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An Extended Ant Colony Algorithm and Its Convergence Analysis

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Abstract. In this work, we propose a stochastic algorithm for solving $\mathcal{NP} - hard$ combinatorial optimization problems. The procedure is formulated within the Ant Colony Optimization (ACO) framework, and extends the so-called Graph-based Ant System with time-dependent evaporation factor, (GBAS/tdev) studied in Gutjahr (2002). In particular, we consider an ACO search procedure which also takes into account the objective function value. We provide a rigorous theoretical study on the convergence of the proposed algorithm. Further, for a toy example, we compare by simulation the rate of convergence of the proposed algorithm with those from the Random Search (RS) and from the corresponding procedure in Gutjahr (2002).

Keywords: ant colony, convergence analysis, simulation, stochastic optimization

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1. Introduction

Several stochastic algorithms have been proposed and successfully used for solving $\mathcal{NP} - hard$ combinatorial optimization problems. These include Simulated Annealing (SA), Evolutionary Algorithms (EA), among them Genetic algorithms (GA) and Evolution Strategies (ES), Tabu search (TS), and Ant Colony (AC) based algorithms; see, for example, Kirkpatrick et al. (1983), Geman and Geman (1984), and Hajek (1988) for SA, Schwefel (1995) for EA and ES, Goldberg (1989) for GA, Glover and Laguna (1997) for TS, and Dorigo et al. (1996) and Dorigo and Gambardella (1997) for AC. The EA, ES, GA and AC algorithms rely on biological mechanisms. In particular, AC algorithms mimic the behaviour of ant colonies. The ants of a colony succeed in finding the shortest path between hill and food via a suitable exchange of information among them. While walking, each ant deposits a chemical substance, called "pheromone.'' Of course, during a given time interval, the concentration of pheromone along each path is a decreasing function of the length. Since paths with high amount of pheromone are naturally followed by the ants, such trajectories are reinforced in the sense that the pheromone concentration on them is increased. On the other hand, since the pheromone evaporates according to a suitable law, the pheromone concentration on long paths decreases with time. These two mechanisms lead the ant colony to find the shortest way between hill and food.

The different proposed AC algorithms gave rise to the Ant Colony Optimization (ACO) meta-heuristic algorithms introduced by Dorigo et al. (1999). Although AC algorithms are broadly used in applications, only a few theoretical work on them has been done. Gutjahr (2000), (2002) and (2003) was the first to prove the convergence of some special classes of ACO algorithms. A relevant work is also that one of Rubinstein (2001), where the so-called "cross-entropy'' AC algorithm is proposed and studied theoretically.

In this paper, we propose a class of ACO algorithms which generalizes the so-called Graph-based Ant System with time-dependent evaporation factor (GBAS/tdev) studied in Gutjahr (2002). In particular, we consider an ACO procedure where the reinforcement mechanism is amplified by including some suitable factors involving the objective function value. In this work, we study in a rigorous way the convergence of the proposed algorithm, and we compare by simulation the proposed algorithm with existing methods. By the results in this paper, the set of converging GBAS/tdev algorithms in Gutjahr (2002) is enlarged. This is true also in the case when the amplification factors are suppressed.

This paper is organized as follows. In Section 2, we describe construction algorithms for arbitrary combinatorial optimization problems. In Section 3, we introduce the proposed algorithm. The theoretical results are given in Section 4, where also some examples are provided. In Section 5, we compare for a toy problem the proposed algorithm with existing methods by simulation. Finally, in Section 6, we discuss some analogies and differences among the proposed ACO algorithm, GBAS/tdev, SA and the "Random Search'' (RS); see, for instance, Papadimitriou and Steiglitz (1982) for RS. The references included in this paper are not exhaustive, indeed there is a huge literature on this subject.

2. Construction Algorithms for Combinatorial Optimization Problems

Let V be a finite set of positive integers. A combinatorial optimization problem consists of minimizing or equivalently maximizing, under some constraints, a given objective function H of $\mathbf{x} = (x_1,\ldots,x_n) \in V^n$. In this paper, we consider the minimization problem. Let us denote by $S = \{\mathbf{x}_1,\ldots,\mathbf{x}_N\} \subseteq V^n$ the set of points satisfying the constraints ("feasible solutions''). Construction algorithms provide feasible solutions for the combinatorial optimization problem under study in the following iterative way. During iteration t, a vector $\mathbf{z}(t) = (z_1(t),\ldots,z_n(t)) \in V^n$ is built in n steps by adding one component at time; see Figure 1. This is done in such a way that the vector $\mathbf{z}_j(t) = (z_1(t),\ldots,z_j(t))$ $(1 \le j \le n)$ is compatible with the constraints, i.e. there exists a vector $\bar{\mathbf{z}}_j(t) \in V^{n-j}$ such that $\left(\mathbf{z}_j(t), \bar{\mathbf{z}}_j(t)\right) \in S$. It is worthwhile to notice that alternative construction algorithms allow $\mathbf{z}(t)$ to be not feasible, penalizing it depending on its degree of "infeasibility.'' The iterative procedure described above can also be represented by a "walk'' of length $n - 1$ on the graph $\mathcal{G} = (V, E)$, where E is the set of the $|V|^2$ "edges'' of ordered pairs of elements of V (vertices); see Figure 1. Other graphs may be considered; see, for instance, Gutjahr (2000) and (2002). We notice that the technique described above can be applied

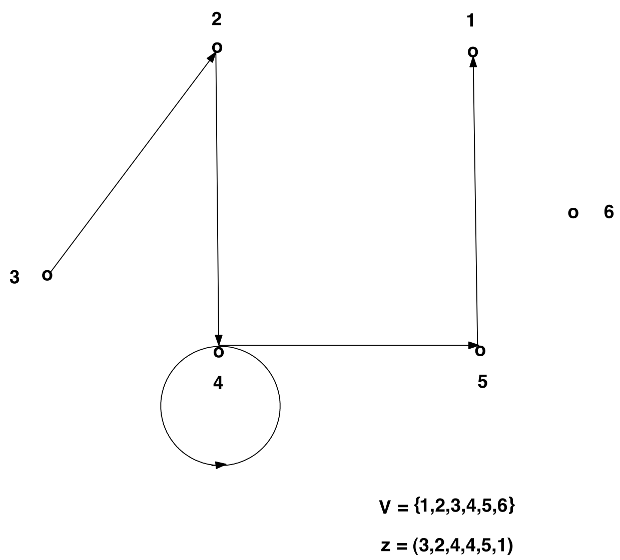

Figure 1. Construction of a feasible solution. A feasible solution $\mathbf{z}$ = (3,2,4,4,5,1) is built by adding one element of V = {1,2,3,4,5,6} at each time step. The first component is 3, and the others are added following directed path.

in practice when an efficient procedure for deciding wether $\mathbf{z}_j(t)$ is compatible with the constraints exists. Once a feasible solution is built, the objective function is evaluated on it. Then, the next iteration is started.

