Algorithm for differential equations for Feynman integrals in general dimensions

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Abstract

We present an algorithm for determining the minimal order differential equations associated with a given Feynman integral in dimensional or analytic regularisation. The algorithm is an extension of the Griffiths–Dwork pole reduction adapted to the case of twisted differential forms. In dimensional regularisation, we demonstrate the applicability of this algorithm by explicitly providing the inhomogeneous differential equations for the multi-loop two-point sunset integrals: up to 20 loops for the equal-mass case, the generic mass case at two- and three-loop orders. Additionally, we derive the differential operators for various infrared-divergent two-loop graphs. In the analytic regularisation case, we apply our algorithm for deriving a system of partial differential equations for regulated Witten diagrams, which arise in the evaluation of cosmological correlators of conformally coupled ϕ^4 theory in four-dimensional de Sitter space.

Contents

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1 Introduction

Feynman integrals are key ingredients in various areas of physics, and their accurate calculation, whether analytically or numerically, remains a significant hurdle in advancing our understanding of physical phenomena. In particular, identifying the specific types of special functions required to evaluate Feynman integrals has been an ongoing challenge since the early days of quantum field theory [1, 2] and continues to be an active research field as recently reviewed, for instance in [3–8].

The set of differential operators acting on a Feynman integral gives important information about its analytic nature. Moreover, the differential equation is important for evaluating physical observables by solving the system of differential equations associated with the Feynman integrals, either analytically, in perturbation with respect to the kinematic parameters or numerically. For instance, the differential operator has real singularities at the position of thresholds and pseudo-thresholds, and the order of the differential operator is connected to the underlying algebraic geometry of the singular locus of the integrand [9]. Intriguingly, there is growing evidence suggesting that certain Feynman integrals correspond to relative period integrals of singular Calabi–Yau geometries, a connection explored in a number of studies, including [10–24]. In addition, it has been remarked that correlation functions [25] and cosmological correlators [26, 27] of a conformally coupled ϕ^4 field in four dimensions can be expressed in terms of flat space Feynman integrals. The regulation of ultraviolet divergences in this case leads to integrals in analytical regularisation.

In this work, we give an algorithmic procedure for deriving such differential equations and the inhomogeneous part without having to go through the integral reduction

to master integrals and the construction of a reducible system of differential equations satisfied by the set of master integrals. Among the motivations for finding a short-cut to derive differential equations without relaying on master integrals reduction is that the integration-by-parts reduction leads to large system of master integrals that may obscure the algebraic geometry underlying the analytic structure of Feynman integrals. Another motivation is the application to cosmological correlators which give rise to analytic regularisation for which the commonly used integration by part algorithms are not developed out-of-the-box. Finding a system of partial differential equations (PDEs) is also useful for generalised Feynman integrals in the context of Gel'fand–Kapranov–Zelevinskiĭ (GKZ) systems, which gives a D-module of differential operators acting on the Feynman integral [16, 28–35]. However, the transition of this D-module to the PDEs of Feynman integrals requires a restriction which is highly non-trivial and still an open problem [29, 36–38].

We work with the regularised parametric representation of a Feynman integral $I_\Gamma^{\epsilon,\kappa} = \int_{x_i \geq 0} \omega_\Gamma^{\epsilon,\kappa} \mathrm{d}x_1 \cdots \mathrm{d}x_n$ attached to a graph Γ (we refer to Sect. 2 for details) with

$$\omega_\Gamma^{\epsilon,\kappa} = \frac{\mathbf{U}_\Gamma^{\nu_1+\cdots+\nu_n-(L+1)D}}{\mathbf{F}_\Gamma^{\nu_1+\cdots+\nu_n-LD}} \prod_{i=1}^{n} x_i^{\nu_i-1}, \tag{1}$$

with $D = 2\delta - 2\epsilon$ with δ a positive integer for dimensional regularisation and $\nu_i = \nu_i + \mu_i \kappa$ for analytic regularisation. In the case when $\epsilon = \kappa = 0$ and δ a positive integer, the exponents in (1) are integers and we have a rational differential form. One may then use the Griffiths–Dwork pole reduction [39–43] applied to the case of Feynman integrals [1, 12, 28, 44–46] for determining the minimal order differential operators associated with a given Feynman integral. In integer space-time dimensions and without analytic regulator, the integrand of the Feynman integral is a rational differential form to which one can apply the generalised Griffith-Dwork algorithm [46]. When working in dimensional regularisation, i.e. $\epsilon \neq 0$, or analytic regularisation, i.e. $\kappa \neq 0$, the integrand is a twisted differential form. One possible approach is a direct application of the Griffiths–Dwork pole reduction [44, 45] or the creative telescoping algorithm [47–50] but this approach leads to large linear systems limiting its use for Feynman graphs with many legs or many loops. Therefore, in this work, we give an extension of the Griffiths–Dwork reduction algorithm which make an essential use of the fact that the twist is built from the Symanzik graph polynomials $\mathbf{U}_\Gamma$ and $\mathbf{F}_\Gamma$. This reduces the size of the linear system to be solved for determining the coefficients of the differential operator. Because this linear system is generically dense and of large size, we use the finite field package `FiniteFlow` [51] to derive analytic solutions.

This way we can analyse how the space-time dimension or the analytic regulator affects the minimal order of the differential operators.

This paper is organised as follows. In Sect. 2, we review the parametric representation of Feynman integrals setting our notation for the dimensionally and analytically regulated integrals in Sect. 2.2. In Sect. 3, we present the algorithmic procedure for deriving the differential equations. In Sect. 3.1, we generalise the Griffiths–Dwork pole reduction to the case of the twisted differential $\omega_\Gamma^{\epsilon,\kappa}$, and explain in Sect. 3.1.3 how to use this iteratively to determine the differential equations. This generalises the algorithm for the rational differential form cases used in [46] to the case of twisted

differential forms appearing in Feynman integrals in general dimensions. In Sect. 4, we illustrate the procedure by working on the dimensionally regulated massless box. We then consider the Witten cross diagram in (A)dS_4 in Sect. 4.2. In Sect. 5, we derive the ϵ-deformed differential equation for the two-loop sunset integral for various mass configurations, the massless double-box and the ice-cream cone graph. We give a Gröbner basis of differential operators for the analytically regulated two-loop ice-cream graph, which arises in the cosmology correlator in dS_4. In Sect. 6, we give the ϵ-deformed differential equation for the equal-mass sunset up to twenty loop orders and for the three-loop massive sunset. In Sect. 7.1, we discuss the question of the minimal order differential operator, and in Sect. 7.3 we analyse how the ϵ parameter arises in the differential equation. Section 7.4 contains a short conclusion. Appendix A is dedicated to a short discussion of the derivation of the differential equation using the Bessel representation with the creative telescoping algorithm.

2 Twisted differential from regulated Feynman integrals

We will apply our formalism in dimensional regularisation for Feynman integrals and analytic regularisation for Witten diagrams. Their treatment is slightly different, so we will consider them separately.

2.1 Review of the parametric representation

We consider the parametric representation of Feynman integrals in D space-time dimensions associated with a graph Γ with n internal edges and L loops. Its derivation can be found for instance in [7, 52–54]. The differential form associated with a Feynman integral is

$$\omega_\Gamma = \frac{\mathbf{U}_\Gamma^{v_1+\cdots+v_n-\frac{(L+1)D}{2}}}{\mathbf{F}_\Gamma^{v_1+\cdots+v_n-\frac{LD}{2}}} \prod_{i=1}^n x_i^{v_i-1}; \qquad \Omega_\Gamma := \omega_\Gamma \, \Omega_0^{(n)}, \tag{2}$$

where we have introduced the canonical differential form on $\mathbb{P}^{n-1}$

$$\Omega_0^{(n)} := \sum_{i=1}^n (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n, \tag{3}$$

and where $\widehat{dx_i}$ means that the term is omitted in the wedge product. We denote collectively the variables attached to all edges of Γ as $\underline{x} := \{x_i | 1 \le i \le n\}$. The exponents v_i are the powers of propagators (internal edges) of Γ. The polynomials $\mathbf{U}_\Gamma$ and $\mathbf{F}_\Gamma$ associated with Γ are defined as follows [7, 52, 54]. The *first Symanzik polynomial* is defined by

$$\mathbf{U}_\Gamma(\underline{x}) = \sum_{\substack{\mathsf{T} \in \text{ Spanning} \\ \text{trees of } \Gamma}} x_\mathsf{T}, \tag{4}$$

where the sum is over all the spanning trees T of Γ, i.e. all sub-graphs T of Γ which contain all vertices of Γ so that the first Betti number (i.e. the number of loops) $b_1(\mathsf{T}) = 0$ and the number of connected component is $b_0(\mathsf{T}) = 1$. The monomial is the product of the variables not in the spanning tree $x_{\mathsf{T}} = \prod_{e \notin \mathsf{T}} x_e$. This is a homogeneous polynomial of $\deg(\mathbf{U}_\Gamma(\underline{x})) = L$. The *second Symanzik polynomial* $\mathbf{F}_\Gamma$ is defined by

$$\mathbf{V}_\Gamma(\underline{x}) = \sum_{\substack{\text{Spanning} \\ \text{2-forests of } \Gamma}} s_{\mathsf{F}} x_{\mathsf{F}}, \qquad \mathbf{F}_\Gamma(\underline{x}) = \mathbf{U}_\Gamma(\underline{x}) \left(\sum_{e \in e(\Gamma)} m_e^2 x_e \right) - \mathbf{V}_\Gamma, \qquad (5)$$

where $x_{\mathsf{F}} = \prod_{e \notin \mathsf{F}} x_i$ to each spanning 2-forest. A 2-forest is a disjoint union of two sub-trees $\mathsf{F} = \mathsf{T}_1 \cup \mathsf{T}_2$. This is a homogeneous polynomial of $\deg(\mathbf{F}_\Gamma(\underline{x})) = L + 1$. With the dot denoting the scalar product on $\mathbb{R}^{1,D-1}$, we define the invariant of T as $s_{\mathsf{F}} = \sum_{(v_1,v_2) \in \mathsf{F} = \mathsf{T}_1 \cup \mathsf{T}_2} p_{v_1} \cdot p_{v_2}$. The second Symanzik polynomial carries all dependence on the physical parameters, that is the internal masses and the external kinematics, which we write as

$$\vec{m} := \{m_1, \ldots, m_n\}, \quad \vec{s} := \{s_{\mathsf{F}} \mid \mathsf{F} \text{ spanning 2-forests of } \Gamma\}, \qquad (6)$$

respectively.

The differential form (2) is defined in the middle cohomology $H^{n-1}(\mathbb{P}^{n-1} \backslash \{\mathbf{U}_\Gamma \mathbf{F}_\Gamma = 0\})$ [10, 55]. The Feynman integral associated with the graph Γ is given by the integral $I_\Gamma = \int_{\Delta_n} \Omega_\Gamma$ of the differential form over the positive orthant

$$\Delta_n := \{(x_1, \ldots, x_n) \in \mathbb{P}^{n-1}, x_i \geq 0 \text{ for } 1 \leq i \leq n\}. \qquad (7)$$

2.2 Twisted differential forms

The Feynman integral I_Γ may diverge for integer values of D and the exponents ν_i, but there is an open subset of $(D, \nu_1, \ldots, \nu_n) \in \mathbb{C}^{n+1}$ where the integral converges. The (unique) value of the Feynman integral is defined by analytic continuation [56]. We work in dimension $D = 2\delta - 2\epsilon$ with $\delta \in \mathbb{N}^*$ and $\epsilon \in \mathbb{R}$ and consider as well the situation where the powers of the propagators are shifted from integer values, that is $\nu_i = \mathsf{v}_i + p_i \kappa$ with $(\mathsf{v}_1, \ldots, \mathsf{v}_n, p_1, \ldots, p_n) \in \mathbb{Z}^{2n}$. The differential form (2) thus becomes a *twisted differential form* $\Omega_\Gamma^{\epsilon,\kappa} = \omega_\Gamma^{\epsilon,\kappa} \Omega_0^{(n)}$ with

$$\omega_\Gamma^{\epsilon,\kappa} := \frac{\mathbf{U}_\Gamma^{\mathsf{v}_1 + \cdots + \mathsf{v}_n - (L+1)\delta}}{\mathbf{F}_\Gamma^{\mathsf{v}_1 + \cdots + \mathsf{v}_n - L\delta}} \left(\frac{\mathbf{U}_\Gamma^{L+1}}{\mathbf{F}_\Gamma^L} \right)^\epsilon \left(\prod_{i=1}^n \left(\frac{x_i \mathbf{U}_\Gamma(\underline{x})}{\mathbf{F}_\Gamma(\underline{x})} \right)^{p_i} \right)^\kappa \prod_{i=1}^n x_i^{\mathsf{v}_i - 1}. \qquad (8)$$

The twists are the ϵ or κ-th powers of homogeneous degree zero rational functions on $\mathbb{P}^{n-1}$.

