MATHEMATICAL NOTES, VOL. 80, NO. 5, 2006, PP. 631–644.
Translated from Matematicheskie Zametki, vol. 80, *no.* 5, 2006, *pp.* 668–682.

Inverse Spectral Reconstruction Problem for the Convolution Operator Perturbed by a One-Dimensional Operator

S. A. Buterin

Received October 4, 2004

Abstract—We consider a one-dimensional perturbation of the convolution operator. We study the inverse reconstruction problem for the convolution component using the characteristic numbers under the assumption that the perturbation summand is known *a priori*. The problem is reduced to the solution of the so-called basic nonlinear integral equation with singularity. We prove the global solvability of this nonlinear equation. On the basis of these results, we prove a uniqueness theorem and obtain necessary and sufficient conditions for the solvability of the inverse problem.

KEY WORDS: *inverse spectral reconstruction problem*, *convolution operator*, *nonlinear integral equation*, *Fredholm alternative*, *Hilbert–Schmidt operator*.

INTRODUCTION

The main results for inverse problems of spectral analysis were obtained for differential operators (see, for example, the surveys in [1], [2]). For inverse problems involving integral operators, see the well-known paper [3] of Yurko, in which an operator of the form

$$Af = Mf + g(x)\int_0^\pi f(t)v(t)\,dt, \qquad Mf = \int_0^x M(x,t)f(t)\,dt, \quad 0 \le x \le \pi, \tag{0.1}$$

was studied. There the author investigated the reconstruction problem for the functions $g(x)$, $v(x)$ using the spectral data of the operator A and assuming that the function $M(x,t)$ is known *a priori*, and showed the connection of this inverse problem with the inverse Sturm–Liouville problem. As is well known, the inverse operator to the Sturm–Liouville differential operator is a special case of an operator of the form (0.1). Inverse operators to Volterra differential and integro-differential operators of higher order with boundary conditions at only one of the endpoints, namely, at the point π, are of the same form. More general boundary conditions are associated with finite-dimensional perturbations. Note that direct problems for finite-dimensional perturbations of Volterra operators were studied by Khromov (see, for example, [4]) and by other authors.

In the present paper, we study a different inverse problem for the operator (0.1), namely, the reconstruction problem for the operator M, assuming that the functions $g(x)$, $v(x)$ are known *a priori*. For the spectral data we use the characteristic numbers of the operator A. The solution of this inverse problem is hampered by the fact that the characteristic function depends nonlinearly on M.

We consider the case in which $M(x,t)$ depends only on the difference of its arguments, i.e., M is a convolution operator. Note that, in the more general case of the function $M(x,t)$, the solution of the inverse problem under consideration, is, in general, nonunique. In Sec. 1, we establish the special form of the kernel of the transformation operator related to M, which allows us to reduce the inverse problem to the solution of the so-called main nonlinear integral equation with singularity (2.1). In Sec. 2, we prove the global solvability of this nonlinear equation. The important part of the proof is the study of the singularity. In Sec. 3, using the results obtained in the previous sections, we establish the uniqueness of the solution of the inverse problem and obtain necessary and sufficient conditions for its solvability. The proof is constructive and allows us to find an algorithm for the solution of the inverse problem. The main results of this paper are contained in Theorems 3.1 and 3.2.

1. PRELIMINARIES

Consider an operator $A = A(M, g, v)$ of the form

$$Af = Mf + g(x)\int_0^\pi f(t)v(t)\,dt, \qquad Mf = \int_0^x M(x-t)f(t)\,dt, \quad 0 \le x \le \pi. \tag{1.1}$$

Suppose that $M(x) \in W_2^2[0,T]$ for all $T \in (0,\pi)$, $(\pi - x)M''(x) \in L_2(0,\pi)$; $M(0) = -i$, $M'(0) = 0$. Under these conditions, the operator M^{-1} can be expressed as

$$M^{-1}y = Dy := iy'(x) + \int_0^x H(x-t)y(t)\,dt, \qquad y(0) = 0,$$

where the functions $N = M''(x)$ and $H(x)$ are related by the relation

$$N(x) = H(x) + i\int_0^x N(t)\,dt\int_0^{x-t} H(\tau)\,d\tau \tag{1.2}$$

and, therefore,

$$(\pi - x)H(x) \in L_2(0,\pi). \tag{1.3}$$

In what follows, we assume that the function $M(x)$ satisfies the conditions indicated above and $g(x), v(x) \in W_2^1[0,\pi]$. If, besides, $a_1 a_2 \ne 0$, where

$$a_1 = 1 + ig(0)v(0) + \int_0^\pi v(x)Dg(x)\,dx, \qquad a_2 = ig(0)v(\pi), \tag{1.4}$$

then we say that the operator A belongs to the class $\mathscr{A}$.

The characteristic numbers λ_k of the operator A of the form (1.1) coincide with the zeros of its characteristic function

$$\mathscr{L}(\lambda) = 1 - \lambda\int_0^\pi v(x)g(x,\lambda)\,dx, \tag{1.5}$$

counting multiplicity, where

$$g(x,\lambda) = (E - \lambda M)^{-1}g(x) = g(x) + \lambda\int_0^x M(x-t,\lambda)g(t)\,dt. \tag{1.6}$$

Here E is the identity operator and $M(x-t,\lambda)$ is the kernel of the integral operator $R_\lambda(M) = (E - \lambda M)^{-1}M$.

Lemma 1.1. *The following representation is valid*:

$$M(x,\lambda) = -i\Big(\exp(-i\lambda x) + \int_0^x P(x,t)\exp(-i\lambda(x-t))\,dt\Big), \tag{1.7}$$

where

$$P(x,t) = \sum_{\nu=1}^\infty i^\nu \frac{(x-t)^\nu}{\nu!} H^{*\nu}(t). \tag{1.8}$$

Here

$$H^{*1}(x) = H(x), \qquad H^{*(\nu+1)}(x) = H * H^{*\nu}(x) = \int_0^x H(x-t)H^{*\nu}(t)\,dt.$$

