel procedures to bombsight data during World War II. Since then many papers dealing with multivariate quality control problems appeared in the literature, see e.g., Alt and Smith (1988), Crosier (1988), Pignatiello and Runger (1990), Lowry and Montgomery (1995).

The most popular multivariate quality control schemes are the multivariate Shewhart, $CUSUM$ and $EWMA$ control charts. For specific applications of these charts, as well as applications of other multivariate methods in quality improvement, the interested reader may consult Crosier (1988), Pignatiello and Runger (1990), Hawkins (1991), Lowry and Montgomery (1995), Ryan (2000).

In the literature of statistical process control and monitoring, two distinct phases of control charting practice have been discussed. In Phase I, charts are used for retrospectively testing whether the process was in control when the first subgroups were being drawn. In general, in this phase the charts are used as aids to the practitioner, in bringing a process into a state of statistical in-control. Once this is accomplished, the control chart is used to define what is meant by statistical in-control. This is referred to as the retrospective use of control charts. During this phase the practitioner is studying the process very intensively. In Phase II, control charts are used for testing whether the process remains in control when future subgroups are drawn. In this phase, the charts are used as aids to the practitioner, in monitoring the process for any change from an in-control state (for more details we refer to Montgomery 2001).

As already mentioned, the interest in the present article is focused in the simultaneous monitoring of m distinct quality characteristics (variables) $x_1, x_2, ..., x_m$ (in the case of Phase II). Thus, we shall be assuming that the in-control joint probability distribution of the vector $\boldsymbol{x} = (x_1, x_2, ..., x_m)^t$ follows the m-variate Normal distribution with mean vector $\boldsymbol{\mu}_0$ and variance-covariance matrix $\boldsymbol{\Sigma}_0$, that is $\boldsymbol{x} \sim N_m(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$. Rational subgroups of size $n > 1$ are inspected and the mean sample vector $\bar{\boldsymbol{x}}_i$ of the ith subgroup is calculated. In a *chi-square control chart* (the term CSCC will be used hereafter) for monitoring the process mean, the test statistic

$$D_i^2 = n(\bar{\boldsymbol{x}}_i - \boldsymbol{\mu}_0)^t \boldsymbol{\Sigma}_0^{-1} (\bar{\boldsymbol{x}}_i - \boldsymbol{\mu}_0), \quad i \geq 1$$

is plotted in the chart; this statistic represents the weighted distance (Mahalanobis distance) between $\bar{\boldsymbol{x}}_i$ and $\boldsymbol{\mu}_0$ in the Euclidean space R^m and follows a (central) chi-square distribution with m degrees of freedom ($D_i^2 \sim \chi_m^2$). The upper control limit of the CSCC is given by UCL $= \chi_{m,a}^2$ where $\chi_{m,a}^2$ denotes the upper a percentage point of the chi-square distribution, i.e., $\Pr[D_i^2 > \chi_{m,a}^2] = a$. If $D_i^2 > \chi_{m,a}^2$ there is evidence that the process is out-of-control due to assignable causes, otherwise the process is assumed to be in-control and no action is deemed necessary.

It is worth noting that, in a CSCC there is no need for a lower control limit since the discrimination between the in-control and out-of-control states is determined by the magnitude of $D_i^2 \geq 0$; extreme values of the statistic D_i^2 indicate that the point $\bar{\boldsymbol{x}}_i$ is far away from $\boldsymbol{\mu}_0$ and therefore the process can be declared out-of-control, while small values of D_i^2 indicate that the point $\bar{\boldsymbol{x}}_i$ is close to $\boldsymbol{\mu}_0$ and therefore it is reasonable to assume that the process is in-control. Apparently, the performance of the CSCC depends solely on the distance of the out-of-control mean from the in-control mean and not on the particular direction where the former is placed as compared to the latter (directional invariance).

It goes without saying that, for single (individual) observations ($n = 1$) the quantity $\bar{\boldsymbol{x}}_i$ should be replaced by $\boldsymbol{x}_i$ in the evaluation of the statistic D_i^2. Another point of interest is that, in real life applications, the parameters $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are unknown, so they have to be estimated from the analysis of preliminary samples. Then the appropriate control chart leads to the celebrated *Hotelling T^2 control chart* in which the upper control limit depends on percentiles of an appropriate F distribution (for more details we refer to Alt and Smith (1988) and Lowry and Montgomery (1995)).

The CSCC is a Shewhart-type control chart since it takes into account only information pertaining to the most recently processed sample. This fact inherits to the CSCC, the classical disadvantage of all Shewhart-type control charts, that is the insensitivity to detect gradual or small shifts in the process mean vector. This handicap was recognized at the very beginning by the Western Electric Company who suggested in 1956 the adoption of runs and scans rules in order to make the detection of small shifts from the target mean more efficient. For recent articles where runs and scans rules are used in quality control, see Shmueli and Cohen (2000), Khoo and Quah (2003), Shmueli and Cohen (2003), Shmueli (2003), and Aparisi et al. (2004). The numerical evaluation of the performance of such rules can be easily achieved by the aid of Markov chains, as indicated in Champ and Woodall (1987) and Fu et al. (2002, 2003).

In this paper we consider a CSCC which signals an out-of-control process if k consecutive values of the test statistic D_i^2 exceed an appropriate upper control limit U_k (k is a positive integer). The procedure is termed as $k|k$ CSCC, and in the special case $k = 1$, it coincides with the standard one. For $k \geq 2$ the new control chart has better average run length performance than the corresponding standard one ($1|1$ CSCC). In addition, we introduce a combined $r|r - k|k$ CSCC which is a control chart with two control limits U_r and U_k that signals an out-of-control process if either k consecutive values of the test statistic D_i^2 exceed U_k or r consecutive values of the test statistic D_i^2 exceed U_r. Our numerical experimentation revealed that the combined $1|1 - k|k$ CSCC with $k \geq 2$ is more effective than both $1|1$ and $k|k$ CSCCs.

The present paper is organized as follows: in "A Run Related Chi-square Control Chart" we introduce the $k|k$ CSCC while in "Comparing Two Runs Related Chi-square Control Charts" we carry out a theoretical study of its characteristics focusing mainly on its performance as compared to the standard CSCC. In "A Chi-square Control Chart with Multiple Limits" we introduce the combined $r|r - k|k$ CSCC and study the special case $r = 1$. In "Numerical Comparisons" we conduct a systematic numerical experimentation in order to investigate the average run length of the two new CSCCs. Finally, "Conclusions" briefly states some concluding remarks on the material presented in this article.