Finally, to clarify notations, we briefly describe the Traveling Salesperson Problem (TSP); see, for instance, Papadimitriou and Steiglitz (1982). A salesperson aims to minimize the length of a tour starting from one of n cities and visiting all the others exactly once before coming back. Let us denote by $x_j \in V = \{1,\ldots,n\}$ the index of the j-th city visited along a tour, which is identified by the vector $\mathbf{x} = (x_1,\ldots,x_n) \in V^n$. By the constraints, all the components of $\mathbf{x}$ must be different from each other. Therefore, the set S of all feasible solutions is composed by the $N = n!$ permutations of V. The component $\mathbf{z}_j(t)$ is chosen in the set $V \setminus \{z_1(t),\ldots,z_{j-1}(t)$.

The objective function for the TSP is given by $H(\mathbf{x}) = \sum_{i=1}^{n-1} d_{x_i, x_{i+1}} + d_{x_n, x_1}$, where the terms $d_{i,j}$ are the mutual distances between two different cities.

3. The Algorithm

In this Section, we describe the proposed class of ACO algorithms, which is obtained by generalizing the GBAS/tdev studied theoretically in Gutjahr (2002).

In ACO algorithms, some initial points in S are drawn according to an assigned distribution. Each point is then updated independently of the others componentwise by means of the same random mechanism, and can be interpreted as the trajectory of an ant of a colony. This random mechanism depends on the so-called "pheromone values" which, in some sense, describe the *common memory* of the ant colony. Such pheromone values are also updated by an iterative rule. To formalize these ideas, let us introduce some notations. Let A be a chosen positive integer identifying the number of ants and let $\mathbf{x}^i(t) = (x_1^i(t),\ldots,x_n^i(t)) \in S$ $(1 \leq i \leq A)$ denote the trajectory of the i-th ant at iteration t. The feasible solution $\mathbf{x}^i(t)$ is obtained by adding one component at time by means of a probabilistic rule based on the "pheromone values.'' At iteration t, for any p, $q \in V$, the pheromone value $\tau_{p,q}^t$ quantifies the belief from common memory regarding the chances that $x_{j+1}^i(t) = q$ if $x_j^i(t) = p$ $(1 \leq i \leq A,\ 1 < j \leq n - 1)$. Analogously, the law $\phi^t(\bullet)$ refers to the chances for the values of $x_1^i(t)$. As it will be described later, the quantities $\tau_{p,q}^t$ and $\phi^t(\bullet)$ will be initialized and then updated after each iteration.

More precisely, the components of $\mathbf{x}^i(t)$ are updated according to the following rule. Chose $\mathbf{x}^i(0)$ according to an assigned initial distribution. At iteration $t \geq 1$:

Step 1. Draw $x_1^i(t)$ from the distribution $\phi^t(\bullet)$, set $j = 1$.

Step 2. For $2 \leq j \leq n$, Propose $x_j^i(t) \in V$ with probability $\tau^t_{x^i_{j-1}(t),\, x^i_j(t)} \Big/ \sum_{l\in V} \tau^t_{x^i_{j-1}(t),\, l}$.

Step 3. Accept $x_j^i(t)$ if the vector $\left(x_1^i(t), \ldots, x_{j-1}^i(t), x_j^i(t)\right)$ is compatible with the constraints. (1)

Step 4. Go to step 2 and, *if* $x_j^i(t)$ is accepted, set $j = j + 1$.

We notice that, when the set of values y such that $(x_1^i(t),\ldots,x_{j-1}^i(t), y)$ is compatible with the constraints can be easily determined, more efficient procedures can be developed. This is the case for the TSP, where the steps 2 and 3 are equivalent to sample $x_j^i(t)$ in the set $V' = V\backslash\{x_1^i(t),\ldots,x_{j-1}^i(t)\}$ with probability $\tau^t_{x^i_{j-1}(t),\, x^i_j(t)} / \sum_{l\in V'} \tau^t_{x^i_{j-1}(t), l}$.

The updating rule for the pheromone is performed by reinforcing its value on feasible solutions recognized as "good'' and by the evaporation principle, which consists in a global reduction of the pheromone value. To quantify these two mechanisms, let us define the best feasible solution $\hat{\mathbf{x}}(t)$ visited by the colony up to time t. At iteration t, after performing steps 1–4 for all ant, we set $\hat{\mathbf{x}}(t) = \mathbf{x}^i(t)$ if $H(\mathbf{x}^i(t)) < H(\hat{\mathbf{x}}(t-1))$ and $H(\mathbf{x}^i(t)) \leq H(\mathbf{x}^j(t))$ $(1 \leq j \leq A)$ for some ant i (if this happens for more than one ant, we choose the ant with the smallest index). Otherwise, we keep $\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}(t-1)$. We then set for any $t \geq 1$ and p, $q \in V$:

$$\begin{aligned} \tau_{p,q}^{t+1} &= (1 - \rho_t a_t)\tau_{p,q}^t + \rho_t a_t e(p, q, \hat{\mathbf{x}}(t)) \\ \phi^{t+1}(p) &= (1 - \rho_t a_t)\phi^t(p) + \rho_t a_t \mathbf{1}\{p = \hat{x}_1(t)\}. \end{aligned} \tag{2}$$