Proof. Since $R_\lambda(M) = M + \lambda M R_\lambda(M)$, we see that the functions $M(x)$ and $M(x,\lambda)$ are related by

$$M(x,\lambda) = M(x) + \lambda \int_0^x M(x-t)M(t,\lambda)\,dt,$$

and hence they are of identical smoothness with respect to x, and $M(0,\lambda) = -i$. Applying the operator $(R_\lambda(M))^{-1} = M^{-1} - \lambda E$ to the function $y(x) = R_\lambda(M)f(x)$, where $f \in L_2(0,\pi)$, we obtain the relation

$$i\int_0^x M'(x-t,\lambda)f(t)\,dt + \int_0^x f(t)\,dt \int_t^x H(x-\tau)M(\tau-t,\lambda)\,d\tau = \lambda \int_0^x M(x-t,\lambda)f(t)\,dt,$$

where the "prime" denotes differentiation with respect to x. Since f is arbitrary, the function $M(x,\lambda)$ is a solution of the Cauchy problem

$$iM'(x,\lambda) + \int_0^x H(x-t)M(t,\lambda)\,dt = \lambda M(x,\lambda), \qquad M(0,\lambda) = -i$$

and hence is of the form (1.7) if

$$P(x,x) + \int_0^x \frac{\partial}{\partial x}P(x,t)\exp(-i\lambda(x-t))\,dt$$
$$= i\int_0^x H(t)\exp(-i\lambda(x-t))\,dt + i\int_0^x \exp(-i\lambda(x-t))\,dt \int_0^t H(t-\tau)P(x-t+\tau,\tau)\,d\tau.$$

Since λ is arbitrary, the last relation is equivalent to the Cauchy problem

$$\frac{\partial}{\partial x}P(x,t) = iH(t) + i\int_0^t H(t-\tau)P(x-t+\tau,\tau)\,d\tau, \qquad P(x,x) = 0, \quad 0 \le t \le x \le \pi,$$

which, in turn, is equivalent to the integral equation

$$P(x,t) = i(x-t)H(t) + i\int_0^{x-t} ds \int_0^t H(t-\tau)P(s+\tau,\tau)\,d\tau, \qquad 0 \le t \le x \le \pi. \tag{1.9}$$

Thus, if the function $P(x,t)$ is a solution of Eq. (1.9), then (1.7) holds. Let us solve (1.9) by the method of successive approximations. Let

$$P_1(x,t) = i(x-t)H(t), \qquad P_{\nu+1}(x,t) = i\int_0^{x-t} ds \int_0^t H(t-\tau)P_\nu(s+\tau,\tau)\,d\tau$$

and, by induction, we obtain the expression

$$P_\nu(x,t) = i^\nu \frac{(x-t)^\nu}{\nu!}H^{*\nu}(t).$$

The series on the right-hand side of (1.8) is uniformly convergent for $0 \le t \le x \le \pi$ and yields the solution of Eq. (1.9). Lemma 1.1 is proved. □

Denote

$$\mu_0(x) = \int_x^\pi v(t)g(t-x)\,dt, \qquad \mu(x) = \mu_0(x) + \int_x^\pi P(t,t-x)\mu_0(t)\,dt. \tag{1.10}$$

It is readily verified that $\mu(x) \in W_2^2[0,\pi]$.

Lemma 1.2. *For the characteristic function of the operator* (1.1), *the following representation is valid*:

$$\mathscr{L}(\lambda) = a_1 - a_2 \exp(-i\lambda\pi) + \int_0^\pi w(x)\exp(-i\lambda x)\,dx, \qquad w(x) \in L_2(0,\pi), \tag{1.11}$$

where a_1, a_2 *are defined in* (1.4) *and*

$$w(x) = -i\mu''(x). \tag{1.12}$$

Proof. Substituting (1.6) into (1.5), substituting $t \to x - t$, and changing the order of integration, we obtain

$$\mathscr{L}(\lambda) = 1 - \mu_0(0)\lambda - \lambda^2 \int_0^\pi \mu_0(x) M(x,\lambda)\,dx.$$

Using (1.7), (1.10), we find

$$\mathscr{L}(\lambda) = 1 - \mu_0(0)\lambda + i\lambda^2 \int_0^\pi \mu(x)\exp(-i\lambda x)\,dx.$$

Twice integrating by parts and using the relations

$$\mu(0) = \mu_0(0), \quad \mu(\pi) = 0, \quad \mu'(0) = i(a_1 - 1), \quad \mu'(\pi) = ia_2,$$

we obtain (1.11). Lemma 1.2 is proved. $\square$

It follows from expression (1.11) (see [3]) that the operator $A \in \mathscr{A}$ has an infinite set of characteristic numbers λ_k, $k = 0, \pm1, \pm2, \dots$, of the form

$$\lambda_k = 2k + \alpha + \kappa_k, \qquad \{\kappa_k\} \in \ell_2, \quad \lambda_k \neq 0. \tag{1.13}$$

Moreover, the function $\mathscr{L}(\lambda)$ is uniquely defined by its zeros:

$$\mathscr{L}(\lambda) = \exp(p\lambda) \prod_{k=-\infty}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right) \exp\left(\frac{\lambda}{\lambda_k}\right), \tag{1.14}$$

where

$$p = \begin{cases} \dfrac{i\pi}{\exp(-i\alpha\pi) - 1} + \displaystyle\sum_{k=-\infty}^{\infty} \left(\frac{1}{\lambda_k^0} - \frac{1}{\lambda_k}\right) & \text{if } \exp(i\alpha\pi) \neq 1, \\ \dfrac{\pi}{2i} - \dfrac{1}{\lambda_{-\alpha/2}} + \displaystyle\sum_{k=-\infty,\ k\neq-\alpha/2}^{\infty} \left(\frac{1}{\lambda_k^0} - \frac{1}{\lambda_k}\right) & \text{if } \exp(i\alpha\pi) = 1. \end{cases} \tag{1.15}$$

Here $\lambda_k^0 = 2k + \alpha$. In [3], the following converse assertion was also proved.

Lemma 1.3. *Let numbers* λ_k, $k = 0, \pm1, \pm2, \dots$, *of the form* (1.13) *be given. Then, for the function* $\mathscr{L}(\lambda)$ *defined by* (1.14), *the following representation holds*:

$$\mathscr{L}(\lambda) = \gamma(1 - \exp(i(\alpha - \lambda)\pi)) + \int_0^\pi w(x)\exp(-i\lambda x)\,dx \tag{1.16}$$

with some function $w(x) \in L_2(0,\pi)$, *where*

$$\gamma = \begin{cases} \dfrac{1}{1 - \exp(i\alpha\pi)} \displaystyle\prod_{k=-\infty}^{\infty} \frac{\lambda_k^0}{\lambda_k} & \textit{if } \exp(i\alpha\pi) \neq 1, \\ \dfrac{i}{\pi\lambda_{-\alpha/2}} \displaystyle\prod_{k=-\infty,\ k\neq-\alpha/2}^{\infty} \frac{\lambda_k^0}{\lambda_k} & \textit{if } \exp(i\alpha\pi) = 1. \end{cases} \tag{1.17}$$

