Methodology and Computing in Applied Probability, 7, 335–352, 2005

Spectral Properties of Superpositions of Ornstein-Uhlenbeck Type Processes

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Received December 23, 2003; Revised March 4, 2005; Accepted June 7, 2005

Abstract. Stationary processes with prescribed one-dimensional marginal laws and long-range dependence are constructed. The asymptotic properties of the spectral densities are studied. The possibility of Mittag-Leffler decay in the autocorrelation function of superpositions of Ornstein-Uhlenbeck type processes is proved.

Keywords: stationary processes, long range dependence, correlation function, Mittag-Leffler function, Ornstein-Uhlenbeck type processes, normal inverse Gaussian distribution

AMS 2000 Subject Classification: Primary 60E07, 60G10, 60G18; Secondary 62M10, 62P05

1. Introduction

In the fields of finance, distributions of logarithmic asset returns can often be fitted extremely well by the normal inverse Gaussian distribution or more general infinitely divisible distributions (see Barndorff-Nielsen (1998a,b), Barndorff-Nielsen and Shephard (2001) and references therein).

Another issue in modelling economic time series is that their sample autocorrelation function may have non-negligible values at large lags. This phenomenon is known as long range dependence (long memory or strong dependence). The volume of Doukhan et al. (2003) contains outstanding surveys of the field. In particular, that volume discusses a different definitions of long range dependence of stationary processes in terms of the autocorrelation function (the integral of the correlation function diverges) or the spectrum (the spectral density has a singularity at zero). Perhaps the definition of long range dependence has to be reconsidered in non-Gaussian contexts. However we will use these features as an indication of long range dependence of stationary processes with infinitely divisible marginal distributions.

On the other hand stochastic processes with infinitely divisible marginal distributions and long range dependence have considerable potential for stochastic modelling of observational series from a wide range of fields, such as turbulence (see Barndorff-Nielsen

*Supported by the Danish National Research Foundation and EPSRC grant RCMT091.
† Partially supported by the ARC grants DP 0345577, DP0559807 and grant EPSRC RCMT091.

et al. (1990) and the references therein) or anomalous diffusion (see Metzler et al. (1999) and the references therein). These ubiquitous phenomena call for development of reasonable models which can be integrated into economic and financial theory as well as theories of turbulence or anomalous diffusion.

This paper is motivated by the papers of Barndorff-Nielsen (1998a,b, 2001) in which stationary processes of Ornstein-Uhlenbeck (OU) type with long-range dependence and infinitely divisible marginal distributions are constructed. These processes may, in particular, have the normal inverse Gaussian distribution as one-dimensional marginal law. The normal inverse Gaussian distributions have considerable potential with respect to modelling in quite different contexts such as finance or turbulence (see, Barndorff-Nielsen et al. (1990), Barndorff-Nielsen (1998a,b), Barndorff-Nielsen and Pérez-Abreu (1999), Barndorff-Nielsen and Shephard (2001) and references therein).

In the present paper we discuss, within the above-mentioned framework, several new instances of continuous time strictly stationary processes whose autocorrelation functions and spectra have a simple explicit form and exhibit long-range dependence and whose marginal laws are simple and tractable. Note that Metzler et al. (1999) reported a Mittag-Leffler decay in the autocorrelation function of the velocity of a particle in anomalous diffusion. We are able to establish analytically that Mittag-Leffler decay is a possible property of the autocorrelation function in our approach (see Example 4).

The general framework of our approach is outlined in Section 2. Section 3 presents new instances of the type of stationary processes in question with emphasis on the structure of the autocorrelation and spectral functions.

For Gaussian models the idea of obtaining long-range dependence by aggregation of Ornstein-Uhlenbeck processes with random coefficients was used in different forms by Okabe (1981), Inoue (1993), Carmona and Coutin (1998), Igloi and Terdik (1999), Oppenheim and Viano (1999) (see also their references).

2. Prerequisites

This section reviews a number of known results, in particular from Barndorff-Nielsen (1998a,b, 2001).

2.1. Infinite Divisibility

As a standard notation we shall write $C\{\zeta \ddagger y\}$ for the cumulant (generating) function of a random vector or random variable y, i.e.,

$$C\{\zeta \ddagger y\} = \log E \exp\{i\langle \zeta, y\rangle\}.$$

Recall that a random variable y is infinitely divisible if its cumulant function has the Lévy-Khintchine representation

$$C\{\zeta \ddagger y\} = ia\zeta - \frac{b}{2}\zeta^2 + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta\tau(x)\right)Q(dx),\ \zeta \in \mathbb{R} \tag{2.1}$$

where $a \in \mathbb{R}, b > 0$ and

$$\tau(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ \dfrac{x}{|x|} & \text{if } |x| > 1 \end{cases} \tag{2.2}$$

and where the Lévy measure Q is a Radon measure on $\mathbb{R}\backslash\{0\}$ such that $Q(\{0\}) = 0$ and

$$\int_R \min\left(1, x^2\right) Q(dx) < \infty. \tag{2.3}$$

A stochastic process $z(s)$, $s \geq 0$ is said to be a Lévy process if it has independent increments and càdlàg sample paths and is continuous in probability. If the increments are stationary z is said to be homogeneous. In the following, unless otherwise stated, we take the term Lévy process to mean a homogeneous Lévy process z such that $z(s) \xrightarrow{p} 0$ as $s \downarrow 0$.

If z is a Lévy process then the cumulant function z satisfies $C\{\zeta \ddagger z(s)\} = sC\{\zeta \ddagger z(1)\}$. Note that to any infinitely divisible random variable y there corresponds a Lévy process z such that $y \overset{L}{=} z(1)$; we speak of z as the Lévy process generated by y, where '$\overset{L}{=}$' means the equality in law. If y has representation (2.1), then (a,b,Q) will be called the characteristic triplet of the Lévy process $z(s)$, $s \geq 0$.