2 A Run Related Chi-square Control Chart

Consider a typical Shewhart-type control chart with one-sided upper control limit U_1. Assume also that the values of a specific statistic W_i are plotted in the chart and the process is declared out-of-control if $W_i > U_1$, otherwise the process is assumed to be in-control. It is well known that the in-control average run length of the plan (abbr. $\mathrm{ARL_{in}}$) is given by

$$\mathrm{ARL_{in}} = \frac{1}{p_1}$$

where $p_1 = \Pr[W_i > U_1]$, the last probability being calculated under the assumption that the process is in-control. Moreover, the out–of-control average run length of the plan (abbr. $\mathrm{ARL_{out}}$) is expressed as

$$\mathrm{ARL_{out}} = \frac{1}{\Pr[W_i > U_1]}$$

with the probability in the denominator being evaluated under the assumption that the process parameter has shifted to a value different than the one specified as in-control value (target value).

As an illustration, let us consider a CSCC with in-control mean vector $\boldsymbol{\mu}_0$. Then, the test statistic is given by

$$W_i = D_i^2 = n(\overline{x}_i - \boldsymbol{\mu}_0)^t \boldsymbol{\Sigma}_0^{-1}(\overline{x}_i - \boldsymbol{\mu}_0), \quad i \geq 1$$

which follows a χ^2 distribution with m degrees of freedom ($W_i \sim \chi^2_m$). On the other hand, if $E(x) = \boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \boldsymbol{\delta}$ denotes a specific out-of-control mean vector ($\boldsymbol{\delta} \neq 0$), the distribution of the same statistic follows a non-central χ^2 with m degrees of freedom and non-centrality parameter

$$\lambda = \lambda(\boldsymbol{\mu}_1) = n(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^t \boldsymbol{\Sigma}_0^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0) = n\boldsymbol{\delta}^t \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\delta} \tag{2.1}$$

($W_i \sim \chi^2_m(\lambda)$). In order to distinguish between the in-control and out-of-control states, we shall be using in the sequel the symbol $D_i^2(\lambda)$ to denote the test statistic when a sample comes from an out-of-control process and $D_i^2 = D_i^2(0)$ when it comes from an in-control process. The probability density function (pdf) and the cumulative distribution function (cdf) of a non-central chi-square variable with m degrees of freedom and non-centrality parameter λ , will be denoted by $f_m(x;\lambda)$ and $F_m(x;\lambda)$ respectively. The corresponding pdf and cdf for a (central) chi-square variable will be denoted by $f_m(x;0) = f_m(x)$ and $F_m(x;0) = F_m(x)$ respectively. Readers who are interested in studying the aforementioned distributions in detail, may consult Johnson et al. (1994, 1995). Making use of the previous notations, the ARLs of the CSCC take the form

$$\mathrm{ARL_{in}} = \frac{1}{\Pr[D_i^2 > U_1]} = \frac{1}{1 - F_m(U_1)}, \quad \mathrm{ARL_{out}} = \frac{1}{\Pr[D_i^2(\lambda) > U_1]} = \frac{1}{1 - F_m(U_1;\lambda)}.$$

Let us go back to the typical Shewhart-type control chart with one-sided upper control limit U_1. As already explained in the introduction, the main disadvantage of the Shewhart-type control charts is the lack of sensitivity in case of gradual or slight shifts from the in-control parameter value. To a certain extent, this handicap stems from the fact that, to reach a decision, only the most recently processed sample is taken into account. In light of this evidence, a remedial measure would be to couch our decision on information carried over by consecutive samples.

A reasonable approach toward these lines is to consider modified control charts which declare a process out-of-control if $k \geq 1$ consecutive values of the test statistic W_i exceed the upper control limit, say U_k, of the chart. Manifestly, for $k = 1$, the above rule leads to the classical (Shewhart) control chart without supplementary sensitizing rules.

In order to study the $\mathrm{ARL_{in}}$ of the modified plan let us introduce the binary variables

$$Y_i = \begin{cases} 1, \text{ if } W_i > U_k \\ 0, \text{ if } W_i \leq U_k \end{cases}$$

for $i = 1, 2, \ldots$ Assuming that the samples are independent and come from the same distribution, we conclude that $Y_1, Y_2, \ldots$ consist a sequence of independent Bernoulli trials with common success (failure) probability $p = \Pr[W_i > U_k]$ ($q = 1 - p = \Pr[W_i \leq U_k]$). Then, the point at which the process will be declared out-of-control (while in fact it is in-control), will be described by the waiting time T_k for the first occurrence of a success run of length k.

The expected value of T_k (see, e.g., Balakrishnan and Koutras (2002) or Fu and Lou (2003)) is given by

$$E(T_k) = h_k(p) = \frac{1 - p^k}{p^k\,(1 - p)} \tag{2.2}$$

with $h_k(p)$ being a monotonically decreasing function of p. Moreover, it is clear that

$$\lim_{p \to 0} h_k(p) = +\infty, \qquad \lim_{p \to 1} h_k(p) = k.$$

It is now evident that one can always achieve a prespecified $\mathrm{ARL_{in}} = c > k$ by adjusting appropriately the upper control limit U_k of the plan. This is easily accomplished by calculating first the unique root $p_k \in (0, 1)$ of the equation

$$h_k(p_k) = \frac{1 - (p_k)^k}{(p_k)^k\,(1 - p_k)} = c, \qquad (c > k)$$

and then identifying U_k by the aid of the condition

$$\Pr[W_i > U_k] = p_k$$

(the evaluation of the last probability is carried out under the assumption that the process is in-control).

In the sequel, the control chart resulting from the aforementioned procedure will be referred to as "a $k|k$ Shewhart-type control chart" with $\text{ARL}_{\text{in}} = c$. An algorithmic description of this chart in discrete steps is as follows:

Step 1: Choose a positive integer k.
Step 2: Set the desired in-control $\text{ARL}_{\text{in}} = c, c > k$.
Step 3: Calculate the unique root p_k of the equation $c = \frac{1-(p_k)^k}{(p_k)^k(1-p_k)}$ in the interval $(0, 1)$.
Step 4: Calculate the upper control limit U_k by the aid of the equality $p_k = \Pr[W_i > U_k]$.
Step 5: Declare the process out-of-control if k consecutive points are plotted above U_k, that is if at the examination of the ith $(i \geq k)$ sample the inequality $W_j > U_k$ turns out to be valid for all $j = i - k + 1, i - k + 2, ..., i$.