2. BASIC NONLINEAR INTEGRAL EQUATION

Relation (1.12) can be regarded as a nonlinear equation with respect to the function $H(x)$. Twice differentiating (1.10) and using (1.12), we obtain the following equation ($a_2 \neq 0$):

$$(\pi - x)H(x) = \varphi(x) + \sum_{\nu=1}^{\infty}\Big(b_\nu(x)H^{*\nu}(x) + \int_0^x B_\nu(x,t)H^{*\nu}(t)\,dt\Big), \qquad 0 < x < \pi, \tag{2.1}$$

where

$$\varphi(x) = \frac{iw(\pi - x) - \check{\mu}_0''(x)}{a_2}, \qquad \check{\mu}_0(x) = \mu_0(\pi - x), \tag{2.2}$$

$$b_1(x) \equiv 0, \qquad b_\nu(x) = i^{\nu+1}\frac{(\pi - x)^\nu}{\nu!}, \quad \nu \geq 2,$$

$$B_\nu(x,t) = -\frac{i^\nu}{a_2}\frac{(\pi - x)^{\nu-2}}{\nu!}\big(\nu(\nu-1)\check{\mu}_0(x-t) - 2\nu(\pi - x)\check{\mu}_0'(x-t) + (\pi - x)^2\check{\mu}_0''(x-t)\big). \tag{2.3}$$

Equation (2.1) is called the *basic nonlinear integral equation*. Note that condition (1.3) ensures the inclusion $\varphi(x) \in L_2(0,\pi)$ and, besides,

$$\int_0^\pi (\pi - x)\varphi(x)\,dx = 0. \tag{2.4}$$

Indeed, by (1.12), (2.2), we have

$$\mu''(x) = \mu_0''(x) + a_2\varphi(\pi - x)$$

and, in view of expressions $\mu'(\pi) = \mu_0'(\pi)$, $\mu(\pi) = \mu_0(\pi)$, we can integrate this expression twice, obtaining

$$\mu(x) = \mu_0(x) + a_2\int_x^\pi dt\int_t^\pi \varphi(\pi - \tau)\,d\tau.$$

Since $\mu(0) = \mu_0(0)$, we obtain (2.4). The present section is devoted to the proof of the following converse assertion.

Theorem 2.1. *For any function $\varphi(x) \in L_2(0,\pi)$ satisfying (2.4), Eq. (2.1) has a unique solution $H(x)$, $(\pi - x)H(x) \in L_2(0,\pi)$.*

Proof. In connection with the proof of a theorem of Sakhnovich, Khromov [5] obtained the solution of the equation

$$f(x) = y(x) + y * y(x), \qquad 0 < x < 1, \tag{2.5}$$

which is described below. First, the solution $y(x)$ is found for $0 < x < \delta$, where δ is sufficiently small, by using the contraction mapping principle. Let us rewrite (2.5) as

$$f(x) = y(x) + \int_0^\delta y(x-t)y(t)\,dt + \int_\delta^x y(x-t)y(t)\,dt. \tag{2.6}$$

Then, for $x < 2\delta$, $y(t)$ in the first integral and $y(x-t)$ in the second integral are known. Therefore, (2.6) is a linear equation for $\delta < x < 2\delta$. Thus, knowing the solution of Eq. (2.5) on $(0,\delta)$ for some $\delta \in (0,1)$, we can find the solution on $(\delta, 2\delta)$ and hence on the whole interval $(0,1)$ in a finite number of steps.

This property of convolution also enables the solution of the following equation of a more general form:

$$y(x) = \xi(x) + \sum_{\nu=1}^{\infty}\Big(\psi_\nu(x)y^{*\nu}(x) + \int_0^x \Psi_\nu(x,t)y^{*\nu}(t)\,dt\Big), \qquad 0 < x < T, \tag{2.7}$$

where $\psi_1(x) = 0$. Suppose that the functions $\psi_\nu(x)$, $\Psi_\nu(x,t)$ are square-integrable and there exist square-integrable functions $u(x)$, $U(x,t)$ such that

$$|\psi_\nu(x)| \le u(x), \quad |\Psi_\nu(x,t)| \le U(x,t), \qquad 0 < t < x < T, \quad \forall\, \nu.$$

Theorem 2.2. *For any function $\xi \in L_2(0,T)$, Eq. (2.7) has a unique solution $y \in L_2(0,T)$.*

Proof. Let us first show that, for a sufficiently small $\delta > 0$, Eq. (2.7) has a unique solution $y(x)$, $0 < x < \delta$, in the ball $B_\delta = \{y : \|y\|_\delta \le 1/2\}$, where $\|\cdot\|_\delta$ is the norm on $L_2(0,\delta)$. Denote

$$\psi_\nu y = \psi_\nu(x) y^{*\nu}(x) + \int_0^x \Psi_\nu(x,t) y^{*\nu}(t)\,dt, \qquad \Psi y = \xi + \sum_{\nu=1}^{\infty} \psi_\nu y.$$

Let $y, \widetilde{y} \in L_2(0,\delta)$. The Cauchy–Bunyakovskii inequality implies

$$|y * \widetilde{y}(x)| \le \|y\|_\delta \|\widetilde{y}\|_\delta \quad \forall\, x \in [0,\delta].$$

For convenience, we assume $\delta \le 1$. Then, by induction, we obtain the estimate

$$|y^{*\nu}(x)| \le \|y\|_\delta^\nu, \qquad \nu \ge 2,$$

and hence

$$\|\psi_\nu y\|_\delta \le C_\delta \|y\|_\delta^\nu, \qquad \text{where} \quad C_\delta = \|u\|_\delta + \left(\int_0^\delta \int_0^x U^2(x,t)\,dt\,dx\right)^{1/2}. \tag{2.8}$$

Also, since

$$y^{*\nu} - \widetilde{y}^{*\nu} = (y - \widetilde{y}) * (y^{*(\nu-1)} + y^{*(\nu-2)} * \widetilde{y}^{*1} + \dots + \widetilde{y}^{*(\nu-1)}), \qquad \nu \ge 2,$$

and $\|y^{*\nu}\|_\delta \le \|y\|_\delta^\nu$, we obtain the estimate

$$\|\psi_\nu y - \psi_\nu \widetilde{y}\|_\delta \le C_\delta \nu (\max\{\|y\|_\delta, \|\widetilde{y}\|_\delta\})^{\nu-1} \|y - \widetilde{y}\|_\delta. \tag{2.9}$$

