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Some Conformal and Projective Scalar Invariants of Riemannian Manifolds

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Abstract—It is proved that, on any closed oriented Riemannian n-manifold, the vector spaces of conformal Killing, Killing, and closed conformal Killing r-forms, where $1 \le r \le n-1$, have finite dimensions t_r, k_r, and p_r, respectively. The numbers t_r are conformal scalar invariants of the manifold, and the numbers k_r and p_r are projective scalar invariants; they are dual in the sense that $t_r = t_{n-r}$ and $k_r = p_{n-r}$. Moreover, an explicit expression for a conformal Killing r-form on a conformally flat Riemannian n-manifold is given.

Key words: *closed oriented Riemannian manifold*, *conformal Killing form*, *Killing form*, *conformal scalar invariant*, *projective scalar invariant*, *conformally flat Riemannian manifold.*

1. INTRODUCTION

Let (M, g) be a Riemannian manifold of dimension $n \ge 2$. For each integer $1 \le r \le n$, we denote the vector space of differential r-forms on (M, g) by $\Omega^r(M, \mathbb{R})$ and consider its subspaces. The best-known subspace is the space $H^r(M, \mathbb{R})$ of *harmonic r-forms*; for a closed oriented Riemannian manifold (M, g), it has finite dimension b_r, which is called the *rth Betti number*. The Betti numbers obey *Poincaré duality* $b_r = b_{n-r}$ and are conformally invariant for $n = 2r$ (see [1, p. 85 (Russian transl.)]). There are also three subspaces not as well known. The first of them is the space $T^r(M, \mathbb{R})$ of *conformal Killing differential r-forms*, or conformal Killing–Yano tensors of rank r, which are a generalization of conformal Killing vector fields to Riemannian manifolds. The second subspace is the space $K^r(M, \mathbb{R})$ of coclosed conformal Killing differential r-forms, which are also known as *Killing r-forms*, or Killing–Yano tensors of rank r; they generalize Killing vector fields. Finally, the subspace $P^r(M, \mathbb{R})$ consists of *closed conformal Killing differential r-forms*, or closed conformal Killing tensors of rank r, which generalize concircular vector fields (see the survey [1]).

The three subspaces of the vector space on a closed oriented Riemannian manifold (M, g) listed above have properties similar to those of the subspace of harmonic forms.

Theorem 1. *On any closed oriented Riemannian n-manifold (M, g), the vector spaces $T^r(M, \mathbb{R})$, $K^r(M, \mathbb{R})$, and $P^r(M, \mathbb{R})$, where $1 \le r \le n-1$, have finite dimensions t_r, k_r, and p_r. The numbers t_r are conformal scalar invariants of the manifold, and the numbers k_r and p_r are projective scalar invariants; they are dual in the sense that $t_r = t_{n-r}$ and $k_r = p_{n-r}$.*

Note that, unlike t_r, k_r, and p_r, the Betti numbers b_r are also topological invariants of the manifold (M, g).

In [2] and [3], this author obtained explicit expressions for the coclosed and closed conformal Killing forms in a local coordinate system on a projectively plane pseudo-Riemannian manifold by using the projective invariance of these forms. This made it possible to write the integrals of the special Maxwell equations and the symmetry operators of the massless Dirac equations on the given manifold.

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In this paper, we apply the conformal invariance of conformal Killing forms to obtain their explicit expression in a local coordinate system on a conformally flat manifold, which, in particular, makes it possible to write the symmetry operators of the Dirac equations with mass (see [3]) on a conformally flat pseudo-Riemannian manifold.

