Methodology and Computing in Applied Probability, 7, 271–283, 2005

Methods for Evaluating Density Functions of Exponential Functionals Represented as Integrals of Geometric Brownian Motion

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Received May 27, 2003; Revised March 9, 2005; Accepted March 17, 2005

Abstract. The purpose of this paper is to present a survey on Yor's formula on the probability densities of the exponential functionals represented as integrals in time of geometric Brownian motions and to present results on numerical computations for the densities. We perform the computations via another formula for the densities obtained by Dufresne and we show numerically the desired coincidence in some cases. As an application, we compute the price of an Asian option.

Keywords: Brownian motion, Asian option, Yor's formula

AMS 2000 Subject Classification: 65C50, 60J65

1. Introduction

Let $B = \{B_t,\ t \geq 0\}$ be a standard Brownian motion and, for $\mu \in \mathbf{R}$, $B^{(\mu)} = \{B_t^{(\mu)} = B_t + \mu t, t \geq 0\}$ be a Brownian motion with constant drift μ. In this paper, we are concerned with a Brownian functional $A_t^{(\mu)}$ defined by

$$A_t^{(\mu)} = \int_0^t \exp(2B_s^{(\mu)})ds, \quad t \geq 0.$$

Such exponential type functionals play important roles in the theory of mathematical finance, in study of diffusion processes in random media and in stochastic analysis on the hyperbolic spaces. See, e.g., Alili et al. (2001), Dufresne (2001), Yor (1992b, 2000), and the references cited therein.

In particular, in mathematical finance, to study $A_t^{(\mu)}$ is equivalent to study the mean in time of the stock process $\{\exp(2B_s^{(\mu)}), 0 \leq s \leq t\}$ in the Black-Scholes model. The functional $A_t^{(\mu)}$ plays a major role in study of Asian options, where it is important to evaluate $E[(A_t^{(\mu)} - K)_+], K > 0$. For this purpose, we need an explicit expression for the density of $A_t^{(\mu)}$.

In the celebrated paper, Yor (1992b) has derived an explicit expression for the density of the joint distribution of $(A_t^{(\mu)}, B_t^{(\mu)})$ for fixed t, given by (1) below. By integration with respect to the second component, we obtain an expression for the density of $A_t^{(\mu)}$. However, in this expression, we have an oscillatory integral and a double integral.

Hence, it is natural to ask whether we can carry out numerical computations efficiently. One of the main results in this paper is to give an affirmative answer to this question when t is not small. In the course of our study, we see that the mode of the density of the distribution of $A_t^{(\mu)}$ does not change fast as t or μ varies. This fact by itself might be of independent interest.

When $\mu < 0$, it is easy to see that $A_t^{(\mu)}$ converges as $t \to \infty$. It is known (cf. Dufresne, 2001; Yor, 1992a) that, letting γ_β be a gamma random variable with parameter $\beta > 0$, $A_t^{(\mu)}$ converges in law to $(2\gamma_{-\mu})^{-1}$. As a byproduct, we obtain numerically the corresponding convergence of the densities.

In order to carry out the numerical computations, we have to evaluate the function $\theta(r, t)$ given by (2) below. This function is closely related to the so-called Hartman-Watson distribution and comes from the modified Bessel function $I_\nu(x)$(see Barrieu et al., 2004; Yor, 1980). For the positive function $\theta(r, t)$, only an integral representation is known and the integral is an oscillatory one. As we see in Figure 2, when t is small, the function $\theta(r, t)$ seems to be negative, from our computations. We have not obtained a nice result for the density of $A_t^{(\mu)}$ in this case. A similar result on the numerical computations has been made by Barrieu et al. (2004).

Since the expression due to Yor is a little complicated, some authors have been trying to obtain simpler expressions (see, e.g., Dufresne, 2000, 2001; Schröder, 2003). In particular, Dufresne (2001) has derived other expressions for the density of $A_t^{(\mu)}$, which are simpler in the case of $\mu = 0$ and $\mu = 1$. The coincidence between Yor's and Dufresne's expressions has been confirmed in Matsumoto and Yor (2003) and, for the cases $\mu = 0$ or $\mu = 1$, we see the coincidence by numerical computations.

As is mentioned above, numerical studies on the functonal $A_t^{(\mu)}$are important in the study of Asian options. As an application of our computations for the density of $A_t^{(\mu)}$, we evaluate numerically the price of an Asian option, which Rogers and Shi (1995) have done by using a numerical solution for a Cauchy problem.