More generally, a stochastic process $x(u)$, $u \in T, T$ an arbitrary index set, is said to be infinitely divisible if all its finite dimensional distributions are infinitely divisible. Any such process generates a generalized Lévy process $z = \{z(s,u), s \in \mathbb{R}_+, u \in T\}$ by the prescription

$$C\{\zeta_1, \ldots, \zeta_m \ddagger z(s,u_1), \ldots, z(s,u_m)\} = sC\{\zeta_1, \ldots, \zeta_m \ddagger x(u_1), \ldots, x(u_m)\}$$

for all the finite dimensional laws.

We review some basic facts about infinitely divisible random measures and integration of non-random functions with respect to such measures (cf. Rajput and Rosinski (1989)).

Let T be a Borel subset of $\mathbb{R}^d$ and S be a σ-ring of T (i.e., countable unions of sets in S belong to S and if A and B are sets in S with $A \subset B$, then $B \backslash A$ is also in S). The σ-algebra generated by S will be denoted by $\sigma(S)$. A collection of random variables $Z = \{Z(A), A \in S\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is said to be an independently scattered random measure (i.s.r.m.) if for every sequence $\{A_n\}$ of disjoint sets in S the random variables $Z(A_n)$, $n = 1, 2, \ldots$ are independent and if

$$Z\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} Z(A_n) \ a.s.$$

whenever $\bigcup_{n=1}^{\infty} A_n \in S$ We shall be interested in the case when Z is infinitely divisible, that is, for each $A \in S$, $Z(A)$ is an infinitely divisible random variable whose cumulant function can be written as

$$C\{\zeta \ddagger Z(A)\} = i\zeta m_0(A) - \frac{\zeta^2}{2} m_1(A) + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta\tau(x)\right)\tilde{q}(A, dx),$$

where m_0 is a signed measure, m_1 is a nonnegative measure, $\tilde{q}(A, dx)$ is (for fixed A) a measure in $\mathfrak{B}(\mathbb{R})$. The class of Borel sets of $\mathbb{R}$, without atom at 0 and such that (2.3) is satisfied, and $\tau(x)$ is defined in (2.2).

In this case we say that Z has the Lévy characteristics $(m_0, m_1, \tilde{q})$ and $\tilde{q}$ is called the Lévy measure. There is one to one correspondence between infinity divisible i.s.r.m. and the characteristics $(m_0, m_1, \overline{q})$.

The control measure m defined as

$$m(A) = |m_0|(A) + m_1(A) + \int_{\mathbb{R}} \min\{1, x^2\} \tilde{q}(A, dx)$$

is such that $m(A_n) \to 0$ implies that $Z(A_n) \to 0$ in probability.

This measure is important in characterizing the class of non-random functions that are integrable with respect to Z. Integration of a function f on T with respect to Z is defined first for real simple functions f_n and then through

$$\int_A f dZ = p - \lim_{n \to \infty} \int_A f_n dZ$$

where $\{f_n\}$ is a sequence of simple functions, such that $f_n \to f$ *a.s*. For details, see Rajput and Rosinski (1989).

If $|\cdot|$ denotes Lebesgue measure in $\mathbb{R}^d$ and if $m_0 \sim |\cdot|$, $m_1 \sim |\cdot|$ and $\tilde{q}(A, dx) = |A| q(dx)$, for q a classical Lévy measure, we say that Z is a *homogeneous Lévy basis* with characteristics $(m_0, m_1, \tilde{q})$.

2.2. *Selfdecomposability*

An infinitely divisible random variable y is selfdecomposable if its characteristic function $\phi(\zeta) = E \exp\{i\zeta y\}, \zeta \in \mathbb{R}$, has the property that for every $c \in (0,1)$ there exists a characteristic function ϕ_c such that $\phi(\zeta) = \phi(c\zeta)\, \phi_c(\zeta)$ for all $\zeta \in \mathbb{R}$. Equivalently, y is selfdecomposable if its Lévy measure is of the form $Q(dx) = q(x)dx$ with $q(x) = |x|^{-1} c(x)$, where $c(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

The selfdecomposability of y implies that y is representable as

$$y = \int_0^\infty e^{-s} d\check{z}(s) \tag{2.4}$$

where $\check{z}$ is a Lévy process whose law is determined uniquely by that of y. The Lévy measure W of $\check{z}(1)$ is related to the Lévy density q of y by the formulae

$$W^+(x) = xq(x) \tag{2.5}$$

for $x > 0$ and

$$W^-(x) = |x| q(x)$$

for $x < 0$, where $W^+(x) = W([x, \infty))$ and $W^-(x) = W((-\infty, x])$.

Furthermore, if the Lévy density q of y is differentiable then W has a density w with respect to Lebesgue measure, and w and q are related by

$$w(x) = -q(x) - xq'(x). \tag{2.6}$$

The process $\grave{z}(u)$, $u \geq 0$ is termed *the background driving Lévy process* or BDLP corresponding to y.

In the following y will stand for a selfdecomposable random variable whose Lévy density q is differentiable and $z(s)$, $s \geq 0$ will denote the Lévy process generated by y (see Section 2.1), i.e., the Lévy process such that $z(1) \stackrel{L}{=} y$. The BDLP determined by $z(1)$ is denoted by $\grave{z}$.

We will use the notation

$$\acute{\kappa}(\zeta) = C\{\zeta \ddagger y\}, \quad \grave{\kappa}(\zeta) = C\{\zeta \ddagger \grave{z}(1)\}.$$

Then

$$\acute{\kappa}(\zeta) = \int_0^\infty \grave{\kappa}(e^{-s}\zeta)ds, \quad \grave{\kappa}(\zeta) = \zeta\acute{\kappa}'(\zeta) \tag{2.7}$$

the last relation holding under the assumption that $\acute{\kappa}$ is differentiable for $\zeta \neq 0$ and provided $\zeta\acute{\kappa}'(\zeta) \to 0$ for $0 \neq \zeta \to 0$.

