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# Remarks on Cotton solitons

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**Abstract**
In this note, we show that the potential vector field of a Cotton soliton $(M, g, V)$ is an infinitesimal harmonic transformation, and we use it to give another proof of the triviality of compact Cotton solitons. Moreover, we extend this triviality result to the complete case by imposing certain regularity conditions on the potential vector field $V$.

**Keywords** Cotton solitons · Infinitesimal harmonic transformation · Locally conformally flat · Killing vector field

**Mathematics Subject Classification** 53C21 · 53C25

## 1 Introduction

The evolution of a Riemannian metric $g$ on a three-dimensional smooth manifold $M$ to a metric $g(t)$ in time $t$ through the equation

$$\frac{\partial}{\partial t} g(t) = \kappa C_{g(t)},$$

where $C_{g(t)}$ is the $(0, 2)$-Cotton tensor of $(M, g)$, is called the Cotton flow and was introduced by Kisisel et al. [9]. For $\kappa = 1$, the corresponding soliton, known as the Cotton soliton, is a three-dimensional Riemannian manifold $(M, g)$ with a vector field $V$ and a real constant $\lambda$ such that

$$L_V g + C = \lambda g, \tag{1.1}$$

where $L_V$ denotes the Lie-derivative operator along the vector field $V$. A Cotton soliton is said to be shrinking, steady, or expanding when $\lambda > 0, = 0, < 0$, respectively. It

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is said to be trivial if $V$ is Killing and $M$ is locally conformally flat. A Lorentzian Lie-group admits a non-trivial left-invariant Cotton soliton if and only if the Cotton operator is nilpotent [2]. If $V = \nabla f$ in (1.1), for a smooth function $f$, where $\nabla$ denotes the gradient operator of $g$, then it is known as a gradient Cotton soliton, in which case (1.1) assumes the form

$$2Hessf + C = \lambda g,$$

where $Hessf$ denotes the Hessian of $f$ with respect to $g$. Lorentzian metrics with nilpotent Ricci operator allow the existence of gradient Cotton solitons [3]. The Cotton tensor is given by

$$C_{ijk} = (\nabla_i S)_{jk} - (\nabla_j S)_{ik},$$

where $S$ is the Schouten tensor given by

$$S_{ij} = R_{ij} - \frac{r}{4} g_{ij},$$

where $R_{ij}$ are the components of the Ricci tensor and $r$ denotes the scalar curvature of $M$. Now, the (0, 2)-Cotton tensor (also known as the Cotton–York tensor) is given by

$$C_{ij} = \frac{1}{2\sqrt{g}} C_{mni} \epsilon^{nml} g_{lj},$$

where $\epsilon^{123} = 1$. The Cotton tensor is trace-free and divergence-free [16] and vanishes if and only if $M$ is locally conformally flat. For more details on Cotton flow and its physical aspects, we refer the reader to [4, 6, 8, 10].

Following Nouhaud [11], we say that a vector field $V$ is an infinitesimal harmonic transformation if trace $(L_V \nabla) = 0$, i.e. $(L_V \nabla)(e_i, e_i) = 0$, where $(e_i)$, $i = 1, ..., n$ is any local orthonormal frame on $M$. It was shown by Stepanov and Shandra [13] that a vector field $V$ is an infinitesimal harmonic transformation on a Riemannian manifold $(M, g)$ if and only if

$$\Delta V = 2QV, \tag{1.2}$$

where $Q$ is the Ricci operator associated with the Ricci tensor $Ric(X, Y) = g(QX, Y)$, and the Laplacian $\Delta$ is determined by the Weitzenbӧck formula

$$\Delta V = \bar{\Delta} V + QV, \tag{1.3}$$

where $\bar{\Delta} V$ is the rough Laplacian given by

$$\sum_i \{\nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i}\} V. \tag{1.4}$$

One can infer from equation (1.2) that $V \in ker\Box$, where $\Box$ denotes the Yano operator (see Yano [14]) given by

$$\Box V = \Delta V - 2QV. \tag{1.5}$$

A Killing vector field is obviously an example of an infinitesimal harmonic transformation [13]. Ghosh [7] proved that an almost Ricci soliton reduces to a Ricci soliton if and only if the potential vector field is an infinitesimal harmonic transformation. In [12], Stepanov and Shelepova showed that the potential vector field of a Ricci soliton is an infinitesimal harmonic transformation and used it to show that there exist no non-trivial expanding Ricci solitons on a compact manifold $M$.