Choose δ so that $C_\delta < 1/4$, $\|\xi\|_\delta \le 1/4$. Then it follows from (2.8), (2.9) that the operator Ψ maps B_δ into B_δ and is a contraction in B_δ. Indeed, suppose that $y, \widetilde{y} \in B_\delta$; then

$$\|\Psi y\|_\delta \le \|\xi\|_\delta + \sum_{\nu=1}^{\infty} \|\psi_\nu y\|_\delta \le \|\xi\|_\delta + C_\delta \sum_{\nu=1}^{\infty} \|y\|_\delta^\nu \le \|\xi\|_\delta + C_\delta < \frac{1}{2},$$

$$\|\Psi y - \Psi \widetilde{y}\|_\delta \le \sum_{\nu=1}^{\infty} \|\psi_\nu y - \psi_\nu \widetilde{y}\|_\delta \le C_\delta \sum_{\nu=1}^{\infty} \nu (\max\{\|y\|_\delta, \|\widetilde{y}\|_\delta\})^{\nu-1} \|y - \widetilde{y}\|_\delta \le \alpha \|y - \widetilde{y}\|_\delta,$$

where

$$\alpha = C_\delta \sum_{\nu=1}^{\infty} \frac{\nu}{2^{\nu-1}} = 4C_\delta < 1.$$

Applying the contraction mapping principle, we establish the unique solvability of Eq. (2.7) in B_δ.

Now, assuming that $y = y_1(x)$ is a solution of (2.7) for $0 < x < \delta$, $\delta \in (0,\pi)$, we can show that (2.7) has a unique solution $y(x)$ in $L_2(0,2\delta)$ coinciding with $y_1(x)$ on $(0,\delta)$. We search for $y(x)$ in the form $y(x) = y_1(x) + y_2(x)$, where $y_1(x) = 0$ for $\delta < x < 2\delta$, and $y_2(x) = 0$ for $0 < x < \delta$. By induction, we can prove the representation

$$y^{*\nu}(x) = (y_1 + y_2)^{*\nu}(x) = y_1^{*\nu}(x) + \sum_{k=1}^{\nu-1} C_\nu^k (y_1^{*(\nu-k)} * y_2^{*k})(x) + y_2^{*\nu}(x), \qquad \nu \ge 2, \tag{2.10}$$

where $C_\nu^k = \nu!/(k!(\nu-k)!)$. Since $y_2(x) = 0$ on $(0,\delta)$, it follows that $y_2^{*2}(x) \equiv 0$ on $[0,\delta]$ and

$$y_2^{*2}(x) = \int_\delta^x y_2(t)y_2(x-t)\,dt = \int_0^{x-\delta} y_2(x-t)y_2(t)\,dt = 0, \qquad \delta \le x \le 2\delta.$$

Hence, for $\nu \ge 2$, we have $y_2^{*\nu}(x) \equiv 0$ on $[0,2\delta]$ and, by (2.10), we obtain the expression

$$y^{*\nu}(x) = y_1^{*\nu}(x) + \nu(y_1^{*(\nu-1)} * y_2)(x), \qquad 0 \le x \le 2\delta, \quad \nu \ge 2.$$

Substituting it into (2.7), we obtain a linear equation with respect to $y_2(x)$:

$$y_2(x) = \zeta(x) + \int_\delta^x \Psi(x,t)y_2(t)\,dt, \qquad \delta < x < 2\delta, \tag{2.11}$$

where the functions

$$\zeta(x) = \xi(x) + \sum_{\nu=1}^\infty \biggl(\psi_\nu(x)y_1^{*\nu}(x) + \int_0^x \Psi_\nu(x,t)y_1^{*\nu}(t)\,dt\biggr),$$
$$\Psi(x,t) = \Psi_1(x,t) + \sum_{\nu=2}^\infty \nu\biggl(\psi_\nu(x)y_1^{*(\nu-1)}(x-t) + \int_0^{x-t} \Psi_\nu(x,t+\tau)y_1^{*(\nu-1)}(\tau)\,d\tau\biggr)$$

are square-integrable. Equation (2.11) has a unique solution and, therefore, the function given by $y(x) = y_1(x) + y_2(x)$ is the unique solution of Eq. (2.7) in $L_2(0,2\delta)$ coinciding with $y_1(x)$ on $(0,\delta)$. Continuing the process, we obtain the solution of (2.7) on the whole interval $(0,T)$ in a finite number of steps. It is unique. Indeed, suppose that $\widetilde{y} \in L_2(0,T)$ is another solution. For a sufficiently small $\delta > 0$, both functions $y(x)$, $\widetilde{y}(x)$, $0 < x < \delta$, will belong to the ball B_δ and, by the first part of the proof, will be equal almost everywhere on $(0,\delta)$. By the uniqueness of the continuation of the solution, we obtain $y(x) = \widetilde{y}(x)$ almost everywhere on $(0,T)$. Theorem 2.2 is proved. □

Let us continue the proof of Theorem 2.1. By Theorem 2.2, there exists a unique square-integrable solution $H = H_1(x)$ of Eq. (2.1) on the interval $(0,\pi/2)$. Just as in its proof, we search for a solution on $(0,\pi)$ in the form $H(x) = H_1(x) + H_2(x)$, where $H_1(x) = 0$ on $(\pi/2,\pi)$, and $H_2(x) = 0$ on $(0,\pi/2)$. We obtain the following equation with respect to $H_2(x)$:

$$(\pi - x)H_2(x) = \zeta(x) + \int_{\pi/2}^x B(x,t)H_2(t)\,dt, \qquad \frac{\pi}{2} < x < \pi, \tag{2.12}$$

where

$$\zeta(x) = \varphi(x) + \sum_{\nu=1}^\infty \biggl(b_\nu(x)H_1^{*\nu}(x) + \int_0^x B_\nu(x,t)H_1^{*\nu}(t)\,dt\biggr),$$
$$B(x,t) = B_1(x,t) + \sum_{\nu=2}^\infty \nu\biggl(b_\nu(x)H_1^{*(\nu-1)}(x-t) + \int_0^{x-t} B_\nu(x,t+\tau)H_1^{*(\nu-1)}(\tau)\,d\tau\biggr).$$