For any $u > 0$ and $\lambda > 0$ we may rewrite the representation (2.4) as

$$y = \int_0^\infty e^{-\lambda s} d\grave{z}(\lambda s) = e^{-\lambda u} \int_0^\infty e^{-\lambda s} d\grave{z}(\lambda(s+u)) + \int_0^u e^{-\lambda s} d\grave{z}(\lambda s)$$

and here, due to homogeneity of $\grave{z}$,

$$\int_0^\infty e^{-\lambda s} d\grave{z}(\lambda(s+u)) \stackrel{L}{=} y.$$

Consequently, y is representable as

$$y = e^{-\lambda u} y_0 + w_u$$

where y_0 and w_u are independent and $y_0 \stackrel{L}{=} y$ while

$$w_u = \int_0^u e^{-\lambda s} d\grave{z}(\lambda s).$$

In fact, a stronger statement is true: for any $\lambda > 0$, the stochastic differential equation

$$dy(u) = -\lambda y(u)du + d\grave{z}(\lambda u)$$

has a stationary solution $y(u)$ such that $y(u) \stackrel{L}{=} y$. This stochastic differential equation is solved by

$$y(u) = e^{-\lambda u} y(0) + \int_0^u e^{-\lambda(u-s)} d\grave{z}(\lambda s)$$

A stationary process $y(u)$ of this kind is said to be an Ornstein-Uhlenbeck type process, or an OU process for short. When $y(u)$ is square integrable with $Ey(u) = 0$ it has correlation function $r(u) = \exp\{-\lambda u\}$, $u > 0$. The stationary process $y(u)$, $u \geq 0$ can be extended to a stationary process on the whole real line. To do this we introduce an independent copy of the process $z(u)$, $u \geq 0$, but modify it to be càdlàg, thus obtaining a process $\grave{z}_1(u)$. Now for $u < 0$ define $\grave{z}(u)$ and $y(u)$ by $\grave{z}(u) = \grave{z}_1(-u)$ and

$$y(u) = e^{-\lambda|u|}y(0) + e^{-\lambda|u|}\int_u^0 e^{\lambda|s|}d\grave{z}(\lambda s).$$

Then $\grave{z}(u), u \in \mathbb{R}$, is a càdlàg Levy process and $y(u), u \in \mathbb{R}$, is a strictly stationary process of Ornstein-Uhlenbeck type with correlation function (if exists)

$$r(u) = e^{-\lambda|u|}, u \in \mathbb{R}. \tag{2.8}$$

The process $y(u)$ is equivalent in law to the stationary process

$$y(u) = \int_{-\infty}^{u} e^{-\lambda(u-s)}d\grave{z}(\lambda s) \stackrel{L}{=} \int_0^{\infty} e^{-\lambda\tau}d\grave{z}(\lambda(u-\tau)).$$

The last formula is the one-sided moving average representation of a stationary random process $y(u)$ with autocorrelation $r(u) = \exp\{-\lambda|u|\}, u \in \mathbb{R}$ and spectral density $f(s) = (\lambda/\pi)(\lambda^2 + s^2)^{-1}$, $s \in \mathbb{R}$, such that $f(s) = |g(s)|^2$. In our case $g(s) = (\lambda + is)^{-1}\sqrt{\lambda/\pi}$. If, however, we use the function $g(s) = \sqrt{\lambda/\pi}/\sqrt{\lambda^2 + s^2}$, which also satisfies the relation $f(s) = |g(s)|^2$, we obtain a two-sided moving-average representation of the form

$$y(u) = (1/\pi)\int_{\mathbb{R}} K_0(\lambda\tau)d\grave{z}(\lambda(u-\tau)),$$

where K_ν is modified Bessel function of the third kind. The last formula follows from Theorem 1 in Gihman and Skorohod (1974), p. 242 using the same arguments as in the Gaussian case (see Yaglom (1987), pp. 454–455).

EXAMPLE 1 *Normal inverse Gaussian (NIG) distribution*. The density function of a $NIG(\alpha, \beta, \mu, \delta)$ random variable y is given by

$$nig(x) = \frac{\alpha}{\pi}\exp\left\{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right\}\frac{K_1(\delta\alpha g(\frac{x-\mu}{\delta}))}{g(\frac{x-\mu}{\delta})},\ x \in \mathbb{R}$$

where $\delta > 0$, $0 \leq |\beta| < \alpha, g(x) = \sqrt{1+x^2}$, and K_ν denotes the modified Bessel function of the third kind of order ν (see Watson (1944)). The distribution is symmetric around μ provided $\beta = 0$. The normal distribution $N(\mu, \sigma^2)$ appears as a limiting case for $\beta = 0$, $\alpha \to \infty$ and $\delta/\alpha = \sigma^2$, and the Cauchy law appears as limiting case of $NIG(\alpha,0,0,1)$ for $\alpha \to 0$. Note that

$$Ey = \mu + \delta\pi/(1-\pi^2)^{1/2},\ \text{Var}\ y = \delta^2/\left\{\bar{\alpha}(1-\pi^2)^{3/2}\right\},$$

where $\pi = \beta/\alpha$, $\bar{\alpha} = \delta\alpha$ are invariant under location-scale changes. If $y_1, \ldots, y_m$ are independent $NIG(\alpha, \beta, \mu_i, \delta_i)$, $i = 1, \ldots, m$, then

$$y_1 + \ldots + y_m \sim NIG(\alpha, \beta, \mu_1 + \ldots + \mu_m, \delta_1 + \ldots + \delta_m).$$

We note that $NIG(\alpha, \beta, \mu, \delta)$ has semiheavy tails, specifically,

$$nig(x) \sim \text{const } |x|^{-3/2} \exp\{-\alpha|x| + \beta x\} \text{ as } x \to \pm\infty.$$

Assuming, for simplicity, that $\mu = 0$ one can show that the cumulant function of the $NIG(\alpha, \beta, \mu, \delta)$ distribution in the form of (2.1) is

$$C\{\zeta \ddagger y\} = \int_{-\infty}^{\infty} (e^{i\zeta x} - 1 - i\zeta x) q(x; \alpha, \beta, \delta) dx + i\zeta 2\pi^{-1}\delta\alpha \int_0^1 \sinh(\beta x) K_1(\alpha x) dx,$$

where

$$q(x; \alpha, \beta, \delta) = \pi^{-1}\delta\alpha|x|^{-1} e^{\beta x} K_1(\alpha|x|).$$