Notice that we do not assume that ϵ or κ are small numbers. When $\kappa = 0$ we have the parametric representation of a Feynman integral in dimensional regularisation, and

we will use the short-hand notation

$$\omega_\Gamma^\epsilon := \omega_\Gamma^{\epsilon,0} = \frac{\mathbf{U}_\Gamma^{\nu_1+\cdots+\nu_n-(L+1)\delta}}{\mathbf{F}_\Gamma^{\nu_1+\cdots+\nu_n-L\delta}} \left(\frac{\mathbf{U}_\Gamma^{L+1}}{\mathbf{F}_\Gamma^L}\right)^\epsilon \prod_{i=1}^n x_i^{\nu_i-1}. \tag{9}$$

The differential forms (9) and (8) are twisted differential of the kind studied in [57–60]. Their relevance to Feynman integrals was already recognised in these works and has been applied in e.g. [61–70] for expanding Feynman integrals on the basis of master integrals. In contrast to these works, we will use the fact that the twist is given by the power of the homogeneous degree 0 rational form which will be essential in the construction presented in this work.

3 Annihilators of Feynman integrals

Feynman integrals are holonomic functions of their physical parameters [71–74]. This means that Feynman integrals satisfy systems of (inhomogeneous) partial differential equations of finite order with respect to their physical parameters $\vec{m}$ and $\vec{s}$.

Let us consider r parameters from the set of internal masses and independent kinematics, $\underline{z} := \{z_1, \ldots, z_r\} \in \vec{m} \cup \vec{s}$. We seek differential operators annihilating the differential form $\Omega_\Gamma^{\epsilon,\kappa}$

$$\left(\sum_{a_1=0}^{o_1}\cdots\sum_{a_r=0}^{o_r} c_{a_1,\ldots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa)\left(\frac{\partial}{\partial z_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial z_r}\right)^{a_r}\right)\Omega_\Gamma^{\epsilon,\kappa} = d\beta_\Gamma^{\epsilon,\kappa}, \tag{10}$$

where $c_{a_1,\ldots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa)$ are rational functions of the physical parameters, but they are independent of the edge variables $x_1, \ldots, x_n$. The inhomogeneous term is a total derivative in x_i's where the only allowed poles are those already present in $\Omega_\Gamma^{\epsilon,\kappa}$ [46]. Because the domain of integration of the Feynman integral does not depend on the physical parameters, we then deduce

$$\left(\sum_{a_1=0}^{o_1}\cdots\sum_{a_r=0}^{o_r} c_{a_1,\ldots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa)\left(\frac{\partial}{\partial z_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial z_r}\right)^{a_r}\right)I_\Gamma^{\epsilon,\kappa} = \mathscr{S}_\Gamma^{\epsilon,\kappa}, \tag{11}$$

where $\mathscr{S}_\Gamma^{\epsilon,\kappa}$ is an inhomogeneous term obtained by integrating $d\beta_\Gamma^{\epsilon,\kappa}$ over the boundary of orthant (7). This is a non-trivial task because one needs to blow-up the intersections between the graph hypersurface and the domain of integration, so the integral is well-defined [10, 12, 45, 55]. For instance, Section 3.2 of [12] gives a detailed derivation of the inhomogeneous term for the two-loop sunset integral along these lines. If the integration is done over a cycle $\mathcal{C}$, like the one defined by the maximal cut $\mathcal{C}_{\max} := \{|x_1| = \cdots = |x_n| = 1\}$, the resulting integral is annihilated by the action of the

differential operator [28]

$$\left(\sum_{a_1=0}^{o_1} \cdots \sum_{a_r=0}^{o_r} c_{a_1,\ldots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) \left(\frac{\partial}{\partial z_1} \right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r} \right)^{a_r} \right) \int_{\mathcal{C}} \Omega_{\Gamma}^{\epsilon,\kappa} = 0. \tag{12}$$

The ideal generated by these differential operators is a differential module (or D-module). Thus, the differential equations we are seeking can be obtained by deriving annihilators of $\Omega_{\Gamma}^{\epsilon,\kappa}$, i.e. partial differential operators that annihilate the integrand by acting on the physical parameter and the edge variables.

3.1 Griffiths–Dwork reduction for twisted differential forms

The differentiation of $\Omega_{\Gamma}^{\epsilon,\kappa}$ leads to expressions of the type

$$\sum_{\substack{a=a_1+\cdots+a_r \\ a_i \geq 0}} c_{a_1,\ldots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) \left(\frac{\partial}{\partial z_1} \right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r} \right)^{a_r} \Omega_{\Gamma}^{\epsilon,\kappa}$$

$$= \sum_{\substack{a=a_1+\cdots+a_r \\ a_i \geq 0}} \frac{c_{a_1,\ldots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) P^{(a_1,\ldots,a_r)}(\underline{x})}{\mathbf{F}_{\Gamma}^{a}} \Omega_{\Gamma}^{\epsilon,\kappa}, \tag{13}$$

where $P^{(a_1,\ldots,a_r)}(\underline{x})$ is a homogeneous polynomial of degree $(L+1)(a_1+\cdots+a_r)$ in the edge variables $\underline{x}$. The sum is over the differential operators of order $a_1 \geq 0, \ldots, a_r \geq 0$ and fixed total order $a := a_1 + \cdots + a_r$. The pole order in the second Symanzik polynomial $\mathbf{F}_\Gamma$ has increased by a. To derive Eq. (10) one needs to find the coefficient $c_{a_1,\ldots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa)$. From now on we consider the case where $\nu_1 = \cdots = \nu_r = 1$ so that $\nu = n$. The case with $\nu_i \neq 1$ is an immediate generalisation.

3.1.1 The pole reduction for dimensional regularisation

We adapt the Griffiths–Dwork pole reduction to the case of the twisted differential form (9) in dimensional regularisation (i.e. $\kappa = 0$ and $\epsilon \neq 0$). The starting point of the algorithm is the reduction of polynomial $P^{(a_1,\ldots,a_r)}(\underline{x})$ in the numerator of (13)

$$P^{(a_1,\ldots,a_r)}(\underline{x}) = \vec{C}_a(\underline{x}) \cdot \vec{\nabla} \mathbf{F}_{\Gamma}, \tag{14}$$

where we have introduced the gradient $\vec{\nabla} \mathbf{F}_\Gamma := \left(\partial_{x_1} \mathbf{F}_\Gamma(\underline{x}), \ldots, \partial_{x_n} \mathbf{F}_\Gamma(\underline{x}) \right)$. The components of the size n vector $\vec{C}_a(\underline{x})$ are homogeneous polynomials of degree $a(L+1)-L$ in $\underline{x}$. We generalise the construction by Griffiths [40, 41] to include the twist factor for $a > 1$

$$\beta^{(a_1,\ldots,a_r)} = \sum_{1 \leq i < j \leq n} \frac{x_i G_a^j(\underline{x}) - x_j G_a^i(\underline{x})}{\mathbf{F}_{\Gamma}^{a-1}} \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n. \tag{15}$$

To take into account the general dimensional case, we have introduced the vectors of twisted forms

$$\vec{G}_{\mathsf{a}}(\underline{x}) := \vec{C}_{\mathsf{a}}\,\omega_{\Gamma}^{\epsilon}\,, \tag{16}$$

whose components are of homogeneous degree $(\mathsf{a}-1)(L+1)+1-n$. Following the same steps as in [39], we have

$$\mathrm{d}\beta_{\Gamma}^{(a_1,\dots,a_r)} = -(\mathsf{a}-1)\sum_{1\le i<j\le n}\frac{x_i\,G_{\mathsf{a}}^j(\underline{x}) - x_j\,G_{\mathsf{a}}^i(\underline{x})}{\mathbf{F}_{\Gamma}^{\mathsf{a}}}\,d\mathbf{F}_{\Gamma}\wedge dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n$$

$$+\sum_{1\le i<j\le n}\frac{d(x_i\,G_{\mathsf{a}}^j(\underline{x}) - x_j\,G_{\mathsf{a}}^i(\underline{x}))}{\mathbf{F}_{\Gamma}^{\mathsf{a}-1}}\wedge dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n\,. \tag{17}$$

From the degree of homogeneity of $\mathbf{F}_{\Gamma}$ and the components of $\vec{G}_{\mathsf{a}}(\underline{x})$

$$\sum_{i=1}^{n} x_i\frac{\partial\mathbf{F}_{\Gamma}(\underline{x})}{\partial x_i} = (L+1)\mathbf{F}_{\Gamma}(\underline{x}),$$

$$\sum_{i=1}^{n} x_i\frac{\partial\vec{G}_{\mathsf{a}}(\underline{x})}{\partial x_i} = ((\mathsf{a}-1)(L+1)+1-n)\vec{G}_{\mathsf{a}}(\underline{x})\,, \tag{18}$$

we find that

$$\mathrm{d}\beta_{\Gamma}^{(a_1,\dots,a_r)} = (\mathsf{a}-1)\frac{\vec{G}_{\mathsf{a}}(\underline{x})\cdot\vec{\nabla}\mathbf{F}_{\Gamma}}{\mathbf{F}_{\Gamma}^{\mathsf{a}}}\Omega_0^{(n)} - \frac{\vec{\nabla}\cdot\vec{G}_{\mathsf{a}}(\underline{x})}{\mathbf{F}_{\Gamma}^{\mathsf{a}-1}}\Omega_0^{(n)}\,. \tag{19}$$

Using the definition of $\vec{G}_{\mathsf{a}}$ in (16) we have reduced the pole order of $\mathbf{F}_{\Gamma}$ in (13)

$$(\mathsf{a}-1)\left(\frac{\partial}{\partial z_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial z_r}\right)^{a_r}\Omega_{\Gamma}^{\epsilon} = \frac{\vec{\nabla}\cdot\vec{G}_{\mathsf{a}}(\underline{x})}{\mathbf{F}_{\Gamma}^{\mathsf{a}-1}}\Omega_0^{(n)} + \mathrm{d}\beta_{\Gamma}^{(a_1,\dots,a_r)}. \tag{20}$$

We now expand the first term in the right-hand-side

$$\vec{\nabla}\cdot\vec{G}_{\mathsf{a}}(\underline{x}) = \vec{\nabla}\cdot\vec{C}_{\mathsf{a}}(\underline{x})\frac{\mathbf{U}_{\Gamma}^{\lambda_U}}{\mathbf{F}_{\Gamma}^{\lambda_F}} + \vec{C}_{\mathsf{a}}(\underline{x})\cdot\vec{\nabla}\left(\frac{\mathbf{U}_{\Gamma}^{\lambda_U}}{\mathbf{F}_{\Gamma}^{\lambda_F}}\right), \tag{21}$$

where we have defined

$$\lambda_U = n - (L+1)(\delta-\epsilon),\qquad \lambda_F = n - L(\delta-\epsilon). \tag{22}$$