Thus, Eq. (2.1) has a unique solution $H(x)$, $0 < x < \pi$, which is square-integrable on any interval $(0,T)$, $T \in (0,\pi)$. However, the question of the integrability of $H(x)$ on the whole $(0,\pi)$ still remains open. Denote $h_2(x) = (\pi - x)H_2(x)$; then, in view of (2.3), the following equation is equivalent to (2.12):

$$h_2(x) = \zeta(x) + 2\int_{\pi/2}^x \frac{h_2(t)\,dt}{\pi - t} + \int_{\pi/2}^x G(x,t)h_2(t)\,dt, \qquad \frac{\pi}{2} < x < \pi, \tag{2.13}$$

where the function

$$G(x,t) = \frac{i}{a_2(\pi - t)}\left\{2\int_t^x \check{\mu}_0''(x-\tau)\,d\tau - (\pi - x)\check{\mu}_0''(x-t)\right\}$$
$$+ \frac{1}{\pi - t}\sum_{\nu=2}^{\infty}\nu\left(b_\nu(x)H_1^{*(\nu-1)}(x-t) + \int_0^{x-t} B_\nu(x, t+\tau)H_1^{*(\nu-1)}(\tau)\,d\tau\right)$$

is square-integrable for $0 < t < x < \pi$. Theorem 2.1 will be proved if we show that its solution satisfies the inclusion $h_2(x) \in L_2(\pi/2, \pi)$. However, as can be seen from Lemma 2.2 (see below), an equation of the form (2.13) with arbitrary square-integrable functions $\zeta(x)$, $G(x,t)$ need not, in general, have a solution in the required class. Let us prove several auxiliary assertions. Denote

$$T_\alpha f = \frac{1}{(b-x)^\alpha}\int_a^x \frac{f(t)\,dt}{(b-t)^{1-\alpha}}, \quad T_\alpha^* f = \frac{1}{(b-x)^{1-\alpha}}\int_x^b \frac{f(t)\,dt}{(b-t)^\alpha}, \qquad a < x < b.$$

Lemma 2.1. *Choose $\alpha < 1/2$. Then the operators T_α, T_α^* map $L_2(a,b)$ into $L_2(a,b)$ and are bounded.*

Proof. The proof is based on the application of the generalized Minkowski inequality

$$\left\{\int_a^b \left|\int_c^d f(x,t)\,dt\right|^2 dx\right\}^{1/2} \le \int_c^d \left\{\int_a^b |f(x,t)|^2\,dx\right\}^{1/2} dt, \tag{2.14}$$

which is understood in the following sense: "if the right-hand side of (2.14) is finite, then the left-hand side is also finite and both sides obey the stated relation" (see [6, p. 179 (Russian transl.)]). Suppose that $f(x) \in L_2(a,b)$. Denote $\check{f}(x) = f(b-x)$. Successively substituting $x \to b - x$, $t \to b - xt$, we obtain

$$\int_a^b |T_\alpha^* f(x)|^2\,dx = \int_0^{b-a}\left|\frac{1}{x^{1-\alpha}}\int_{b-x}^b \frac{f(t)\,dt}{(b-t)^\alpha}\right|^2 dx = \int_0^{b-a}\left|\int_0^1 \frac{\check{f}(xt)\,dt}{t^\alpha}\right|^2 dx,$$

and, by (2.14), we obtain the estimate

$$\|T_\alpha^* f\| \le \int_0^1 \frac{1}{t^\alpha}\left\{\int_0^{b-a} |\check{f}(xt)|^2\,dx\right\}^{1/2} dt,$$

where $\|\cdot\|$ is the norm on $L_2(a,b)$. Further, substituting $x \to (b-x)/t$, we find

$$\|T_\alpha^* f\| \le \int_0^1 \frac{1}{t^{\alpha+1/2}}\left\{\int_{b(1-t)+at}^b |f(x)|^2\,dx\right\}^{1/2} dt \le \frac{2}{1-2\alpha}\|f\|,$$

i.e., the operator T_α^* acts from $L_2(a,b)$ to $L_2(a,b)$ and is bounded. By Fubini's theorem (see [7, p. 208 (Russian transl.)]) the operator T_α is adjoint to T_α^*. Lemma 2.1 is proved. □

Lemma 2.2. *The solution $y(x)$ of the equation*

$$y(x) = f(x) + 2\int_a^x \frac{y(t)\,dt}{b-t}, \qquad a < x < b, \tag{2.15}$$

satisfies the condition $(b-x)^\theta y(x) \in L_2(a,b)$ if and only if one of the following conditions holds, depending on the value of the parameter θ:

1) $(b-x)^\theta f(x) \in L_2(a,b)$, *for $\theta > 3/2$;*

2) $(b-x)^{\theta} f(x) \in L_2(a,b)$,

$$\int_a^b (b-x) f(x)\,dx = 0 \qquad \textit{for} \ \ 0 \le \theta < 3/2. \tag{2.16}$$

Proof. Performing the substitution, we can readily see that the solution of Eq. (2.15) is of the form

$$y(x) = f(x) + \frac{2}{(b-x)^2} \int_a^x (b-t) f(t)\,dt. \tag{2.17}$$

Suppose that $\theta > 3/2$. We can easily verify that

$$(b-x)^{\theta} y(x) = f_0(x) + 2T_{\alpha} f_0(x),$$

where

$$f_0(x) = (b-x)^{\theta} f(x) \in L_2(a,b), \quad \alpha = 2 - \theta < 1/2.$$

Applying Lemma 2.1, we obtain $(b-x)^{\theta} y(x) \in L_2(a,b)$. Now, suppose that $\theta < 3/2$. Using (2.16), we transform (2.17) as follows:

$$y(x) = f(x) - \frac{2}{(b-x)^2} \int_x^b (b-t) f(t)\,dt,$$

whence, by multiplying by $(b-x)^{\theta}$, we obtain

$$(b-x)^{\theta} y(x) = f_0(x) - 2T_{\alpha}^{*} f_0(x),$$

where $\alpha = \theta - 1 < 1/2$. Using Lemma 2.1 again, we find that $(b-x)^{\theta} y(x) \in L_2(a,b)$. The *sufficiency* is proved; let us pass to the proof of the *necessity*. By (2.15), we have

$$(b-x)^{\theta} |f(x)| \le |y_0(x)| + 2 \int_a^x \frac{|y_0(t)|\,dt}{b-t} \in L_2(a,b),$$

where $y_0(x) = (b-x)^{\theta} y(x) \in L_2(a,b)$. Suppose that $\theta < 3/2$. Multiplying both sides of (2.15) by $b-x$ and integrating from 0 to π, we obtain (2.16). Lemma 2.2 is proved. □

Let $\beta(x) = (x-b)^5/5$ and denote by $L_{2,\beta}(a,b)$ the space of functions $f(x)$, $a < x < b$, with the norm

$$\|f\|_{L_{2,\beta}} = \left(\int_a^b |f(x)|^2\,d\beta(x) \right)^{1/2} = \left(\int_a^b (b-x)^4 |f(x)|^2\,dx \right)^{1/2}.$$

Lemma 2.3. *For any function* $f(x) \in L_{2,\beta}(a,b)$, *the equation*

$$y(x) = f(x) + 2 \int_a^x \frac{y(t)\,dt}{b-t} + \int_a^x G(x,t) y(t)\,dt, \qquad a < x < b, \tag{2.18}$$

where

$$\int_a^b \int_a^x |G(x,t)|^2\,dt\,dx < \infty,$$

has a unique solution $y(x) \in L_{2,\beta}(a,b)$.