On observing that $|x|q(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(\infty, 0)$ one sees that the $NIG(\alpha, \beta, \mu, \delta)$ distribution is selfdecomposable. Thus, there exists a stationary OU process $y(u), u \in \mathbb{R}$, such that $y(u) \sim NIG(\alpha, \beta, \mu, \delta)$ for every $u \in \mathbb{R}$, whatever the value of the parameter $\lambda > 0$, and with correlation function (2.8). For $\mu = \beta = 0$ we have $Ey(u) = 0$. For $NIG(\alpha, 0, 0, \delta)$ we obtain from (2.7)

$$\acute{\kappa}(\zeta) = \alpha\delta - \delta(\alpha^2 + \zeta^2)^{1/2}, \quad \grave{\kappa}(\zeta) = \delta\zeta^2(\alpha^2 + \zeta^2)^{-1/2}.$$

2.3. Superpositions

2.3.1. Discrete Type Case

Let $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}, s > 1$ be the Riemann zeta-function.

THEOREM 1 *Let $y^{(k)}(u)$, $u \in \mathbb{R}, k = 1, 2, \ldots$ be a sequence of independent OU processes such that for all k and $u \in R$ the marginal distribution of $y^{(k)}(u)$ is $NIG(\alpha, \beta, \delta_k, 0)$ with*

$$\delta_k = 1/k^{1+2(1-H)}, \quad k = 1, 2, \ldots$$

where $H \in (0,1)$ and $c > 0$ is some constant, and let

$$\delta = \sum_{k=1}^{\infty} \delta_k = \zeta(1 + 2(1 - H)). \tag{2.9}$$

Suppose moreover that the processes $y^{(k)}(u)$ all have the correlation function (2.8).
Then the process

$$y(u) = \sum_{k=1}^{\infty} y^{(k)}(k^{-1}u), \; u \in \mathbb{R} \tag{2.10}$$

is stationary and well defined as an L_2 limit, the marginal distribution of $y(u)$ is NIG(α, β, δ, 0) where δ is defined in (2.9), the correlation function of $y(u)$ is of the form

$$\begin{aligned} \bar{r}(\tau) &= [\zeta(1+2(1-H))]^{-1} \sum_{k=1}^{\infty} \frac{e^{-|\tau|\lambda/k}}{k^{1+2(1-H)}} \\ &= \frac{1}{|\tau|^{2(1-H)}} \frac{\Gamma(2(1-H))}{\lambda^{2(1-H)}\zeta(1+2(1-H))}(1+o(1)), \end{aligned} \tag{2.11}$$

as $|\tau| \to \infty$, and the normalized spectral density is given by

$$\bar{f}(\mu) = \frac{(\lambda/\pi)}{\zeta(1+2(1-H))} \sum_{k=1}^{\infty} \frac{1}{k^{2(1-H)}} \frac{1}{(\lambda^2+\mu^2k^2)}, \; \mu \in \mathbb{R}. \tag{2.12}$$

Thus, if $H \in \left(\frac{1}{2}, 1\right)$ the process exhibits long-range dependence with Hurst exponent H.

REMARK 1 The spectral density (2.12) satisfies the following asymptotic relations:

$$\bar{f}(\mu) = \frac{c_1}{|\mu|^{1-2(1-H)}}(1+o(1)), \; |\mu| \to 0, \frac{1}{2} < H < 1 \tag{2.13}$$

and

$$\bar{f}(\mu) = \frac{c_2}{|\mu|^2}(1+o(1)), \; \text{as } |\mu| \to \infty. \tag{2.14}$$

where c_1, c_2 are positive constants.

REMARK 2 Similar stationary processes of the form (2.10) with autocorrelation function (2.11) and normalized spectral density (2.12) can be constructed for any parametric class of selfdecomposable distributions which is closed under convolution with respect to at least one parameter δ, and with variance proportional to this parameter δ. Thus, it can be shown that there exists a stationary process of discrete superposition type with long-range dependence and the following marginal distributions: Gaussian, Gamma, inverse Gaussian, normal inverse Gaussian, variance Gamma, Meixner and symmetric Gamma. These distributions have considerable potential with respect to modeling in quite different contexts such as turbulence and finance. For further details we refer to Barndorff-Nielsen (1998a,b, 2001), Barndorff-Nielsen and Shephard (2001) and the references therein.

2.3.2. Continuous Type Case

Let $Z = \{Z(A);\, A \in S\}$ be an i.s.r.m. on $T \subseteq \mathbb{R}^d$ which is infinitely divisible with Lévy characteristics $(m_0, m_1, \tilde{q})$ (see Section 2.1). We restrict the discussion to the case where

$$m_0 = m_1 = 0 \tag{2.15}$$

and where $\widetilde{q}$ factorizes as

$$\tilde{q}(A, dx) = M(A)W(dx) \tag{2.16}$$

for some σ-finite measure M on T and some Lévy measure W on $\mathbb{R}$. We denote the cumulant function associated with W by $\grave{\kappa}$, i.e.,

$$\grave{\kappa}(\zeta) = \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta\tau(x)\right) W(dx).$$

Formally, then

$$C\{\zeta \ddagger Z(du)\} = \grave{\kappa}(\zeta)M(du). \tag{2.17}$$

The following two propositions can be found in Barndorff-Nielsen (2001).

Proposition 1 *A function f on T is Z-integrable if and only if*

1. $\int_T U_0(f(u))M(du) < \infty$;
2. $\int_T |U(f(u))|M(du) < \infty$

where

$$U_0(y) = \int_{\mathbb{R}} \min\left\{1, (yx)^2\right\} W(dx),\; U(y) = \int_{\mathbb{R}} (\tau(yx) - y\tau(x))W(dx).$$

Proposition 2

$$C\left\{\zeta \ddagger \int_A f dZ\right\} = \int_A \kappa(\zeta f(u))M(du). \tag{2.18}$$

Let now $T = \mathbb{R} \times \mathbb{R}_+$, with points $w = (s, \xi)$, and let Z be an i.s.r.m. on $(T, \mathcal{B})$ with Lévy characteristics $(0, 0, \widetilde{q})$ and $\widetilde{q}$ of the form (2.16).