This paper is organized as follows. In Section 2, we recall Yor's and Dufresne's expressions and sketch a proof of Yor's formula. We see that all the integrals in Yor's formula are obtained from the properties of modified Bessel functions. In Sections 3 and 4, we present the results of our numerical computations for the densities of $A_t^{(\mu)}$ and for the pricing formula on Asian options, respectively.

2. Expressions for the Density

2.1. *Yor's Results*

For the joint density of $(A_t^{(\mu)}, B_t^{(\mu)})$, Yor (1992b) has obtained the following result:

THEOREM 2.1 *For $a > 0$ and $x \in \mathbf{R}$, it holds that*

$$P(A_t^{(\mu)} \in da, B_t^{(\mu)} \in dx) = e^{\mu x - \mu^2 t/2} \exp\left(-\frac{1 + e^{2x}}{2a}\right) \theta(e^x/a, t) \frac{da}{a}\ dx, \tag{1}$$

where

$$\theta(r,t) = \frac{r}{(2\pi^3 t)^{1/2}} \int_0^\infty e^{(\pi^2-\xi^2)/2t} e^{-r\cosh(\xi)} \sinh(\xi) \sin(\frac{\pi\xi}{t}) d\xi. \tag{2}$$

Recall that $\theta(r, t)$ is related to the modified Bessel function $I_\nu(r)$ (for details, see Yor, 1980):

$$\int_0^\infty e^{-\lambda t}\theta(r,t)dt = I_{\sqrt{2\lambda}}(r), \quad \lambda \geq 0. \tag{3}$$

We give a proof of this theorem, following Alili et al. (2001), which shows that we indeed live in the world of modified Bessel functions.

Proof: For $\lambda > 0$, we consider the fundamental solution $q(t, \xi, \eta)$ of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} - \frac{\lambda^2}{2} e^{2\xi} u.$$

It is easy to show by the general theory of the Sturm-Liouville operators that the Green function G is given by

$$G(\xi, \eta; \frac{\alpha^2}{2}) \equiv \int_0^\infty e^{-\alpha^2 t/2} q(t,\xi,\eta)dt = 2I_\alpha(\lambda e^\xi) K_\alpha(\lambda e^\eta), \quad \xi < \eta,$$

where $I_\nu(r)$ and $K_\nu(r)$ are the usual modified Bessel functions. For the product of the modified Bessel functions, the following integral representation is known (cf. Gradshteyn and Ryzhik, 2000 6.653 at page 705):

$$I_\alpha(a)K_\alpha(b) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}u - \frac{a^2+b^2}{2u}\right) I_\alpha\left(\frac{ab}{u}\right)\frac{du}{u}, \quad 0 < a < b. \tag{4}$$

From (3) and (4), we get for $\alpha > 0$,

$$\int_0^\infty e^{-\alpha^2 t/2} q(t,\xi,\eta)dt = \int_0^\infty e^{-\alpha^2 t/2} dt \times \int_0^\infty \exp\left(-\frac{u}{2} - \frac{\lambda^2(e^{2\xi}+e^{2\eta})}{2u}\right)\theta\left(\lambda^2 e^{\xi+\eta}/u, t\right)\frac{du}{u}.$$

From the uniquness of Laplace transform, we get

$$q(t,\xi,\eta) = \int_0^\infty \exp\left(-\frac{\lambda^2 u}{2} - \frac{e^{2\xi}+e^{2\eta}}{2u}\right)\theta\left(e^{\xi+\eta}/u, t\right)\frac{du}{u}.$$

On the other hand, the fundamental solution $q(t, \xi, \eta)$ is written, in a probabilistic way, as

$$q(t,\xi,\eta) = E[e^{-\lambda^2 e^{2\xi} A_t/2} | B_t = \eta - \xi] \frac{1}{\sqrt{2\pi t}} e^{-(\eta-\xi)^2/2t}$$
$$= \int_0^\infty e^{-\lambda^2 e^{2\xi} u/2} P(A_t \in du | B_t = \eta - \xi) \frac{1}{\sqrt{2\pi t}} e^{-(\eta-\xi)^2/2t}.$$

Letting $\xi = 0$ and applying the uniquness of Laplace transform again, we get

$$P(A_t \in du | B_t = x) \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} = \exp\left(-\frac{1+e^{2x}}{2u}\right) \theta(e^x/u, t) \frac{du}{u},$$

which is equivalent to (1) in the case $\mu = 0$. The general formula is easily shown by combining the result in the case $\mu = 0$ with the Cameron-Martin theorem. □

For later use, we denote the integral of (1) with respect to x by $g_Y^{(\mu)}(a,t)da$:

$$g_Y^{(\mu)}(a,t) = P(A_t^{(\mu)} \in da)/da$$
$$= \int_{-\infty}^{+\infty} e^{\mu x - \mu^2 t/2} \frac{1}{a} \exp\left(-\frac{1+e^{2x}}{2a}\right) \theta(e^x/a, t) dx. \quad (5)$$