The second term in this equation can be evaluated using

$$\vec{C}_{\mathsf{a}}(\underline{x})\cdot\vec{\nabla}\left(\frac{\mathbf{U}_{\Gamma}^{\lambda_U}}{\mathbf{F}_{\Gamma}^{\lambda_F}}\right) = \left(\lambda_U\vec{C}_{\mathsf{a}}\cdot\vec{\nabla}\log\mathbf{U}_{\Gamma} - \lambda_F\vec{C}_{\mathsf{a}}\cdot\vec{\nabla}\log\mathbf{F}_{\Gamma}\right)\frac{\mathbf{U}_{\Gamma}^{\lambda_U}}{\mathbf{F}_{\Gamma}^{\lambda_F}}$$

$$= \left[-\lambda_F \frac{P^{(a_1,\dots,a_r)}(\underline{x})}{\mathbf{F}_\Gamma} + \lambda_U \vec{C}_\mathsf{a} \cdot \vec{\nabla} \log \mathbf{U}_\Gamma \right] \frac{\mathbf{U}_\Gamma^{\lambda_U}}{\mathbf{F}_\Gamma^{\lambda_F}}, \qquad (23)$$

where we have used Eq. (14) in the second equality. Therefore,

$$\left(\frac{\partial}{\partial z_1} \right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r} \right)^{a_r} \Omega_\Gamma^\epsilon = \frac{\vec{\nabla} \cdot \vec{C}_\mathsf{a}(\underline{x}) + \lambda_U \vec{C}_\mathsf{a} \cdot \vec{\nabla} \log \mathbf{U}_\Gamma}{(\mathsf{a} - 1 + \lambda_F)\, \mathbf{F}_\Gamma^{\mathsf{a}-1}} \Omega_\Gamma^\epsilon + \frac{1}{\mathsf{a} - 1 + \lambda_F} \mathrm{d}\beta_\Gamma^{(a_1,\dots,a_r)}.$$
$$(24)$$

This expression involves the term $\vec{C}_\mathsf{a} \cdot \vec{\nabla} \log \mathbf{U}_\Gamma$ which has a pole in $\mathbf{U}_\Gamma$. We then perform a second reduction by demanding that

$$\vec{C}_\mathsf{a}(\underline{x}) \cdot \vec{\nabla} \mathbf{U}_\Gamma = c_\mathsf{a}(\underline{x})\, \mathbf{U}_\Gamma, \qquad (25)$$

where $c_\mathsf{a}(\underline{x})$ is a homogeneous polynomial of degree $(\mathsf{a}-1)(L+1)$. This is equivalent to the computation of syzygies of $\mathrm{Jac}(\mathbf{U}_\Gamma) := \langle \vec{\nabla}\mathbf{U}_\Gamma(\underline{x}) \rangle$. Indeed, using the homogeneity of $\mathbf{U}_\Gamma$ we can rewrite the previous equation as

$$\left(L\vec{C}_\mathsf{a}(\underline{x}) - c_\mathsf{a}(\underline{x})\vec{x} \right) \cdot \vec{\nabla}\mathbf{U}_\Gamma = 0, \qquad (26)$$

which are examples of syzygies of the Jacobian of $\mathbf{U}_\Gamma$. Using this reduction in Eq. (24) leads to

$$\left(\frac{\partial}{\partial z_1} \right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r} \right)^{a_r} \Omega_\Gamma^\epsilon = \frac{M^{(a_1,\dots,a_r)}(\underline{x})}{\mathbf{F}_\Gamma^{\mathsf{a}-1}} \Omega_\Gamma^\epsilon + \frac{\mathsf{a}-1}{\mathsf{a} + n - L(\delta - \epsilon)} \mathrm{d}\beta_\Gamma^{(a_1,\dots,a_r)}$$
$$(27)$$

with the numerator given by the polynomial of homogeneous degree $(\mathsf{a}-1)(L+1)$

$$M^{(a_1,\dots,a_r)}(\underline{x}) := \frac{\vec{\nabla} \cdot \vec{C}_\mathsf{a}(\underline{x}) + \lambda_U\, c_\mathsf{a}(\underline{x})}{\mathsf{a} - 1 + \lambda_F}, \qquad (28)$$

with λ_U and λ_F the powers of the $\mathbf{U}_\Gamma$ and the $\mathbf{F}_\Gamma$ polynomials respectively given in (22).

To perform the pole reduction, we have to solve the linear system

$$\begin{cases} \vec{C}_\mathsf{a}(\underline{x}) \cdot \vec{\nabla}\mathbf{F}_\Gamma = P^{(a_1,\dots,a_r)}(\underline{x}) \\ \vec{C}_\mathsf{a}(\underline{x}) \cdot \vec{\nabla}\mathbf{U}_\Gamma = c_\mathsf{a}(\underline{x})\mathbf{U}_\Gamma \end{cases}, \qquad (29)$$

for determining the coefficients of $\vec{C}_\mathsf{a}(\underline{x})$ and $c_\mathsf{a}(\underline{x})$. The system (29) has a solution when its rank is positive. We have a linear system of the n components of $\vec{C}_\mathsf{a}(\underline{x})$ which are homogeneous polynomial of degree $\deg(C) = \deg(P^{(a_1,\dots,a_r)}) - L$ in $\underline{x}$ and $c_\mathsf{a}(\underline{x})$ which is a polynomial of homogeneous degree $\deg(C) - 1$. Since the number of coefficients of a homogeneous polynomial of degree d in n variables is $\binom{d+n-1}{d}$,

the system has

$$
n\begin{pmatrix} \deg(P^{(a_1,\ldots,a_r)}) - L + n - 1 \\ \deg(P^{(a_1,\ldots,a_r)}) - L \end{pmatrix} + \begin{pmatrix} \deg(P^{(a_1,\ldots,a_r)}) - L + n - 2 \\ \deg(P^{(a_1,\ldots,a_r)}) - L - 1 \end{pmatrix} \tag{30}
$$

unknown variables for

$$
\begin{pmatrix} \deg(P^{(a_1,\ldots,a_r)}) + n - 1 \\ \deg(P^{(a_1,\ldots,a_r)}) \end{pmatrix} + \begin{pmatrix} \deg(P^{(a_1,\ldots,a_r)}) + n - 2 \\ \deg(P^{(a_1,\ldots,a_r)}) - 1 \end{pmatrix} \tag{31}
$$

equations. Since the $\deg(P^{(a_1,\ldots,a_r)}) = \mathsf{a}(L+1)$, the rank of the system (29) is

$$
\text{rank} = (30) - (31) \tag{32}
$$
$$
= n\begin{pmatrix} (L+1)(\mathsf{a}-1) + n \\ (L+1)(\mathsf{a}-1) + 1 \end{pmatrix} + \begin{pmatrix} (L+1)(\mathsf{a}-1) + n - 1 \\ (L+1)(\mathsf{a}-1) \end{pmatrix}
$$
$$
- \begin{pmatrix} (L+1)\mathsf{a} + n - 1 \\ (L+1)\mathsf{a} \end{pmatrix} - \begin{pmatrix} (L+1)\mathsf{a} + n - 2 \\ (L+1)\mathsf{a} - 1 \end{pmatrix}
$$

For fixed values of loops L and number of edges n there is always a value of the number of derivatives a such that the system has positive rank.

A few comments are in order. In practice for Feynman integrals, the polynomial $P^{(a_1,\ldots,a_r)}(\underline{x})$ is not a generic homogeneous polynomial, so the number of equations is smaller or equal than (31). We remark that this way of solving the linear system includes implicitly the freedom given by the syzygies of $\mathrm{Jac}(\mathbf{F}_\Gamma) := \langle \vec{\nabla}\mathbf{F}_\Gamma(\underline{x}) \rangle$ and $\mathrm{Jac}(\mathbf{U}_\Gamma)$ since they belong to the kernel of equation (14) and (25) respectively.[1] One important property of that reduction is that the differential form $\beta_\Gamma^{(a_1,\ldots,a_n)}$ is that it does not have poles that are not poles of $\mathbf{F}_\Gamma$ which is guaranteed by construction. We refer to Section 3 of [46] for a discussion of the pole constraints.

The system of linear equation (29) is dense since, in general, all coefficients in $\vec{C}_\mathsf{a}(\underline{x})$ and $c_\mathsf{a}(\underline{x})$ are non-vanishing. Moreover, we are interested in analytic solutions of these systems. We thus benefit from the dense solver implemented in the Mathematica package FiniteFlow, described in detail in Sec.4 of [51]. Specifically, in Mathematica have used the command FFDenseSolve.

3.1.2 The pole reduction for analytic regularisation

We give an adaption of the Griffiths–Dwork pole reduction to the case of the twisted differential form (8) from analytical regularisation. Since most of the steps are similar to the one presented in the previous section, we only give the main equations.

As before we reduce the polynomial $P^{(a_1,\ldots,a_r)}(\underline{x})$ in the Jacobian of $\mathbf{F}_\Gamma$ using Eq. (14) and introduce the differential forms (15) $\beta_\Gamma^{(a_1,\ldots,a_r)}$ and the vector of differential forms $\vec{G}_\mathsf{a}(\underline{x})$ as in Eq. (16), leading to the pole reduction Eq. (20). Because the

---

[1] It was noticed in [46], that in the rational case, only the first order syzygies are needed to take into account the non-isolated singularities of Feynman integrals.

twist is different the expansion of the right-hand-side (recall that $D = 2\delta - 2\epsilon$)

$$\vec{\nabla} \cdot \vec{G}_a(\underline{x}) = \vec{\nabla} \cdot \vec{C}_a(\underline{x}) \frac{\mathbf{U}_\Gamma^{\lambda_U}}{\mathbf{F}_\Gamma^{\lambda_F}} Q(\underline{x})^\kappa + \vec{C}_a(\underline{x}) \cdot \vec{\nabla} \left(\frac{\mathbf{U}_\Gamma^{\lambda_U}}{\mathbf{F}_\Gamma^{\lambda_F}} Q(\underline{x})^\kappa \right), \quad (33)$$

where we have set (we have assumed that $\nu_1 = \cdots = \nu_n = 1$ the generic case of integer values is an easy generalisation)

$$Q(\underline{x}) := \prod_{i=1}^{n} x_i^{p_i} \quad (34)$$

and defined the powers of the various polynomials by

$$\lambda_U = n - (L+1)(\delta - \epsilon) + \kappa \sum_{i=1}^{n} p_i, \quad \lambda_F = n - L(\delta - \epsilon) + \kappa \sum_{i=1}^{n} p_i, \quad \lambda_Q = \kappa. \quad (35)$$

This is evaluated using

$$\vec{C}_a(\underline{x}) \cdot \vec{\nabla} \left(\frac{\mathbf{U}_\Gamma^{\lambda_U} Q(\underline{x})^\kappa}{\mathbf{F}_\Gamma^{\lambda_F}} \right) = \left(\lambda_U \vec{C}_a \cdot \vec{\nabla} \log \mathbf{U}_\Gamma - \lambda_F \vec{C}_a \cdot \vec{\nabla} \log \mathbf{F}_\Gamma + \lambda_Q \vec{C}_a \cdot \vec{\nabla} \log Q(\underline{x}) \right) \frac{\mathbf{U}_\Gamma^{\lambda_U} Q(\underline{x})^\kappa}{\mathbf{F}_\Gamma^{\lambda_F}}, \quad (36)$$

so that

$$\left(\frac{\partial}{\partial z_1} \right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r} \right)^{a_r} \Omega_\Gamma^{\epsilon,\kappa} = \frac{\vec{\nabla} \cdot \vec{C}_a(\underline{x}) + \lambda_U \vec{C}_a \cdot \vec{\nabla} \log \mathbf{U}_\Gamma + \lambda_Q \vec{C}_a \cdot \vec{\nabla} \log Q(\underline{x})}{(a - 1 + \lambda_F) \mathbf{F}_\Gamma^{a-1}} \Omega_\Gamma^{\epsilon,\kappa}$$
$$+ \frac{1}{a - 1 + \lambda_F} d\beta_\Gamma^{(a_1,\ldots,a_r)}. \quad (37)$$

This time we need to reduce the pole in $\mathbf{U}_\Gamma$ from the term $\vec{C}_a \cdot \vec{\nabla} \log \mathbf{U}_\Gamma$ and the new pole in $1/x_i$ arising from the propagators. As before we impose the following conditions

$$\begin{cases} \vec{C}_a(\underline{x}) \cdot \vec{\nabla} \mathbf{F}_\Gamma = P^{(a_1,\ldots,a_r)}(\underline{x}) \\ \vec{C}_a(\underline{x}) \cdot \vec{\nabla} \mathbf{U}_\Gamma = c_a(\underline{x}) \mathbf{U}_\Gamma \\ \vec{C}_a(\underline{x}) \cdot \vec{\nabla} Q(\underline{x}) = q_a(\underline{x}) Q(\underline{x}) \end{cases}, \quad (38)$$

leading to the pole reduction in Eq. (13)

$$\left(\frac{\partial}{\partial z_1} \right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r} \right)^{a_r} \Omega_\Gamma^{\epsilon} = \frac{M^{(a_1,\ldots,a_r)}(\underline{x})}{\mathbf{F}_\Gamma^{a-1}} \Omega_\Gamma^{\epsilon} + \frac{1}{a - 1 + \lambda_F} d\beta_\Gamma^{(a_1,\ldots,a_r)}. \quad (39)$$

The numerator is now given by the polynomial of homogeneous degree $(a-1)(L+1)$

$$M^{(a_1,\ldots,a_r)}(\underline{x}) := \frac{\vec{\nabla} \cdot \vec{C}_a(\underline{x}) + \lambda_U c_a(\underline{x}) + \lambda_Q q_a(\underline{x})}{a - 1 + \lambda_F}. \quad (40)$$

3.1.3 Determination of the differential equations

We turn to the derivation of the differential equation (10) by iterating the generalised Griffiths–Dwork reduction given in the previous sections. We present an algorithm for a derivation of a linear ordinary differential equation with respect to a single variable differentiation z (either an internal mass, or a kinematic variable or a scaling parameter as used in [9, 46]) so that $r = 1$ and $\mathsf{a} = a_1$. The generalisation to the many variable case is immediate.