Theorem 2. *On any conformally flat Riemannian n-manifold (M, g), every point has a neighborhood with local coordinate system $x^1, \dots, x^n$ in which any conformal Killing r-form ω with $1 \le r \le n-1$ has local components*

$$\omega_{i_1\dots i_r} = e^{-(r+1)\sigma}(A_{kji_1\dots i_r}x^k x^j + B_{ji_1\dots i_r}x^j + C_{i_1\dots i_r}). \tag{1}$$

The coefficients $A_{kji_1\dots i_r}$, $B_{i_1\dots i_r}$, and $C_{i_1\dots i_r}$ on the right-hand sides of equalities (1) *are constant and skew-symmetric in $i_1, \dots, i_r$, and they satisfy the conditions*

$$A_{kjii_2\dots i_r} + A_{kiji_2\dots i_r} = 2A_{ki_2\dots i_r}\bar g_{ij} - \sum_{a=2}^{r}(-1)^a(A_{kii_2\dots\hat i_a\dots i_r}\bar g_{ji_a} + A_{kji_2\dots\hat i_a\dots i_r}\bar g_{ji_a}),$$

$$B_{jii_2\dots i_r} + B_{iji_2\dots i_r} = 2B_{i_2\dots i_r}\bar g_{ij} - \sum_{a=2}^{r}(-1)^a(B_{ii_2\dots\hat i_a\dots i_r}\bar g_{ji_a} + B_{ji_2\dots\hat i_a\dots i_r}\bar g_{ji_a}),$$

where

$$A_{ki_2\dots i_r} = \frac{1}{n-p+1}\bar g^{jl}A_{kjli_2\dots i_r}, \qquad B_{i_2\dots i_r} = \frac{1}{n-p+1}\bar g^{jl}B_{jli_2\dots i_r},$$

and the $\bar g_{ij}$ are the constant components of the metric tensor $\bar g$ for which $g = e^{-2\sigma}\bar g$.

2. PROOFS

For an arbitrary smooth map $f\colon (M, g) \to (\overline{M}, \bar g)$ of Riemannian manifolds, we denote its differential by f_* and the transpose of f_* by f^*. Any differential r-form $\overline{\omega}$ on $(\overline{M}, \bar g)$ determines the differential r-form $\omega := f^*\bar\omega$ on (M, g). The following lemma is valid.

Lemma 1. *Let $f\colon (M, g) \to (\overline{M}, \bar g)$ be the conformal diffeomorphism of Riemannian n-manifolds defined by $f^*\bar g = e^{2\sigma}g$, and let ω be a conformal Killing r-form on (M, g) $(1 \le r \le n-1)$. Then the r-form $\widetilde{\omega}$ on $(\overline{M}, \bar g)$ defined by $f^*\widetilde{\omega} = e^{(r+1)\sigma}\omega$ is conformally Killing.*

Proof. Let $f\colon (M, g) \to (\overline{M}, \bar g)$ be a diffeomorphism. Then, for any $x \in M$ and $\bar x = f(x) \in \overline{M}$, we can choose charts (U, φ) (with $x \in U$) and $(\overline{U}, \bar\varphi)$ (with $\bar x \in \overline{U}$) so that the diffeomorphism f is determined in these charts by equations of the form $\bar x^1 = x^1$, …, $\bar x^n = x^n$ (see [4, p. 67 (Russian transl.)]). We say that the local coordinates $x^1, \dots, x^n$ are common for the manifolds (M, g) and $(\bar M, \bar g)$ with respect to the given diffeomorphism f. In these coordinates, the conformal diffeomorphism $f\colon (M, g) \to (\overline{M}, \bar g)$ is determined by the condition $\bar g = e^{2\sigma}g$, which implies the equalities (see [1, pp. 84–85 (Russian transl.)])

$$\bar\nabla_X Y = \nabla_X Y + d\sigma(X)Y + d\sigma(Y)X - g(X, Y)\operatorname{grad}\sigma, \tag{2}$$

$$\bar d^*\omega = e^{-2\sigma}\{d^*\omega - (n-2r)i_{\operatorname{grad}\sigma}\omega\}, \tag{3}$$

where ∇ and $\bar\nabla$ are the Levi-Civita connections, d^* and $\bar d^*$ are the codifferentiation operators on the Riemannian manifolds (M, g) and $(\overline{M}, \bar g)$, respectively, $X = X^i\partial/\partial x^i$ and $Y = Y^k\partial/\partial x^k$ are vector fields, while $i_{\operatorname{grad}\sigma}\omega$ stands for inner multiplication (that is, the convolution of the r-form ω with the vector $\operatorname{grad}\sigma$).