2.2. *Dufresne's Results*

Next we present the expression $f_D^{(\mu)}(a,t)$ for the density of $1/2A_t^{(\mu)}$ due to Dufresne (2001). In particular, $f_D^{(0)}$ and $f_D^{(1)}$ are quite simple:

$$f_D^{(0)}(a,t) = C_t \frac{1}{\sqrt{a}} \int_0^\infty \exp[-a(\cosh(\eta))^2] \mathrm{e}^{-\eta^2/2t} \cosh(\eta) \cos(\frac{\pi\eta}{2t}) d\eta, \quad (6)$$

$$f_D^{(1)}(a,t) = e^{-t/2} C_t \frac{1}{\sqrt{a}}$$
$$\times \int_0^\infty \exp[-a(\cosh(\eta))^2] e^{-\eta^2/2t} \cosh(\eta) \sinh(\eta) \cos(\frac{\pi\eta}{2t}) d\eta, \quad (7)$$

where $C_t = \sqrt{2}(\pi\sqrt{t})^{-1} e^{\pi^2/8t}$. After showing some interesting recursion relations with respect to the drift μ (see also Matsumoto and Yor, 2003), Dufresne has given the following expressions for a general μ: if $\mu \neq -1, -3, \cdots$,

$$f_D^{(\mu)}(a,t) = \frac{C_t}{\sqrt{\pi}} e^{-\mu^2 t/2} \Gamma(\frac{\mu+1}{2}) a^{-(\mu+1)/2} \int_0^\infty \exp[-a(\cosh(\eta))^2] \mathrm{e}^{-\eta^2/2t}$$
$$\times \cosh(\eta) \cos(\frac{\pi\eta}{2t}) \, {}_1F_1(-\frac{\mu}{2}, \frac{1}{2}; a(\sinh(\eta))^2) d\eta,$$

if $\mu \neq -2, -4, \cdots$,

$$f_D^{(\mu)}(a,t) = \frac{2C_t}{\sqrt{\pi}} e^{-\mu^2 t/2} \Gamma(\frac{\mu+2}{2}) a^{-\mu/2} \int_0^\infty \exp[-a(\cosh(\eta))^2] e^{-\eta^2/2t}$$
$$\times \cosh(\eta) \sinh(\eta) \sin(\frac{\pi\eta}{2t}) \, {}_1F_1(\frac{1-\mu}{2}, \frac{3}{2}; a \, (\sinh(\eta))^2) d\eta,$$

where ${}_1F_1$ is the (Kummer) confluent hypergeometric function (see Lebedev, 1972).

3. Numerical Computations

In the numerical computations we employ the software package of NetNUMPAC (http://netnumpac.fuis.fukui-u.ac.jp/). We use a subroutine AQOSCD which is used to compute finite Fourier integrals.

3.1. The Function θ(r, t)

First, we draw the graphs of the function $t \mapsto \theta(r, t)$. For the computations of infinite integrals, we have the subroutine INFIND in NetNUMPAC. However, as is mentioned in its instruction (http://netnumpac.fuis.fukui-u.ac.jp/), it is inadequate for functions that oscillate violently near the origin, and therefore it is inappropriate for our purposes. Therefore we use AQOSCD.

We draw the graphs of the function $t \mapsto \theta(r, t)$ for various fixed values of r in Figure 1. The integral on the right hand side of (2) is an oscillatory integral and, in Figure 1, we see unstableness near $t = 0$. The detailed numerical computation near $t = 0$ when $r = 0.5$ is given in Figure 2, where we see some error. The probability density $\theta(r, t)$ looks negative for some small values of t.

By integrating both hand sides of (1) with respect to da, we get the density of $B_t^{(\mu)}$. We know that the density of $B_t^{(\mu)}$ is $(2\pi t)^{-1/2}\exp(-(x-\mu t)^2/2t)$, and we carry out numerical computations and draw the graphs of them. In Figure 3, we present the result in the case of $\mu = 0$. We obtain similar results in the cases μ or t takes other values, but we omit them.

3.2. Densities of $A_t^{(\mu)}$

Next, we draw the graphs of the probability densities $g_Y^{(\mu)}(a,t)$ of $A_t^{(\mu)}$ given by (5).