A key result can be formulated as follows (see Barndorff-Nielsen (2001) for the proof).

Theorem 2 *Suppose that the measure M factorizes as*

$$M(dw) = ds\, \nu(d\xi) \tag{2.19}$$

where ν is a probability measure on $\mathbb{R}_+$. Assume furthermore that the Lévy measure W is such that W^- and W^+ are of the form

$$W^-(x) = |x|q(x) \text{ and } W^+(x) = xq(x) \tag{2.20}$$

q being the Lévy density of a selfdecomposable distribution on $\mathbb{R}_+$.

Define, for $u \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R}_+)$,

$$y(u, B) = \int_B e^{-\xi u} \int_{-\infty}^{\xi u} e^s Z(ds, d\xi) \tag{2.21}$$

and let

$$y(u) = y(u, \mathbb{R}_+). \tag{2.22}$$

Then y(u), $u \in \mathbb{R}$, is a well-defined, infinitely divisible and strictly stationary process, and the cumulant transforms of the finite dimensional distributions of y are given by

$$C\{\zeta_1, \ldots, \zeta_m \ddagger y(u_1), \ldots, y(u_m)\} \tag{2.23}$$

$$= \int_{\mathbb{R}_+} \int_{\mathbb{R}} \grave{\kappa}\left(\sum_{j=1}^{m} 1_{[0,\infty)}(u_j - s)\zeta_j e^{-\xi(u_j - s)}\right)\xi ds \nu(d\xi) \tag{2.24}$$

where $\grave{\kappa}$ is the cumulant function corresponding to the Lévy measure W, that is

$$\grave{\kappa}(\zeta) = \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta\tau(x)\right) W(dx)$$

and $u_1 < \ldots < u_m$.

REMARK 3 Formal calculation from the formulae (2.21) and (2.22) gives

$$dy(u) = \int_{\mathbb{R}_+} \{-\xi y(u, d\xi)du + Z(\xi du, d\xi)\} \tag{2.25}$$

showing that y is a superposition of perhaps infinitesimally determined OU type processes.

We shall refer to any such a process as a sup-OU process.

Note that conditions (2.20) imply that W is the Lévy measure of the BDLP corresponding to the selfdecomposable law whose Lévy density is q.

COROLLARY 1 *We have*

$$C\{\zeta \ddagger y(u)\} = \acute{\kappa}(\zeta) = \int_0^\infty \grave{\kappa}(\zeta e^{-s}) ds$$

where $\acute{\kappa}$ is the cumulant function of the selfdecomposable law with Lévy density q.

COROLLARY 2 *Assuming that y is square integrable, the correlation function r of y is for $\tau \geq 0$ given by*

$$r(\tau) = \int_0^\infty e^{-\tau\xi}\nu(d\xi) \tag{2.26}$$

Then the covariance function is equal to $\kappa_2 r(|\tau|)$, where κ_2 is the variance of a random variable with cumulant function $\grave{\kappa}(\zeta)$.

REMARK 4 Corollaries 1 and 2 together show that to any selfdecomposable distribution D with finite second moment and to any Laplace transform of a distribution ν on $(0, \infty)$ there exists a stationary process $y(u)$, $u \in \mathbb{R}$, whose one-dimensional marginal law is D and whose autocorrelation function is of the form (2.26). Note that, in contrast to the descrete type construction considered in Section 2.3.1, we are here not requiring existence of a parameter δ of kind discussed in Remark 2.

REMARK 5 It is well-known that correlation functions of stationary processes are non-negative definite, that is for every $n \geq 1, \zeta_j \in C, j = 1, \ldots, n$, the correlation function (2.26) satisfies

$$\sum_{j,k=1}^{n} \zeta_j r(\tau_j - \tau_k)\bar{\zeta}_k \geq 0, \tau_j \in \mathbb{R},\ j = 1, \ldots, n.$$

Following Ostervalder and Schrader (1973), we say that stationary process has reflection positivity (also known as T-positiveness) if its correlation function $r(\tau), \tau \in \mathbb{R}$, satisfies

$$\sum_{j,k=1}^{n} \zeta_j r(\tau_j + \tau_k)\bar{\zeta}_k \geq 0, \tau_j \in \mathbb{R}, j = 1, \ldots, n,$$

for any $n \geq 1, \zeta_j \in C,\ j = 1, \ldots, n$. Hida and Streit (1977) showed that a stationary process has reflection positivity if and only if its autocorrelation function is representable in the form (2.26). From Theorem 2 we obtain that a square integrable sup-OU process (2.21) is a T-positive process. Furthermore, by Bernstein's Theorem (see Feller (1971), p. 426) we have that (2.26) is equivalent to complete monotonicity of the function $r(\tau)$, $\tau \in (0, \infty)$, that is $(-1)^k \frac{d^k}{dt^k} r(\tau) \geq 0, \tau \in (0,\infty), k = 0, 1, \ldots$.

3. Correlation and Spectrum Structures

The spectral density of the sup-OU process with correlation function (2.26) is of the form:

$$f(\lambda) = \frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2 + \lambda^2}\nu(d\xi). \tag{3.1}$$

In fact, we have

$$r(|\tau|) = \int_0^\infty e^{-|\tau|\xi}\nu(d\xi) = \int_0^\infty \left[\int_R e^{i\lambda\tau}\frac{\xi}{\pi(\xi^2+\lambda^2)}d\lambda\right]\nu(d\xi)$$
$$= \int_{-\infty}^\infty e^{i\lambda\tau}\left[\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2+\lambda^2}\nu(d\xi)\right]d\lambda\cdot$$

Example 2 Suppose that ν is the Gamma law with the density function

$$p(x) = [1/\Gamma(2\bar{H})]x^{2\bar{H}-1}e^{-x}1_{(0,\infty)}(x),$$

where $\bar{H} > 0$. Then

$$r_L(\tau) = \frac{1}{(1+|\tau|)^{2\bar{H}}},\ \tau \in \mathbb{R},\ 0 < 2\bar{H} \le 1. \tag{3.2}$$

In particular, the process $y(u)$, $u \in \mathbb{R}$, exhibits long-range dependence if $H \in \left(\frac{1}{2}, 1\right)$, where $\bar{H} = 1 - H$.