On the manifold (M, g), the conformal Killing r-forms constitute the kernel of the natural first-order differential operator D, which acts on an arbitrary form $\omega \in C^\infty\Lambda^r M$ as

$$D\omega = \nabla\omega - (r+1)^{-1}\,d\omega - (n-r+1)^{-1}g \wedge d^*\omega,$$

where

$$d\colon C^\infty\Lambda^r M \to C^\infty\Lambda^{r+1}M$$

is the exterior differentiation operator,

$$d^*\colon C^\infty\Lambda^r M \to C^\infty\Lambda^{r-1}M$$

is its conjugate codifferentiation operator mentioned above, and $g\wedge d^*\omega$ is the exterior product of the differential $(r-1)$-form $d^*\omega$ and the metric tensor g, which is defined by

$$(g\wedge d^*\omega)(X_0,X_1,\dots,X_r)=\sum_{a=2}^{r}(-1)^a g(X_0,X_a)(d^*\omega)(X_1,\dots,\widehat{X}_a,\dots,X_r)$$

for any $X_0,\dots,X_r\in C^\infty TM$ (see [2], [5], [6]). According to (2) and (3), in local coordinates $x^1,\dots,x^n$ common with respect to the given conformal diffeomorphism f, the operator D is related to the corresponding operator $\overline{d}$ on the manifold $(\overline{M},\bar g)$ by

$$\overline{D}\omega = D\omega-(r+1)\{d\sigma\otimes\omega-(r+1)^{-1}d\sigma\wedge\omega-(n-r+1)^{-1}g\wedge i_{\operatorname{grad}\sigma}\omega\},$$

where $\overline{\omega}=\omega$ in the coordinate system under consideration. Therefore, the form $\widetilde{\omega}=e^{(r+1)\sigma}\omega$ satisfies the relation $\overline{D}\widetilde{\omega}=e^{(r+1)\sigma}D\omega$, which proves the lemma, because this relation implies

$$\omega\in\operatorname{Ker}D \iff \widetilde{\omega}\in\operatorname{Ker}\overline{D}. \quad \square$$

Proof of Theorem 1. In [6], for a closed oriented Riemannian manifold (M,g), the conjugate D^* of the natural operator D and the rough Laplacian $D^*D\colon C^\infty\Lambda^p M\to C^\infty\Lambda^p M$ were constructed; moreover, it was proved that $\operatorname{Ker}D^*D$ is exhausted by the r-forms from the space $T^r(M,\mathbb{R})$. All operators of the form D^*D are elliptic (see, e.g., [7]); therefore, the kernel $\operatorname{Ker}D^*D$ of any such operator is a finite-dimensional vector space (see [1, pp. 631–632 (Russian transl.)] and [8, Ch. 11]). In particular, the space $T^r(M,\mathbb{R})$ is finite-dimensional; we set $t_r=\dim T^r(M,\mathbb{R})$ (see also [1, p. 49 (Russian transl.)]).

Under the conformal transformation $\bar g=e^{2\sigma}g$ of the metric, the conformal Killing r-forms ω on the manifold (M,g) correspond to the conformal Killing r-forms $\overline{\omega}:=e^{(r+1)\sigma}\omega$ on the Riemannian manifold $(M,\bar g)$; therefore, the vector space of conformal Killing r-forms on $(M,\bar g)$ has a basis consisting of normalized basis r-forms from the space $T^r(M,\mathbb{R})$. Thus, the numbers $t_r=\dim T^r(M,\mathbb{R})$ are conformal invariants.

The Hodge operator

$$*\colon C^\infty\Lambda^r M\cong C^\infty\Lambda^{n-r}M$$

(see [1, p. 52 (Russian transl.)]) establishes an isomorphism $*\colon T^r(M,\mathbb{R})\cong T^{n-r}(M,\mathbb{R})$ (see [2, p. 46], [9]), which allows us to assert that $t_r=t_{n-r}$.