We have computed $g_Y^{(\mu)}(a,t)$ by trapezoidal rule, using the numerical computations for $\theta(r, t)$ presented in the previous subsection. In Figure 4, we draw the graphs of the densities of $A_t^{(0)}$ for various fixed values of t. Similarly, we draw the graphs of the densities of $A_t^{(1)}$ and $A_t^{(-1)}$ in Figures 5 and 6, respectively. When $\mu < 0$, $A_t^{(\mu)}$ converges in law as $t \to \infty$ to $1/2\gamma_{-\mu}$ (see Dufresne, 1990; Yor, 1992a). We put the graph of the

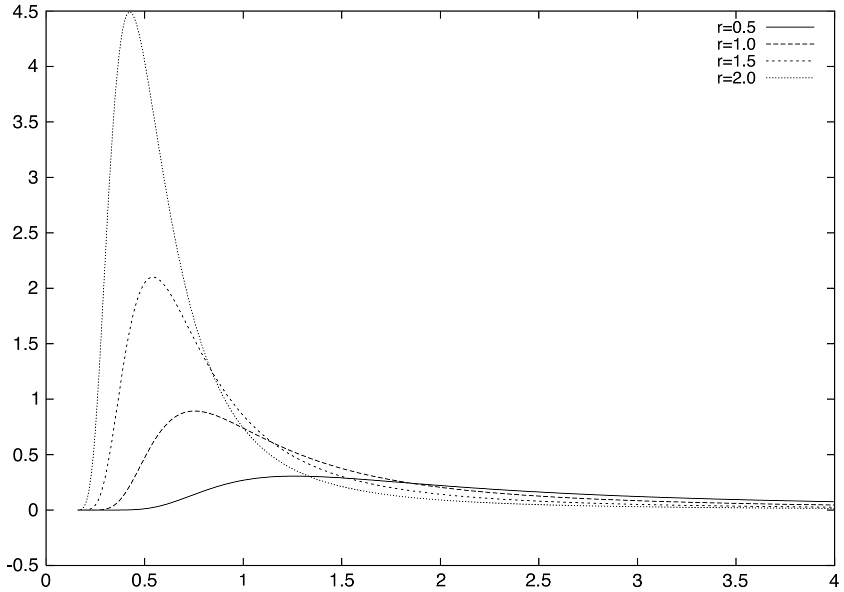

Figure 1. The function $t \mapsto \theta(r, t)$.

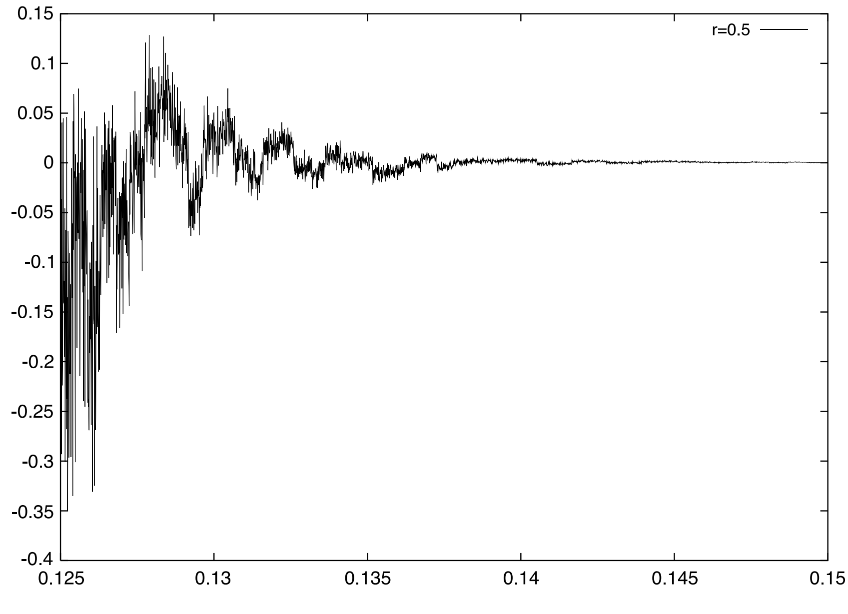

Figure 2. The function $t \mapsto \theta(0.5, t)$ for $t \leq 0.15$.

