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On the Linearization of Hamiltonian Systems on Poisson Manifolds

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Abstract—The linearization of a Hamiltonian system on a Poisson manifold at a given (singular) symplectic leaf gives a dynamical system on the normal bundle of the leaf, which is called the first variation system. We show that the first variation system admits a compatible Hamiltonian structure if there exists a transversal to the leaf which is invariant with respect to the flow of the original system. In the case where the transverse Lie algebra of the symplectic leaf is semisimple, this condition is also necessary.

Key words: *Hamiltonian system, Poisson bracket, linearization, Poisson coupling, normal bundle, first variation system, Hamiltonian vector field.*

1. INTRODUCTION

Let (M, Ψ) be a Poisson manifold with Poisson bracket

$$\{F, G\}_\Psi = \Psi(dF, dG) = \Psi^{JI}(y)\,\frac{\partial F}{\partial y^I}\,\frac{\partial G}{\partial y^J}$$

(here and in the following, the summation is taken with respect to repeated indices). Suppose we are given a Hamiltonian system (M, Ψ, H), corresponding to the Hamiltonian vector field

$$X_H = \Psi^\natural\,dH = -\Psi^{JI}(y)\,\frac{\partial H}{\partial y^I}\,\frac{\partial}{\partial y^J}\,;$$

here $\Psi^\natural\colon T^*M \to TM$ is a vector bundle morphism generated by the Poisson tensor Ψ. According to the general scheme [1, 2], the linearization procedure for the dynamical system (M, X_H) defines a vector field $\operatorname{Var}(X_H)$ on the tangent bundle TM. The dynamical system corresponding to $\operatorname{Var}(X_H)$ is called the *first variation system* of X_H on TM. In local coordinates (y^I, u^J) on TM, the first variation system is represented as

$$\dot y^I = -\Psi^{JI}(y)\,\frac{\partial H}{\partial y^I}, \tag{1.1}$$

$$\dot u^J = -\biggl(\frac{\partial \Psi^{JS}(y)}{\partial y^I}\,\frac{\partial H}{\partial y^S} + \Psi^{JS}\,\frac{\partial^2 H}{\partial y^S \partial y^I}\biggr)u^I. \tag{1.2}$$

In invariant terms, $\operatorname{Var}(X_H)$ can be defined by means of the canonical involution on TTM [1]. As is known, system (1.1), (1.2) is Hamiltonian relative to the tangent Poisson structure on TM [1].

Our goal is to study the linearized Hamiltonian dynamics at a given (singular) symplectic leaf. Suppose we are given a closed symplectic leaf (B, ω) of (M, Ψ) with symplectic structure

$$\omega = \frac{1}{2}\omega_{ij}(\xi)\,d\xi^i \wedge d\xi^j. \tag{1.3}$$

The restriction of X_H to B is a Hamiltonian vector field on (B,ω),

$$v_f = X_H|_B = \omega^{is}(\xi)\,\frac{\partial f}{\partial \xi^s},$$

where $f = H|_B$. Let $T_B M$ be the restriction of the tangent bundle TM to the leaf B. The *normal bundle* $E = T_B M/TB$ of B is a vector bundle $\pi\colon E \to B$ over B whose fiber E_ξ over ξ is a quotient space $E_\xi = T_\xi M/T_\xi B$.

Let $p\colon T_B M \to E$ be the natural projection. Since the submanifold B is invariant with respect to the flow of X_H, the vector field $\mathrm{Var}(X_H)$ has two invariant submanifolds TB and $T_B M$ in TM. One can show that the vector field $\mathrm{Var}(X_H)$ is projectible under p, i.e., there exists a unique vector field $\mathrm{var}_B(X_H)$ on E such that

$$(d_u p)\,\mathrm{Var}(X_H)(u) = \mathrm{var}_B(X_H)(p(u))$$

for every $u \in T_\xi M$ and $\xi \in B$. The dynamical system $(E, B, \mathrm{var}_B(X_H))$ is called the *first variation system* of X_H at B. The corresponding phase space E (the total space of the normal bundle of B) is of the same dimension as the original manifold, $\dim E = \dim M$. Moreover, $B \subset E$ (as the zero section of E) is invariant with respect to the flow of $\mathrm{var}_B(X_H)$. So, the first variation system represents a natural linearized model for the original Hamiltonian system X_H at B. We are interested in the following question: When is $\mathrm{var}_B(X_H)$ Hamiltonian relative to a natural Poisson structure on E? This problem appears as a first step in the study of the (nonlinear) Hamiltonian dynamics near a (singular) symplectic leaf in the context of perturbation theory. In general, the linearization procedure may destroy the Hamiltonian property for $\mathrm{var}_B(X_H)$. In the symplectic case, this effect was studied in [3–5]. In this paper, we formulate some results on the existence of a Hamiltonian structure for $\mathrm{var}_B(X_H)$, which are based on the notion of linearized Poisson structure of a symplectic leaf [6, 7].