Here $e(p, q, \hat{\mathbf{x}}(t)) = 1/L(\hat{\mathbf{x}}(t))$, if (p,q) is one of the $L(\hat{\mathbf{x}}(t))$ distinct pairs of adjacent components of $\hat{\mathbf{x}}(t)$. Otherwise, we define $e(p, q, \hat{\mathbf{x}}(t)) = 0$. It is worthwhile to notice that for the TSP $L(\hat{\mathbf{x}}(t)) = n$. The sequence $\{\rho_t\}_{t \geq 1}$ of the evaporation factors is fixed in advance and it is contained in the interval $[0, 1)$. We also remark that, if $\phi^1(\bullet)$ is a probability distribution, then so is $\phi^t(\bullet)$ for any t. In the pheromone updating rule of Equation (2) we have introduced the factors a_t, which are not included in the GBAS/tdev studied in Gutjahr (2002). These factors are defined by $a_t = f(H^*_{Max} - H(\hat{\mathbf{x}}(t)))$, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that $f(0) = 1$, and $f(\Delta H^*_{Max}) \sup\{\rho_t : t \geq 1\} < 1$. Here, H^*_{Max} denotes a suitable upper bound on the maximum of H. Similarly, ΔH^*_{Max} is an upper bound on the difference between the maximal and minimal value of H. We also notice that the above conditions ensure that $\rho_t\, a_t < 1$ for all t. By definition, the a_t's depend on $H(\hat{\mathbf{x}}(t))$ and therefore are not a priori fixed. Moreover, since $a_t \geq 1$, each term $\rho_t a_t / L(\hat{\mathbf{x}}(t))$, which quantifies the reinforcement, is larger than or equal to $\rho_t / L(\hat{\mathbf{x}}(t))$ which describes the reinforcement in the GBAS/tdev algorithm. For this reason, the a_t's can be interpreted as amplification factors for the reinforcement.

As it happens in GBAS/tdev, we assume $\tau^1_{p,q} > 0$, $\phi^1(p) > 0$ for all p, $q \in V$, and $\sum_{p,q \in V} \tau^1_{p,q} = 1$. This implies, for all t, $\tau^t_{p,q} > 0$, $\phi^t(p) > 0$, and

$$\sum_{p,q \in V} \tau^t_{p,q} = 1. \tag{3}$$

It is worthwhile to notice that the so-called "visibility" constants can be taken into account by replacing the probability in Equation (1) with

$$\left[\tau^t_{x^i_{j-1}(t), x^i_j(t)}\right]^\alpha \left[\nu_{x^i_{j-1}(t), x^i_j(t)}\right]^\beta \Bigg/ \sum_{l \in V} \left[\tau^t_{x^i_{j-1}(t), l}\right]^\alpha \left[\nu_{x^i_{j-1}(t), l}\right]^\beta;$$

see for instance, Dorigo et al. (1999), and Gutjahr (2000) and (2003). Here, $\nu_{p,q}$, $p,q \in V$ are the "visibility" constants, and α, $\beta \geq 0$. The constants $\nu_{p,q}$ can be choosen heuristically and are related to the objective function (e.g. the mutual distances in the TSP). In this paper, we do not consider "visibility" for the sake of simplicity. However, the proofs in this work can be easily adapted to the case when "visibility" appears in Equation (1).

4. Results

In this Section, we give sufficient conditions for the convergence of the proposed ACO algorithm to an optimal solution, and we provide some examples.

4.1. Convergence

Letting $\mathcal{O}$ denote the set of the m optimal solutions, the following Theorem holds.

Theorem 1 *Let us assume*

(i) $\prod_{t \geq 1}(1 - \rho_t f(\Delta H^*_{Max})) = C \in (0, 1]$
or
(ii) *there exists a sequence* $\{b_t\}_{t \geq 1} \subset (0, \infty)$ *such that* $\sum_{t \geq 1} b_t^{-n} = +\infty$ *and* $1 - \rho_t f\left(\Delta H^*_{Max}\right) \geq \frac{b_t}{b_{t+1}}$ *for all t large enough.*

Then, we have

$$\lim_{t\to+\infty} P\big(\hat{\mathbf{X}}(t) \in \mathcal{O}\big) = 1.$$

This result ensures that the probability that the ant colony enters the set $\mathcal{O}$ of optimal solutions tends to one as the number of iterations goes to infinity.

We notice that, after the colony has hit an optimal solution $\tilde{\mathbf{o}} \in \mathcal{O}, \hat{\mathbf{x}}(t)$ becomes constant, while the $\mathbf{x}^i(t)$'s may vary. Furthermore, under suitable conditions, the pheromone values converge as stated by the following Theorem.

THEOREM 2 *Under conditions* (*ii*) *of Theorem* 1, *if moreover* $\sum_{t\geq 1} \rho_t = +\infty$, *then*

$$P\left(\bigcup_{\mathbf{o}\in\mathcal{O}}\bigcap_{p,q}\left\{T^t_{p,q} \to e(p,q,\mathbf{o})\right\}\right) = 1 \text{ and } P\left(\bigcup_{\mathbf{o}\in\mathcal{O}}\bigcap_{p}\{\mathbf{\Phi}^t(p) \to \mathbf{1}\{p = o_1\}\}\right) = 1,$$

where $T^t_{p,q}$ *and* $\mathbf{\Phi}^t(p)$ *are random variables whose values are* $\tau^t_{p,q}$ *and* $\phi^t(p)$, *respectively*.

Theorem 2 provides further insight about the convergence analysis of the proposed ACO algorithm. Indeed, from a practical point of view, we can be confident to have found an optimal solution when $\hat{\mathbf{x}}(t)$ remains constant in t and the pheromone values $\tau^t_{p,q}$ approach more and more the values $e(p,q,\hat{\mathbf{x}}(t))$, for all $p,q \in V$.