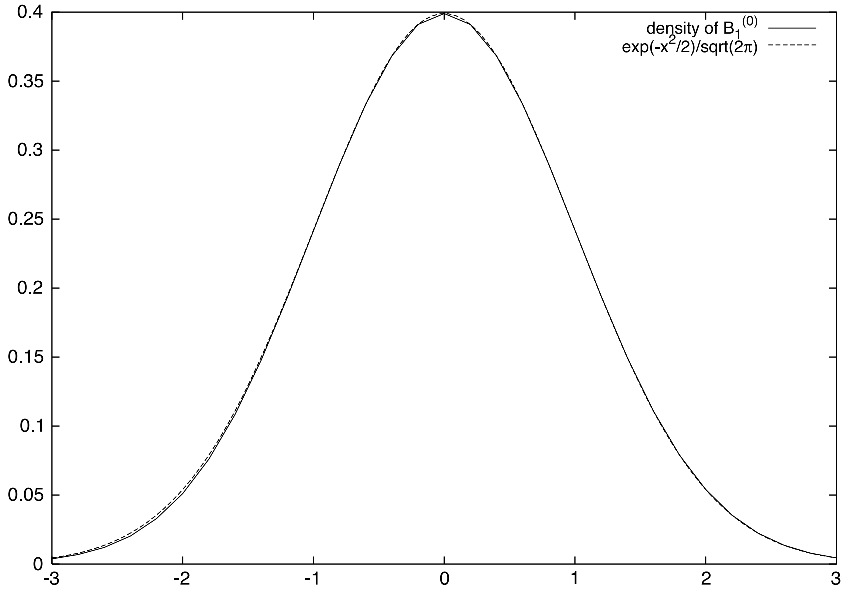

Figure 3. The density of $B_1^{(0)}$ by integration of Yor's expression and the density of $N(0,1)$.

density of $1/2\gamma_1$ together in Figure 6. We can see the convergence of $g_Y^{(-1)}(a,t)$ to the density of $1/2\gamma_1$ as $t \to \infty$. We also draw the graphs of densities of $A_1^{(\mu)}$ for various values of μ in Figure 7.

When $\mu = 0$ and $\mu = 1$, Dufresne (2001) has derived some other simple expressions (6) and (7). Formula (6) is an expression for the density $f_D^{(0)}(a, t)$ of $1/2A_t^{(0)}$. We draw the graphs of $(2a^2)^{-1} f_D^{(0)}(1/2a,1)$, the density of $A_1^{(0)}$, in Figure 8 and compare with the graphs of $g_Y^{(0)}(a, 1)$ presented in Figure 4. We also obtain a similar result in the case $\mu = 1$, but we omit it.

3.3. Change of Modes

In the graphs of the densities of $A_t^{(\mu)}$ given in Subsection 3.2, we find that the modes of the densities do not change fast as t or μ varies. When $\mu < 0$, it is easy to understand that the mode converges because $A_t^{(\mu)}$ converges in law to $1/2\gamma_{-\mu}$ as t tends to ∞.

We consider only the case $\mu \geq 0$ and denote the mode of the density of $A_t^{(\mu)}$ by $\alpha_{\mu,t}$:

$$g_Y^{(\mu)}(\alpha_{\mu,t}, t) = \max_a g_Y^{(\mu)}(a, t).$$

We draw the graphs of $t \mapsto \alpha_{\mu,t}$ in Figure 9 and the graphs of $t \mapsto \max_a g_Y^{(\mu)}(a, t)$ in Figure 10.

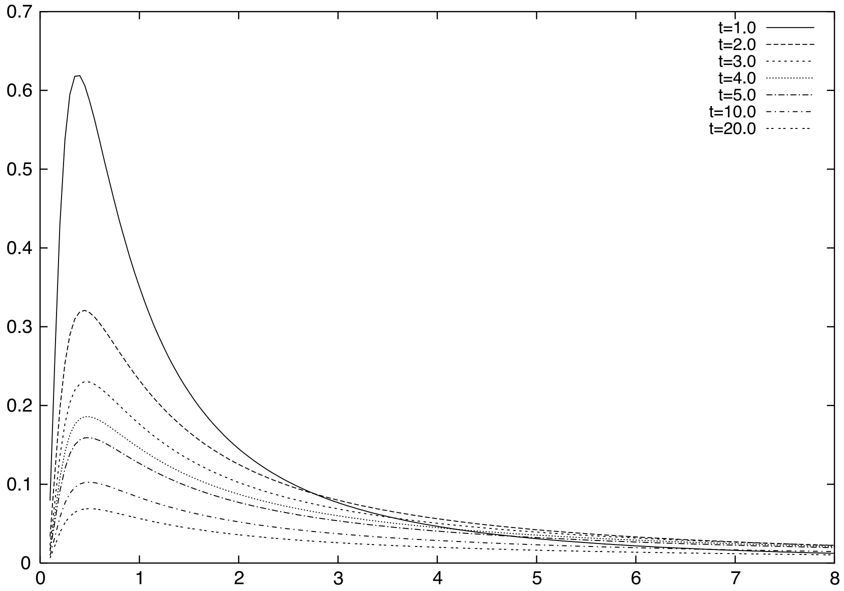

Figure 4. The densities of $A_t^{(0)}$ from Yor's expression.