2. EXISTENCE OF HAMILTONIAN STRUCTURES

Consider the dual bundle $E^* \subset T_B^* M$ of E, which is called the *conormal* bundle of the leaf B and coincides with the kernel of the bundle morphism $\Psi_B^\sharp\colon T_B^* M \to T_B M$, $\ker \Psi_B^\sharp = E^*$. Then E^* carries an intrinsic fiberwise Lie algebra structure $[\,\cdot\,,\,\cdot\,]_{\mathrm{fib}}$, which is uniquely determined by the condition: for arbitrary functions k and $\tilde{k}$ on M constant along the leaf B, we have

$$[\eta,\tilde{\eta}]_{\mathrm{fib}} = d(\{k,\tilde{k}\}_\Psi)|_B, \qquad \text{where} \quad \eta = dk|_B,\ \tilde{\eta} = d\tilde{k}|_B.$$

The Lie algebra bundle E^* is locally trivial with typical fiber $\mathfrak{g}$ called the *transverse Lie algebra* of the leaf B. So, the normal bundle E becomes a locally trivial Lie–Poisson bundle over the symplectic base (B,ω_B). The corresponding fiberwise Lie–Poisson structure induces the vertical Poisson tensor Λ on E called the *linearized transverse Poisson structure* of Ψ at B [8, 9].

Definition 2.1. A Poisson structure Π on E is said to be *compatible* if it is well defined in a neighborhood of the zero section B and satisfies the conditions:

(i) (B,ω_B) is a symplectic leaf of Π;

(ii) the linearized transverse Poisson structure of Π at B coincides with Λ.

By a *transversal* $\mathcal{L}$ to B we mean a subbundle of $T_B M$ which is complementary to TB,

$$T_B M = TB \oplus \mathcal{L}. \tag{2.1}$$

Let Fl_H^t be the flow of the Hamiltonian vector field X_H. Since $B \subset E$ is an invariant submanifold of X_H, the differential $d\,\mathrm{Fl}_H^t$ acts on $T_B M$ leaving TB invariant.

Theorem 2.2. *If the flow of the Hamiltonian vector field X_H admits an invariant transversal $\mathcal{L}$ to B,*

$$d\,\mathrm{Fl}_H^t(\mathcal{L}) = \mathcal{L}, \tag{2.2}$$

then the first variation system $\mathrm{var}_B(X_H)$ of X_H at B is Hamiltonian relative to a certain compatible Poisson structure Π on E and a function $F \in C^\infty(E)$,

$$\mathrm{var}_B(X_H) = \Pi^\natural(dF). \tag{2.3}$$

Below, to prove this theorem, we give a construction of the compatible Poisson structure Π and the Hamiltonian function F.

3. COMPATIBLE POISSON STRUCTURES

Recall a procedure [6, 7] which allows us, starting from the triple (M, Ψ, B), to construct a class of compatible Poisson structures on E parametrized by the transversals to B.

We have a natural decomposition

$$T_B E = TB \oplus E. \tag{3.1}$$

Pick a transversal $\mathcal{L}$ to B. It is clear that $\mathcal{L}$ is a subbundle of $T_B E$ isomorphic to E. The restriction of the projection p to the fiber $\mathcal{L}_\xi$ gives an isomorphism onto E_ξ.

By an *exponential map* associated with a transversal $\mathcal{L}$ to B we mean a diffeomorphism $\mathbf{f}$ from a neighborhood of B in E onto a neighborhood of B in M satisfying the conditions

$$\mathbf{f}|_B = \mathrm{id}_B, \qquad (d_\xi \mathbf{f})(e) = p^{-1}(e)$$

for every $e \in E_\xi$ and $\xi \in B$. In particular, $(d_\xi \mathbf{f})(E_\xi) = \mathcal{L}_\xi$. An exponential map exists because of the tubular neighborhood theorem.