For $1\le r\le n-1$, the Killing r-forms on a closed oriented Riemannian manifold (M,g), which form the vector space $K^r(M,\mathbb{R})$, exhaust $\operatorname{Ker}D^*D\cap\operatorname{Ker}d^*$ (see [6]); hence we have

$$k_r=\dim K^r(M,\mathbb{R})=\dim_{\mathbb{R}}(\operatorname{Ker}D^*D\cap\operatorname{Ker}d^*)<\infty.$$

By a projective diffeomorphism $f\colon(M,g)\to(\overline{M},\bar g)$ we mean a map which takes the geodesics of the manifold (M,g) to geodesics of the manifold $(\bar M,\bar g)$. Note that if f is a projective diffeomorphism, then the map $f^{-1}\colon(\overline{M},\bar g)\to(M,g)$ is a projective diffeomorphism as well.

The following lemma, which was proved in [3], describes a property of projective diffeomorphisms necessary for the proof of the theorem.

Lemma 2. *Let* $f\colon (M,g) \to (\overline{M},\bar g)$ *be a projective diffeomorphism of Riemannian* n*-manifolds,* $1 \le r \le n-1$*, and let* ω *be a Killing* r*-form on the manifold* (M,g)*. Then the* r*-form* $\widetilde\omega$ *on the manifold* $(\overline{M},\bar g)$ *defined by*

$$f^*\widetilde\omega = e^{(p+1)\psi}\omega \qquad \textit{for} \quad \psi = \frac{\ln(\det\bar g\det g)}{2(n+1)}$$

is Killing.

It follows from this lemma that, under any projective transformation $g \to \bar g$ of the metric, each Killing r-form ω on the manifold (M,g) generates the Killing r-form $\overline\omega := e^{(r+1)\psi}\omega$ on the Riemannian manifold $(M,\bar g)$. Thus, the vector space of Killing r-forms on $(M,\bar g)$ has a basis consisting of normalized basis r-forms from the space $K^r(M,\mathbb{R})$. Therefore, the number k_r is projectively invariant.

On the other hand, for $1 \le r \le n-1$, the r-forms that constitute the vector space $P^r(M,\mathbb{R})$ exhaust $\operatorname{Ker} D^*D \cap \operatorname{Ker} d$ (see [6]); hence

$$p_r = \dim P^r(M,\mathbb{R}) = \dim_{\mathbb{R}}(\operatorname{Ker} D^*D \cap \operatorname{Ker} d) < \infty.$$

The duality property $k_r = p_{n-r}$ follows from the isomorphism $*\colon K^r(M,\mathbb{R}) \cong P^{n-r}(M,\mathbb{R})$ (see [2], [5]). Since the numbers k_r are projectively invariant, it follows that so are the numbers p_r. This completes the proof of Theorem 1. $\square$

Lemma 3. *On a flat Riemannian* n*-manifold* (M,g)*, each point has a neighborhood with a local coordinate system* $x^1,\dots,x^n$ *in which any conformal Killing* r*-form* ω*, where* $1 \le r \le n-1$*, has local components*

$$\omega_{i_1\dots i_r} = A_{kji_1\dots i_r}x^kx^j + B_{ji_1\dots i_r}x^j + C_{i_1\dots i_r}. \tag{4}$$

The coefficients $A_{kji_1\dots i_r}$*,* $B_{ji_1\dots i_r}$*, and* $C_{i_1\dots i_r}$ *on the right-hand sides of these equalities are constant and skew-symmetric in* $i_1,\dots,i_r$*, and they satisfy the conditions*

$$A_{kjii_2\dots i_r} + A_{kiji_2\dots i_r} = 2A_{ki_2\dots i_r}g_{ij} - \sum_{a=2}^{r}(-1)^a(A_{kii_2\dots\hat i_a\dots i_r}g_{ji_a} + A_{kji_2\dots\hat i_a\dots i_r}g_{ji_a}),$$

$$B_{jii_2\dots i_r} + B_{iji_2\dots i_r} = 2B_{i_2\dots i_r}g_{ij} - \sum_{a=2}^{r}(-1)^a(B_{ii_2\dots\hat i_a\dots i_r}g_{ji_a} + B_{ji_2\dots\hat i_a\dots i_r}g_{ji_a}),$$

where

$$A_{ki_2\dots i_r} = \frac{1}{n-p+1}g^{jl}A_{kjli_2\dots i_r}, \qquad B_{i_2\dots i_r} = \frac{1}{n-p+1}g^{jl}B_{jli_2\dots i_r}.$$