We are seeking the differential operator

$$\mathscr{L}_\Gamma^{\epsilon,\kappa} = \sum_{\mathsf{a}=0}^{N(\Gamma,\epsilon,\kappa)} c_\mathsf{a}(\vec{m}, \vec{s}, \epsilon, \kappa) \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mathsf{a}} \tag{41}$$

with $c_\mathsf{a}(\vec{m}, \vec{s}, \epsilon, \kappa)$ polynomials in the internal masses $\vec{m}$ and the (independent) kinematic variables $\vec{s}$ and the regulators ϵ and κ, such that

$$\mathscr{L}_\Gamma^{\epsilon,\kappa} \Omega_\Gamma^{\epsilon,\kappa} = \mathrm{d}\beta_\Gamma^{\epsilon,\kappa}. \tag{42}$$

Holonomicity of Feynman integrals gives an upper bound on the order of the differential operator, which is determined by the number of master integrals. For a graph Γ the minimal order of the differential operator depends in general of the regulators. Let $N(\Gamma, \epsilon, \kappa)$ be the starting order of the reduction. We then apply the results of Sects. 3.1.1 or 3.1.2 so that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{N(\Gamma,\epsilon,\kappa)} \Omega_\Gamma^\epsilon = \frac{M^{N(\Gamma,\epsilon,\kappa)}(\underline{x})}{\mathbf{F}_\Gamma^{N(\Gamma,\epsilon,\kappa)-1}} \Omega_\Gamma^{\epsilon,\kappa} + \mathrm{d}\beta_\Gamma^{N(\Gamma,\epsilon,\kappa)}. \tag{43}$$

In the next step we add the lowest-order derivative

$$\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{N(\Gamma,\epsilon,\kappa)} + q_{N(\Gamma,\epsilon,\kappa)-1}(t, \epsilon, \kappa) \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{N(\Gamma,\epsilon,\kappa)-1}\right] \Omega_\Gamma^{\epsilon,\kappa}$$

$$= \frac{M^{N(\Gamma,\epsilon,\kappa)}(\underline{x}) + q_{N(\Gamma,\epsilon,\kappa)-1}(t, \epsilon, \kappa) P^{N(\Gamma,\epsilon,\kappa)-1}(\underline{x})}{\mathbf{F}_\Gamma^{N(\Gamma,\epsilon,\kappa)-1}} \Omega_\Gamma^{\epsilon,\kappa} + \mathrm{d}\beta_\Gamma^{N(\Gamma,\epsilon,\kappa)}, \tag{44}$$

where the rational coefficient $q_{N(\Gamma,\epsilon,\kappa)-1}(t, \epsilon) = c_{N(\Gamma,\epsilon,\kappa)-1}(t, \epsilon)/c_{N(\Gamma,\epsilon,\kappa)}(t, \epsilon)$ is an unknown rational function of t and the regulators ϵ and κ. The polynomial $P^{N(\Gamma,\epsilon,\kappa)-1}(\underline{x})$ is the numerator factor obtained by taking the $N(\Gamma, \epsilon, \kappa) - 1$ derivative of the differential form.

We then apply the reduction of Sect. 3.1.1 or 3.1.2 to the polynomial in the numerator $M^{N(\Gamma,\epsilon,\kappa)}(\underline{x}) + q_{N(\Gamma,\epsilon,\kappa)-1}(t, \epsilon) P^{N(\Gamma,\epsilon,\kappa)-1}(\underline{x})$. The resolution of the system (29) determines the rational coefficient $q_{N(\Gamma,\epsilon,\kappa)-1}(t, \epsilon)$ and $M^{N(\Gamma,\epsilon,\kappa)-1}(x)$ computed using (40). One iterates the reduction until the power of the second Symanzik polynomial $\mathbf{F}_\Gamma$ is $n - L - 1$ so that

$$\left[\left(\frac{d}{dt}\right)^{N(\Gamma,\epsilon,\kappa)} + \sum_{a=1}^{N(\Gamma,\epsilon,\kappa)-1} \frac{c_{N(\Gamma,\epsilon,\kappa)-a}(t,\epsilon)}{c_{N(\Gamma,\epsilon,\kappa)}(t,\epsilon)}\left(\frac{d}{dt}\right)^{N(\Gamma,\epsilon,\kappa)-a}\right]\Omega_\Gamma^{\epsilon,\kappa} = M^0\,\Omega_\Gamma^{\epsilon,\kappa} + d\beta_\Gamma^{\epsilon,\kappa},$$
(45)

where M^0 is of degree 0 so that

$$q_0(t,\epsilon,\kappa) = \frac{c_0(t,\epsilon,\kappa)}{c_{N(\Gamma,\epsilon,\kappa)}(t,\epsilon,\kappa)} = -M^0.$$
(46)

The inhomogeneous term $\beta_\Gamma^{\epsilon,\kappa}$ is the sum of β^a with $1 \le a \le N(\Gamma,\epsilon,\kappa)$ contributions with their multiplicative factor as given in (24)

$$\beta_\Gamma^{\epsilon,\kappa} = \sum_{1\le i<j\le n} (x_i B_\Gamma^j - x_j B_\Gamma^i)\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$
(47)

with

$$\vec{B}_\Gamma^{\epsilon,\kappa} := \left(\sum_{a=1}^{N(\Gamma,\epsilon,\kappa)} \frac{\vec{C}_a(\underline{x})}{(a-1+n-L(\delta-\epsilon))\mathbf{F}_\Gamma^{a-1}}\right)\omega_\Gamma^{\epsilon,\kappa},$$
(48)

which is a vector of degree of homogeneity $1-n$ in the edge variables $\underline{x}$. Since $d\beta_\Gamma^{\epsilon,\kappa} = -\vec{\nabla}\cdot\vec{B}_\Gamma\,\Omega_0^{(n)}$, from (45) we have the ordinary differential equation satisfied by the integrand of the Feynman integral

$$\left[\left(\frac{d}{dt}\right)^{N(\Gamma,\epsilon,\kappa)} + \sum_{r=1}^{N(\Gamma,\epsilon,\kappa)} \frac{c_{N(\Gamma,\epsilon,\kappa)-r}(t,\epsilon)}{c_{N(\Gamma,\epsilon,\kappa)}(t,\epsilon)}\left(\frac{d}{dt}\right)^{N(\Gamma,\epsilon,\kappa)-r}\right]\omega_\Gamma^{\epsilon,\kappa} = -\vec{\nabla}\cdot\vec{B}_\Gamma^{\epsilon,\kappa}.$$
(49)

Integrating this expression over the positive orthant leads to the inhomogeneous differential equation satisfied by the Feynman integrals

$$\left[\left(\frac{d}{dt}\right)^{N(\Gamma,\epsilon,\kappa)} + \sum_{r=1}^{N(\Gamma,\epsilon,\kappa)} \frac{c_{N(\Gamma,\epsilon,\kappa)-r}(t,\epsilon)}{c_{N(\Gamma,\epsilon,\kappa)}(t,\epsilon)}\left(\frac{d}{dt}\right)^{N(\Gamma,\epsilon,\kappa)-r}\right]I_\Gamma^{\epsilon,\kappa}$$
$$= -\int_{x_i\ge0}\vec{\nabla}\cdot\vec{B}_\Gamma^{\epsilon,\kappa}\,dx_1\cdots dx_n.$$
(50)

The inhomogeneous term is a total derivative reflecting the fact that it is spanned by Feynman integral with collapsed edge of the original graph Γ. The evaluation of this inhomogeneous term is delicate and requires taking into account the various blow-ups of the domain of integration so the integral is well-defined [10, 12, 45, 55].

Let us comment on how to determine the order of the differential operators. An upper bound on the order can be computed from the number of master integrals. The number of master integrals can be computed from the Euler characteristic of complement of

the graph polynomial [72] or counting the critical points of the Euler representation of the integral in projective space [65, 74] or [62, 75] (see [33] for a discussion of the relation between the different ways of computing the number of master integrals). On the other hand, determining the minimal order is a difficult question. Our pragmatical approach is to increase the order starting from lower orders until the system (29) has a solution, which is the spirit of the Griffiths–Dwork reduction applied to Feynman integrals in [44].

In practice, for determining the minimal order it is enough to run the step of the Griffiths–Dwork reduction for fixed generic numerical values for the physical parameters (the internal masses and kinematic parameters), because all the reduction amounts to solve a linear system in the projective space of the edge parameters $\underline{x}$. This allows to determine the smallest order for which the algorithm closes to give a differential operator. The Griffiths–Dwork algorithm does not automatically lead to an irreducible differential operator. The factorisation of a linear ordinary differential operator can be done with the `DFactor` routine from `Maple` up to the order 4 for generic parameters [76, 77], and to any orders for linear ordinary differential operators with numerical coefficients using the `facto` algorithm in `sagemath` [78, 79].

In the rational case $\epsilon = \kappa = 0$, it was noticed in [11, 12, 46, 80] that the order of the minimal differential operator is smaller than the number of irreducible master integrals. When the regulator take integer values the minimal order is smaller than the number of master integrals. In the various cases studied below, we will see that the order of the minimal differential operator can saturate the upper bound given by the number of masters for generic values of the regulators ϵ and κ. This will be discussed further in Sect. 7.1.

4 One-loop examples

We start in Sect. 4.1 with the simple example of the massless one-loop box in dimensional regularisation, which will serve as an illustration of where the main features of the algorithm emerge (Fig. 1).

We then work out a Gröbner basis of differential operators associated with the Witten cross diagram for conformally coupled ϕ^4 in four dimensional de Sitter space. It was shown in [25, 26] that the cosmological correlators of conformally coupled ϕ^4 can be organised as dimensionally regulated flat space Feynman integrals in position space. Because of the measure of integration in (anti)-de Sitter the resulting integrals fall into the category of the analytic regularisation of Sect. 2.2.