Proof of Theorem 1: By the definition of $\hat{\mathbf{X}}(t)$ and the following inclusion between events: $\{\hat{\mathbf{X}}(t) \in \mathcal{O}\} \subset \{\hat{\mathbf{X}}(t+1) \in \mathcal{O}\}$, we have

$$P\left(\bigcup_{t\geq 1}\{\hat{\mathbf{X}}(t) \in \mathcal{O}\}\right) = \lim_{t\to+\infty} P\big(\hat{\mathbf{X}}(t) \in \mathcal{O}\big) = \lim_{t\to+\infty} P\left(\bigcup_{1\leq i\leq A}\ \bigcup_{1\leq s\leq t}\{\mathbf{X}^i(s) \in \mathcal{O}\}\right).$$

Therefore, the conclusion follows if we show that the probability

$$p(t) = P\left(\bigcap_{1\leq i\leq A}\ \bigcap_{1\leq s\leq t}\{\mathbf{X}^i(s) \notin \mathcal{O}\}\right)$$

goes to zero as t goes to infinity. For this, we start noticing that, by Lemma 1, proved in the Appendix, for any fixed $\mathbf{o}^* \in \mathcal{O}$, the following inequality holds

$$p(t) \leq \exp\left\{-\eta_n(\mathbf{o}^*)\sum_{j=2}^{t}\left(\prod_{s=1}^{j-1}\left(1 - \rho_s f(\Delta H^*_{Max})\right)\right)^n\right\}, \tag{4}$$

where $\eta_n(\mathbf{o}^*) = \phi^1(o^*_1)\prod_{j=2}^n \tau^1_{o^*_{j-1},o^*_j}$.

Now, we first show that, under assumption (i), the r.h.s. of Equation (4) goes to zero as t tends to infinity. We notice that, for all $j \geq 2$, we have

$$\prod_{t=1}^{j-1}\big(1 - \rho_t f\big(\Delta H^*_{Max}\big)\big) \geq \prod_{t\geq 1}\big(1 - \rho_t f\big(\Delta H^*_{Max}\big)\big) = C \in (0,1].$$

Therefore,

$$p(t) \leq \exp[-\eta_n(\mathbf{o}^*)(t-1)C^n]. \tag{5}$$

Since $\eta_n(\mathbf{o}^*) > 0$, the conclusion follows by letting t tend to infinity.

Finally, we prove that, under assumption (ii), the r.h.s. of Equation (4) goes to zero as t tends to infinity. For this, we notice that, for some t_0 and and all $j > t_0$, it holds

$$\prod_{t=1}^{j-1}\left(1-\rho_t f\left(\Delta H^*_{Max}\right)\right) \geq \prod_{t=1}^{t_0-1}\left(1-\rho_t f\left(\Delta H^*_{Max}\right)\right)\prod_{t=t_0}^{j-1}\frac{b_t}{b_{t+1}} = \frac{K}{b_j}, \tag{5}$$

where $K = b_{t_0}\prod_{t=1}^{t_0-1}\left(1-\rho_t f\left(\Delta H^*_{Max}\right)\right)$. Therefore,

$$\begin{aligned} p(t) &\leq \exp\left\{-\eta_n(\mathbf{o}^*)\sum_{j=t_0+1}^{t}\left(\prod_{s=1}^{j-1}\left(1-\rho_s f\left(\Delta H^*_{Max}\right)\right)\right)^n\right\} \\ &\leq \exp\left[-\eta_n(\mathbf{o}^*)K^n\sum_{j=t_0+1}^{t} b_j^{-n}\right]. \end{aligned}$$

The conclusion follows by letting t tend to infinity. ■

REMARK 1 We point out that, assuming (i) or (ii) of Theorem 1, it follows $\inf_{t\geq 1}\rho_t = 0$. Indeed, reasoning by contradiction if $\inf_{t\geq 1}\rho_t = \gamma > 0$, we have $\prod_{s=1}^{t}\left(1-\rho_s f\left(\Delta H^*_{Max}\right)\right) < \left(1-\gamma f\left(\Delta H^*_{Max}\right)\right)^t$ for all t, which is impossible under condition (i). On the other hand, when (ii) holds, we have $\left(\frac{b_t}{b_{t+1}}\right)^n \leq \left(1-\rho_t f\left(\Delta H^*_{Max}\right)\right)^n \leq \left(1-\gamma f\left(\Delta H^*_{Max}\right)\right)^n < 1$, which implies the convergence of $\sum_{t\geq 1} b_t^{-n}$.

Proof of Theorem 2: By the definition of the updating rule for the best feasible solution $\hat{\mathbf{X}}(t)$ up to time t, it follows

$$\Omega \equiv \bigcup_{k\geq 1}\bigcup_{\mathbf{o}\in\mathcal{O}}\bigcap_{t\geq k}\left\{\hat{\mathbf{X}}(t) = \mathbf{o}\right\} = \bigcup_{t\geq 1}\left\{\hat{\mathbf{X}}(t)\in\mathcal{O}\right\}.$$

Therefore, by Theorem 1, $P(\Omega) = 1$. Hence, the conclusion follows by proving that

$$\Omega \subseteq \bigcup_{\mathbf{o}\in\mathcal{O}}\bigcap_{p,q}\left\{T^t_{p,q}\to e(p,q,\mathbf{o})\right\} \ and \ \Omega \subseteq \bigcup_{\mathbf{o}\in\mathcal{O}}\bigcap_{p}\{\mathbf{\Phi}^t(p)\to \mathbf{1}\{p=o_1\}\}.$$

We start showing the first inclusion. Let $\omega\in\Omega$ be arbitarily fixed, and let $\tilde{\mathbf{o}}\in\mathcal{O}$ be such that $\hat{\mathbf{X}}(t,\omega) = \tilde{\mathbf{o}}$ for all t large enough. We observe that, for all $p, q, \in V, \tilde{t}\geq 1$, and $u\geq 2$, we have

$$\begin{aligned} \tau^{\tilde{t}+u}_{p,q} &= (1-\rho_{\tilde{t}+u-1}a_{\tilde{t}+u-1})\tau^{\tilde{t}+u-1}_{p,q} + \rho_{\tilde{t}+u-1}a_{\tilde{t}+u-1}e(p,q,\tilde{\mathbf{o}}) \\ &= \tau^{\tilde{t}}_{p,q}\prod_{t=\tilde{t}}^{\tilde{t}+u-1}(1-\rho_t a_t) + e(p,q,\tilde{\mathbf{o}})\prod_{t=\tilde{t}}^{\tilde{t}+u-1}\rho_t a_t\prod_{s=t+1}^{\tilde{t}+u-1}(1-\rho_s a_s), \end{aligned} \tag{6}$$

where the latter identity can be easily checked by induction. Summing up on all $p,q \in V$, by Equation (3), we have

$$1 = \sum_{p,q \in V} \tau_{p,q}^{\tilde{t}+u} = \prod_{t=\tilde{t}}^{\tilde{t}+u-1} (1 - \rho_t\, a_t) + \sum_{t=\tilde{t}}^{\tilde{t}+u-1} \rho_t\, a_t \prod_{s=t+1}^{\tilde{t}+u-1} (1 - \rho_s\, a_s). \tag{7}$$