It is noteworthy that the ARL_{out} of the $k|k$ Shewhart-type control chart can be evaluated through $h_k(p)$ as well by replacing p with the probability of the event $\{W_i > U_k\}$ under the assumption that the process parameter has shifted from its in-control value.

Let us next examine the application of the $k|k$ Shewhart-type control chart in a CSCC (the term $k|k$ CSCC will be used hereafter). In this case we have

$$\text{ARL}_{\text{in}} = E(T_k) = h_k(p_k)$$

with $h_k(\cdot)$ given by Eq. 2.2 and

$$p_k = \Pr[D_i^2 > U_k] = 1 - F_m(U_k).$$

Therefore, in order to achieve a prespecified ARL_{in} level, say c, it suffices to calculate the unique root $p_k \in (0, 1)$ of the equation $h_k(p_k) = c$ and then set $U_k = \chi^2_{m,p_k}$.

By way of example let us consider a multivariate normal process with $m = 3$, variance-covariance matrix $\boldsymbol{\Sigma}_0$ and mean vector $\boldsymbol{\mu}_0 = (100, 100, 100)$. In Fig. 1 we have plotted the value of the test statistic $W_i = D_i^2$ for 30 simulated individual obser-

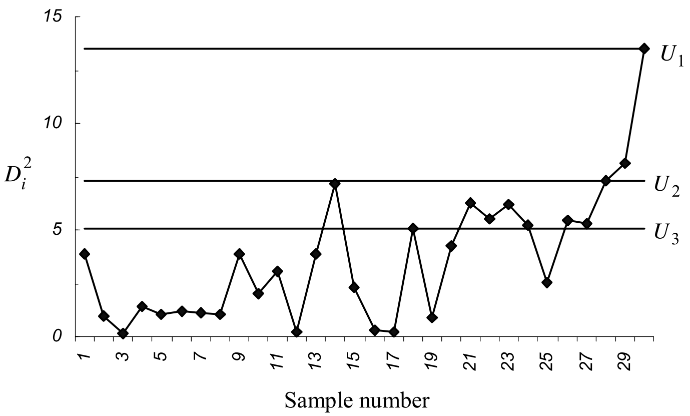

Fig. 1 1|1, 2|2 and 3|3 chi-square control charts

vations from the process, assuming that it stays in-control for the first 20 individual observations and shifts to an out-of-control mean $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \boldsymbol{\delta} = (105, 105, 105)$ at the 21st observation (matrix $\boldsymbol{\Sigma}_0$ remains constant for all 30 observations). Three different control limits U_1, U_2, U_3 are drawn, corresponding to the $1|1$, $2|2$ and $3|3$ CSCCs with common $\text{ARL}_{\text{in}} = 250$.

From Fig. 1, we observe that the standard $1|1$ CSCC gives an out-of-control signal at sample 30, while the $2|2$ and $3|3$ CSCCs give an earlier detection at samples 29 and 23 respectively.

In the next paragraph we give a number of theoretical results regarding to the behavior and the performance of the $k|k$ CSCC.

3 Comparing Two Runs Related Chi-square Control Charts

Before stating the core of the results pertaining to the performance of the $k|k$ CSCC, we shall develop some auxiliary tools that will be proved useful in achieving our goal.

Note first that for the upper control limits U_1, U_2, U_3 of the three CSCCs of the previous example we have $U_1 > U_2 > U_3$ (see Fig. 1). Such an ordering holds true in general, provided that the level of ARL_{in} is kept constant. To verify that, let r, k be two positive integers such that $r < k$ and consider the respective $r|r$ and $k|k$ CSCCs each one having the same $\text{ARL}_{\text{in}} = c$. It can be easily checked that $h_r(x) < h_k(x)$ for all $x \in (0, 1)$, and therefore the unique roots p_r and p_k of the equations

$$c = h_r(p_r), \quad c = h_k(p_k), \quad c > k$$

satisfy the inequality $p_r < p_k$ which in turn leads to $U_r > U_k$ (recall that $p_r = \Pr[D_i^2 > U_r]$ and $p_k = \Pr[D_i^2 > U_k]$). A direct consequence of the previous outcome is that, the upper control limit U_k of a $k|k$ CSCC with $k \geq 2$ is less than the upper control limit U_1 of the standard CSCC with the same ARL_{in}.

Let us next examine how the ARL_{out} of a $k|k$ CSCC can be evaluated. As already mentioned earlier, ARL_{out} is the mean of T_k under the condition that the control parameter $\boldsymbol{\mu}$ has shifted to an out-of-control value $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 + \boldsymbol{\delta}$. Hence,

$$\text{ARL}_{\text{out}}(\lambda) = h_k(p_k(\lambda)) = \frac{1 - (p_k(\lambda))^k}{(p_k(\lambda))^k\,(1 - p_k(\lambda))}$$

where $p_k(\lambda) = \Pr[D_i^2(\lambda) > U_k] = 1 - F_m(U_k; \lambda)$ and $\lambda = \lambda(\boldsymbol{\mu}_1)$ is given by Eq. 2.1.