The function (3.2) is known as the characteristic function of the generalized Linnik distribution (see Erdoğăn and Ostrowski (1998)). Thus, the spectral density of our sup-OU process (or probability density of a generalized Linnik distribution) is of the form

$$\begin{aligned} f_L(\lambda) &= \frac{1}{\pi}\int_0^\infty \frac{\cos(\lambda\tau)}{(1+|\tau|)^{2\bar{H}}}d\tau = \frac{1}{\pi}\mathrm{Im}\int_0^\infty \frac{e^{-\lambda\tau}d\tau}{(1+e^{-i\pi/2}\tau)^{2\bar{H}}} \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g_{2\bar{H}}(z)\lambda^{z-1}dz \end{aligned} \tag{3.3}$$

for $\lambda > 0$, $-1 < c < 2\bar{H}$ and where

$$g_{2\bar{H}}(z) = \frac{1}{\Gamma(2\bar{H})}\frac{\sin(\pi z/2)}{\sin(\pi z)}\Gamma(2\bar{H} - z).$$

The formula (3.3) is a special case of Theorems 1 and 2 of the paper Erdoğăn and Ostrowski (1998), from which and Theorems 6–7 we obtain the following asymptotic properties of the spectral density $f_L(\lambda)$:

$$f_L(\lambda) = \frac{c(2\bar{H})}{\lambda^{1-2\bar{H}}}(1+o(1)),\ \lambda \to 0 \tag{3.4}$$

for $0 < 2\bar{H}$, $c(2\bar{H}) = [2\Gamma(2\bar{H})\cos(\bar{H}\pi)]^{-1}$ while for $2\bar{H} = 1$, from Kotz et al. (1995), we have

$$f_L(\lambda) = \frac{1}{\pi}\log\frac{1}{|\lambda|} - \frac{\gamma}{\pi} + \frac{1}{2}|\lambda| - \frac{1}{2\pi}|\lambda|^2\log\frac{1}{|\lambda|} + O(\lambda^2) \tag{3.5}$$

as $\lambda \downarrow 0$, where $\gamma = -\Gamma'(1) \simeq 0,5772\ldots$ is Euler's constant. For $|\lambda| \to \infty$ we have $f_L(\lambda) = \frac{\bar{H}}{\pi} \frac{1}{\lambda^2}(1 + o(1)), 0 < 2\bar{H} \leq 1$.

EXAMPLE 3 Suppose that ν is the Mittag-Leffler distribution (see Pillai (1990) or Lin (1998)) with distribution function

$$F_{2\bar{H}}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k\bar{H}}}{\Gamma(1 + 2k\bar{H})},\ u \geq 0,\ 0 < 2\bar{H} \leq 1. \tag{3.6}$$

For $2\bar{H} = 1$, $F_{2\bar{H}}(x)$ reduces to an exponential distribution.

Note that

$$F_{2\bar{H}}(x) = 1 - E_{2\bar{H}}\left(-x^{2\bar{H}}\right),\ x \geq 0,$$

where the function

$$E_\alpha(\zeta) = \sum_{k=0}^{\infty} \zeta^k / \Gamma(1 + \alpha k),\ \alpha > 0$$

is known as the Mittag-Leffler function of a complex variable ζ. Pillai (1990) proved that the Laplace transform of the Mittag-Leffler distribution (3.6) is

$$r_{ML}(\tau) = \frac{1}{1 + \tau^{2\bar{H}}},\ \tau \geq 0. \tag{3.7}$$

By Corollary 2, this is also the correlation function of a sup-OU process.

Kotz et al. (1995) proposed to term the distribution with characteristic function

$$r_{ML}(\tau) = \frac{1}{1 + |\tau|^{2\bar{H}}},\ \tau \in \mathbb{R},\ 0 < 2\bar{H} < 2 \tag{3.8}$$

the Linnik distribution because it was introduced by Linnik in 1953.

For the density function with characteristic function (3.8) the following formula is known (see Kotz et al. (1995)):

$$f_{ML}(\lambda) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos(\lambda t) dt}{1 + t^{2\bar{H}}} = \frac{\sin(\bar{H}\pi)}{\pi} \int_0^{\infty} \frac{e^{-\eta\lambda} \eta^{2\bar{H}} d\eta}{\left|1 + \eta^{2\bar{H}} e^{i\pi\bar{H}}\right|^2},\ \lambda > 0. \tag{3.9}$$

For any $2\bar{H} \in (0, 2)$, the density (3.9) decreases at ∞ at the rate of a power function:

$$f_{ML}(\lambda) = \frac{1}{\lambda^{1+2\bar{H}}} \left\{ \frac{1}{\pi} \Gamma(1 + 2\bar{H}) \sin(\pi\bar{H}) \right\} (1 + o(1)),\ \lambda \to +\infty \tag{3.10}$$

and for $0 < 2\bar{H} < 1$

$$f_{ML}(\lambda) = \frac{c(2\bar{H})}{\lambda^{1-2\bar{H}}} (1 + o(1)),\ \lambda \to 0+. \tag{3.11}$$

The formula (3.11) reflects the long-range dependence embodied in (3.9), while the formula (3.10) indicates the degree of fractality of a path, which is an interesting feature in relation to turbulence and finance processes.