Since it holds

$$\prod_{t=\tilde{t}}^{\tilde{t}+u-1} (1 - \rho_t\, a_t) \leq \exp\left(- \sum_{t=\tilde{t}}^{\tilde{t}+u-1} \rho_t \right), \tag{8}$$

by the hypothesis, it follows

$$\lim_{u \to +\infty} \prod_{t=\tilde{t}}^{\tilde{t}+u-1} (1 - \rho_t\, a_t) = 0. \tag{9}$$

By the Equation (9), we have that the second addend in Equation (7) tends to one as $u \to \infty$. The thesis follows by letting u tend to infinity in Equation (6).

The proof of the second inclusion follows replacing τ by ϕ, and $e(\bullet)$ by $\mathbf{1}\{\bullet\}$. ■

REMARK 2 Conditions (i) or (ii) of Theorem 1 impose upper bounds on ρ_t. As it will be discussed in Section 6, the extreme case $\rho_t = 0$ corresponds to RS. In this case, convergence of $\hat{\mathbf{X}}(t)$ holds, but the pheromone values do not converge anymore. By adding to condition (ii) of Theorem 1 the lower bound type condition $\sum_{t \geq 1} \rho_t = +\infty$, the convergence property of Theorem 2 for the pheromone values holds. Thus, there is a trade-off: to achieve pheromone values convergence, ρ_t has to be decreased towards zero neither too "fast" nor too "slowly."

REMARK 3 We notice that the conclusion of Theorem 2 implies that $\prod_{t \geq 1}\left(1 - \rho_t f\left(\Delta H^*_{Max}\right)\right) = 0$, which is the opposite of condition (i) of Theorem 1. Indeed, by Equation (6) with (p,q) such that $e(p, q, \tilde{\mathbf{o}}) = 0$ and the conclusion of Theorem 2, we have $\prod_{t \geq 1} (1 - \rho_t a_t) = 0$

REMARK 4 We point out that by the updating rule of Equation (1) and Bayes' formula, for any $\mathbf{y} \in S$, we have

$$P\left(\mathbf{X}^i(t) = \mathbf{y} | \mathbf{X}(t-1)\right) \propto \Phi^t(y_1) \prod_{j=2}^{n} \frac{T^t_{y_{j-1}, y_j}}{\sum_{l \in V} T^t_{y_{j-1}, l}}, \tag{10}$$

where the proportionality factor is no smaller than one, and $\mathbf{X}(t)$ is the random vector obtained by stacking $\mathbf{X}^i(s)$, $1 \leq i \leq A$, $1 \leq s \leq t$. In particular, for the TSP, since $e(p,q,\mathbf{x}) = 1/n$ if (p,q) is one of the n distinct pairs of adjacent components of $\mathbf{x}$ and zero otherwise, by Theorem 2 it follows that

$$P\left(\bigcup_{\mathbf{o} \in \mathcal{O}} \bigcap_{1 \leq i \leq A} \left\{ P\left(\mathbf{X}^i(t) = \mathbf{y} | \mathbf{X}(t-1)\right) \to \mathbf{1}\{\mathbf{y} = \mathbf{o}\} \right\} \right) = 1.$$

This means that for the TSP, as t goes to infinity, the conditional probability in Equation (10) approaches one if **y** is one optimal solution.

4.2. *Examples*

Finally, we give three examples of the proposed ACO algorithm for which conditions of Theorem 1 or Theorem 2 are met. We preliminary observe that the proposed ACO algorithm is completely determined if we specify the sequence of evaporation factors, the values of $\tau^1_{p,q}$ and $\phi^1(p)$, and the initial distribution of $\mathbf{X}(0)$. However, the assumptions of the Theorems only involve the evaporation factors.

EXAMPLE 1 The infinite product in condition (i) of Theorem 1 can be rewritten as the exponential of $\sum_{t\geq 1}\ln\left(1-\rho_t f\left(\Delta H^*_{max}\right)\right)$. Therefore, to satisfy assumption (i) of Theorem 1, it suffices to model the terms $\ln(1-\rho_t f(\Delta H^*_{max}))$ in such a way that the above series converges. To this aim, setting $\ln\left(1-\rho_t f\left(\Delta H^*_{Max}\right)\right)=-t^{-\alpha}$, with $\alpha>1$, we have

$$\prod_{t\geq 1}\left(1-\rho_t f\left(\Delta H^*_{Max}\right)\right)=\prod_{t\geq 1}e^{-t^{-\alpha}}=e^{-\sum\limits_{t\geq 1}t^{-\alpha}}=C.$$

Moreover, it holds $f(\Delta H^*_{max})\ \sup\{\rho_t : t\geq 1) = f(\Delta H^*_{max})\rho_1 = (1-e^{-1}) < 1$.

EXAMPLE 2 We first consider condition (ii) of Theorem 1, where ρ_t and b_t appear. We model b_t in such a way that it is increasing and $\sum_{t\geq 1} b_t^{-n}=+\infty$. Moreover, we define $\rho_t = a(1 - b_t/b_{t+1})$ with $a < 1$. Therefore, the inequality in assumption (ii) of Theorem 1 is satisfied assuming $f(\Delta H^*_{max})\,a<1$. This inequality ensures also that $f(\Delta H^*_{max})\sup\{\rho_t : t\geq 1\}<1$. The particular choice $b_t=t^{\beta/n}$, with $\beta\in(0,1)$, guarantees $\sum_{t\geq 1}\rho_t=+\infty$. Indeed,

$$\begin{aligned}\sum_{t\geq 1}\rho_t &= a\sum_{t\geq 1}(t+1)^{-\beta/n}\left[(t+1)^{\beta/n}-t^{\beta/n}\right]\\ &= a\beta/n\sum_{t\geq 1}(t+1)^{-\beta/n}\xi_t^{\beta/n-1}\geq a\beta/n\sum_{t\geq 1}1/(t+1)=+\infty\end{aligned}$$

where $t\leq\xi_t\leq t+1$, and we applied the mean value theorem to the function $t^{\beta/n}$.