The purpose of this note is to show that the potential vector field of a Cotton soliton is an infinitesimal harmonic transformation and then apply it to discuss the triviality of Cotton solitons in the compact and non-compact cases, respectively.

**Theorem 1.1** *Let $(M, g, V)$ be a Cotton soliton. Then, $V$ is an infinitesimal harmonic transformation.*

This fact can be used to give an alternative demonstration of the following assertion for a compact Cotton soliton [3].

**Proposition 1.1** *Let $(M, g, V)$ be a compact Cotton soliton. Then, $(M, g)$ is locally conformally flat and $V$ is a Killing vector field.*

***Remark 1.1*** Our proof of Proposition 1.1 is a consequence of the fact that $V$ is an infinitesimal harmonic transformation for a Cotton soliton, and is different from the proof of Calviño-Louzao et al. [3] which uses the following formula

$$\langle L_\xi g, \varphi \rangle = 2div(i_\xi \varphi) - 2(div\varphi)(\xi)$$

for a symmetric $(0, 2)$-tensor field $\varphi$ and an arbitrary smooth vector field $\xi$ on $M$, where $i_\xi \varphi(.) = \varphi(\xi, .)$.

Calviño-Louzao et al. [3] showed the existence of a complete, non-trivial shrinking Cotton soliton on the Heisenberg group. Recently, Cunha and Silva Junior [5] studied complete, non-compact Cotton solitons under various assumptions to infer the triviality of Cotton solitons. This motivates us to extend Proposition 1.1 to the complete case by using the result proved in Theorem 1.1 and imposing certain regularity conditions on $V$. More precisely, we prove the following theorem.

**Theorem 1.2** *Let $(M, g, V)$ be a complete, non-compact Cotton soliton with closed $V$. Then $M$ is locally conformally flat and $V$ is parallel if any one of the following conditions are met*

(i) *$M$ is parabolic and $|V| \in L^\infty(M)$.*
(ii) *$|V| \in L^p(M)$ for $p > 1$.*
(iii) *$|V|$ converges to zero at infinity.*

A vector field $V$ is said to be closed if its metrically associated one-form is closed (i.e. $dv = 0$). Also, we use the notation $L^p(M) = \{u : M \to \mathbb{R}; \int_M |u|^p < +\infty\}$ for each $p \geq 1$.

***Remark 1.2*** Theorem 1.2 is different from Theorems 4 and 6 of Cunha and Silva Junior [5] in the sense that by limiting our scope to the case when $V$ is closed, we avoid imposing restrictions on the curvature of the manifold.

***Remark 1.3*** Theorem 1.2 generalizes Theorems 2 and 5 of Cunha and Silva Junior [5] in the sense that $V$ closed need not imply $V$ gradient, unless $M$ is simply connected.

## 2 Preliminaries

In this section, we prepare a lemma and recall some useful results from the literature, which eventually will be used in proving our main assertions.

**Lemma 2.1** *Let $(M, g, V)$ be a Cotton soliton. Then,*

*(a)* $\nabla_X V = -\frac{1}{2}\mathcal{C}X + \frac{1}{2}\lambda X + FX$, *where* $C(X, Y) = g(\mathcal{C}X, Y)$, $\mathcal{C}$ *is a tensor field of type* $(1, 1)$ *associated with* $C$, *and* $F$ *is a skew-symmetric tensor field of type* $(1, 1)$ *such that* $g(FX, Y) = -g(X, FY)$.
*(b)* $divV = \frac{n}{2}\lambda$.
*(c)* $Ric(X, V) = (divF)X$, *where* $Ric$ *denotes the Ricci tensor.*

***Proof*** Equation (1.1) can be written as

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + C(X, Y) = \lambda g(X, Y). \tag{2.1}$$

The exterior derivative $dv$ of the one-form $v$ metrically associated with $V$ is given by

$$\frac{1}{2}g(\nabla_X V, Y) - \frac{1}{2}g(\nabla_Y V, X) = (dv)(X, Y). \tag{2.2}$$

As $dv$ is skew-symmetric, we define a tensor field $F$ of type $(1, 1)$ by

$$(dv)(X, Y) = g(FX, Y).$$

Thus, equation (2.2) assumes the form

$$g(\nabla_X V, Y) - g(\nabla_Y V, X) = 2g(FX, Y). \tag{2.3}$$

Adding equations (2.1) and (2.3) side by side leads us to

$$2g(\nabla_X V, Y) + C(X, Y) = \lambda g(X, Y) + 2g(FX, Y),$$

i.e.