Proof. Suppose that a linear bounded operator F bijectively maps a Banach space $\mathscr{B}$ into itself and the linear operator $G\colon \mathscr{B} \to \mathscr{B}$ is completely continuous. Then if the operator $S = F + G$ is an injection, it follows that it is also a bijection of $\mathscr{B}$ onto $\mathscr{B}$. Indeed, denoting $S_1 = SF^{-1}$, we obtain $S_1 = E + G_1$, where $G_1 = GF^{-1}$. Then S_1 is an injection and G_1 is completely continuous.

By the Fredholm alternative (see [7, p. 649 (Russian transl.)]), S_1 is a bijection. Hence, S is also a bijection. Denote $F = E - 2T_0$,

$$Gy = \int_a^x G(x,t)y(t)\,dt = \int_a^x G_\beta(x,t)y(t)\,d\beta(t), \qquad G_\beta(x,t) = \frac{G(x,t)}{(b-t)^4},$$

and write Eq. (2.18) in the operator form $f = Fy - Gy$. We can easily show that the operator F is bounded in $L_{2,\beta}(a,b)$ and, by Lemma 2.2 for $\theta = 2$, it is a bijection of $L_{2,\beta}(a,b)$ onto $L_{2,\beta}(a,b)$. The operator G maps $L_{2,\beta}(a,b)$ into itself. Since

$$\int_a^b \int_a^x |G_\beta(x,t)|^2\,d\beta(t)\,d\beta(x) \le \int_a^b \int_a^x |G(x,t)|^2\,dt\,dx < \infty,$$

G is a Hilbert–Schmidt operator, and hence it is completely continuous (see, for example, [8, p. 382 (Russian transl.)]). It remains to note that $F - G$ is an injection, and hence a bijection of $L_{2,\beta}(a,b)$ onto itself. Lemma 2.3 is proved. $\square$

In view of Lemma 2.3, Eq. (2.13) has a unique solution $h_2(x)$ such that $(\pi - x)^2 h_2(x) \in L_2(\pi/2,\pi)$. Hence Eq. (2.12) has a unique solution $H_2(x)$, $(\pi - x)^3 H_2(x) \in L_2(\pi/2,\pi)$. Thus, Eq. (2.1) has a unique solution $H(x)$, $(\pi - x)^3 H(x) \in L_2(0,\pi)$. It remains to show that (2.4) implies (1.3). Denote $h(x) = (\pi - x)H(x)$ and, by (2.3), we can rewrite (2.1) in the form

$$h(x) = \varphi(x) + 2\int_0^x \frac{h(t)\,dt}{\pi - t} + \alpha_1(x) + \alpha_2(x), \tag{2.19}$$

where

$$\alpha_1(x) = i\int_0^x \frac{h(t)}{a_2(\pi - t)}\left\{2\int_0^{x-t} \check{\mu}_0''(\tau)\,d\tau - (\pi - x)\check{\mu}_0''(x-t)\right\}dt,$$
$$\alpha_2(x) = \sum_{\nu=2}^{\infty}\left(b_\nu(x)H^{*\nu}(x) + \int_0^x B_\nu(x,t)H^{*\nu}(t)\,dt\right).$$

Our next move is to use Lemma 2.2 to verify that $h(x) \in L_2(0,\pi)$. To do this, let us prove two more lemmas. By the symbol C we shall denote different constants in estimates independent of the arguments of the functions.

Lemma 2.4. *If* $(\pi - x)^l H(x) \in L_2(0,\pi)$, *where* $3/2 \le l \le 3$, *then* $(\pi - x)^{l-3/2}\alpha_k(x) \in L_2(0,\pi)$, $k = 1, 2$.

Proof. Denote $h_0(x) = (\pi - x)^{l-1}h(x)$. Then, by the assumption of the lemma, $h_0(x) \in L_2(0,\pi)$. The Cauchy–Bunyakovskii inequality implies

$$|\alpha_1(x)| \le C\int_0^x \frac{|h_0(t)|\,dt}{(\pi - t)^{l-1/2}} + C(\pi - x)\left(\int_0^x \frac{dt}{(\pi - t)^{2l}}\right)^{1/2}\left(\int_0^x |h_0(t)|^2|\check{\mu}_0''(x-t)|^2\,dt\right)^{1/2};$$

hence we obtain the relation for $\alpha_1(x)$, because the convolution of integrable functions is an integrable function. Further, we have $\alpha_2(x) = \alpha_{2,1}(x) + \alpha_{2,2}(x)$, where

$$\alpha_{2,1}(x) = \sum_{\nu=2}^{\infty}\left(b_\nu(x)H_1^{*\nu}(x) + \int_0^x B_\nu(x,t)H_1^{*\nu}(t)\,dt\right), \qquad \alpha_{2,2}(x) = \int_0^x \Phi(x,t)H_2(t)\,dt,$$
$$\Phi(x,t) = \sum_{\nu=2}^{\infty}\nu\left(b_\nu(x)H_1^{*(\nu-1)}(x-t) + \int_0^{x-t} B_\nu(x,t+\tau)H_1^{*(\nu-1)}(\tau)\,d\tau\right).$$

Obviously, $|\alpha_{2,1}(x)| < C$. Also, it is easy to obtain the estimate

$$|\alpha_{2,2}(x)| \le C \int_0^x |H_1(x-t)| \frac{|h_2^0(t)|\,dt}{(\pi-t)^{l-2}} + C \int_0^x \frac{|h_2^0(t)|\,dt}{(\pi-t)^{l-1}} + C,$$

where $h_2^0(x) = (\pi-x)^l H_2(x)$. This yields $(\pi-x)^{l-3/2}\alpha_{2,2}(x) \in L_2(0,\pi)$. Lemma 2.4 is proved. $\square$

Lemma 2.5. *If $(\pi-x)^l H(x) \in L_2(0,\pi)$ for some $l < 3$, then*

$$\int_0^\pi (\pi-x)\alpha_k(x)\,dx = 0, \qquad k = 1,\ 2. \tag{2.20}$$

Proof. We have $\alpha_1(x) = \alpha_{1,1}(x) + \alpha_{1,2}(x)$, where

$$\alpha_{1,1}(x) = \frac{2i}{a_2}\int_0^x \frac{h(t)}{\pi-t}\,dt \int_0^{x-t} \check\mu_0''(\tau)\,d\tau, \qquad \alpha_{1,2}(x) = -i\frac{\pi-x}{a_2}\int_0^x \frac{h(t)\check\mu_0''(x-t)\,dt}{\pi-t}.$$