EXAMPLE 3 As in the latter example, we choose b_t increasing and such that $\sum_{t\geq 1} b_t^{-n}=+\infty$. In particular we set $b_t=\ln(t+1)$, thus we have $\sum_{t\geq 1}[\ln(t+1)]^{-n}=+\infty$ since $\lim_{t\to+\infty}(\ln t)^n t^{-1}=0$, by De l'Hospital's rule. Defining $\rho_t = a/((t+1)\ln(t+2))$, with $a<1/2$, it is easily seen that the inequality in assumption (ii) of Theorem 1 is satisfied when $f(\Delta H^*_{Max})\ a<1/2$. This inequality implies also that $f(\Delta H^*_{Max})$ $\sup\{\rho_t : t\geq 1)<1$. Finally, condition $\sum_{t\geq 1}\rho_t=+\infty$ in Theorem 2 holds, indeed $\sum_{t\geq 1}1\big/[(t+1)\ln(t+2)]\geq\int_1^{+\infty}\frac{dx}{(x+2)\ln(x+2)}=+\infty$.

5. Simulation Study

In this Section we study the performance of the proposed ACO algorithm, and we compare the results with those from existing procedures. As in Droste et al. (2002),

Table 1. Numerical simulation for the ACO algorithms in the toy example. The first column contains the different values of $n = |V|$ considered. In the second column are reported the values of the theoretical mean (tm) of W for RS. In the remaining columns, there are the values of the sample mean (sm) and the standard deviation of the mean (se) of W for the ACO algorithms.

	RS	GBAS/tdev		Proposed ACO	
n	tm	sm	se	sm	se
3	3	3	0	3	0
4	12	12	0	13	0
5	60	56	0	60	0
6	360	277	2	270	3
7	2,520	1,513	16	1,369	16
8	20,160	8,656	102	6,815	96
9	181,440	53,503	642	35,987	554
10	1,814,400	366,697	5,016	204,386	4,043

we consider the expected value of the waiting time $W = \min\{t : \hat{\mathbf{X}}(t) \in \mathcal{O}\}$ until an optimum is reached. For a toy problem we provide estimates of $E[W]$ by simulation for different values of n. Particularly, we study the proposed algorithm with the choice $\rho_t = \frac{1}{9(t+1)\ln(t+2)}$ (see example 3), $f(x) = e^{\alpha x}$, with $\alpha = \ln(9/2)/(2\Delta H^*_{Max})$, $\tau^1_{p,q} = 1/|V|^2, \phi^1(p) = 1/|V|$ and $\mathbf{X}(0)$ drawn according to the uniform law on S. Moreover, we focus on the case of a single ant ($A = 1$), which means that the algorithm is based only on the memory effect. We compare the performance of this ACO procedure with that one obtained by setting $\alpha = 0$, corresponding to a particular GBAS/tdev algorithm. This analysis is performed for the following "open" TSP with a specific distance matrix. We define $V = \{1, 2, \dots, n\}$ and we assume S to be the set of all permutations of V. The objective function is $H(\mathbf{x}) = \sum_{i=1}^{n-1} |x_i - x_{i+1}|$. In such a case the set of optimal solutions $\mathcal{O}$ is equal to $\{(1, 2, \dots, n), (n, n-1, \dots, 1)\}$. Furthermore, we choose $H^*_{Max} = \Delta H^*_{Max} = (n-1)^2$. The numerical results presented in this paper are obtained taking $n = 3, \dots, 10$. For the ACO algorithms of above, for each value of n, the sample mean (sm) and the standard deviation of the mean (se) of W have been computed performing 10,000 independent algorithm runs. We also compare the considered ACO algorithms with the RS, for which $E[W] = |S|/|\mathcal{O}| = n!/2$. These results are reported in Table 1. We notice that the ACO algorithm introduced in this paper provides a faster convergence than the GBAS/tdev. Indeed, as Table 1 and Figure 2 show, the proposed ACO algorithm gives a relevant gain in terms of number of iterations starting from $n = 7$.

6. Discussion

We start noticing that the proposed ACO and the GBAS/tdev algorithms are generalizations of RS. In fact, if we set $\rho_t = 0$ for all t in Equation (2), and we assume $\tau^1_{p,q} = 1/|V|^2$ and $\phi^1(p) = 1/|V|$, then the probability in Equation (1) reduces to $1/|V|$. This

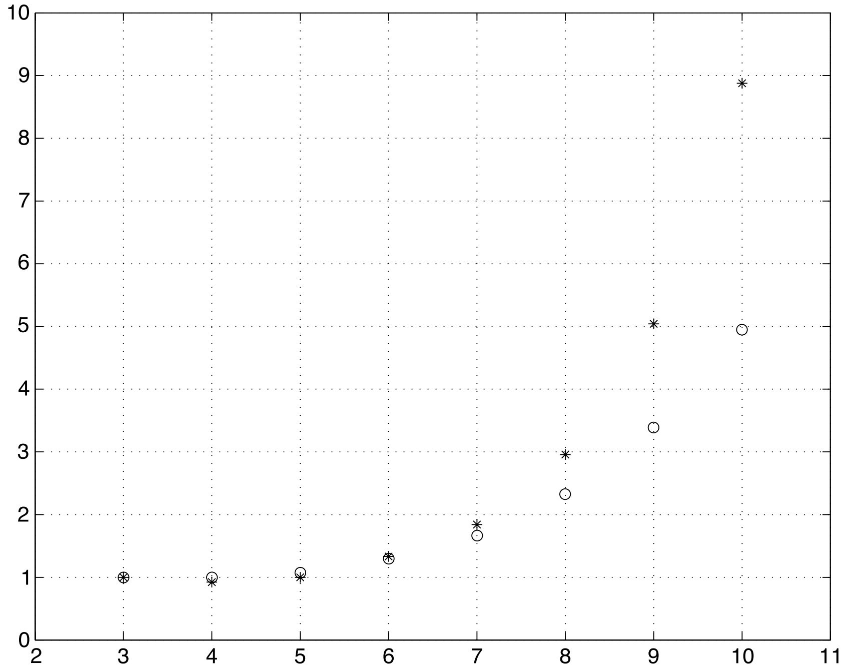

Figure 2. A comparison by simulation between RS and ACO algorithms for the toy example. The different values of $n = |V|$ are reported along the horizontal axis. The corresponding ratios between the sample mean of W for the RS and the ACO algorithms are represented on the vertical axis. These ratios quantify speedup factors. The symbols * and o refer to the proposed ACO and GBAS/tdev algorithms, respectively.