Proof. Let (M,g) be a flat Riemannian manifold of dimension $n \ge 2$; by definition, each of its points x has a neighborhood U isometric to an open set in $\mathbb{R}^n$. In any Cartesian coordinates $x^1,\dots,x^n$ in U, the local components g_{ij} of the metric tensor g are constant, and the Christoffel symbols of the Levi-Civita connection ∇ vanish. Therefore, in U, the covariant differentiation $\nabla_j := \nabla_{\partial_j}$ coincides with the partial differentiation $\partial_j = \partial/\partial x^j$, and the equations determining the conformal Killing r-forms (see [10]) are

$$\partial_j\omega_{ii_2\dots i_r} + \partial_i\omega_{ji_2\dots i_r} = 2\theta_{i_2\dots i_r}g_{ij} - \sum_{a=2}^{r}(-1)^a(\theta_{ii_2\dots\hat i_a\dots i_r}g_{ji_a} + \theta_{ji_2\dots\hat i_a\dots i_r}g_{ii_a}), \tag{5}$$

where the $\omega_{i_1i_2\dots i_r}$ are the local components of the r-form ω in the neighborhood U and the $\theta_{i_2\dots i_r} = (n-r+1)^{-1}g^{ij}\partial_i\omega_{ji_2\dots i_r}$ are their components, which, in turn, satisfy the equations (see [10])

$$\partial_j\theta_{i_2i_3\dots i_r} + \partial_{i_2}\theta_{ji_3\dots i_r} = 0. \tag{6}$$

Since the integrals of Eqs. (6) have the form $\theta_{i_2\dots i_r} = A_{ki_2\dots i_r}x^r + B_{i_2\dots i_r}$ for any constants $A_{i_1 i_2\dots i_r}$ and $B_{i_2\dots i_r}$ skew-symmetric in all of their indices (see [5]), it follows from (5) that

$$\begin{aligned}\partial_k\partial_j\omega_{ii_2\dots i_r} = A_{ki_2\dots i_r}g_{ji} + A_{ji_2\dots i_r}g_{ki} - A_{ii_2\dots i_r}g_{kj} \\ + 2^{-1}\sum_{a=2}^{r}(-1)^a(A_{iki_2\dots\hat{i}_a\dots i_r}g_{ji_a} + A_{iji_2\dots\hat{i}_a\dots i_r}g_{ki_a} \\ - A_{kji_2\dots\hat{i}_a\dots i_r}g_{ii_a} - A_{jii_2\dots\hat{i}_a\dots i_r}g_{ki_a} - A_{jki_2\dots\hat{i}_a\dots i_r}g_{ii_a}),\end{aligned}$$

and therefore $\partial_l\partial_k\partial_j\omega_{ii_2\dots i_r=0}$. This means that the local components of ω have the required form (4). The conditions on the coefficients in (4) are obtained by substituting these components into Eq. (5). This completes the proof of Lemma 3. □

Proof of Theorem 2. A Riemannian manifold (M, g) is said to be *conformally flat* (see [1, p. 86 (Russian transl.)]) if, for any point $x \in M$, there exists its neighborhood U and a function σ on U such that $(\overline{M}, \bar{g})$, where $\overline{M} = U$ and $\bar{g} = e^{2\sigma}g$, is a flat Riemannian manifold. According to Lemma 3, on the flat manifold $(\overline{M}, \bar{g})$, the conformal Killing r-form $\overline{\omega}$ has local components

$$\overline{\omega}_{i_1\dots i_r} = A_{kji_1\dots i_r}x^k x^j + B_{ji_1\dots i_r}x^j + C_{i_1\dots i_r}.$$

Under the pointwise conformal transformation $g = e^{2\sigma'}\bar{g}$ of the metric, the form $\widetilde{\omega} = e^{(r+1)\sigma'}\overline{\omega}$, where $\sigma' = -\sigma$, is conformal Killing by Lemma 1 . This proves Theorem 2. □

In conclusion, note that the proofs of Lemma 1 and Theorem 2 suggested above can be carried over to pseudo-Riemannian manifolds without modifications; this justifies our remark concerning the possibility of applying the results in theoretical physics.

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