$$\nabla_X V = -\frac{1}{2}\mathcal{C}X + \frac{1}{2}\lambda X + FX, \tag{2.4}$$

which proves (a). Contracting (2.4) with respect to $X$ and noting that the Cotton tensor is trace-free give

$$divV = \frac{n}{2}\lambda, \tag{2.5}$$

which proves (b). Using (2.4), we compute $R(Y, X)V$ to get

$$R(Y, X)V = \frac{1}{2}(\nabla_X \mathcal{C})Y - \frac{1}{2}(\nabla_Y \mathcal{C})X + (\nabla_Y F)X - (\nabla_X F)Y. \tag{2.6}$$

Contracting (2.6) with respect to $Y$ and using the trace-freeness and divergence-freeness of the Cotton tensor provide

$$Ric(X, V) = (divF)X, \tag{2.7}$$

which proves part (c) of the assertion, thereby completing the proof. □

Let us recall a very useful lemma due to Yau [15].

**Lemma 2.2** (Yau [15]) *Let $u$ be a non-negative smooth subharmonic function on a complete Riemannian manifold $M$. If $u \in L^p(M)$ for $p > 1$, then $u$ is constant.*

A function $f$ is said to be subharmonic if $\Delta f \geq 0$. Further, a Riemannian manifold $(M, g)$ is said to be parabolic if the unique subharmonic functions on $M$ which are bounded from above are constant functions, i.e. if $u \in C^\infty(M)$ with $\Delta u \geq 0$ and $sup_M u < +\infty$, then $u$ is constant.

We also recall the maximum principle at infinity due to Alías et al. [1] and state it as the following lemma.

**Lemma 2.3** (Alías et al. [1]) *Let $(M, g)$ be a complete non-compact Riemannian manifold and $X$ be an arbitrary smooth vector field on $M$. Assume that there exists a non-negative, non-identically vanishing function $u \in C^\infty(M)$ which converges to zero at infinity and $g(\nabla u, X) \geq 0$. If $divX \geq 0$ on $M$, then $g(\nabla u, X) \equiv 0$ on $M$.*

A continuous function $u \in C^0(M)$ is said to converge to zero at infinity if it satisfies the condition

$$\lim_{d(x,x_0)\to\infty} u(x) = 0,$$

where $d(., x_0) : M \to [0, \infty)$ denotes the Riemannian distance of a complete non-compact Riemannian manifold $M$ measured from a fixed point $x_0 \in M$.

Next, we would require the well-known Bochner formula.

**Lemma 2.4** *For any vector field $V$, we have*

$$\frac{1}{2}\Delta|V|^2 = Ric(V, V) - g(\Delta V, V) + |\nabla V|^2. \tag{2.8}$$

## 3 Proofs of the results

***Proof of Theorem 1.1.*** Taking the covariant derivative of (1.1) along an arbitrary vector field $X$, we acquire

$$(\nabla_X L_V g)(Y, Z) + (\nabla_X C)(Y, Z) = 0. \tag{3.1}$$

Now, the commutation formula (Yano [14]):

$$(L_V \nabla_X g - \nabla_X L_V g - \nabla_{[V,X]} g)(Y, Z) = -g((L_V \nabla)(X, Y), Z) - g((L_V \nabla)(X, Z), Y).$$

reduces to

$$(\nabla_X L_V g)(Y, Z) = g((L_V \nabla)(X, Y), Z) + g((L_V \nabla)(X, Z), Y). \tag{3.2}$$

Comparing (3.1) and (3.2) gives

$$g((L_V \nabla)(X, Y), Z) + g((L_V \nabla)(X, Z), Y) + (\nabla_X C)(Y, Z) = 0. \tag{3.3}$$

Permuting (3.3) cyclically twice over $X$, $Y$ and $Z$, we get

$$g((L_V \nabla)(Y, Z), X) + g((L_V \nabla)(Y, X), Z) + (\nabla_Y C)(Z, X) = 0. \tag{3.4}$$

$$g((L_V \nabla)(Z, X), Y) + g((L_V \nabla)(Z, Y), X) + (\nabla_Z C)(X, Y) = 0. \tag{3.5}$$

Subtracting the sum of (3.4) and (3.5) from (3.3), we achieve

$$2g((L_V \nabla)(Y, Z), X) + (\nabla_Y C)(Z, X) + (\nabla_Z C)(X, Y) - (\nabla_X C)(Y, Z) = 0. \tag{3.6}$$

where we used the fact that $(L_V \nabla)(X, Y) = (L_V \nabla)(Y, X)$. Equation (3.6) can also be written as

$$2g((L_V \nabla)(X, Y), Z) + (\nabla_X C)(Y, Z) + (\nabla_Y C)(Z, X) - (\nabla_Z C)(X, Y) = 0. \tag{3.7}$$