4.1 The massless box graph in dimensional regularisation

We define the usual Mandelstam invariants as $t = (k_1 + k_3)^2$, $s = (k_1 + k_2)^2$. The graph polynomials are given by

$$\mathbf{U}_{\Box} = x_1 + \cdots + x_4, \qquad \mathbf{F}_{\Box}(s, t) = -t x_2 x_4 - s x_1 x_3 . \tag{51}$$

Fig. 1 The box graph with massless external and internal states. The outgoing external momenta are k_i with $k_1 + \cdots + k_4 = 0$. The labels of the graph give the index of the edge variable x_i
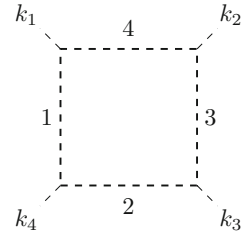
The twisted differential in Eq. (9) for the box graph in $D = 4 - 2\epsilon$ in the projective space $\mathbb{P}^3$ reads $\Omega_\Box^\epsilon = \omega_\Box^\epsilon \, \Omega_0^{(4)}$, where

$$\omega_\Box^\epsilon = \frac{1}{\mathbf{F}_\Box(s,t)^2} \left(\frac{\mathbf{U}_\Box^2}{\mathbf{F}_\Box(s,t)} \right)^\epsilon . \tag{52}$$

This is a single scale function depending only on the ratio $X = t/s$ of the kinematic invariants, so we scale the integral obtaining $\tilde{\Omega}_\Box^\epsilon(X) = (-s)^{2+\epsilon} \Omega_\Box^\epsilon(s, Xs)$. The application of the procedure given in Sect. 3.1.3 needs only to start at the first order. Computing the derivative with respect to X, we obtain

$$P^{(1)} = (2 + \epsilon)x_2 x_4 , \tag{53}$$

which we will reduce with respect to

$$\vec{\nabla}(\mathbf{F}_\Box) = \left(\partial_{x_1}\mathbf{F}_\Box = -x_3, \, \partial_{x_2}\mathbf{F}_\Box = -x_4 X, \, \partial_{x_3}\mathbf{F}_\Box = -x_1, \, -X x_2 \right). \tag{54}$$

The vector $\vec{C}_1(\underline{x})$ has components

$$C_1^i(\underline{x}) = \sum_{e \in m_{2,4}} \lambda_e^i x^e, \tag{55}$$

where $m_{i,j}$ denote the set of exponent vectors of degree i in j variables. Therefore, $m_{2,4} = \{(a_1, a_2, a_3, a_4) | a_1 + a_2 + a_3 + a_4 = 2, a_1, a_2, a_3, a_4 \geq 0\}$. Since $\deg(C_1^i(\underline{x})) = 2$, $\deg(c_1(\underline{x})) = 1$, i.e. $c_1(\underline{x}) = \sum_{e \in m_{1,4}} q_e x^e$, and $m_{1,4} = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$. Hence the linear system becomes

$$\begin{cases} \sum_{i=1}^4 \sum_{e \in m_{2,4}} \lambda_e^i x^e \partial_{x_i}(\mathbf{F}_\Box) = (2 + \epsilon)x_2 x_4 \\ \sum_{i=1}^4 \sum_{e \in m_{2,4}} \lambda_e^i x^e \partial_{x_i}(\mathbf{U}_\Box) = \sum_{e \in m_{1,4}} q_e x^e \mathbf{U}_\Box \end{cases}, \tag{56}$$

which leads to

$$c_0 = \frac{1 + X + \epsilon}{X(X + 1)}. \tag{57}$$

We then derive

$$\mathscr{L}_\Box^\epsilon = (X + 1)X \frac{\mathrm{d}}{\mathrm{d}X} + (1 + X + \epsilon). \tag{58}$$

In this simple case, the algorithm requires only one iteration. The boundary contribution in Eq. (48) is given by

$$
\begin{aligned}
\vec{B}_{\Box}^{\epsilon} = & \left\{ X\frac{x_4}{2}, -\frac{x_1 + x_2 + x_3}{2}, X\frac{x_4}{2}, -\frac{x_4(2X + 1)}{2} \right\} \\
& + \lambda_{0,0,1,0}^{3} \{x_4 X - x_1, x_1 - x_3, x_3 - x_4 X, 0\} \\
& + \lambda_{0,0,1,0}^{4} \{x_4 - x_2, x_2 X - x_3, 0, x_3 - x_4 X\} \\
& + \lambda_{1,0,0,0}^{4} \{0, x_2 X - x_1, (x_4 - x_2)X, x_1 - x_4 X\} .
\end{aligned}
\tag{59}
$$

The boundary vector depends on the free coefficients $\lambda_{n_1,n_2,n_3,n_4}^{i}$ from the reduction. This freedom arises from the kernel of the linear system (56) on the unknown coefficients $\lambda_{n_1,\dots,n_4}^{i}$. The gradient of this vector does not depend on the free coefficients as it reads

$$
\vec{\nabla} \cdot \vec{B}_{\Box}^{\epsilon} = \frac{x_2 x_4 X(X\epsilon + X + 1) - x_1 x_3(X + \epsilon + 1)}{X(X + 1)(x_2 x_4 X + x_1 x_3)} \omega_{\Box}^{\epsilon},
\tag{60}
$$

so that

$$
\mathscr{L}_{\Box}^{\epsilon} \omega_{\Box}^{\epsilon} + \vec{\nabla} \cdot \vec{B}_{\Box}^{\epsilon} = 0 .
\tag{61}
$$

By integrating $\vec{\nabla} \cdot \vec{B}_{\Box}^{\epsilon}$ over the $\Delta_4 = \{x_i \geq 0, 1 \leq i \leq 4\}$ we get the inhomogeneous term

$$
\mathscr{S}_{\Box}^{\epsilon} = \frac{(\epsilon + 1)\Gamma(-\epsilon - 1)^2}{\Gamma(-2\epsilon)} \left((-s)^{-1-\epsilon} + (-t)^{-\epsilon-1} \right) .
\tag{62}
$$

This simple example illustrates the general procedure as we will see with more loops. In general the system of equations is dense.

4.2 The Witten cross diagram in AdS₄ in dimensional regularisation

The dimensionally regulated Witten cross diagram of Fig. 2 considered in Section 4.1 of [26] is given by the integral over the bulk point X

$$
\mathcal{W}_0^{1,4-4\varepsilon}(\zeta, \bar{\zeta}) = \frac{1}{2} \frac{\zeta\bar{\zeta}}{(v_{12}v_{34})^2} \int_{\mathbb{R}^4} \frac{\mathrm{d}^{4-4\varepsilon} X}{\|X\|^2 \|X - u_1\|^{2(1-4\varepsilon)} \|X - u_\zeta\|^2} .
\tag{63}
$$

Fig. 2 Witten cross diagram, between four states on boundary of dS_4
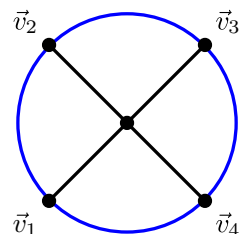
with the vectors $u_1 = (1, 0, 0, 0)$ and $u_\zeta = \left(\frac{\zeta+\bar{\zeta}}{2}, \frac{\zeta-\bar{\zeta}}{2i}, 0, 0\right)$ where ζ and $\bar{\zeta}$ parameterise the cross ratios $\zeta\bar{\zeta} = v_{12}^2 v_{34}^2/(v_{14}^2 v_{23}^2)$ and $(1 - \zeta)(1 - \bar{\zeta}) = v_{13}^2 v_{24}^2/(v_{14}^2 v_{23}^2)$ defined by the position on the boundary of (A)dS_4.

The parametric representation is given by $I_\times^\varepsilon(\zeta, \bar{\zeta}) = \int_{\Delta_3} \Omega_\times^{2\varepsilon, -4\varepsilon}(\zeta, \bar{\zeta})$ the integration over the twisted differential form

$$\Omega_\times^{2\varepsilon, 2\varepsilon}(\zeta, \bar{\zeta}) = \frac{\pi^{2-2\varepsilon}\Gamma(1 - 2\varepsilon)}{\Gamma(1 - 4\varepsilon)} \frac{1}{\mathbf{U}_\times \mathbf{F}_\times(\zeta, \bar{\zeta})} \left(\frac{\mathbf{F}_\times(\zeta, \bar{\zeta})^2}{x_2^4}\right)^\varepsilon \Omega_0^{(3)} \qquad (64)$$

over the domain $\Delta_3 = \{x_i \geq 0, 1 \leq i \leq 3\}$ and with the graph polynomials

$$\mathbf{U}_\times = x_1 + x_2 + x_3, \qquad \mathbf{F}_\times(\zeta, \bar{\zeta}) = x_1 x_2 + \zeta\bar{\zeta} x_1 x_3 + (1 - \zeta)(1 - \bar{\zeta}) x_2 x_3. \quad (65)$$

We can apply the algorithm to the case of the analytic regularisation of Sect. 3.1.2 with $\delta = 2$, $\epsilon = \kappa = 2\varepsilon$, $(p_1, p_2, p_3) = (0, -2, 0)$. We find the following set of differential operators acting on $I_\times(\zeta, \bar{\zeta})$

$$\mathscr{L}_{\times,1} = (\zeta - 1)(\zeta - \bar{\zeta})\zeta^2 \frac{\partial^2}{\partial\zeta^2} + \left(\zeta(3\zeta - \bar{\zeta} - 2) - 2\varepsilon\left(\zeta^2 + \zeta\bar{\zeta} - 2\bar{\zeta}\right)\right)\zeta\frac{\partial}{\partial\zeta}$$
$$+ (2\varepsilon - 1)\left(2\varepsilon(\zeta + \bar{\zeta}) - \zeta^2\right) \quad (66)$$

and

$$\mathscr{L}_{\times,2} = (\zeta - 1)\zeta\bar{\zeta}\frac{\partial}{\partial\zeta} + (\bar{\zeta} - 1)\zeta\bar{\zeta}\frac{\partial}{\partial\bar{\zeta}} + \zeta\bar{\zeta} - 2\varepsilon(\zeta + \bar{\zeta}). \quad (67)$$

We have checked that they form a Gröbner basis with respect to the lexicographical ordering of $\zeta, \bar{\zeta}$ using the command `OreGroebnerBasis`, which is part of the package `HolonomicFunctions` [50].

The algorithm determines the boundary terms such that[2]

$$\mathscr{L}_{\times,r}\left(\frac{1}{\mathbf{U}_\times \mathbf{F}_\times(\zeta, \bar{\zeta})}\left(\frac{\mathbf{F}_\times(\zeta, \bar{\zeta})^2}{x_2^4}\right)^\varepsilon\right) + \vec{\nabla} \cdot \vec{B}_{\times,r}^{2\varepsilon, 2\varepsilon} = 0. \qquad r = 1, 2. \quad (68)$$

For computing the boundary contribution to say, $\mathscr{S}_{\times,1}$ we need to evaluate

$$\mathscr{S}_{\times,1} = -\int_{\Delta_3} \vec{\nabla} \cdot \vec{B}_{\times,1}^{2\varepsilon, 2\varepsilon}. \quad (69)$$

Integrating by parts we have the expression

$$\mathscr{S}_{\times,1} = \iint_0^\infty \left(\lim_{x_1 \to \infty} (B_{\times,1}^{2\varepsilon,2\varepsilon})_1 - \lim_{x_1 \to 0} (B_{\times,1}^{2\varepsilon,2\varepsilon})_1 \right) dx_2 dx_3$$

$$+ \iint_0^\infty \left(\lim_{x_2 \to \infty} (B_{\times,1}^{2\varepsilon,2\varepsilon})_2 - \lim_{x_2 \to 0} (B_{\times,1}^{2\varepsilon,2\varepsilon})_2 \right) dx_1 dx_3$$

$$+ \iint_0^\infty \left(\lim_{x_3 \to \infty} (B_{\times,1}^{2\varepsilon,2\varepsilon})_3 - \lim_{x_3 \to 0} (B_{\times,1}^{2\varepsilon,2\varepsilon})_3 \right) dx_1 dx_2 . \quad (70)$$

The components of $\vec{B}_{\times,1}^{2\varepsilon,2\varepsilon}$ depend on seven free parameters

$$\left\{ \lambda_{1,0,0}^1, \lambda_{1,1,1}^3, \lambda_{1,2,0}^3, \lambda_{2,1,0}^3, \lambda_{0,0,1}^3, \lambda_{0,1,0}^3, \lambda_{1,0,0}^3 \right\}. \quad (71)$$

For all values of these parameters we have for $\varepsilon > 0$

$$\lim_{x_2 \to \infty} (B_{\times,1}^{2\varepsilon,2\varepsilon})_2 = \lim_{x_2 \to 0} (B_{\times,1}^{2\varepsilon,2\varepsilon})_2 = 0. \quad (72)$$