Consider the pull-back $\mathbf{f}^*\Psi$ of the original Poisson structure Ψ via an exponential map $\mathbf{f}$. Fix a basis $\{e^\sigma\}$ of local sections of E^*. Let $\{e_\sigma\}$ be the dual basis of E. Consider a coordinate system (ξ^i, x^σ) on E, where the (ξ^i) are coordinates along B and $\{x^\sigma\}$ are the normal coordinates to B associated with the basis $\{e_\sigma\}$, $B = \{x = 0\}$. Then $\mathbf{f}^*\Psi$ is a compatible Poisson tensor on E whose bracket relations have the following decompositions at B:

$$\{\xi^i, \xi^j\}_{\mathbf{f}^*\Psi} = -\omega^{ij} - \omega^{is}\mathcal{R}_{sm\nu}\omega^{mj}x^\nu + O_2, \tag{3.2}$$

$$\{\xi^i, x^\sigma\}_{\mathbf{f}^*\Psi} = \omega^{ij}\theta^\sigma_{j\nu}x^\nu + O_2, \tag{3.3}$$

$$\{x^\alpha, x^\beta\}_{\mathbf{f}^*\Psi} = \lambda^{\alpha\beta}_\nu x^\nu + O_2. \tag{3.4}$$

Here $\omega^{is}(\xi)\omega_{sj}(\xi) = \delta^i_j$, the (ω_{sj}) are the coefficients of the symplectic form (1.3), and the $\lambda^{\alpha\beta}_\nu$, $\theta^\sigma_{\nu j}$, and $\mathcal{R}_{sm\nu}$ are smooth functions on B. The symbol O_k denotes a term of order k in the formal Taylor expansion of a function at $x = 0$. Note that the functions $\lambda^{\alpha\beta}_\nu = \lambda^{\alpha\beta}_\nu(\xi)$ are the structural constants of the Lie bracket on the fiber E with respect to the basis $\{e^\sigma(\xi)\}$.

Denote by $\Omega^k(B, E)$ the space of vector-valued k-forms on B with values in the space of smooth sections of E. In particular, $\Omega^0(B, E) = C^\infty(B; E)$ is the space of vector-valued functions on E. Let us introduce the matrix-valued 1-form $\theta^{\mathcal{L}} = (\theta^\alpha_\beta)$ and the vector-valued 2-form $\mathcal{R}^{\mathcal{L}} = (\mathcal{R}_\sigma) \in \Omega^2(B, E^*)$ with components

$$\theta^\alpha_\beta = \theta^\alpha_{i\beta}(\xi)\, d\xi^i, \qquad \mathcal{R}_\sigma = \frac{1}{2}\mathcal{R}_{ij\sigma}(\xi)\, d\xi^i \wedge d\xi^j$$

relative to the basis $\{e^\alpha\}$. It can be shown [10] that there exists a linear connection $\nabla^{\mathcal{L}}$ on E whose connection form relative to the basis $\{e_\sigma\}$ is given precisely by $\theta^{\mathcal{L}}$. The main feature of $\nabla^{\mathcal{L}}$ is that the parallel transport of $\nabla^{\mathcal{L}}$ preserves the fiberwise Lie–Poisson structure of E. Moreover, the curvature of $\nabla^{\mathcal{L}}$ is related to $\mathcal{R}^{\mathcal{L}}$ by

$$\operatorname{Curv}^{\nabla^{\mathcal{L}}} = d\theta^{\mathcal{L}} + \theta^{\mathcal{L}} \wedge \theta^{\mathcal{L}} = -\operatorname{ad}^* \circ \mathcal{R}^{\mathcal{L}}.$$

Here ad^* is the co-adjoint operator on the fibers of E. Denote by

$$\operatorname{hor}_i \overset{\text{def}}{=} \frac{\partial}{\partial \xi^i} - \theta^\sigma_{i\nu}(\xi) x^\nu \frac{\partial}{\partial x^\sigma} \tag{3.5}$$

the horizontal lift of the basic vector field $\partial/\partial\xi^i$, $i = 1, \dots, \dim B$. Let us also define the scalar 2-form $\mathcal{F} = \frac{1}{2}\mathcal{F}_{ij}(\xi, x)\, d\xi^i \wedge d\xi^j$ on E with coefficients

$$\mathcal{F}_{ij}(\xi, x) \overset{\text{def}}{=} \omega_{ij}(\xi) - x^\nu \mathcal{R}_{ij\nu}(\xi). \tag{3.6}$$