is the updating rule of RS, whose convergence can be proved trivially. When ρ_t is not identically equal to zero, the proposed ACO algorithm has higher probability to visit "neighbours" of the current best feasible solution, keeping reasonable chances of visiting all the other points of S. This is the main difference between RS and the proposed ACO algorithm, the same being true for the GBAS/tdev algorithm. We also point out that in RS, we simulate a sequence of i.i.d. random vectors. On the contrary, the proposed ACO algorithm with non vanishing evaporation factors allows non-Markovian dependency of the current state from the past, as it arises by Equations (1) and (2). However, arguing as in Lemma 3.1 of Gutjahr (2002), the process $\mathbf{M} = \{(\hat{\mathbf{X}}(t), \mathbf{T}^t)\}_{t\geq 1}$ is a non-homogeneous Markov chain, where $\mathbf{T}^t$ is the vector with components $T^t_{p,q}$. Therefore, to run the ACO algorithm we only need at each iteration the realization of the process $\mathbf{M}$ at the previous one.

Another stochastic algorithm, which is broadly applied to solve $\mathcal{NP}-$*hard* combinatorial optimization problems, is SA. Here, again the optimal solution is the limit of a stochastic process $\{\mathbf{Y}(t)\}_{t\geq 1}$ with values in S. However, three important facts have to be noticed. First, in SA $\{\mathbf{Y}(t)\}_{t\geq 1}$ is a non-homogeneous Markov chain. Secondly, the sequence of kernels which governs the one step transitions in SA is given and it is defined by means of a sequence of real numbers called "cooling schedule." In the proposed ACO algorithm the kernels are not fixed in advance, as it is clear by Equations (1) and (2). Finally, exploring S by a colony of ants is in the same spirit of the multi-start version of SA, where independent replications of $\{\mathbf{Y}(t)\}_{t\geq 1}$ are simulated, starting from different initial values. However, the main difference between these two procedures is that in ACO algorithms, the A stochastic processes $\{\mathbf{X}^i(t)\}_{t\geq 1}$ $(i = 1, \ldots, A)$ are not independent. Convergence analysis of SA is based on the theory of non-homogeneous Markov chains; see, for instance, Geman and Geman (1984) and Hajek (1988). More precisely, convergence in distribution of $\{\mathbf{Y}(t)\}_{t\geq 1}$ to the uniform distribution on the set of optimal solutions is proved. This is done assuming suitable sufficient conditions on the cooling schedule. The convergence result in Theorem 1 is of a similar kind as the one for SA, but its proof is based on similar techniques as in Gutjahr (2002).

We now point out some differences between the proposed ACO and the GBAS/tdev algorithms. As already mentioned in Section 3, the ACO algorithm studied here involves amplification factors which are not present in the GBAS/tdev. Such factors allow to take into account the objective function value during reinforcement. In some cases, this may improve the speed of convergence of the algorithm. For the toy problem considered in Section 5, the simulation results on the waiting time until an optimum is reached provide a numerical evidence in this direction. We also notice that the pheromone values at the beginning are not all equal as in the GBAS/tdev algorithm. Furthermore, also when amplification factors are not considered, we extend the set of converging GBAS/tdev algorithms of Gutjahr (2002). Indeed, the conditions of Theorem 2 involve a generic sequence $\{b_t\}_{t\geq 1}$, while Theorem 4.1 of Gutjahr (2002) only deals with the particular sequence $\{\ln t\}_{t\geq 1}$. Moreover, the sufficient condition (i) of Theorem 1 was not considered in Gutjahr (2002).

The theoretical study on the waiting time until an optimum is reached for the proposed ACO algorithm is presently under investigation by the authors. This kind of analysis has already been performed for the (1 + 1) EA; see, for instance, Droste et al. (2002). The results obtained in Droste et al. (2002) refer to specific classes of objective functions and provide in some cases very good asymptotic rates. The same strategy of restricting the class of objective functions could be adopted in the context of ACO algorithms.

Appendix

In this Appendix we prove the following Lemma 1, which is exploited in the proof of Theorem 1.

LEMMA 1 *Setting* $p(t) = P\left(\bigcap_{1\leq i\leq A}\bigcap_{1\leq s\leq t}\left\{\mathbf{X}^i(s)\notin\mathcal{O}\right\}\right)$, *for any fixed* $\mathbf{o}^* \in \mathcal{O}$, *the following inequality holds*

$$p(t) \leq \exp\left\{-\eta_n\left(\mathbf{o}^*\right)\sum_{j=2}^{t}\left(\prod_{s=1}^{j-1}\left(1-\rho_s f(\Delta H^*_{Max})\right)\right)^n\right\},$$

where $\eta_n\left(\mathbf{o}^*\right) = \phi^1\left(o_1^*\right)\prod_{j=2}^{n}\tau^1_{o^*_{j-1},o^*_j}$.

Proof: For any fixed $\mathbf{o}^* \in \mathcal{O}$, it follows

$$p(t) \leq P\left(\bigcap_{1 \leq i \leq A} \bigcap_{1 \leq s \leq t} \left\{\mathbf{X}^i(s) \neq \mathbf{o}^*\right\}\right) \leq P\left(\bigcap_{1 \leq s \leq t} \left\{\mathbf{X}^1(s) \neq \mathbf{o}^*\right\}\right). \tag{A.1}$$