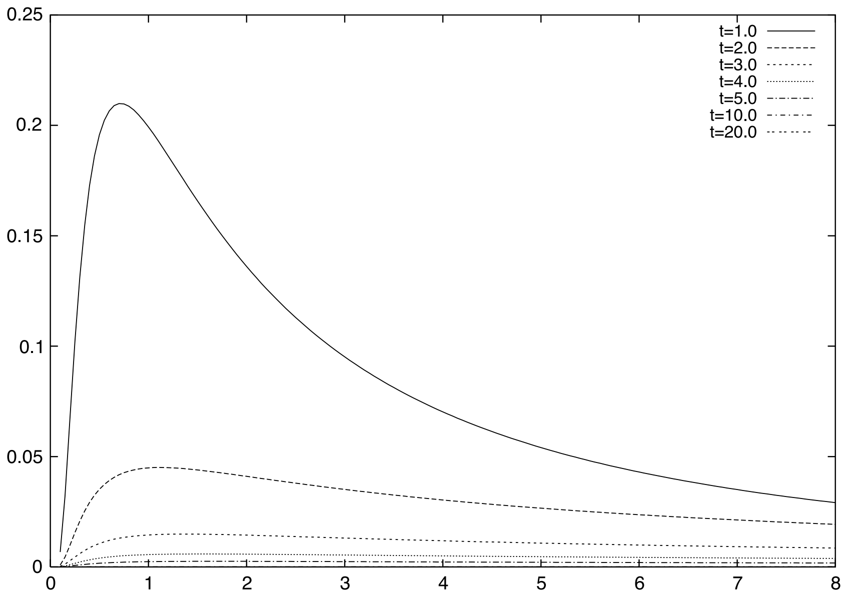

Figure 5. The densities of $A_t^{(1)}$ from Yor's expression.

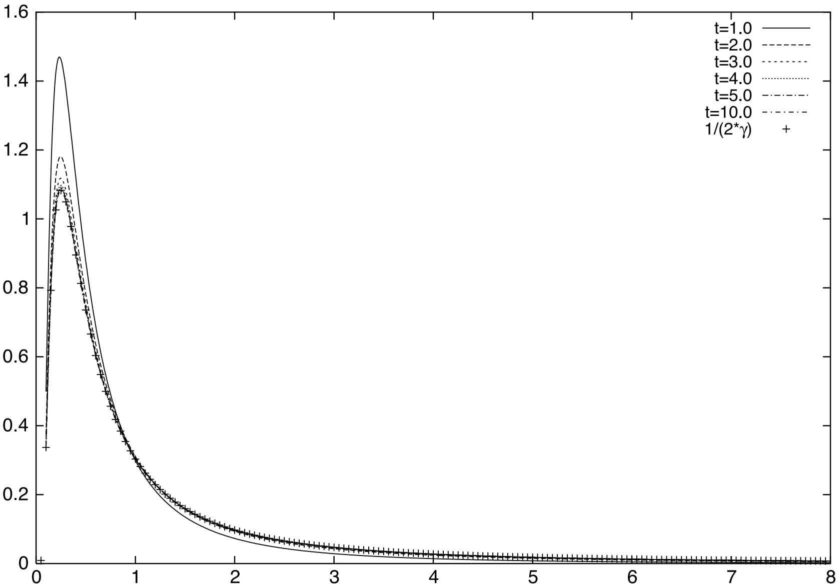

Figure 6. The densities of $A_t^{(-1)}$ from Yor's expression and the density of $(2\gamma_1)^{-1}$.

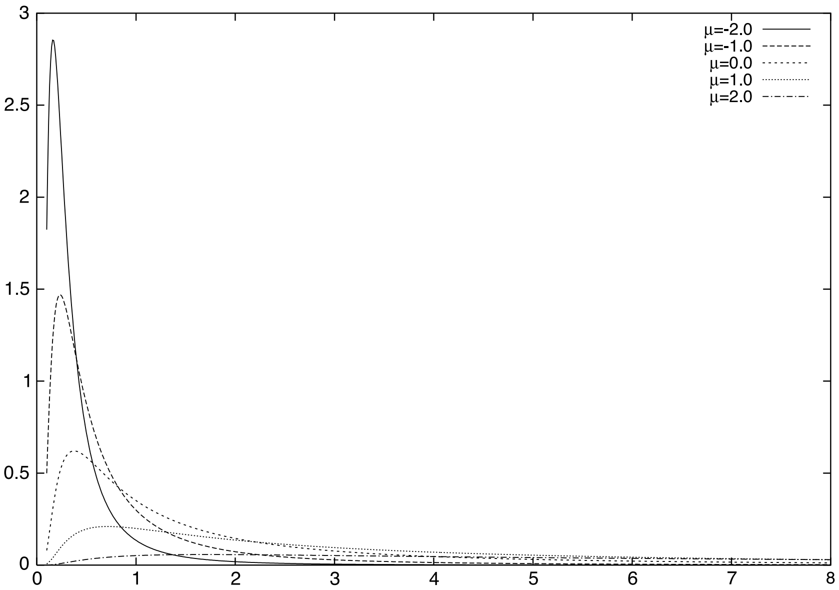

Figure 7. The densities of $A_1^{(\mu)}$ from Yor's expression.