Note that $\mathcal{F}_{ij}(\xi, 0) = \omega_{ij}(\xi)$ and hence $\det[\mathcal{F}_{ij}(\xi, 0)] \neq 0$ for all $\xi \in B$. So, the 2-form $\mathcal{F}$ is *nondegenerate* in the open domain

$$\mathcal{N} = \{(\xi, x) \in E \mid \det[\mathcal{F}_{ij}(\xi, x)] \neq 0\}$$

containing B. The elements of the inverse of $[\mathcal{F}_{ij}]$ will be denoted by

$$\mathcal{F}^{ij} = \mathcal{F}^{ij}(\xi, x), \qquad \mathcal{F}^{is}\mathcal{F}_{sj} = \delta^i_j.$$

Introduce the bivector field $\Pi_{\mathcal{L}}$ on $\mathcal{N} \subseteq E$ associated to the data $(\nabla^{\mathcal{L}}, \mathcal{R}^{\mathcal{L}})$, which is defined by

$$\Pi_{\mathcal{L}} \overset{\text{def}}{=} -\frac{1}{2}\mathcal{F}^{ij} \operatorname{hor}_i \wedge \operatorname{hor}_j + \Lambda. \tag{3.7}$$

Here the bivector field Λ on E is given by

$$\Lambda = \frac{1}{2}\lambda^{\alpha\beta}_\nu(\xi) x^\nu \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}$$

and defines the linearized transverse Poisson structure of Ψ at B.

Proposition 3.1 [8]**.** *For every transversal $\mathcal{L}$, the bivector field $\Pi_{\mathcal{L}}$ in* (3.7) *determines a compatible Poisson tensor on $\mathcal{N} \subseteq E$.*

As is known, the Poisson structure $\Pi_{\mathcal{L}}$ is independent of the choice of a transversal $\mathcal{L}$ up to an isomorphism in a neighborhood of B (see [6, 7]). This fact gives rise to the notion of linearized Poisson structure of a given symplectic leaf.

Suppose we are given a *linear vector field* $\mathcal{V}_f$ on E, which descends to a Hamiltonian vector field v_f on (B, ω),

$$\mathcal{V}_f = v^i_f(\xi)\frac{\partial}{\partial \xi^i} + V^\alpha_\nu(\xi) x^\nu \frac{\partial}{\partial x^\alpha},$$

where $V = (V^\alpha_\nu(\xi))$ is a matrix-valued function on B.

Proposition 3.2. *A linear vector field $\mathcal{V}$ is Hamiltonian relative to the Poisson structure $\Pi_{\mathcal{L}}$ and to a fiberwise linear function $F = \pi^* f - \langle x, \eta\rangle$, $\eta \in C^\infty(B, E)$ if and only if the pair (η, V) satisfies the equations on B:*

$$d\eta - (\theta^{\mathcal{L}})^T \eta = -\mathbf{i}_{v_f}\mathcal{R}^{\mathcal{L}}, \tag{3.8}$$

$$V = -(\mathbf{i}_{v_f}\theta^{\mathcal{L}} + \operatorname{ad}^* \circ \eta). \tag{3.9}$$

Here $\mathbf{i}_{v_f}$ denotes the inner product of the vector field v_f and a differential form on B.

4. INVARIANT TRANSVERSALS AND DYNAMICAL TORSION

Let $\mathbf{f}$ be an exponential map associated with a transversal $\mathcal{L}$. Consider the pull-back Poisson structure $\mathbf{f}^*\Psi$ on E. The symplectic leaf (B,ω) is also the symplectic leaf of $\mathbf{f}^*\Psi$ with normal bundle identified with E. The Taylor expansion at $x=0$ for the Hamiltonian system $(E,\mathbf{f}^*\Psi,\mathbf{f}^*H=H\circ\mathbf{f})$ by using (3.2)–(3.4) gives

$$\frac{d\xi^i}{dt}=\omega^{is}(\xi)\,\frac{\partial f(\xi)}{\partial\xi^s}+\Upsilon^i_\nu(\xi)x^\nu+O_2, \tag{4.1}$$

$$\frac{dx^\sigma}{dt}=[\lambda^{\sigma\beta}_\nu(\xi)\eta_\beta(\xi)-\theta^\sigma_{j\nu}(\xi)\omega^{js}(\xi)\frac{\partial f(\xi)}{\partial\xi^s}]x^\nu+O_2. \tag{4.2}$$