Integrating by parts, we obtain

$$\begin{aligned}\int_0^s (\pi-x)\alpha_{1,1}(x)\,dx = &\left\{-i\frac{(\pi-x)^2}{a_2}\int_0^x \frac{h(t)\,dt}{\pi-t}\int_0^{x-t}\check\mu_0''(\tau)\,d\tau\right\}\Bigg|_{x=0}^{s} \\ &+ i\int_0^s \frac{(\pi-x)^2}{a_2}\,dx \int_0^x \frac{h(t)\check\mu_0''(x-t)\,dt}{\pi-t},\end{aligned}$$

i.e.,

$$\int_0^s (\pi-x)\alpha_1(x)\,dx = -i\frac{(\pi-s)^2}{a_2}\int_0^s \frac{h_0(t)\,dt}{(\pi-t)^3}\int_0^{s-t}\check\mu_0''(\tau)\,d\tau,$$

where $h_0(x) = (\pi-x)^3 H(x)$. Choose an $\varepsilon > 0$ and an $s_1 \in (0,\pi)$ so that

$$\left(\int_0^{\pi-s_1} |\check\mu_0''(\tau)|^2\,d\tau\right)^{1/2} < \varepsilon.$$

Suppose that $s > s_1$; then, by the Cauchy–Bunyakovskii inequality, we obtain the estimates

$$\begin{gathered}\left|\int_0^s (\pi-x)\alpha_1(x)\,dx\right| \le C(\pi-s)^2\int_0^{s_1}\frac{|h_0(t)|\,dt}{(\pi-t)^3} + C\varepsilon(\pi-s)^2\int_{s_1}^s \frac{|h_0(t)|\,dt}{(\pi-t)^{5/2}}, \\ \int_{s_1}^s \frac{|h_0(t)|\,dt}{(\pi-t)^{5/2}} \le C\frac{1}{(\pi-s)^2}.\end{gathered} \tag{2.21}$$

The second summand in (2.21) can be made arbitrarily small under an appropriate choice of s_1, while the first summand tends to zero as $s \to \pi$. By Lemma 2.4, we have $(\pi-x)\alpha_1(x) \in L(0,\pi)$; hence we obtain (2.20) for $k = 1$. Further, denote

$$\begin{aligned}\mu_1(x) &= \sum_{\nu=2}^\infty i^\nu \frac{(\pi-x)^\nu}{\nu!}\int_0^x H^{*\nu}(t)\check\mu_0(x-t)\,dt \\ &= \sum_{\nu=2}^\infty i^\nu \frac{(\pi-x)^\nu}{\nu!}\left\{\int_0^x H_1^{*\nu}(t)\check\mu_0(x-t)\,dt\right. \\ &\qquad \left. + \nu\int_0^x \frac{h_2^0(t)\,dt}{(\pi-t)^3}\int_0^{x-t}\check\mu_0'(x-t-\tau)\,d\tau \int_0^\tau H_1^{*(\nu-1)}(\xi)\,d\xi\right\}.\end{aligned}$$

where $h_2^0(x) = (\pi - x)^3 H_2(x) \in L_2(0, \pi)$. We have the estimate

$$|\mu_1(x)| \le C(\pi - x)^2 \left(1 + \int_0^x \frac{|h_2^0(t)|\,dt}{(\pi - t)^2}\right). \tag{2.22}$$

Also, for all $T \in (0, \pi)$ we have $\mu_1(x) \in W_2^2[0, T]$. By differentiation, we can easily verify that $\mu_1''(x) = -a_2\alpha_2(x)$. Since $\mu_1(0) = \mu_1'(0) = 0$, we have

$$\mu_1(x) = -a_2 \int_0^x (x - t)\alpha_2(t)\,dt, \qquad 0 \le x < \pi.$$

Using $(\pi - x)\alpha_2(x) \in L(0, \pi)$, letting x tend to π, and invoking (2.22), we obtain (2.20) also for $k = 2$. Lemma 2.5 is proved. □

Let us return to the proof of Theorem 2.1. Consider relation (2.19). Since $(\pi - x)^3 H(x) \in L_2(0, \pi)$, Lemma 2.4 implies $(\pi - x)^{3/2}\alpha_k(x) \in L_2(0, \pi)$, $k = 1, 2$. By Lemma 2.2, for $\theta = 8/5$ we find that $(\pi - x)^{8/5} h(x) \in L_2(0, \pi)$, or, equivalently, $(\pi - x)^{13/5} H(x) \in L_2(0, \pi)$. Further, by Lemma 2.5, we have (2.20). Since Eq. (2.4) also holds, applying Lemma 2.4 together with Lemma 2.2 for $0 \le \theta < 3/2$ four more times, we finally verify that $h(x) \in L_2(0, \pi)$, or, equivalently, $(\pi - x)H(x) \in L_2(0, \pi)$. Theorem 2.1 is proved. □

3. INVERSE PROBLEM

Consider the following inverse problem.

Problem 3.1. Using the characteristic numbers $\{\lambda_k\}$ of the operator $A(M, g, v)$ of the form (1.1), find the operator M under the assumption that the functions $g(x)$, $v(x)$ are known *a priori*.

To be definite, let us solve problem 3.1 for the class $\mathscr{A}$. Besides the operator $A = A(M, g, v) \in \mathscr{A}$, also consider the operator $\widetilde{A} = A(\widetilde{M}, g, v) \in \mathscr{A}$. Let us agree that if a symbol χ denotes an object belonging to the operator A, then this symbol equipped with a tilde $\widetilde{\chi}$ denotes a similar object corresponding operator $\widetilde{A}$. The following uniqueness theorem for the solution of the inverse problem holds.

Theorem 3.1. *If* $\{\lambda_k\} = \{\widetilde{\lambda}_k\}$, *then* $M = \widetilde{M}$. *In other words, the characteristic numbers of the operator* $A(M, g, v) \in \mathscr{A}$ *determine the operator* M *uniquely under the assumption that the functions* $g(x)$, $v(x)$ *are known a priori.*

Proof. In view of (1.14), it follows from the coincidence of the spectra of the operators A and $\widetilde{A}$ that $\mathscr{L}(\lambda) \equiv \widetilde{\mathscr{L}}(\lambda)$. Hence, by virtue of (1.11), we obtain $w(x) = \widetilde{w}(x)$, and hence $\varphi(x) = \widetilde{\varphi}(x)$ almost everywhere on $(0, \pi)$. Thus, both functions $H(x)$ and $\widetilde{H}(x)$ satisfy Eq. (2.1). By Theorem 2.1, we have $H(x) = \widetilde{H}(x)$ and, by (1.2), $M''(x) = \widetilde{M}''(x)$ almost everywhere on $(0, \pi)$; hence $M = \widetilde{M}$. Theorem 3.1 is proved. □

Let us present necessary and sufficient conditions for the solvability of the inverse problem.