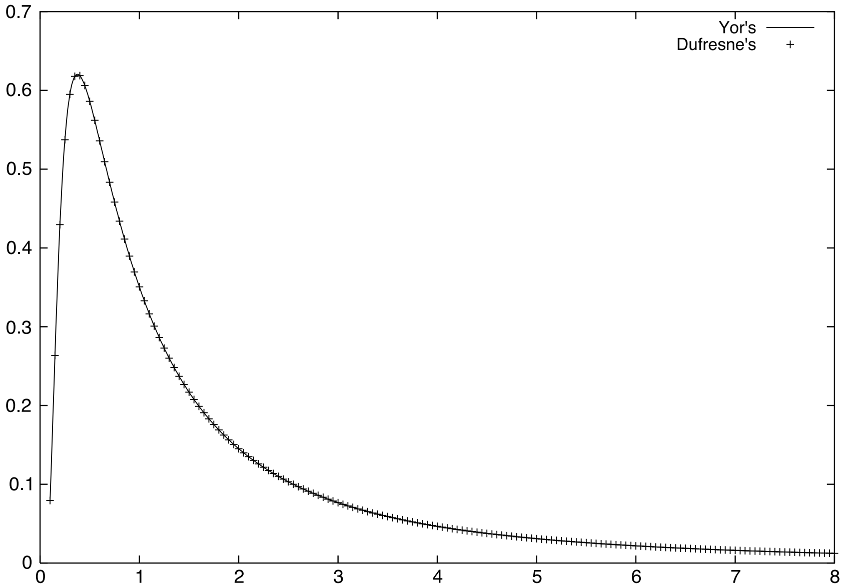

Figure 8. Comparison of Yor's expression of the density of $A_1^{(0)}$ with Dufresne's.

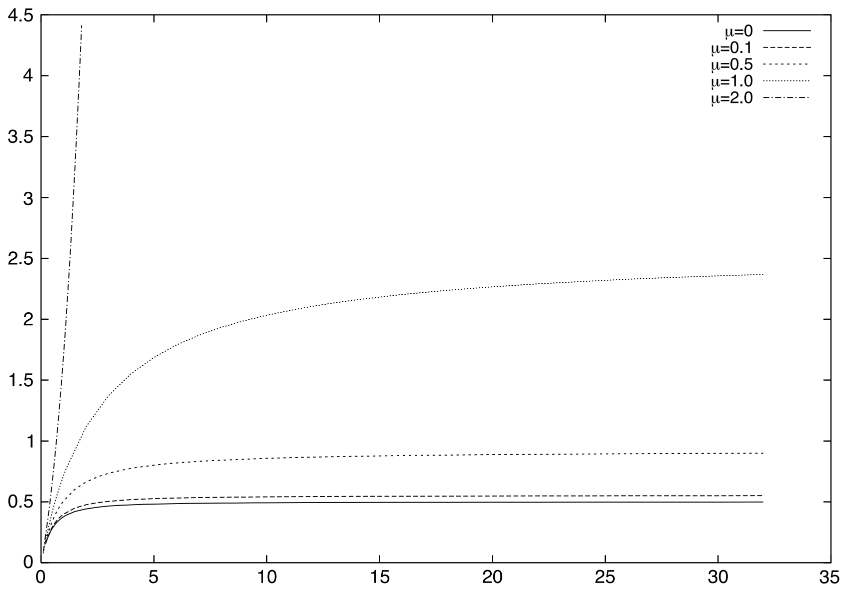

Figure 9. The mode of the density of $A_t^{(\mu)}$ (Graphs of $t \mapsto \alpha_{\mu,t}$).