REMARK 6 Schneider (1996) proved that the generalized Mittag-Leffler function of negative real argument

$$E_{\alpha,\beta}(-x) = \Gamma(\beta)\sum_{k=0}^{\infty}\frac{(-x)^k}{\Gamma(\alpha k+\beta)},\ x \geq 0,\ \alpha,\beta > 0 \tag{3.12}$$

is infinitely differentiable and completely monotone if and only if $0 < \alpha \leq 1$, $\beta \geq \alpha$. Thus, by Bernstein's theorem, $E_{\alpha,\beta}$ is the Laplace transform of a probability measure $\mu_{\alpha,\beta}$ supported by $\mathbb{R}_+$. Apart from the trivial case $\alpha = \beta = 1$ these measures are absolutely continuous with respect to the Lebesgue measure with densities

$$f_{\alpha,\beta}(x) = \Gamma(\beta)\sum_{k=0}^{\infty}\frac{(-x)^k}{k!\Gamma(\beta-\alpha-\alpha k)},\ x \geq 0, \tag{3.13}$$

for $\alpha \leq 1$, $\beta \geq \alpha$, while for $\alpha = 1$, $\beta > 1$

$$f_{1,\beta}(x) = \begin{cases} (\beta-1)(1-x)^{\beta-2} & 0 \leq x \leq 1 \\ 0 & 1 < x < \infty. \end{cases} \tag{3.14}$$

For $\alpha = \beta = 1$ the measure $\mu_{1,1}$ is the Dirac measure at the point 1.

For properties of Mittag-Leffler type functions see the book by Djrbashian (1993). In particular,

$$E_{1,1}(-x) = e^{-x}, E_{1,2}(-x) = (1-e^{-x})/x, E_{1,3}(-x) = 2(e^{-x}-1+x)/x^2$$
$$E_{1/2,1}(-x) = e^{x^2}(1-\tfrac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds).$$

For $\alpha < 1$, $\beta < 1 + \alpha$ the following formula is true

$$E_{\alpha,\beta}(-x^\alpha) = x^{1-\beta}\frac{\Gamma(\beta)}{\pi}\int_0^\infty e^{-x\tau}\frac{\sin\{\pi(\beta-\alpha)\}+\tau^\alpha\sin\{\pi\beta\}}{1+2\tau^\alpha\cos\{\pi\alpha\}+\tau^{2\alpha}}\tau^{\alpha-\beta}d\tau.$$

The last two formulae are particular cases of Theorems 1.3–5 and 1.3–6 of Djrbashian (1993). From Theorem 1.3–4 of the same book we obtain the following asymptotic relation:

$$E_{\alpha,\beta}(-x) = -\Gamma(\beta)\sum_{k=1}^{N}(-1)^k x^{-k}/\Gamma(\beta-k\alpha) + O(|x|^{-N-1}) \tag{3.15}$$

as $x \to \infty$, $\alpha < 2$, $\alpha \neq 1$.

From Feller (1971) we have that

$$E_{\alpha,1}(-u) = \int_0^\infty e^{-ux}\zeta_\alpha(x)dx, \tag{3.16}$$

where $u > 0$ and $\zeta_\alpha(x), x > 0$ is a probability density on $(0, \infty)$ such that

$$\zeta_\alpha(x) = \frac{1}{\alpha}x^{-1-\frac{1}{\alpha}}\rho_\alpha(x^{-1/\alpha}), \tag{3.17}$$

where here and below $\rho_\alpha(x), x > 0$, is the one-sided stable probability density with Laplace transform $\exp\{-p^\alpha\}, p > 0$.

EXAMPLE 4 Suppose that the probability measure ν in (2.26) has the density function (3.13), (3.14) or (3.17). Then (see Remark 3) for $\tau > 0$ the correlation function of the sup-OU process is of the form

$$r(\tau) = E_{\alpha,\beta}(-\tau), \tag{3.18}$$

where the Mittag-Leffler type function $E_{\alpha,\beta}$ is defined in (3.12). Note that $E_{\alpha,\beta}(0) = 1$. In particular, for $\alpha = 1$, $\beta = 1$, we have $r(\tau) = e^{-\tau}, \tau \geq 0$. In general, the correlation function of the sup-OU process is

$$r(\tau) = E_{\alpha,\beta}(-|\tau|), \quad \tau \in \mathbb{R}, \quad \alpha \neq 1 \tag{3.19}$$

and from (3.15) we find

$$r(\tau) = \frac{\Gamma(\beta)}{|\tau|\Gamma(\beta-\alpha)} + O\left(\frac{1}{|\tau|^2}\right) \tag{3.20}$$

as $|\tau| \to \infty$.

Thus our sup-OU process exhibits long range dependence. This result may be generalized as follows. Besides $E_{\alpha,1}$ also $G_{\alpha,\gamma}$ is completely monotonic (see Feller, 1971)), where

$$G_{\alpha,\gamma}(u) = E_{\alpha,1}(-u^{\gamma}), \quad 0 < \gamma < 1, \quad u \geq 0, \tag{3.21}$$

and $E_{\alpha,1}$ is defined in (3.12).

From Schneider (1996) we obtain that the function (3.21) is the Laplace transform of the probability density

$$g_{\alpha,\gamma}(x) = \frac{\gamma}{\alpha} x^{-1-\frac{\gamma}{\alpha}} \int_0^{\infty} y^{\gamma/\alpha} \rho_{\gamma}(y) \rho_{\gamma}\left(\left(\frac{y}{x}\right)^{\gamma/\alpha}\right) dy, \tag{3.22}$$

in terms of the stable densities ρ_α and ρ_γ (see (3.17)) with Laplace transforms $e^{-p^{\alpha}}$ and $e^{-p^{\gamma}}$ respectively or, more explicitly,

$$g_{\alpha,\gamma}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\gamma+\gamma k)\Gamma(1-\alpha-\alpha k)} x^{\gamma(k+1)-1}, \quad x > 0, \quad \gamma \geq \alpha, \tag{3.23}$$

and

$$g_{\alpha,\gamma}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(-\gamma k)\Gamma(1+\alpha k)} x^{-\gamma k-1}, \quad x > 0, \quad \gamma < \alpha. \tag{3.24}$$