The latter term of (A.1) can be rewritten as

$$p\left(\bigcap_{1 \leq s \leq t} \left\{\mathbf{X}^1(s) \neq \mathbf{o}^*\right\}\right) = \sum_{\mathbf{x}(t-1) \in F} P\left(\mathbf{X}^1(t) \neq \mathbf{o}^* | \mathbf{X}(t-1) = \mathbf{x}(t-1)\right) \cdot P(\mathbf{X}(t-1) = \mathbf{x}(t-1)), \tag{A.2}$$

where $F = \{\mathbf{y}(t-1) : \mathbf{y}^1(s) \in S \backslash \{\mathbf{o}^*\}, \mathbf{y}^i(s) \in S,\ 1 \leq s \leq t-1,\ 2 \leq i \leq A\}$. Moreover, by Equation (10), we have

$$P\left(\mathbf{X}^1(t) = o^* | \mathbf{X}(t-1) = \mathbf{x}(t-1)\right) \geq \phi^t\left(o_1^*\right) \prod_{j=2}^{n} \frac{\tau^t_{o^*_{j-1}, o^*_j}}{\sum_{l \in V} \tau^t_{o^*_{j-1}, l}}. \tag{A.3}$$

Since $a_t \leq f(\Delta H^*_{Max})$ for all t, by the pheromone updating rule of Equation (2), for all $t \geq 2$ and $p, q \in V$, it follows

$$\tau^t_{p,q} \geq (1 - \rho_{t-1} a_{t-1}) \tau^{t-1}_{p,q} \geq \tau^1_{p,q} \prod_{s=1}^{t-1} (1 - \rho_s a_s) \geq \tau^1_{p,q} \prod_{s=1}^{t-1} (1 - \rho_s f(\Delta H^*_{Max})) \tag{A.4}$$

$$\phi^t(p) \geq \phi^1(p) \prod_{s=1}^{t-1} (1 - \rho_s f(\Delta H^*_{Max})).$$

Therefore, by Equations (A.3) and (A.4), and the identity $\sum_{p,q \in V} \tau^1_{p,q} = 1$, it follows

$$P\left(\mathbf{X}^1(t) = \mathbf{o}^* | \mathbf{X}(t-1) = \mathbf{x}(t-1)\right) \geq \eta_n(\mathbf{o}^*) \left[\prod_{s=1}^{t-1} (1 - \rho_s f(\Delta H^*_{Max}))\right]^n, \tag{A.5}$$

where $\eta_n(\mathbf{o}^*) = \phi^1(o^*_1) \prod_{j=2}^{n} \tau^1_{o^*_{j-1}, o_j *}$.

Therefore, by equations (A.2) and (A.5) we have

$$P\left(\bigcap_{1 \leq s \leq t} \left\{\mathbf{X}^1(s) \neq \mathbf{o}^*\right\}\right)$$
$$\leq \left[1 - \eta_n(\mathbf{o}^*) \left(\prod_{s=1}^{t-1} (1 - \rho_s f(\Delta H^*_{Max}))\right)^n\right] \sum_{\mathbf{x}(t-1) \in F} P(\mathbf{X}(t-1) = \mathbf{x}(t-1))$$
$$= \left[1 - \eta_n(\mathbf{o}^*) \left(\prod_{s=1}^{t-1} (1 - \rho_s f(\Delta H^*_{Max}))\right)^n\right] P\left(\bigcap_{1 \leq s \leq t-1} \left\{\mathbf{X}^1(s) \neq \mathbf{o}^*\right\}\right). \tag{A.6}$$

By iterating inequality (A.6) we obtain, for all $t \geq 2$,

$$P\left(\bigcap_{1 \leq s \leq t} \left\{\mathbf{X}^1(s) \neq \mathbf{o}^*\right\}\right) \leq \prod_{j=2}^{t} \left[1 - \eta_n(\mathbf{o}^*)\left(\prod_{s=1}^{j-1} (1 - \rho_s f(\Delta H^*_{Max}))\right)^n\right]. \quad \text{(A.7)}$$

The conclusion follows by Equations (A.1) and (A.7), and $\ln(1 - x) \leq -x$, where $x < 1$. ■

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References

M. Dorigo, V. Maniezzo, and A. Colorni, "The ant system: Optimization by a colony of cooperating agents," *IEEE Transactions on Systems, Man and Cybernetics Part B* vol. 26 pp. 29–41, 1996.

M. Dorigo and L. M. Gambardella, "Ant colony systems: A cooperative learning approach to the traveling salesman problem," *IEEE Transactions on Evolutionary Computation* vol. 1 pp. 53–66, 1997.

M. Dorigo, G. Di Caro, and L. M. Gambardella, "Ant algorithms for discrete optimization," *Artificial Life* vol. 5 pp. 137–172, 1999.

S. Droste, T. Jansen, and I. Wegener, "On the analysis of the (1 + 1) evolutionary algorithm," *Theoretical Computer Science* vol. 276 pp. 51–81, 2002.

S. Geman and D. Geman, "Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images," *IEEE Transactions on Pattern Analysis and Machine Intelligence* vol. 6 pp. 721–741, 1984.

F. Glover and M. Laguna, *Tabu Search*, Kluwer Academic Publishers: Boston, 1997.

D. Goldberg, *Genetic Algorithms in Search, Optimization and Machine Learning*, Addison Wesley: Reading, 1989.

W. J. Gutjahr, "A Graph-based ant system and its convergence," *Future Generation Computer Systems* vol. 16 pp. 873–888, 2000.

W. J. Gutjahr, "ACO algorithms with guaranteed convergence to the optimal solution," *Information Processing Letters* vol. 82 pp. 145–153, 2002.

W. J. Gutjahr, "A generalized convergence result for the graph-based ant system metaheuristic," *Probability in the Engineering and Informational Sciences* vol. 17 pp. 545–569, 2003.

B. Hajek, "Cooling schedules for optimal annealing," *Mathematics of Operations Research* vol. 13 pp. 311–329, 1988.

S. Kirkpatrick, C. D. Jr. Gellat, and M. P. Vecchi, "Optimization by simulated annealing," *Science* vol. 220 pp. 671–680, 1983.

Ch. D. Papadimitriou and K. Steiglitz, *Combinatorial Optimization: Algorithms and Complexity*, Prentice-Hall: Englewood Cliffs, 1982.

R. Y. Rubinstein, "Combinatorial optimization, cross-entropy, ants and rare events," In S. Uryasev and P. M. Pardalos (eds.), *Stochastic Optimization: Algorithms and Applications*, pp. 1–57, Kluwer Academic Publishers: Boston, 2001.

H. P. Schwefel, *Evolution and Optimum Seeking*, Wiley: New York, 1995.