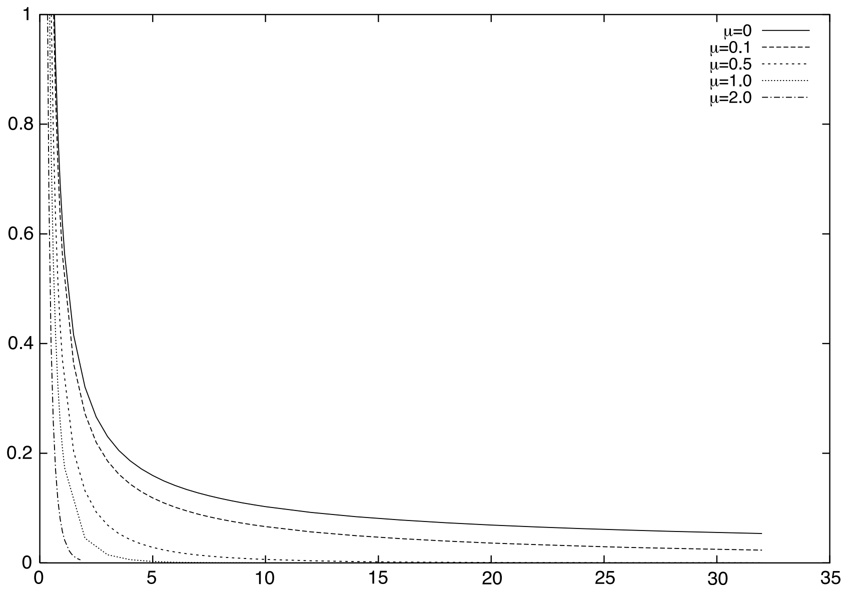

Figure 10. The maximum of the density of $A_t^{(\mu)}$ (Graphs of $t \mapsto \max_{a\in\mathbf{R}} g_Y^{(\mu)}(a,t)$).

4. Pricing of an Asian Option

In mathematical finance, the process $\{A_t^{(\mu)}, t \geq 0\}$ is related to an Asian option. In the Black-Scholes model, the theoretical price of an Asian option can be obtained from the density of $A_t^{(\mu)}$ for fixed t.

We consider a financial market following the Black-Scholes model with maturity T. One asset is a riskless asset whose price at time $t \in [0, T]$ is equal to $S_t^0 = e^{rt}$. Letting $\{S_t =$

Table 1. The price of Asian option where $K = 100$, $S_0 = 100$, $T = 1$.

Voratility σ	Interest rate r	Asian option price V_0
0.7	0.06	17.52
	0.10	18.01
	0.14	18.52
0.9	0.06	21.96
	0.10	22.33
	0.14	22.69
1.1	0.06	26.29
	0.10	26.53
	0.14	26.77

Table 2. The price of Asian option where $r = 0.10$, $S_0 = 100$, $T = 1$.

Voratility σ	Strike price K	Asian option price V_0
0.7	90	22.34
	100	18.02
	110	14.62
0.9	90	26.19
	100	22.33
	110	19.19
1.1	90	30.00
	100	26.53
	110	23.54

$S_0 \exp(\sigma B_t + (r - \sigma^2/2)t)$, $0 \le t \le T\}$, $\sigma > 0$, be a geometric Brownian motion which gives the stock price process, we consider the option whose payoff is equal to

$$h = \left(\frac{1}{T}\int_0^T S_t dt - K\right)_+,$$

where $K > 0$. We can show that the price of this Asian option is given by

$$V_t = e^{-r(T-t)} S_t F(t, \xi_t)$$

(see Rogers and Shi, 1995), where

$$F(t,\xi) = E\left[\left(\xi + \frac{1}{T}\int_t^T S_u/S_t du\right)_+\right]$$

and

$$\xi_t = \frac{1}{S_t}\left(\frac{1}{T}\int_0^t S_u du - K\right).$$

It is easy to see that

$$F(t,\xi) = \int_0^{+\infty}\left(\xi + \frac{4}{\sigma^2 T}a\right)_+ g_Y^{(\frac{2r}{\sigma^2}-1)}\left(a, \frac{\sigma^2}{4}(T-t)\right)da. \tag{8}$$

We compute the price of this Asian option by using (8) and our numerical computations for $g_Y^{(\mu)}$. We present the results in Tables 1 and 2.

5. Conclusions

For the densities of exponential Brownian functionals represented as integrals in time of geometric Brownian motions, several kinds of integral representations are known. We have carried out numerical computations for the representations due to Yor and Dufresne. Although oscillatory integrals appear, we have succeeded in drawing the graphs of the densities when time parameter is not small. By applying the numerical computations, we present a new method of computing the price of an Asian option.

Acknowledgments

A part of the results presented in this article are included in the author's master thesis, Graduate School of Human Informatics, Nagoya University. He would like to thank Professors Marc Yor and Hiroyuki Matsumoto for their interest in the topics of this paper and for valuable suggestions.

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