In the sequel we adopt the notation $\text{ARL}_k(\lambda)$ for the ARL_{out} of a $k|k$ CSCC, while the respective $\text{ARL}_{\text{in}} = \text{ARL}_k(0)$ will be denoted simply by ARL_k, $k \geq 1$. In the light of the foregone discussion we may write

$$\text{ARL}_k(\lambda) = \frac{1 - [1 - F_m(U_k; \lambda)]^k}{F_m(U_k; \lambda)[1 - F_m(U_k; \lambda)]^k} = \frac{1}{1 - H_k(F_m(U_k; \lambda))} \tag{3.1}$$

where

$$H_k(x) = \frac{1 - (1 + x)(1 - x)^k}{1 - (1 - x)^k} = 1 - \frac{x(1 - x)^k}{1 - (1 - x)^k}.$$

Consider now two $r|r$, $k|k$ CSCCs with $r < k$ and common $\mathrm{ARL}_{\mathrm{in}} = c$ (i.e., $\mathrm{ARL}_r = \mathrm{ARL}_k = c$). We observe that Eq. 3.1 yields

$$\mathrm{ARL}_r(\lambda) - \mathrm{ARL}_k(\lambda) = \frac{S_{r,k}(\lambda)}{[1 - H_r(F_m(U_r;\lambda))][1 - H_k(F_m(U_k;\lambda))]} \tag{3.2}$$

where $S_{r,k}(\lambda)$ was used to denote the difference

$$S_{r,k}(\lambda) = H_r(F_m(U_r;\lambda)) - H_k(F_m(U_k;\lambda)). \tag{3.3}$$

In the rest of the section we focus on the problem of identifying conditions on λ guaranteeing that

$$\mathrm{ARL}_r(\lambda) < \mathrm{ARL}_k(\lambda) \quad (\text{or } \mathrm{ARL}_r(\lambda) > \mathrm{ARL}_k(\lambda)).$$

Should such a condition hold true for all $\lambda > 0$, one of the CSCCs will be preferable to the other no matter how far apart is the out-of-control value from the respective in-control value. Our numerical experimentation revealed that the condition $\mathrm{ARL}_r(\lambda) < \mathrm{ARL}_k(\lambda)$ does not hold true for all λ values. However, for large values of the parameter λ the $r|r$ *CSCC* has always better performance (in terms of the $\mathrm{ARL}_{\mathrm{out}}$ values) than the $k|k$ *CSCC* ($r < k$), as the next proposition states.

PROPOSITION 1 *Let r, k be two positive integers such that $r < k$. Then, there exists a real positive number λ_0 such that the out-of-control ARLs of the $r|r$ CSCC and $k|k$ CSCC with the same in-control ARL value c ($ARL_r = ARL_k = c$) satisfy the inequality $ARL_r(\lambda) < ARL_k(\lambda)$ for all $\lambda > \lambda_0$.*

Proof: It can be easily verified that, for any positive integer k, we have

$$\lim_{x\to 0} H_k(x) = 1 - \frac{1}{k}.$$

Therefore, recalling the well-known result (see, e.g., Johnson et al. 1995)

$$\lim_{\lambda\to\infty} F_m(x;\lambda) = 0$$

we may readily conclude that

$$\lim_{\lambda\to\infty} (\mathrm{ARL}_r(\lambda) - \mathrm{ARL}_k(\lambda)) = rk \lim_{\lambda\to\infty} S_{r,k}(\lambda) = r - k < 0$$

which completes the proof of the proposition. ■

An immediate consequence of Proposition 1 is that, for large values of λ (i.e., large shifts from the in-control parameter value) the standard CSCC ($r = 1$) is superior to the $k|k$ CSCC with $k \geq 2$.

There is still another case where the standard CSCC is better than the $k|k$ CSCC with $k \geq 2$. More specifically we shall prove that if the prespecified $\mathrm{ARL}_{\mathrm{in}}$ value c is small enough, then $\mathrm{ARL}_1(\lambda) < \mathrm{ARL}_k(\lambda)$ for all $\lambda > 0$ ($k \geq 2$). Before stating and proving this assertion we shall present a useful lemma which is instrumental in the justification of our approach.

Lemma 1 (*a*) *The quantity* $S_{1,k}(\lambda)$ *is a decreasing function of* $\lambda > 0$ *if and only if the following inequality holds true*

$$H'_k(F_m(U_k;\lambda)) < \frac{f_{m+2}(U_1;\lambda)}{f_{m+2}(U_k;\lambda)}. \tag{3.4}$$

(*b*) *If* $k \geq 2$ *we have* $H'_k(x) < 1$ *for all* $0 < x < 1$.

Proof: (*a*) The well known formula for the cdf of the non-central chi-square distribution

$$\frac{\partial}{\partial\lambda}F_m(x;\lambda)) = -f_{m+2}(x;\lambda),$$

(see, e.g., Johnson et al. 1995) may be used in conjuction with the chain rule of calculus to write

$$\frac{\partial}{\partial\lambda}H_k(F_m(U_k;\lambda)) = -f_{m+2}(U_k;\lambda)\cdot H'_k(x)\mid_{x=F_m(U_k;\lambda))}.$$

Differentiating Eq. 3.3 with respect to λ we deduce

$$\begin{aligned} S'_{r,k}(\lambda) &= \frac{\partial}{\partial\lambda}[H_r(F_m(U_r;\lambda)) - H_k(F_m(U_k;\lambda))] \\ &= f_{m+2}(U_k;\lambda)\cdot H'_k(F_m(U_k;\lambda)) - f_{m+2}(U_r;\lambda)\cdot H'_r(F_m(U_r;\lambda)) \end{aligned}$$

and taking into account that $H'_1(x) = 1$ for all $0 < x < 1$, the following expression for $S'_{1,k}(\lambda)$ is established

$$S'_{1,k}(\lambda) = f_{m+2}(U_k;\lambda)\left(H'_k(F_m(U_k;\lambda)) - \frac{f_{m+2}(U_1;\lambda)}{f_{m+2}(U_k;\lambda)}\right). \tag{3.5}$$

In view of the last expression, condition (3.4) implies that $S'_{1,k}(\lambda) < 0$, which in turn guarantees that $S_{1,k}(\lambda)$ is monotonically decreasing (and vise versa).

(*b*) Let $k \geq 2$. Since

$$H'_k(x) = \frac{(1-x)^{k-1}[(1-x)^{k+1} + x(k+1) - 1]}{(1-(1-x)^k)^2},$$

the condition $H'_k(x) < 1$ for $0 < x < 1$ is equivalent to

$$(1-x)^{k-1}[x(k-1)+1] < 1.$$

The last inequality is valid for $k = 2$ and it may be checked by induction with respect to k that it holds true as well for any positive integer $k > 2$. This completes the proof of part (*b*). ■

We are now ready to prove the superiority of the standard CSCC over $k|k$ CSCC with $k \geq 2$ for small ARL$_{\text{in}}$ values.