For all values of the parameters, the limits $\lim_{x_1 \to 0} (B_{\times,1}^{2\varepsilon,2\varepsilon})_1$ and $\lim_{x_3 \to 0} (B_{\times,1}^{2\varepsilon,2\varepsilon})_3$ are finite for $\varepsilon > 0$. But the limits $\lim_{x_1 \to \infty} (B_{\times,1}^{2\varepsilon,2\varepsilon})_1$ and $\lim_{x_3 \to \infty} (B_{\times,1}^{2\varepsilon,2\varepsilon})_3$ are not finite for $\varepsilon > 0$. With the choice of the parameters

$$\lambda_{1,0,0}^3 = -\frac{4\varepsilon - 1}{2(\varepsilon - 1)} \lambda_{2,1,0}^3,$$

$$\lambda_{0,1,0}^3 = \frac{(2\varepsilon - 1)\lambda_{1,1,1}^3}{\zeta\bar{\zeta}(\varepsilon - 1)} + \frac{2(2\varepsilon - 1)\lambda_{0,0,1}^3}{\zeta\bar{\zeta}(4\varepsilon - 1)} - \frac{(2\varepsilon - 1)\lambda_{1,2,0}^3}{\varepsilon - 1}$$

$$- \frac{\lambda_{2,1,0}^3(2\zeta + 2\bar{\zeta} - \zeta\bar{\zeta} - 2 + 4(\zeta\bar{\zeta} - \zeta - \bar{\zeta} + 1)\varepsilon)}{2\zeta\bar{\zeta}(\varepsilon - 1)}$$

$$+ \frac{(2\varepsilon - 1)^2 \left(\zeta + \bar{\zeta} - 3\zeta^2 + 2(4\zeta^2 - \zeta - \bar{\zeta})\varepsilon \right)}{(4\varepsilon - 1)(\zeta - 1)\zeta^3\bar{\zeta}(\zeta - \bar{\zeta})}, \quad (73)$$
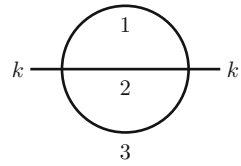
we have $\lim_{x_1 \to \infty} (B_{\times,1}^{2\varepsilon,2\varepsilon})_1 = \lim_{x_3 \to \infty} (B_{\times,1}^{2\varepsilon,2\varepsilon})_3 = 0$ for $\varepsilon > 0$. The boundary term is given by

$$\mathscr{S}_{\times,1} = -\iint_0^\infty \lim_{x_1 \to 0} (B_{\times,1}^{2\varepsilon,2\varepsilon})_1 \, dx_2 dx_3 - \iint_0^\infty \lim_{x_3 \to 0} (B_{\times,1}^{2\varepsilon,2\varepsilon})_3 \, dx_1 dx_2,$$

$$= -\frac{\pi(2\varepsilon - 1) \left((\zeta - 1)^{2\varepsilon}(\bar{\zeta} - 1)^{2\varepsilon} - 1 \right)}{\sin(2\pi\varepsilon)}. \quad (74)$$

It was shown in [26] that

$$I_\times^\varepsilon(\zeta, \bar{\zeta}) = 2\frac{\mathrm{Li}_2(\zeta) - \mathrm{Li}_2(\bar{\zeta}) - \frac{1}{2}\log(\zeta\bar{\zeta})\log\left(\frac{1-\zeta}{1-\bar{\zeta}}\right)}{\zeta - \bar{\zeta}} + O(\varepsilon), \quad (75)$$

Fig. 3 The two-loop sunset
graph. The labels of the graph
give the index of the edge
variable x_i

where $\mathrm{Li}_2\,(z) = -\int_0^z \log(1-u)\,\mathrm{d}\log u$ is the dilogarithm. One easily checks that

$$\mathscr{L}_{\times,1} I^\varepsilon_\times(\zeta, \bar{\zeta}) + \mathscr{S}_{\times,1} = 0 + O(\varepsilon). \tag{76}$$

This analysis shows that the boundary term produced by the Griffiths–Dwork reduction is not guaranteed to be integrable over the positive orthant. The choice of free parameters we have made is equivalent to add a total derivative (or an exact form) to get a convergent integral. Although the algorithm produces the integrand for computing it according (50), this simple case shows that evaluating the inhomogeneous term is not an easy task, which will not carry for most of the remaining cases studied in this paper.

5 Two-loop examples

In this section, we apply the algorithm to the case of dimensionally regulated two loop integrals. We derive the ϵ-deformation of the equal-mass sunset integral in Sect. 5.1.1, the general mass configuration sunset integral in Sect. 5.1.2, the two point kite integral in Sect. 5.2, the ice-cream cone graph in Sect. 5.3, the non-planar triangle-box graph in Sect. 5.4 and the four points planar and non-planar boxes Sect. 5.5. We conclude in Sect. 5.6 with the ice-cream cone graph in analytic regularisation in four dimensions, which arises in the two-loop correction to cosmological correlators of conformally coupled ϕ^4 in de Sitter space [27].

5.1 The two-point two-loop sunset graph

We now turn to the two-loop sunset graph of Fig. 3 and show how to adapt the Griffiths-Dwork reduction used in [12, 46] to the ϵ-dependent integrand from dimensional regularisation. For the two-loop case the differential Eq. (9), setting $t = k^2$, is

$$\Omega^\epsilon_\ominus(t) = \frac{1}{\mathbf{F}_\ominus(t)} \left(\frac{\mathbf{U}^3_\ominus}{\mathbf{F}_\ominus(t)^2} \right)^\epsilon \Omega^{(3)}_0 \tag{77}$$

with the graph polynomials

$$\mathbf{U}_\ominus = x_1 x_2 + x_1 x_3 + x_2 x_3,$$
$$\mathbf{F}_\ominus(t) = -t x_1 x_2 x_3 + (m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3)\mathbf{U}_\ominus. \tag{78}$$

The twisted differential form $\Omega_{\ominus}^{\epsilon}$ is defined on the complement of the sunset elliptic curve $\mathbf{F}_{\ominus}(t) = 0$. We discuss the equal-mass and the general case separately. We apply the algorithm by starting to seek an operator of second order which is enough for the equal-mass case, but for the three-mass case we find that the minimal order is four, in agreement with previous results [81–83].

5.1.1 The equal-mass case

We derive the differential operator satisfied by the two-loop all equal-mass in general dimensions. We start with at $N(\ominus, \epsilon) = 2$ finding

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \Omega_{\ominus}^{\epsilon}(t, \epsilon) = \frac{\Gamma(3 + 2\epsilon)}{\Gamma(1 + 2\epsilon)} \frac{(x_1 x_2 x_3)^2}{\mathbf{F}_{\ominus}(t)^3} \left(\frac{\mathbf{U}_{\ominus}^3}{\mathbf{F}_{\ominus}(t)^2}\right)^{\epsilon} \Omega_0^{(3)}, \tag{79}$$

from which we can extract $P^{(2)} = 2(\epsilon + 1)(2\epsilon + 1)(x_1 x_2 x_3)^2$. Accordingly, we label the unknowns with the upper index so $S^{(k)}$ indicates the k-th reduction. We then perform the Jacobian reduction of $P^{(2)}$ as

$$2(\epsilon + 1)(2\epsilon + 1)(x_1 x_2 x_3)^2 = \sum_{i=1}^{3} \sum_{e \in m_{4,3}} \lambda_e^{(2),i} x^e \partial_{x_i}(\mathbf{F}_{\ominus}(t)), \tag{80}$$

where we have written explicitly the components of $\vec{C}^{(2)}$ as $C_i^{(2)} = \sum_{e \in m_{4,3}} \lambda_e^{(2),i} x^e$. We have to solve the following equation for the coefficients $\lambda_e^{(2),i}$ coupled to the equations generated by

$$\sum_{i=1}^{3} \sum_{e \in m_{4,3}} \lambda_e^{(2),i} x^e \partial_{x_i}(\mathbf{U}_{\ominus}) = \sum_{e \in m_{3,3}} q_e^{(2)} x^e \mathbf{U}_{\ominus}, \tag{81}$$

where $c^{(2)} = \sum_{e \in m_{3,3}} q_e^{(2)} x^e$ is an unknown homogeneous degree 3 polynomial in $\underline{x}$. The unknowns will be fully determined at the end of algorithm. For this case, the differential form given in Eq. (15) reads

$$\beta_{\ominus}^{\epsilon} = \frac{x_2 C_3^{(2)} - x_3 C_2^{(2)}}{\mathbf{F}_{\ominus}(t)^2} \left(\frac{\mathbf{U}_{\ominus}^3}{\mathbf{F}_{\ominus}(t)^2}\right)^{\epsilon} \mathrm{d}x_1 + \frac{x_3 C_1^{(2)} - x_1 C_3^{(2)}}{\mathbf{F}_{\ominus}(t)^2} \left(\frac{\mathbf{U}_{\ominus}^3}{\mathbf{F}_{\ominus}(t)^2}\right)^{\epsilon} \mathrm{d}x_2$$

$$+ \frac{x_1 C_2^{(2)} - x_2 C_1^{(2)}}{\mathbf{F}_{\ominus}(t)^2} \left(\frac{\mathbf{U}_{\ominus}^3}{\mathbf{F}_{\ominus}(t)^2}\right)^{\epsilon} \mathrm{d}x_3, \tag{82}$$

which leads to

$$(2 + 2\epsilon)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \Omega_{\ominus}^{\epsilon}(t, \epsilon) = \left(\frac{\sum_{i=1}^{3} \partial_i C_i^{(2)}}{\mathbf{F}_{\ominus}(t)^2} + 3\epsilon \frac{\sum_{i=1}^{3} C_i^{(2)} \partial_i \log \mathbf{U}_{\ominus}}{\mathbf{F}_{\ominus}(t)^2}\right)$$

$$\times \left(\frac{\mathbf{U}_\ominus^3}{\mathbf{F}_\ominus(t)^2} \right)^\epsilon \Omega_0^{(3)} - d(2\beta_\ominus^\epsilon). \quad (83)$$

Therefore, using Eq.(28), we define

$$M^{(2)} := \frac{\sum_{i=1}^3 \partial^i C_i^{(2)} + 3\epsilon c^{(2)}}{2 + 2\epsilon}, \quad (84)$$

which upon using (81) leads to the reduction of the pole

$$\left(\frac{d}{dt} \right)^2 \Omega_\ominus^\epsilon(t, \epsilon) = \frac{M^{(2)}}{\mathbf{F}_\ominus^2} \left(\frac{\mathbf{U}_\ominus^3}{\mathbf{F}_\ominus(t)^2} \right)^\epsilon \Omega_0^{(3)} + d\beta_\ominus^\epsilon. \quad (85)$$

We now add the first derivative with an unknown rational coefficient $q_1(t, \epsilon)$

$$\left(\frac{d}{dt} \right)^2 \Omega_\ominus^\epsilon(t, \epsilon) + q_1(t, \epsilon) \left(\frac{d}{dt} \right) \Omega_\ominus^\epsilon(t, \epsilon) = \frac{M^{(1)}}{\mathbf{F}_\ominus(t)^2} \left(\frac{\mathbf{U}_\ominus^3}{\mathbf{F}_\ominus(t)^2} \right)^\epsilon \Omega_0^{(3)} + d\beta_\ominus^\epsilon, \quad (86)$$

where $M^{(1)} := M^{(2)} + q_1(t, \epsilon) x_1 x_2 x_3 (1 + 2\epsilon)$. We then reduce the pole a second time by writing

$$M^{(1)} = \sum_{i=1}^3 C_i^{(1)} \partial^i \mathbf{F}_\ominus(t), \quad (87)$$

where $C_i^{(1)}$ are unknown homogeneous degree 1 polynomials. Thus,

$$\left(\frac{d}{dt} \right)^2 \Omega_\ominus^\epsilon(t, \epsilon) + q_1(t, \epsilon) \left(\frac{d}{dt} \right) \Omega_\ominus^\epsilon(t, \epsilon) = \frac{\sum_{i=1}^3 C_i^{(1)} \partial^i \mathbf{F}_\ominus(t)}{\mathbf{F}_\ominus(t)^2} \left(\frac{\mathbf{U}_\ominus^3}{\mathbf{F}_\ominus(t)^2} \right)^\epsilon \Omega_0^{(3)} + d\beta_\ominus^\epsilon. \quad (88)$$