Here $f(\xi)=\mathbf{f}^*H(\xi,0)$ and

$$\eta_\nu(\xi)=-\frac{\partial(\mathbf{f}^*H)}{\partial x^\nu}(\xi,0). \tag{4.3}$$

Moreover, the smooth functions Υ^i_ν on B are given by

$$\Upsilon^i_\nu\overset{\text{def}}{=}-\omega^{ij}\,\frac{\partial\eta_\nu}{\partial\xi^j}+\omega^{ij}\theta^\alpha_{j\nu}\eta_\alpha+\omega^{is}\mathcal{R}_{sm\nu}\omega^{mj}\,\frac{\partial f}{\partial\xi^j}. \tag{4.4}$$

We can associate with η_ν the global vector function $\eta^{\mathcal{L}}=\eta_\nu(\xi)\otimes e^\nu(\xi)$ on B (a section of E^*). Moreover, one can show that Υ^i_ν define the vector field

$$\Upsilon^{\mathcal{L}}=\Upsilon^i_\nu(\xi)x^\nu\,\frac{\partial}{\partial\xi^i}$$

on E, which can be called the *torsion* of the flow of $X_{\mathbf{f}^*H}$ relative to the splitting (2.1). The vanishing of the torsion $\Upsilon^{\mathcal{L}}=0$ means that the transversal $\mathcal{L}$ is invariant with respect to $d\,\mathrm{Fl}^t_{\mathbf{f}^*H}$. We will see below how $\eta^{\mathcal{L}}$ and $\Upsilon^{\mathcal{L}}$ depend on the choice of $\mathcal{L}$.

Now, it follows from (4.2) that the first variation system of the Hamiltonian vector field $X_{\mathbf{f}^*H}$ at B has the form

$$\frac{d\xi}{dt}=v_f, \tag{4.5}$$

$$\frac{dx}{dt}=-(\mathrm{ad}^*_\eta+\mathbf{i}_{v_f}\theta)x. \tag{4.6}$$

The corresponding vector field can be represented as

$$\mathrm{var}_B(X_{\mathbf{f}^*H})=\mathrm{hor}_{v_f}-\left\langle\mathrm{ad}^*_\eta\,x,\,\frac{\partial}{\partial x}\right\rangle. \tag{4.7}$$

Here hor_{v_f} is the horizontal lift of the Hamiltonian vector field v_f relative to the connection $\nabla^{\mathcal{L}}$.

Next, let us consider two transversals $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ to B. Let $l\colon T_BM\to\widetilde{\mathcal{L}}$ be the projection along TB according to decomposition (3.1). Given a basis $\{n_\sigma\}$ of (local) sections of $\mathcal{L}$, we define the basis $\{\tilde n_\sigma\}$ of sections of $\widetilde{\mathcal{L}}$ by $\tilde n_\sigma(\xi)=l_\xi(n_\sigma(\xi))$. Then we have

$$\tilde n_\sigma(\xi)=n_\sigma+u_\sigma(\xi), \tag{4.8}$$

where the u_σ are some vector fields on B, $u_\sigma(\xi)\in T_\xi B$. Using these vector fields and the symplectic 2-form ω on B, we define the vector-valued 1-form $\varrho\in\Omega^1(B,E^*)$ by

$$\varrho\overset{\text{def}}{=}-(\mathbf{i}_{u_\nu}\omega)\otimes e^\nu, \tag{4.9}$$

or, in coordinates, by $\varrho_{i\nu} = \omega_{ij} u^j_\nu$. Since ω is nondegenerate, for a fixed $\mathcal{L}$, formula (4.9) gives a one-to-one correspondence between the set of all transversals to B and the space $\Omega^1(B, E^*)$ of vector-valued 1-forms on B. Direct computations show that the data corresponding to $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are related by

$$\nabla^{\widetilde{\mathcal{L}}} = \nabla^{\mathcal{L}} - \operatorname{ad}^* \circ \varrho, \tag{4.10}$$

$$\mathcal{R}^{\widetilde{\mathcal{L}}} = \mathcal{R}^{\mathcal{L}} + (\nabla^{\mathcal{L}})^* \varrho^{\mathcal{L}} + \frac{1}{2}[\varrho^{\mathcal{L}} \wedge \varrho^{\mathcal{L}}]. \tag{4.11}$$

Moreover,

$$\eta^{\widetilde{\mathcal{L}}}_\sigma = \eta^{\mathcal{L}}_\sigma - L_{u_\sigma} f. \tag{4.12}$$

These relations imply the following fact.