PROPOSITION 2 *Let c be a positive number such that $\chi^2_{m,1/c} \leq m$. Then for all $k \geq 2$ the out-of-control ARLs of the* $1|1$ *CSCC and $k|k$ CSCC with the same in-control ARL value c ($ARL_1 = ARL_k = c$) satisfy the inequality $ARL_1(\lambda) < ARL_k(\lambda)$, for all $\lambda > 0$.*

Proof: Let M_λ be the mode of the distribution $\chi^2_{m+2}(\lambda)$. The pdf $f_{m+2}(x;\lambda)$ is increasing for $x < M_\lambda$, while M_λ is an increasing function of λ (see, e.g., Johnson et al. 1995). Moreover, the mode of the central chi-square distribution with $m+2$ degrees of freedom equals $m = M_0$. In the light of the foregoing arguments we conclude that $f_{m+2}(x;\lambda)$ is an increasing function of x for $x < m = M_0 < M_\lambda$. Observe next that, the condition $\chi^2_{m,1/c} \leq m$ leads to $U_1 < m$ and make use of the monotonicity of $f_{m+2}(x;\lambda)$ with respect to x to yield the inequality

$$\frac{f_{m+2}(U_1;\lambda)}{f_{m+2}(U_k;\lambda)} > 1$$

(recall also that $U_k < U_1$). By virtue of Lemma 3.1(b) we obtain

$$H'_k(F_m(U_k;\lambda)) < 1 < \frac{f_{m+2}(U_1;\lambda)}{f_{m+2}(U_k;\lambda)}$$

and therefore (c.f. Lemma 1(a)) $S_{1,k}(\lambda)$ is decreasing in λ. The desired result is now easily derived if we take into account that $S_{1,k}(\lambda)$ is a continuous decreasing function with $S_{1,k}(0) = 0$, $\lim_{\lambda\to\infty} S_{1,k}(\lambda) = \frac{1}{k} - 1 < 0$ (note that $\lim_{\lambda\to\infty} F_m(x;\lambda) = 0$) and that the sign of $S_{1,k}(\lambda)$ coincides to that of $\text{ARL}_1(\lambda) - \text{ARL}_k(\lambda)$. ■

Although the outcome of Proposition 2 seems to attribute to the standard CSCC uniformly better performance, this is not quite so, at least as far as situations of practical importance are concerned. The condition $\chi^2_{m,1/c} \leq m$ is in fact fulfilled only for extremely low values of c (close to 2) which are of no interest for the quality control practitioners (as a matter of fact, it is intuitively clear that if you want a signal within an average of a single point ($c < 2$), $k = 1$ will be the only rule that can do it). It should also be stressed that the condition $\chi^2_{m,1/c} \leq m$ is not necessary and sufficient and therefore one might suspect that the (uniform) superiority of the standard CSCC could emerge even for large values of c; however, as our extensive numerical experimentation revealed, this situation does not appear for c values of practical importance ($c \geq 200$).

In view of Propositions 1, 2 and the above discussion, should we wish to construct appropriate $k|k$ CSCCs (with $k \geq 2$) that will improve the standard CSCC, we should further elucidate the performance of $\text{ARL}_1(\lambda) - \text{ARL}_k(\lambda)$ for small values of λ

and/or large $\mathrm{ARL_{in}}$ values. The next two results provide some insight towards this direction.

Proposition 3 *If the following inequality holds true for a prespecified $k \geq 2$*

$$H_k'(F_m(U_k)) > \left(\frac{U_1}{U_k}\right)^{m/2} \exp[(U_k - U_1)/2] \tag{3.6}$$

then there exists a real positive number λ_0 such that $ARL_1(\lambda) > ARL_k(\lambda)$ for all $\lambda < \lambda_0$.

Proof: Applying Eq. 3.5 for $\lambda = 0$ and replacing $f_{m+2}(x; 0)$ by

$$f_{m+2}(x) = \frac{1}{2^{(m/2)+1}\Gamma((m/2)+1)} x^{m/2} e^{-x/2}$$

we have

$$\begin{aligned} S_{1,k}'(0) &= f_{m+2}(U_k)\left(H_k'(F_m(U_k)) - \frac{f_{m+2}(U_1)}{f_{m+2}(U_k)}\right) \\ &= f_{m+2}(U_k)\left(H_k'(F_m(U_k)) - \left(\frac{U_1}{U_k}\right)^{m/2} \exp[(U_k - U_1)/2]\right) \end{aligned}$$

and making use of condition (3.6) we conclude that $S_{1,k}'(0) > 0$. The continuity of $S_{1,k}(\lambda)$ as a function of λ guarantees that there exists an interval of the form $(0, \lambda_0)$, $\lambda_0 > 0$, such that for $\lambda \in (0, \lambda_0)$ the function $S_{1,k}(\lambda)$ is an increasing function of λ. Since $S_{1,k}(0) = 0$, it follows that $S_{1,k}(\lambda) > 0$ for $\lambda \in (0, \lambda_0)$, that is $\mathrm{ARL}_1(\lambda) > \mathrm{ARL}_k(\lambda)$ for $\lambda \in (0, \lambda_0)$. This completes the proof of the proposition. ■

Corollary 1 *Let c be a positive number such that*

$$1 - \frac{1}{(1+s)^2} > \left(\frac{\chi^2_{m,1/c}}{\chi^2_{m,s}}\right)^{m/2} \exp[(\chi^2_{m,s} - \chi^2_{m,1/c})/2]$$

where $s = (1 + \sqrt{1+4c})/2c$. Then there exists a real positive number λ_0 such that $ARL_1(\lambda) > ARL_2(\lambda)$ for all $\lambda < \lambda_0$.

Proof: Note that, in the special case $k = 2$, s is the solution of the equation $h_2(s) = c$ in the interval $(0, 1)$ and that $H_2'(x) = 1 - 1/(x-2)^2$. The proof of the corollary follows immediately by applying Proposition 3. ■

The obvious conclusion to be drawn from the foregone analysis is that, if the practitioner wishes to work with large $\mathrm{ARL_{in}}$ values (which is usually the case) and/or expects only small shifts of the control parameter from the in-control level, he/she should initiate a $k|k$ CSCC with $k \geq 2$ instead of a standard CSCC. In the opposite case a standard CSCC is a better choice.

4 A Chi-square Control Chart with Multiple Limits

A technique commonly used in applied quality control when two non-uniformly ordered procedures are available, is to consider a combined chart which exploits the control limits of both procedures. Such a chart results in improved performance, as compared to the performances of each individual procedure. This will be the subject of the present section.