Since $C(X, Y) = g(\mathcal{C}X, Y)$, equation (3.7) can be exhibited as

$$2g((L_V \nabla)(X, Y), Z) + g((\nabla_X \mathcal{C})Y, Z) + g(\nabla_Y \mathcal{C})Z, X) - g((\nabla_Z \mathcal{C})X, Y) = 0. \tag{3.8}$$

Now, using the formula

$$g((L_V \nabla)(X, Y), Z) = g(\nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V - R(X, V)Y, Z)$$

in equation (3.8) and subsequently setting $X = Y = e_i$, where $(e_i)$ is an orthonormal frame field on $M$, and using the definition of rough Laplacian given by (1.4), along with the fact that the Cotton tensor is trace-free and divergence-free, we get

$$Ric(V, Z) - g(\bar{\Delta} V, Z) = 0. \tag{3.9}$$

The use of equation (1.3) in (3.9) and factoring out $Z$ entails

$$\Delta V = 2QV.$$

Hence, $V$ is an infinitesimal harmonic transformation. This completes the proof. □

***Proof of Proposition 1.1*** Since $V$ is an infinitesimal harmonic transformation, in view of (1.5), we have

$$\square V = 0.$$

Integrating the equation in Lemma 2.1 (b) and using divergence theorem give $\lambda = 0$, and hence,

$$divV = 0.$$

At this stage, we appeal to Theorem 3.4 of Yano [14], which states that, “A necessary and sufficient condition for a vector field $V$ on a compact Riemannian manifold to be Killing is that $\square V = 0$ and $divV = 0$", in order to infer that $V$ is Killing, and hence $C = 0$, i.e. $M$ is locally conformally flat. This completes the proof. □

***Proof of Theorem 1.2*** Since the potential vector field of a Cotton soliton is an infinitesimal harmonic transformation, i.e. $\Delta V = 2QV$, equation (2.8) can be written as

$$\frac{1}{2}\Delta|V|^2 = |\nabla V|^2 - Ric(V, V). \tag{3.10}$$

Further, by assumption, $V$ is closed; therefore, using (2.2) and (2.3) provides $F \equiv 0$. The use of this in (2.7) shows that $Ric(V, V) = 0$ for the choice $X = V$. This finding reduces (3.10) to

$$\frac{1}{2}\Delta|V|^2 = |\nabla V|^2 \geq 0. \tag{3.11}$$

To prove part (i), we observe from (3.11) that $|V|^2$ is a subharmonic function. Since $M$ is parabolic, $|V|^2$ is constant on $M$. Therefore, (3.11) entails $V$ is parallel. This implies $V$ is Killing, and hence, $divV = 0$. The last equality and Lemma 2.1 (b) give $\lambda = 0$. The use of these findings in (1.1) leads us to $C = 0$, i.e. $M$ is locally conformally flat.

For part (ii), $|V|^2$ is subharmonic in view of (3.11). Since $|V|^2 \in L^p(M)$ for $p > 1$, we invoke Lemma 2.2 to infer that $|V|^2$ is constant on $M$. Thus, (3.11) shows that $V$ is parallel. With the same reasoning as elucidated in the last part of the proof of part (i), we conclude that $V$ is Killing and $M$ is locally conformally flat.

Finally, we prove part (iii) by contradiction. Suppose $|V|^2$ is non-constant on $M$. We consider the function $u = |V|^2$. We observe that $u$ is non-negative and non-identically vanishing and converges to zero at infinity. Let us consider the smooth vector field $X = \nabla|V|^2$ on $M$. For this vector field, we have

$$g(\nabla u, X) = |\nabla|V|^2|^2 \geq 0.$$

Also, $divX = div(\nabla|V|^2) = \Delta|V|^2 = 2|\nabla V|^2 \geq 0$. That is, $divX \geq 0$ on $M$. Hence, by Lemma 2.3, $|V|^2$ is constant on $M$, thus arriving at a contradiction. Therefore, $|V|$ is constant, and in view of (3.11), $V$ is parallel. Again, appealing to the same arguments as in the last part of the proof of part (i), we arrive at our conclusion. This completes the proof. □

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## Declarations

**Competing interests** The authors have no relevant financial or non-financial interests to disclose.

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