For $\gamma = \alpha$, (3.23) reduces to

$$g_{\alpha,\gamma}(x) = \frac{\sin \pi\alpha}{\pi} \frac{x^{\alpha-1}}{1+2\cos(\pi\alpha)x^{\alpha}+x^{2\alpha}}. \tag{3.25}$$

Let now the measure ν in (2.26) have density function $g_{\alpha,\gamma}$ defined by (3.22)–(3.25). Then for $\tau > 0$ the correlation function (2.25) of the sup-OU process is of the form

$$r(\tau) = E_{\alpha,1}(-\tau^{\gamma}), \quad \tau > 0,$$

where $E_{\alpha,1}$ is defined in (3.12). From (3.15) we obtain that the correlation function

$$r(\tau) = E_{\alpha,1}(-|\tau|^{\gamma}), \quad \tau \in \mathbb{R}, \tag{3.26}$$

of such a sup-OU process satisfies the following asymptotic expansion:

$$r(\tau) = \frac{1}{|\tau|^{\gamma}\Gamma(1-\alpha)} + O\left(\frac{1}{|\tau|^{2\gamma}}\right), \quad \tau \to \infty.$$

Thus, for $0 < \gamma < 1$ the stationary sup-OU process exhibits long range dependence.

Note that Metzler et al. (1999) reported a Mittag-Leffler decay in the autocorrelation function of the velocity of a particle in an anomalous diffusion. As we have just established, the autocorrelation function (3.26) exhibits Mittag-Leffler decay. Thus, our sup-OU process is a possible dynamic model of anomalous diffusion (see also Anh et al. (2002)).

The spectral densities of stationary processes with correlation functions (3.18), (3.26) of Mittag-Leffler type can be expressed in the form of Fox's H-functions (see for example Schneider (1996)) and they are Green functions of fractional diffusion equations (see Anh and Leonenko (2000, 2001, 2002), and Anh et al. (2002) and references therein).

We summarize our results in the following two Tables:

Table 1. Correlation functions and spectral densities of some sup-OU processes.

	$\nu(dx) = p(x)dx$	$r(\tau),\ \tau \in R$	$f(\lambda),\ \lambda > 0$
Model	$p(x),\ x > 0$	$\int_0^{\infty} e^{-\lvert\tau\rvert x}\nu(dx)$	$\frac{1}{\pi}\int_0^{\infty}\frac{x}{x^2+\lambda^2}\nu(dx)$
Linnik	$2\bar{H}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}kx^{2k\bar{H}-1}}{\Gamma(1+2k\bar{H})}$, $0 < 2\bar{H} \le 1$	$(1+\lvert\tau\rvert^{2\bar{H}})^{-1}$	$\frac{\sin(\bar{H}\pi)}{\pi}\int_0^{\infty}\frac{e^{-\eta\lambda}\eta^{2\bar{H}}d\eta}{\lvert 1+\eta^{2\bar{H}}e^{i\pi\bar{H}}\rvert^2}$
Generalized Linnik	$\frac{1}{\Gamma(2\bar{H})}x^{2\bar{H}-1}e^{-x}$, $0 < 2\bar{H} \le 1$	$(1+\lvert\tau\rvert)^{-2\bar{H}}$	$\frac{1}{\pi}\mathrm{Im}\int_0^{\infty}\frac{e^{-\lambda\tau}d\tau}{(1+e^{-i\pi/2}\tau)^{2\bar{H}}}$
Mittag-Leffler	$\frac{1}{\alpha}x^{-1-\frac{1}{\alpha}}\rho_{\alpha}(x^{-1/\alpha})$, $0 < \alpha < 1$	$E_{\alpha,1}(-\lvert\tau\rvert)$	
Two-Parameter Mittag-Leffler	$\Gamma(\beta)\sum_{k=0}^{\infty}\frac{(-x)^k}{k!\Gamma(\beta-\alpha-\alpha k)}$, $x \ge 0, 0 < \alpha < 1 \beta \ge \alpha$	$E_{\alpha,\beta}(-\lvert\tau\rvert)$	
Generalized Mittag-Leffler	$\frac{\gamma}{\alpha}x^{-1-\frac{\gamma}{\alpha}}\int_0^{\infty}y^{\gamma/\alpha}\rho_{\gamma}(y)\times$ $\times\rho_{\gamma}\left(\left(\frac{y}{x}\right)^{\gamma/\alpha}\right)dy$, $0 < \gamma < 1, \alpha > 0$	$E_{\alpha,1}(-\lvert\tau\rvert^{\gamma})$	

In Table 1 $\rho_{\alpha}(x), x > 0$, is the one-sided stable probability density with Laplace transform $\exp\{-\mathrm{p}^{\alpha}\}, \mathrm{p} > 0$.

Table 2. Asymptotics of correlation functions and spectral densities of sup-OU processes.

Model	$r(\tau) \propto, \tau \to \infty$	$f(\lambda) \propto, \lambda \downarrow 0$	$f(\lambda) \propto, \lambda \uparrow \infty$
Linnik	$\tau^{-2\bar{H}}, 0 < 2\bar{H} < 1$	$\lambda^{2\bar{H}-1}, 0 < 2\bar{H} < 1$	$\lambda^{2\bar{H}+1}, 0 < 2\bar{H} < 2$
Generalized Linnik	$\tau^{-2\bar{H}}, 0 < 2\bar{H} < 1$	$\lambda^{2\bar{H}-1}, 0 < 2\bar{H} < 1$	$\lambda^{-2}, 0 < 2\bar{H} < 1$
Mittag-Leffler	τ^{-1}		
Two-Parameter Mittag-Leffler	τ^{-1}		
Generalized Mittag-Leffler	$\tau^{-\gamma}, 0 < \gamma < 1$		

In Table 2 we use the notation: $f(\lambda) \propto g(\lambda)$ if there exists a constant c such that $f(\lambda) \sim cg(\lambda)$.

Acknowledgment

We thank both referees and the Associate Editor for helpful comments.

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