Let us consider two $r|r$ and $k|k$ CSCCs ($r < k$) with the same in-control ARL value c ($\mathrm{ARL}_r = \mathrm{ARL}_k = c$) and respective control limits U_r and U_k ($U_r > U_k$). The term $r|r - k|k$ chi-square control chart (abbr. $r|r - k|k$ CSCC) will be used to indicate a control chart which signals an out-of-control process if either k consecutive values of the test statistic $W_i = D_i^2$ are plotted above U_k or r consecutive values are plotted above U_r.

In an $r|r - k|k$ CSCC three regions are defined: one consisting of the points below the control limit U_k (region 0), one containing the points above the control limit U_r (region 2), and a central region extending between the two limits (region 1). For an in-control process, the probability that a single point falls in regions 0, 1, 2 are $1 - p_k$, $p_k - p_r$ and p_r respectively, where

$$p_k = \Pr[D_i^2 > U_k], \quad p_r = \Pr[D_i^2 > U_r],$$

while for an out-of-control process the respective probabilities become $1 - p_k(\lambda)$, $p_k(\lambda) - p_r(\lambda)$ and $p_r(\lambda)$, where

$$p_k(\lambda) = \Pr[D_i^2(\lambda) > U_k], \quad p_r(\lambda) = \Pr[D_i^2(\lambda) > U_r].$$

By way of example consider once more the process illustrated in Fig. 1 and place the limits U_2, U_3 as shown in Fig. 2. Then the use of the $2|2 - 3|3$ CSCC for monitoring the process mean will give an out-of-control signal as soon as 2 consecutive plotted values of the test statistic D_i^2 exceed level U_2 (region 2) or 3

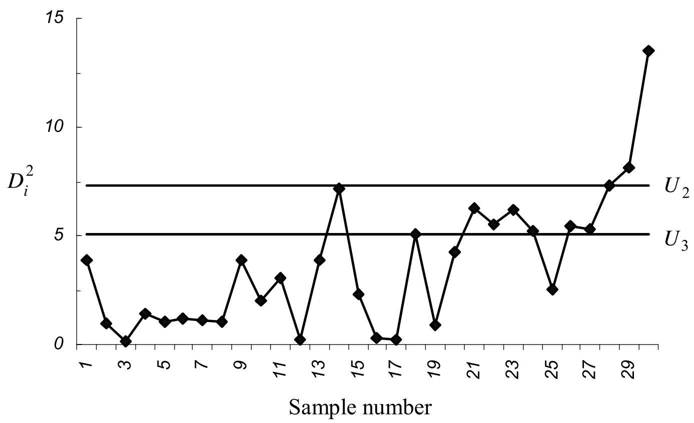

Fig. 2 The $2|2 - 3|3$ chi-square control chart

consecutive plotted values of the test statistic D_i^2 exceed level U_3 (regions 1 and 2). Apparently for the data depicted in Fig. 2 this will take place at sample 23.

Consider a sequence of independent trials $Y_1, Y_2, \ldots$ with three possible outcomes, say 0, 1, 2 and assume that $q = \Pr[Y_i = 0]$, $p = \Pr[Y_i = 1]$ and $p^* = \Pr[Y_i = 2]$ for $i \geq 1$. Denote by $T_{r,k}$ the waiting time for the occurrence of run of 2's of length r, or the occurrence of a strand of k consecutive trials consisting of 1's or 2's, whichever comes sooner. Manifestly, the ARL of the combined chart coincides with the mean value of the random variable $T_{r,k}$. Even though there are some tools in the literature for the study of $T_{r,k}$ (see, e.g., Aki 1992; Antzoulakos 2001; Fu and Chang 2003; Koutras 1997), for typographical convenience we shall not pursue here this general scheme. Instead, we shall restrict ourselves to the special case $r = 1$ ($1|1 - k|k$ CSCC) whose ARL can be calculated by the aid of formula

$$h_{1,k}(p^*, p) = E(T_{1,k}) = \left(p + p^* - \frac{p - p^k}{1 - p^k} \right)^{-1} \tag{4.1}$$

(see e.g., Page 1955). More specifically, if we denote the $\mathrm{ARL}_{\mathrm{out}}$ and the $\mathrm{ARL}_{\mathrm{in}}$ of the $1|1 - k|k$ CSCC by $\mathrm{ARL}_{1,k}(\lambda)$ and $\mathrm{ARL}_{1,k} = \mathrm{ARL}_{1,k}(0)$ respectively, we may write

$$\mathrm{ARL}_{1,k}(\lambda) = h_{1,k}(p_1(\lambda), p_k(\lambda) - p_1(\lambda)), \quad \mathrm{ARL}_{1,k} = h_{1,k}(p_1, p_k - p_1).$$

Another point of interest is that for the construction of a $1|1 - k|k$ CSCC we may use different values for the in-control ARL values of the individual $1|1$ and $k|k$ CSCCs, say $c_1 = \mathrm{ARL}_1$ and $c_k = \mathrm{ARL}_k$, as long as the ordering $U_1 > U_k$ remains valid (the condition $k < c_k < h_k(1/c_1)$ guarantees this ordering). An algorithmic description of the $1|1 - k|k$ CSCC in discrete steps is as follows:

Step 1: Choose a positive integer $k \geq 2$.
Step 2: Set the desired in-control values $c_1 = \mathrm{ARL}_1$ and $c_k = \mathrm{ARL}_k$ for the individual $1|1$ and $k|k$ CSCCs ($k < c_k < h_k(1/c_1)$) and calculate the respective individual control limits U_1 and U_k.
Step 3: Declare the process out-of-control if either k consecutive points are plotted above U_k or a single point is plotted above U_1.

One may use the $\min\{c_1, c_k\}$ as a rough estimate of the $\mathrm{ARL}_{1,k}$ of the $1|1 - k|k$ CSCC; as a matter of fact this is an upper bound of $\mathrm{ARL}_{1,k}$. However a much better approximation for $\mathrm{ARL}_{1,k}$ can be deduced by the aid of the classical formula

$$\frac{1}{\mathrm{ARL}_{1,k}} \cong \frac{1}{\mathrm{ARL}_1} + \frac{1}{\mathrm{ARL}_k} = \frac{1}{c_1} + \frac{1}{c_k} \tag{4.2}$$

which is quite commonly used by the quality control practitioners.