The last step to compute the differential operator is to derive the constant term, which must reduce the poler order of the right-hand-side of this equation. We thus impose

$$\sum_{i=1}^3 C_i^{(1)} \partial^i \mathbf{F}_\ominus(t) + q_0(t, \epsilon) \mathbf{F}_\ominus(t) = 0. \quad (89)$$

Solving all the equations needed for the pole reduction leads to the unique solution for the coefficients q_1 and q_0. The solutions are

$$q_1(t, \epsilon) = \frac{(3t^2 - 10t - 9)\,\epsilon}{(t-9)(t-1)t} + \frac{3t^2 - 20t + 9}{(t-9)(t-1)t},$$

$$q_0(t, \epsilon) = \frac{\epsilon^2(2t+2)}{(t-9)(t-1)t} + \frac{\epsilon(3t-5)}{(t-9)(t-1)t} + \frac{t-3}{(t-9)(t-1)t}, \quad (90)$$

leading to the ϵ-deformed differential operator

$$\mathscr{L}_{\ominus}^{(2),\epsilon,1-mass} = \frac{\mathrm{d}}{\mathrm{d}t}\left(t(t-1)(t-9)\frac{\mathrm{d}}{\mathrm{d}t}\right) + (t-3)$$

$$+ \epsilon\left((3t^2 - 10t - 9)\frac{\mathrm{d}}{\mathrm{d}t} + 3t - 5\right) + \epsilon^2 2(t+1). \quad (91)$$

Collecting the inhomogeneous contributions into the vector

$$\vec{B}_{\ominus}^{\epsilon} = \sum_{a=1}^{2} \frac{\vec{C}_{a}}{(a+2\epsilon)\mathbf{F}_{\ominus}(t)^{a-1}}\, \Omega_{\ominus}^{\epsilon} \quad (92)$$

one can check that the action of this differential operator on $\Omega_{\ominus}^{\epsilon}(t)$ is

$$\mathscr{L}_{\ominus}^{(2),\epsilon,1-mass}\Omega_{\ominus}^{\epsilon}(t) + \vec{\nabla}\cdot\vec{B}_{\ominus}^{\epsilon} = 0 \quad (93)$$

as it should be from the general considerations of Sect. 3.1.3.

From the solutions, we can compute the inhomogeneous term is given by evaluating the integral

$$\mathscr{S}_{\ominus}(t,\epsilon) = -\int_{\Delta_3} \vec{\nabla}\cdot\vec{B}_{\ominus}^{\epsilon}\,. \quad (94)$$

Because the denominator of $B_{\ominus}^{\epsilon}$ has a pole at the coordinate point $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$ one needs to consider the blow-up of the domain of integration Δ_3. This is done by inserting a small $\mathbb{P}^1$ of radius ρ (see Eq. (3.47) of [12])

$$\mathscr{S}_{\ominus}(t,\epsilon) = \lim_{\rho\to 0}\sum_{i=1}^{3}\int_{\partial\tilde{\Delta}_3|_{x_i=0}}\sum_{1\le j\ne i\le 3}(B_{\ominus}^{\epsilon})_j\mathrm{d}x_j\,. \quad (95)$$

A computation identical to the one performed in [12] leads to

$$\mathscr{S}_{\ominus}(t,\epsilon) = -6\frac{\Gamma(1+\epsilon)^2}{\Gamma(1+2\epsilon)}\,. \quad (96)$$

The piece of order ϵ^0 reproduces the differential operator for the two-loop equal-mass sunset in $D=2$ given in [53]. This differential equation in general dimensions can be obtained by applying the results [82, 84] to the all-equal-mass case.

5.1.2 The different mass case

For the non-equal-mass case, the order of the differential equation is 4 with the following ϵ expansion

$$\mathscr{L}_{\ominus}^{\epsilon} = \mathscr{L}_{1}^{(1)}\mathscr{L}_{1}^{(2)}\mathscr{L}_{\ominus}^{3-mass} + \epsilon\mathscr{L}_{4}^{(3)} + \epsilon^2\mathscr{L}_{3}^{(4)} + \epsilon^3\mathscr{L}_{2}^{(5)} + \epsilon^4\mathscr{L}_{1}^{(6)} + \epsilon^5\mathscr{L}_{0}^{(7)}, \quad (97)$$

where $\mathscr{L}_i^{(r)}$ are irreducible differential operator of order i and $\mathscr{L}_\ominus^{3-mass}$ is the differential operator for the three-mass two-loop sunset integral in two dimensions. This differential equation reproduces the one derived in [82, 84]. The ϵ deformation does not change the non-apparent singularities of the differential operator as can be seen from the coefficient of the highest order term

$$\mathscr{L}_\ominus^\epsilon \Big|_{(d/dt)^4} = t^3 \prod_{i=1}^{4} (t - \mu_i^2) \Big(- (2\epsilon + 5) t^2 - 2 \left(m_1^2 + m_2^2 + m_3^2 \right) (1 + 2\epsilon) t$$

$$+ (7 + 6\epsilon) \prod_{i=1}^{4} \mu_i \Big), \quad (98)$$

where $\mu_i = \{m_1 + m_2 + m_3, -m_1 + m_2 + m_3, m_1 - m_2 + m_3, m_1 + m_2 - m_3\}$ are the thresholds. The ϵ deformation is only affecting the apparent singularities, since the ϵ factor in (77) does not change the nature of the singular locus which is still given by the same elliptic curve as in the $\epsilon = 0$ case.

The action of $\mathscr{L}_\ominus^\epsilon$ on the Feynman integral is given by

$$\mathscr{L}_\ominus^\epsilon I_\ominus^\epsilon(\underline{m}, t, \epsilon) = \mathscr{S}_\ominus(\vec{m}, t, \epsilon) \qquad (99)$$

with the source term

$$\mathscr{S}_\ominus(\vec{m}, t, \epsilon) = \frac{c_{23}(t, \epsilon)\Gamma(\epsilon + 1)^2}{(m_2 m_3)^{2\epsilon}\Gamma(1 + 2\epsilon)} + \frac{c_{13}(t, \epsilon)\Gamma(\epsilon + 1)^2}{(m_1 m_3)^{2\epsilon}\Gamma(1 + 2\epsilon)} + \frac{c_{12}(t, \epsilon)\Gamma(\epsilon + 1)^2}{(m_1 m_2)^{2\epsilon}\Gamma(1 + 2\epsilon)}, \qquad (100)$$

where $c_{12}(t, \epsilon)$, $c_{13}(t, \epsilon)$ and $c_{23}(t, \epsilon)$ are polynomials of degree 4 in t and degree 2 in ϵ, respectively. The contribution to the inhomogeneous term arise from each boundary contributions located at $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$. They are given by the two-bouquet Feynman graphs

$$(101)$$

The coefficients match the one derived from the general dimension results of [82, 84].

The results are provided on the SageMath worksheet Sunset-Twoloop-3mass-Epsilon.ipynb. Expanding in powers of ϵ, we have

$$\mathscr{S}_\ominus(\vec{m}, t, \epsilon) = \mathscr{S}_\ominus^0(\vec{m}, t) + \left(c_0^{(1)}(\vec{m}) + \sum_{i=1}^{3} c_i^{(1)}(\vec{m}) \log(m_i) \right) \epsilon + O(\epsilon^2) \quad (102)$$

with the leading term given by

$$\mathscr{S}_\ominus^0(\vec{m}, t) = 60t^4 + 56 \left(m_1^2 + m_2^2 + m_3^2\right) t^3 - 308 \prod_{i=1}^{4} \mu_i. \tag{103}$$

For $\epsilon = 0$ the two-loop sunset integral satisfies the differential equation [12, 83]

$$\mathscr{L}_\ominus^{3-mass} f_\ominus^{(0)}(t) = s_0(\vec{m}, t) + \sum_{i=1}^{3} s_i(\vec{m}, t) \log(m_i^2) \tag{104}$$

with

$$s_0(\vec{m}, t) = \tag{105}$$
$$18t^4 - 24 \left(m_1^2 m_2^2 + m_3^2\right) t^3 - 4 \left(m_1^4 + m_2^4 + m_3^4 + 10(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2)\right) t^2$$
$$+ 8 \left(m_1^2 + m_2^2 + m_3^2\right) \prod_{i=1}^{4} \mu_i t + 2 \prod_{i=1}^{4} \mu_i^2,$$

$$s_1(\vec{m}, t) = \tag{106}$$
$$\left(4m_1^2 - 2m_2^2 - 2m_3^2\right) t^3 + \left(-12m_1^4 + 14m_1^2 m_2^2 + 14m_1^2 m_3^2 + 6m_2^4 - 28m_2^2 m_3^2 + 6m_3^4\right) t^2$$
$$+ \left(12m_1^6 - 22m_1^4 m_2^2 - 22m_1^4 m_3^2 + 16m_1^2 m_2^4 + 16m_1^2 m_3^4 - 6m_2^6 + 6m_2^4 m_3^2 + 6m_2^2 m_3^4\right.$$
$$\left. - 6m_3^6\right) t - 2 \left(2m_1^4 - m_1^2 m_2^2 - m_1^2 m_3^2 - m_2^4 + 2m_2^2 m_3^2 - m_3^4\right) \prod_{i=1}^{4} \mu_i,$$

$$s_2(\vec{m}, t) = \tag{107}$$
$$\left(-2m_1^2 + 4m_2^2 - 2m_3^2\right) t^3 + \left(6m_1^4 + 14m_1^2 m_2^2 - 28m_1^2 m_3^2 - 12m_2^4 + 14m_2^2 m_3^2 + 6m_3^4\right) t^2$$
$$+ \left(-6m_1^6 + 16m_1^4 m_2^2 + 6m_1^4 m_3^2 - 22m_1^2 m_2^4 + 6m_1^2 m_3^4 + 12m_2^6 - 22m_2^4 m_3^2 + 16m_2^2 m_3^4\right.$$
$$\left. - 6m_3^6\right) t + 2 \left(m_1^4 + m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^4 + m_2^2 m_3^2 + m_3^4\right) \prod_{i=1}^{4} \mu_i,$$

with

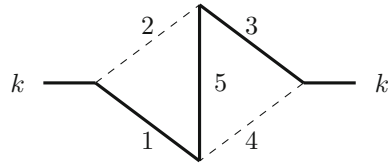$$s_3(\vec{m}, t) = -s_1 - s_2. \tag{108}$$

It can be checked that

$$\mathscr{S}_\ominus^0(\vec{m}, t) = \mathscr{L}_1^{(1)} \mathscr{L}_1^{(2)} \mathscr{L}_\ominus^{3-mass} f_\ominus^{(0)}(t). \tag{109}$$

The logarithmic dependence on the masses arise at the order ϵ and

$$c_0^{(1)}(\vec{m}) = 114t^4 + 168 \left(m_1^2 + m_2^2 + m_3^2\right) t^3 + \left(-552(m_1^4 + m_2^4 + m_3^4)\right.$$

$$+ 1136(m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2)\big)t^2 - 64\left(m_1^2 + m_2^2 + m_3^2\right)\prod_{i=1}^{4}\mu_i t + 14\prod_{i=1}^{4}\mu_i^2,$$

$$(110)$$

and

$$c_1^{(1)}(m_1, m_2, m_3) = \qquad\qquad (111)$$

$$- 80t^4 - \left(88m_1^2 + 68m_2^2 + 68m_3^2\right)t^3 + \left(360m_1^4 - 780m_1^2 m_2^2 - 780m_1^2 m_3^2\right.$$

$$+ 436m_2^4 - 904m_2^2 m_3^2 + 436m_3^4\big)t^2 + \Big(-136m_1^6 + 324m_1^4 m_2^2 + 324m_1^4 m_3^2 - 256m_1^2 m_2^4$$

$$- 256m_1^2 m_3^4 + 68m_2^6 - 68m_2^4 m_3^2 - 68m_2^2 m_3^4 + 68m_3^6\Big)t - 28\prod_{i=1}^{4}\mu_i$$

$$\times \left(2m_1^4 - m_1^2 m_2^2 - m_1^2 m_3^2 - m_2^4 + 2m_2^2 m_3^2 - m_3^4\right)$$

with $c_2^{(1)}(m_1, m_2, m_3) = c_1^{(1)}(m_2, m_1, m_3)$, and $c_3^{(1)}(m_1, m_2, m_3) = c_1^{(1)}(m_3, m_2, m_1)$, and $c_1^{(1)}(\vec{m}) + c_2^{(1)}(\vec{m}) + c_3^{(1)}(\vec{m}) = 4\mathscr{S}_\ominus^0(\vec{m}, t)$.