Proposition 4.1. *The first variation system of the Hamiltonian system $X_{\mathbf{f}^*H}$ is independent of the choice of the exponential map $\mathbf{f}$,*

$$\operatorname{var}_B(X_{\mathbf{f}^*H}) = \operatorname{var}_B(X_H).$$

The comparison of (3.8) and (4.7) with (4.4) shows that the condition $\Upsilon^{\mathcal{L}} = 0$ is equivalent to Eqs. (3.8), (3.9) for $\eta^{\mathcal{L}}$ in (4.3). So, by Proposition 3.2, we derive the main result.

Theorem 4.2. *Assume that there is a transversal $\mathcal{L}$ invariant with respect to $d\,\mathrm{Fl}^t_H$. Let $\Pi^{\mathcal{L}}$ be the corresponding Poisson structure in* (3.7). *Then $\operatorname{var}_B(X_H)$ is Hamiltonian relative to $\Pi^{\mathcal{L}}$ and to the function*

$$F^{\mathcal{L}}(\xi, x) = f(\xi) - \langle x, \eta^{\mathcal{L}}(\xi)\rangle.$$

In the particular case, by analyzing Eqs. (3.8), (3.9), one can derive the following criterion.

Theorem 4.3. *If the transverse Lie algebra $\mathfrak{g}$ of the symplectic leaf B is semisimple, then the existence of an X_H-invariant transversal $\mathcal{L}$ to B is a necessary and sufficient condition for the first variation system $\operatorname{var}_B(X_H)$ to be Hamiltonian in the class of compatible Hamiltonian structures on E.*

Using this criterion, we can describe the possible obstructions to the existence of a Hamiltonian structure in the following simple situation.

Example 4.4. Let $B = (a, b) \times \mathbb{S}^1$ be the 2-cylinder equipped with the canonical symplectic structure $\omega = ds \wedge d\tau$, where $s \in (a, b)$ and $\tau \pmod{2\pi}$ is the angle variable on the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Consider the cyclic Poisson brackets on $\mathbb{R}^3$ associated with the Lie algebra so(3),

$$\{x^1, x^2\} = x^3, \qquad \{x^2, x^3\} = x^1, \qquad \{x^3, x^1\} = x^2.$$

Let us think of $M = (a, b) \times \mathbb{S}^1 \times \mathbb{R}^3$ as a Poisson manifold equipped with product Poisson structure. Clearly, $B = (a, b) \times \mathbb{S}^1$ is a symplectic leaf of M. Consider the Hamiltonian system on M corresponding to the function

$$H = 1 - \langle x, \boldsymbol{\phi}(s, \tau)\rangle + O_2,$$

where $\boldsymbol{\phi}(s, \tau) = \boldsymbol{\phi}(s, \tau + 2\pi)$ is a smooth vector-function 2π-periodic in τ. The corresponding first variation system of X_H at B is of the form

$$\dot{s} = 0, \qquad \dot{\tau} = 1, \tag{4.13}$$

$$\frac{d\mathbf{x}}{d\tau} = \boldsymbol{\phi}(s, \tau) \times \mathbf{x}. \tag{4.14}$$

Then (4.14) presents a one-parameter family of periodic linear systems on $\mathbb{R}^3$. Let $M(s)$ be the corresponding monodromy matrix smoothly depending in s. One can show that system (4.13), (4.14) admits a compatible Hamiltonian structure if and only if $M(s)$ satisfies the Lax type equation

$$\frac{d\mathcal{M}(s)}{ds} = [\mathcal{M}(s), \mathbb{A} \circ \boldsymbol{\mu}(s)]$$

for a certain smooth vector-function $\boldsymbol{\mu}(s)$. Here $\mathbb{A} \circ \boldsymbol{\mu}(s)$ denotes the 3×3 skew-symmetric matrix of the cross product on $\mathbb{R}^3$. This implies that the *spectrum of* $\mathcal{M}(s)$ *is independent of* s (i.e., the monodromy has the property of *isospectral deformation*). Otherwise, $\operatorname{spec} \mathcal{M}(s)$ varies with s, system (4.13), (4.14) does not possess a *compatible Hamiltonian* structure.

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