Theorem 3.2. *Suppose that functions* $g(x), v(x) \in W_2^1[0, \pi]$, $g(0)v(\pi) \ne 0$, *and a sequence of complex numbers* $\{\lambda_k\}$, $k = 0, \pm 1, \pm 2, \dots$ *are given. Then, for the existence of the operator* $A(M, g, v) \in \mathscr{A}$ *with characteristic numbers* $\{\lambda_k\}$, *it is necessary and sufficient that these numbers have the form* (1.13) *and the following matching conditions be satisfied*:

$$p = -\int_0^\pi g(x)v(x)\,dx, \qquad \gamma \exp(i\alpha\pi) = ig(0)v(\pi), \tag{3.1}$$

where the numbers p *and* γ *are defined by* (1.15) *and* (1.17), *respectively.*

Proof. *Necessity*. The asymptotics of (1.13) was established earlier. Taking the logarithm and then differentiating (1.5) and (1.14), for $\lambda = 0$ we obtain

$$\mathscr{L}'(0) = -\int_0^\pi g(x)v(x)\,dx, \qquad \mathscr{L}'(0) = p;$$

this yields the first of relations (3.1). The second relation is obtained by comparing of expressions (1.11) and (1.16).

Sufficiency. Using the given sequence $\{\lambda_k\}$, we construct the function $\mathscr{L}(\lambda)$ by formula (1.14). By Lemma 1.3, $\mathscr{L}(\lambda)$ satisfies relation (1.16) with some function $w(x) \in L_2(0,\pi)$. Let us show that if conditions (3.1) are satisfied, then the function $\varphi(x)$ defined by (2.2) with this function $w(x)$ satisfies condition (2.4). Denote

$$w_1(x) = -a_2 + \int_x^\pi w(t)\,dt, \qquad w_2(x) = -\int_x^\pi w_1(t)\,dt. \tag{3.2}$$

Twice integrating by parts in (1.16), we obtain

$$\begin{aligned}\mathscr{L}(\lambda) = \gamma + w_1(0) + i\lambda w_2(0) - (\gamma\exp(i\alpha\pi) - a_2)\exp(-i\lambda\pi)\\ + \lambda^2\int_0^\pi w_2(x)\exp(-i\lambda x)\,dx;\end{aligned}$$

hence

$$\mathscr{L}'(0) = iw_2(0) + i\pi(\gamma\exp(i\alpha\pi) - a_2). \tag{3.3}$$

On the other hand, by (1.14), we have $\mathscr{L}'(0) = p$, and by by (2.2), (3.2),

$$w_2(x) = i\mu_0(x) + ia_2\int_x^\pi dt\int_t^\pi \varphi(\pi-\tau)\,d\tau.$$

Therefore, relation (3.3) can be rewritten as

$$p = -\int_0^\pi g(x)v(x)\,dx - a_2\int_0^\pi(\pi - x)\varphi(x)\,dx + i\pi(\gamma\exp(i\alpha\pi) - ig(0)v(\pi)),$$

and since $a_2 \neq 0$, by (3.1) we have (2.4). By Theorem 2.1, Eq. (2.1) has a unique solution $H(x)$ satisfying condition (1.3). Next, we obtain the function $N(x)$, $(\pi-x)N(x) \in L_2(0,\pi)$, from Eq. (1.2) and construct $M(x)$ by the formula

$$M(x) = -i + \int_0^x (x-t)N(t)\,dt.$$

Consider an operator $A = A(M, g, v)$ of the form (1.1). Suppose that $\mathscr{L}_1(\lambda)$ is its characteristic function. Then, by Lemma 1.2, we have

$$\mathscr{L}_1(\lambda) = a_1 - a_2\exp(-i\lambda\pi) + \int_0^\pi w(x)\exp(-i\lambda x)\,dx. \tag{3.4}$$

Subtracting (1.16) from (3.4) and taking into account the second relation in (3.1), we can write

$$\mathscr{L}_1(\lambda) - \mathscr{L}(\lambda) = a_1 - \gamma.$$

Since $\mathscr{L}(0) = \mathscr{L}_1(0) = 1$, we have $a_1 = \gamma \neq 0$ and $\mathscr{L}_1(\lambda) \equiv \mathscr{L}(\lambda)$. Hence, A is an operator of class $\mathscr{A}$, and its characteristic numbers coincide with $\{\lambda_k\}$. Theorem 3.2 is proved. □

Remark 3.1. Similar results are also valid for the case in which M^{-1} is an integro-differential operator of arbitrary natural order.

ACKNOWLEDGMENTS

The author wishes to express gratitude to Professor V. A. Yurko for posing the problem and supervising the research work.

This research was supported by the Ministry of Education under grant no. E02-1.0-186, by the program "Universities of Russia" under grant no. UR.04.01.042, by the Russian Foundation for Basic Research under grant no. 04-01-00007, and by the Presidential grant of the Russian Federation for the support of leading scientific schools under grant no. NSh-1295.2003.1.

BIBLIOGRAPHY

1. V. A. Yurko, *Inverse Spectral Problems and Their Applications* [in Russian], SPI, Saratov, 2001.
2. V. Yurko, *Method of Spectral Mappings in the Inverse Problem Theory*, Inverse and Ill-posed Problems Series, VSP, Utrecht, 2002.
3. V. A. Yurko, "An inverse problem for integral operators," *Mat. Zametki* [*Math. Notes*], **37** (1985), no. 5, 690–701.
4. A. P. Khromov, "Finite-dimensional perturbations of Volterra operators," Summary of doctorate thesis in the physico-mathematical sciences, *Mat. Zametki* [*Math. Notes*], **16** (1974), no. 4, 669–680.
5. A. P. Khromov, "On generating functions of Volterra integral operators," in: *The Theory of Functions and Approximations* [in Russian], pt. 1, Izdat. Saratov Univ., Saratov, 1987, pp. 90–96.
6. G. Hardy, D. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952; Russian transl.: Inostr. Lit., Moscow, 1948.
7. N. Dunford and J. T. Schwartz, *Linear Operators. General Theory*, Interscience Publ., New York–London, 1958; Russian transl.: Inostr. Lit., Moscow, 1962.
8. K. Yoshida, *Functional Analysis*, Springer-Verlag, Berlin, 1965; Russian transl.: Mir, Moscow, 1967.

S. A. Buterin
N. G. Chernyshevskii Saratov State University
E-mail: buterinsa@info.sgu.ru