The superiority of the last formula is elucidated in Table 1. As formula (4.2) implies, if one wishes to work with a prespecified $\mathrm{ARL}_{1,k} = c$, he may calculate the values of the control limits U_1, U_k by considering the individual $1|1$ and $k|k$ CSCCs at $\mathrm{ARL}_1 = \mathrm{ARL}_k = 2c$.

In closing we mention that, although we focused only on the mean of the run length, the whole distribution of the statistic in use is also available (see e.g., Balakrishnan and Koutras 2002). One might suspect that the price paid for the

Table 1 In-control ARL values for the $1|1-k|k$ CSCC

k	ARL_1	ARL_k	$\min(ARL_1, ARL_k)$	$ARL_{1,k}$	
				Formula (4.2)	Exact value
2	600	600	600	300	312
3	600	600	600	300	306
4	600	600	600	300	304
2	300	200	200	120	127
2	200	300	200	120	128
3	400	300	300	171	176
3	300	400	300	171	176
4	500	200	200	143	145
4	200	500	200	143	146
2	970	970	970	485	500
3	987	987	987	493	500
4	990	990	990	495	500

improvement of the ARL will be a less smooth behavior of the resulting run length distribution. However, as our extensive numerical experimentation revealed, there is no evidence of such a change; surprisingly, in most cases besides the improvement of the ARL, a simultaneous improvement on the variance has been observed.

5 Numerical Comparisons

In the present section we are addressing some problems of practical importance. More specifically, by the aid of numerical experimentation, we discuss some procedures which will facilitate the practitioner to choose the most reasonable control plan for his/her needs. Assume first, that we wish to work with $k|k$ CSCCs and the problem we are focusing on is finding the most appropriate k value to be used. Since there is no uniform ordering between $ARL_1(\lambda)$ and $ARL_k(\lambda)$, one will ultimately need to compare them as a function of λ. In Fig. 3 we have graphed the percentage of improvement of ARL_{out} of several $k|k$ CSCCs over the standard CSCCs ($k=1$), under the same $ARL_{in}=500$. Thus, the curve that corresponds to $k=4$ is the graph of the quantity

$$I(\lambda)=\frac{ARL_1(\lambda)-ARL_4(\lambda)}{ARL_1(\lambda)}, \quad \lambda\geq 0$$

(the same quantity has also been used for the comparison of several statistical process control procedures by Aparisi et al. 2004).

By the aid of Eq. 3.1, four different graphs have been provided, each one referring to different data dimensionality m. Our extensive numerical experimentation revealed that, as m increases, not only the range of λ values in which $I(\lambda)$ remains positive becomes wider, but its maximum value increases as well. Another point of interest is that, for large m values, the curve with the maximum value, is the one that corresponds to the largest k values (note however that this curve attains negative values, much faster than the others).

It is evident from the foregoing arguments, that the optimal choice of the k value depends heavily on the magnitude of the shift we wish to detect. Should we have an idea about that, we can resort to the corresponding graph and pick the appropriate k value. By way of example consider the case $m = 20$ and $\text{ARL}_{\text{in}} = 500$. Then, according to the last graph in Fig. 3, for $\lambda < 6.52$ the best performance (among $k = 1, 2, 3, 4$) is achieved by choosing $k = 4$, for $6.52 < \lambda < 9.89$ the best choice is $k = 3$, for $9.89 < \lambda < 17.71$ it is advisable to use $k = 2$, while for $\lambda > 17.71$ the standard CSCC offers the best performance. Similar conclusions can be stated for the other m values included in Fig. 3. Additional graphs and the computer program that produces the percentage of improvement curve, is available for any interested reader from the authors upon request.

In the same spirit, Fig. 4 depicts plots of the percentage of improvement in the ARL_{out} of the $1|1-2|2$, $1|1-3|3$, and $1|1-4|4$ CSCCs over the standard CSCC with $\text{ARL}_{\text{in}} = 500$. The numerical calculations were carried out by the aid of Eq. 4.1. To achieve the same ARL_{in} value for the combined charts we used the results of the last three rows of Table 1.

Note that, the ARL performance of the $1|1-k|k$ CSCC has the same behavior and similar features as the $k|k$ CSCC. However, a comparison of Figs. 3 and 4 reveals that the percentage of improvement achieved by $1|1-k|k$ CSCC is considerably higher than the improvement gained by the respective $k|k$ CSCC. In addition, the interval of the non-centrality parameter λ in which a $1|1-k|k$ CSCC has better performance is much wider. It should also be stressed that, when huge shifts in the mean vector are likely to occur, the $1|1-k|k$ CSCC is a more natural choice than the $k|k$ CSCC with $k > 1$.

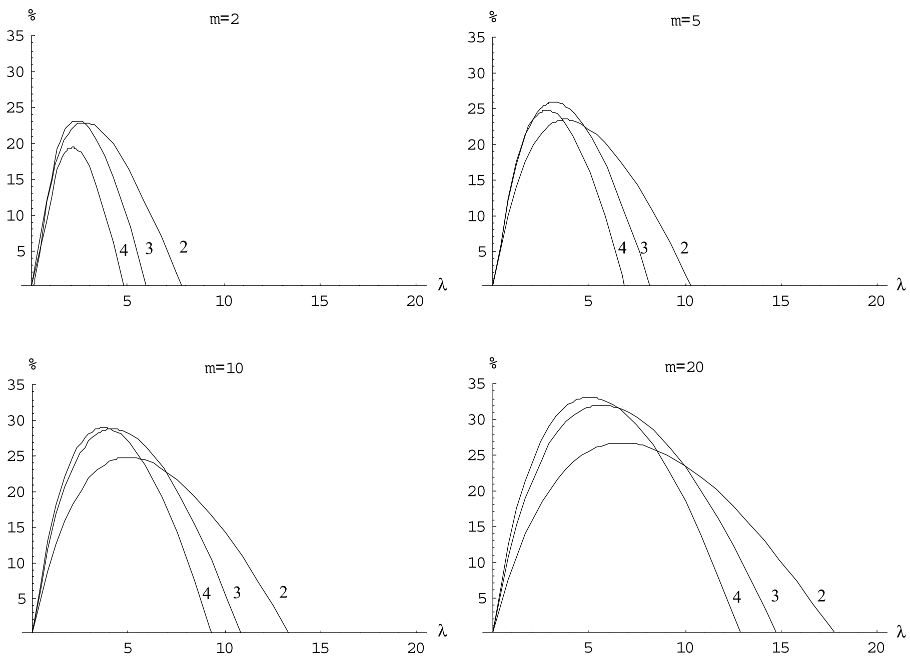

Fig. 3 Percentage of improvement in the ARL_{out} of the $k|k$ CSCC for $k = 2, 3, 4$ over $1|1$ CSCC

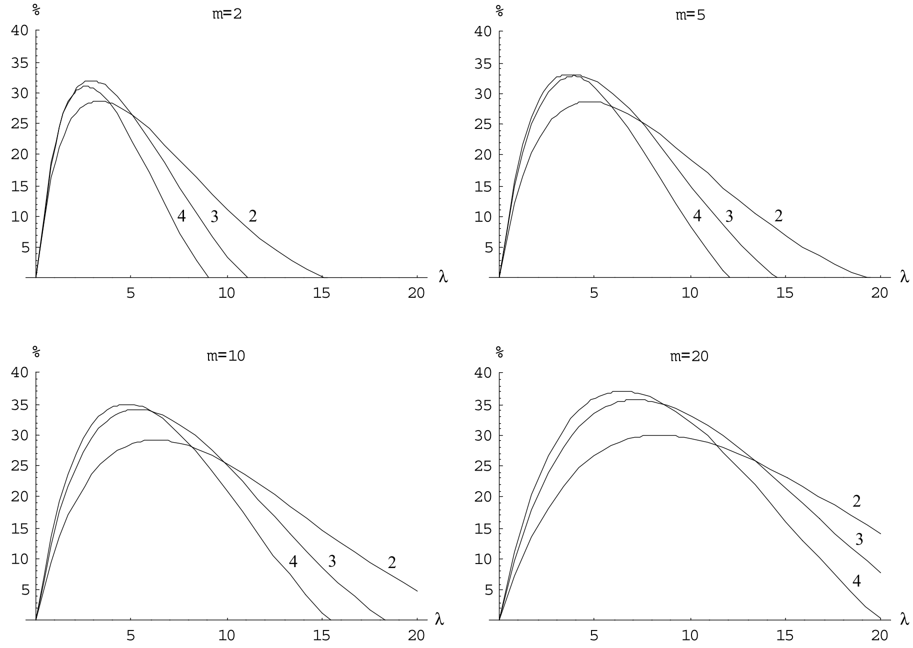

Fig. 4 Percentage of improvement in the ARL_{out} of the $1|1-k|k$ CSCC for $k = 2, 3, 4$ over $1|1$ CSCC

In closing, we mention that, as our extensive experimentation revealed, the new CSCCs become much more attractive as compared to the standard CSCC, as the value of ARL_{in} increases.

Table 2 ARL values for several chi-square control charts

m	$\lambda^{1/2}$	CSCC	$k\|k$		$1\|1-k\|k$		SRR
2	0	200	200		200		200.6
	1	41.93	38.54	(2)	35.16	(3)	34.16
	2	6.88	6.36	(2)	5.49	(2)	5.68
	3	2.16	2.71	(2)	2.01	(2)	2.21
3	0	200	200		200		199.7
	1	52.64	48.25	(2)	44.49	(3)	42.01
	2	8.82	7.90	(2)	6.93	(3)	7.06
	3	2.55	3.01	(2)	2.30	(2)	2.52
10	0	200	200		200		199.77
	1	92.70	83.69	(4)	80.80	(4)	78.06
	2	20.62	17.27	(3)	15.44	(3)	14.65
	3	5.21	4.99	(2)	4.23	(2)	4.56

6 Conclusions

In the present article the combined use of the theory of runs and of the classical Shewhart-type multivariate CSCC led to a procedure which improves the (weak) performance of the latter in the case of relatively small mean vector shifts. The smooth performance of the suggested variation may be attributed, on the one hand to the increased sensitivity of the runs statistic in detecting clustering of similar results and on the other hand to the substantial descriptive power of the CSCC.

In a recent article, Aparisi et al. (2004) investigated the performance of the CSCC with supplementary runs rules. More specifically, besides the classical out-of-control criterion (one point above the UCL), they suggested using three additional rules based on two out of three scans and runs of length seven and eight. As indicated there, for moderate shifts, the combined use of all supplementary runs rules improves the $\mathrm{ARL}_{\mathrm{out}}$ values of the CSCC by approximately 25% (for $\mathrm{ARL}_{\mathrm{in}} = 200$). Our extensive numerical experimentation revealed that the simple approach suggested in the present article leads to $\mathrm{ARL}_{\mathrm{out}}$ values very close to the ones achieved in Aparisi et al. (2004). For $\mathrm{ARL}_{\mathrm{in}} = 200$ and $m = 2, 3, 10$, Table 2 displays the ARL values of the standard CSCC, the $k|k$ CSCC, the $1|1-k|k$ CSCC and the respective values of Aparisi et al. (2004) (SRR column). The value in parentheses under the columns labeled "$k|k$" and "$1|1-k|k$" gives the k value ($k = 2$, 3 or 4) that produces the largest ARL decrease.

When our data sets require the simultaneous analysis of two or more quality characteristics, one could also use a multivariate $CUSUM$ or a multivariate $EWMA$ control chart (see Alwan 1986; Lowry and Montgomery 1995). It goes without saying that, like their univariate counterparts, these charts are more effective than the Shewhart-type charts in detecting small to medium process shifts but worse for large deviations. Moreover, they exhibit notable robustness to deviations from the normality assumption. However, they are much more involved and difficult to apply and, needless to say, the laws governing them may be so intricate as to preclude any theoretical analysis.

On the basis of the points mentioned above, we may summarize the key features of the new method as follows: (a) it preserves the simplicity of the standard CSCC and offers a manageable environment for establishing results of theoretical interest c.f. "Comparing Two Runs Related Chi-square Control Charts," (b) it improves significantly the performance of the standard CSCC, (c) unlike the multivariate $CUSUM$ and $EWMA$ procedures, it does not require the estimation of parameters through exhaustive mathematical search or time consuming numerical calculations (see e.g., Crosier 1988; Hawkins 1991; Ridgon 1995).

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