5.2 The two-point one-mass kite

We consider the two points kite graph of Fig. 4 with three massive and two massless propagators. Its Symanzik polynomials read
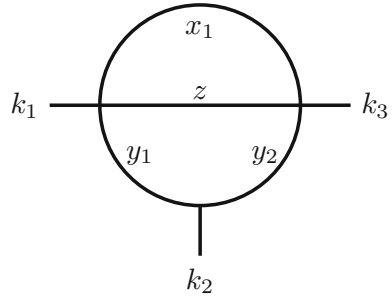
$$\mathbf{U}_\diamondsuit = (x_1 + x_2)(x_3 + x_4) + (x_1 + x_2 + x_3 + x_4)x_5, \qquad (112)$$

$$\mathbf{F}_\diamondsuit(m^2, p^2) = k^2(x_1 x_2 x_3 x_4 \sum_{i=1}^{4} x_i^{-1} + (x_1 + x_4)(x_2 + x_3)x_5) - m^2(x_1 + x_3 + x_5)\mathbf{U}_\diamondsuit.$$

$$(113)$$

Now setting $k^2 = Xm^2$, we have single scale problem and now set $m = 1$ so

$$\Omega_\diamondsuit^\epsilon = \frac{\mathbf{U}_\diamondsuit^{5-3\delta}}{\mathbf{F}_\diamondsuit(1, X)^{5-2\delta}}\left(\frac{\mathbf{U}_\diamondsuit^3}{\mathbf{F}_\diamondsuit^2}\right)^\epsilon \Omega_0^{(5)}. \qquad (114)$$

Fig. 5 The ice cream cone
graph. The massive external
momenta are k_i satisfy
$k_1 + k_2 + k_3 = 0$. We have
labelled the graph with the edges
variables

It was shown in [46, 85, 86] that the two point Kite graph with generic masses satisfies
a first order differential equation in four dimensions. The integrand of the Feynman
integral is the twisted differential form

$$\Omega_{\diamondsuit} = \frac{\mathbf{U}_{\diamondsuit}^{5-3\delta}}{(F_{\diamondsuit}(X))^{5-2\delta}} \left(\frac{\mathbf{U}_{\diamondsuit}^3}{\mathbf{F}_{\diamondsuit}(X)^2} \right)^{\epsilon} \Omega_0^{(5)} \tag{115}$$

and the result of the reduction gives the differential operator

$$\mathscr{L}_{\diamondsuit}^{\epsilon} = X(X-1)\frac{\mathrm{d}}{\mathrm{d}X} + X - 1 + (1+X)\epsilon . \tag{116}$$

5.3 The three-point ice-cream cone graph

In this section, we give the result for the ϵ-deformed differential equation for the ice-
cream cone graph of Fig. 5 generalising the result for $\epsilon = 0$ given in [9, 46]. The
two-loop (one scoop) ice-cream cone differential form in $D = 2 - 2\epsilon$ dimensions is
given by

$$\Omega_{\ominus}^{\epsilon}(t) = \frac{\mathbf{U}_{\ominus}}{\mathbf{F}_{\ominus}^2} \left(\frac{\mathbf{U}_{\ominus}^3}{\mathbf{F}_{\ominus}^2} \right)^{\epsilon} \Omega_0^{(4)}, \tag{117}$$

with

$$\begin{aligned}
\mathbf{U}_{\ominus} &:= (y_1 + y_2)(x_1 + z) + zx_1, \\
\mathbf{V}_{\ominus} &:= k_2^2 y_1 y_2(z + x_1) + zx_1(k_1^2 y_1 + k_3^2 y_2), \\
\mathbf{F}_{\ominus}(t) &:= (\mu_1^2 y_1 + \mu_2^2 y_2^2 + m_1^2 x_1 + m_2^2 z)\mathbf{U}_{\ominus} - t\mathbf{V}_{\ominus}.
\end{aligned} \tag{118}$$

We find the following results (some numerical cases are accessible on the SageMath
worksheet IceCream-Epsilon.ipynb).

- **The equal-kinematics case** $\mu_1 = \mu_2 = m_1 = m_2 = k_1^2 = k_2^2 = k_3^2 = 1$**:** the differential operator has order 3 and reads

$$\mathscr{L}_{\ominus}^{[7],\epsilon} = \sum_{r=0}^{4} \epsilon^r \, \mathscr{L}_{\ominus}^{[7],r} \tag{119}$$

with

$$\mathscr{L}_{\ominus}^{[7],0} = 2t^3 \,(t-1)\,(t-3)\,(t-4) \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^3 + 2t^2\,(t-2)\left(11t^2 - 44t + 15\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2$$
$$+ 2t^2 \left(29t^2 - 116t + 89\right)\frac{\mathrm{d}}{\mathrm{d}t} + 32t^2\,(t-2)\,, \tag{120}$$

$$\mathscr{L}_{\ominus}^{[7],1} = t^3\,(t-1)\,(t-3)\,(t-4)\,(t+1)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^3$$
$$+ t^2\left(10t^4 - 37t^3 - 26t^2 + 95t + 18\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2$$
$$+ t\left(24t^4 + 5t^3 - 242t^2 + 53t + 48\right)\frac{\mathrm{d}}{\mathrm{d}t} + 12t^4 + 64t^3 - 112t^2 - 48t - 12, \tag{121}$$

$$\mathscr{L}_{\ominus}^{[7],2} = t^2\,(t+1)\left(5t^3 - 22t^2 + 5t + 24\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 + t^2\left(28t^3 - 23t^2 - 130t - 71\right)\frac{\mathrm{d}}{\mathrm{d}t}$$
$$+ 26t^4 + 56t^3 - 48t^2 - 64t - 18, \tag{122}$$

$$\mathscr{L}_{\ominus}^{[7],3} = 4t\left(2t^2 - 4t - 3\right)(t+1)^2\,\frac{\mathrm{d}}{\mathrm{d}t} + 6\left(3t^2 - 1\right)(t+1)^2\,, \tag{123}$$

$$\mathscr{L}_{\ominus}^{[7],4} = 4t\,(t+1)^3\,. \tag{124}$$

The ϵ^0 term factorises as

$$L_{\ominus}^{[7],0} = \left(\left(2t^6 - 16t^5 + 38t^4 - 24t^3\right)\frac{\mathrm{d}}{\mathrm{d}t} + 4\left(2t^3 - 12t^2 + 19t - 6\right)t^2\right)$$
$$\circ \left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{5t^3 - 30t^2 + 49t - 18}{(t-4)\,t\,(t-1)\,(t-3)}\right) \circ \left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{2t - 4}{(t-1)\,(t-3)}\right). \tag{125}$$

The rightmost operator is the minimal differential equation for the $\epsilon = 0$ case [46]

$$L_{\ominus}^{[7]} = \frac{\mathrm{d}}{\mathrm{d}t} + \frac{2(t-2)}{(t-1)(t-3)}. \tag{126}$$

- **The equal-mass case** $\mu_1 = \mu_2 = m_1 = m_2 = 1$ **and generic momenta** $k_1^2 \neq k_2^2 \neq k_3^2 \neq 1$**:** the differential operator is of order 3 reads

$$\mathscr{L}_{\ominus}^{[41^3],\epsilon} = \sum_{r=0}^{1} \mathscr{L}_{\ominus,3}^{[41^3],r}\,\epsilon^r + \sum_{r=0}^{2} \mathscr{L}_{\ominus,2-r}^{[41^3],r}\,\epsilon^{2+r}, \tag{127}$$

where $\mathscr{L}_{\ominus,n}^{[41^3],r}$ is of order n. The ϵ^0 term factorises as

$$\mathscr{L}_{\ominus,3}^{[41^3],0} = \mathscr{L}_{a,1}^0 \circ \mathscr{L}_{\ominus}^{[41^3],0}, \tag{128}$$

where $\mathscr{L}_{a,1}^0$ is a first order operator and the second order differential operator $\mathscr{L}_{\ominus}^{[41^3],0}$ matches the mass specialisation of the differential operator derived algorithmically in Section 5.2 of [46] and using Hodge theory in Section 7.3 of [9]. The highest order coefficient factorises as

$$\mathscr{L}_{\ominus}^{[41^3],\epsilon}\bigg|_{(d/dt)^3} = t^3(tk_2^2 - (m_1 + m_2)^2)(tk_2^2 - (m_1 - m_2)^2)c_1(t)c_2(t)c_3^{[41^3]}(t,\epsilon) \tag{129}$$

with

$$\begin{aligned}
c_1(t) &= k_1^2 k_2^2 k_3^2 t^2 + t\Big(m_1^2\left(-k_1^2 k_3^2 - k_2^2 k_3^2 + k_3^4\right) + m_2^2\left(k_1^4 - k_1^2 k_2^2 - k_1^2 k_3^2\right) \\
&\quad + (m_3 + m_4)^2\left(-k_1^2 k_2^2 + k_2^4 - k_2^2 k_3^2\right)\Big) \\
&\quad + m_1^4 k_3^2 + m_1^2 m_2^2\left(-k_1^2 + k_2^2 - k_3^2\right) + m_1^2(m_3 + m_4)^2\left(k_1^2 - k_2^2 - k_3^2\right) \\
&\quad + m_2^4 k_1^2 + m_2^2(m_3 + m_4)^2\left(-k_1^2 - k_2^2 + k_3^2\right) + k_2^2(m_3 + m_4)^4, \tag{130}
\end{aligned}$$

and

$$\begin{aligned}
c_2(t) &= t^2 k_1^2 k_2^2 k_3^2 + t\Big(m_1^2\left(-k_1^2 k_3^2 - k_2^2 k_3^2 + k_3^4\right) + m_2^2\left(k_1^4 - k_1^2 k_2^2 - k_1^2 k_3^2\right) \\
&\quad + (m_3 - m_4)^2\left(-k_1^2 k_2^2 + k_2^4 - k_2^2 k_3^2\right)\Big) \\
&\quad + m_1^4 k_3^2 + m_1^2 m_2^2\left(-k_1^2 + k_2^2 - k_3^2\right) + m_1^2(m_3 - m_4)^2\left(k_1^2 - k_2^2 - k_3^2\right) + m_2^4 k_1^2 \\
&\quad + m_2^2(m_3 - m_4)^2\left(-k_1^2 - k_2^2 + k_3^2\right) + k_2^2(m_3 - m_4)^4, \tag{131}
\end{aligned}$$

and $c_3^{[41^3]}(t,\epsilon)$ a polynomial of degree 5 in t and 1 in ϵ. We recognise the physical thresholds of the ice-cream cone graph given in Section 5.2 of [46] (and given on this page PF-icecream-2loop.ipynb). The ϵ deformation only affects the position of the apparent singularities.

- **The equal-mass case for the scoop $m_1 = m_2 = 1$ and generic masses $\mu_1 \neq \mu_2 \neq 1$ and generic momenta $k_1^2 \neq k_2^2 \neq k_3^2 \neq 1$:** the differential operator has order 3 and has the ϵ expansion

$$\mathscr{L}_{\ominus}^{[21^5],\epsilon} = \sum_{r=0}^{1} \mathscr{L}_{\ominus,3}^{[21^5],r} \epsilon^r + \sum_{r=0}^{2} \mathscr{L}_{\ominus,2-r}^{[21^5],r} \epsilon^{2+r}, \tag{132}$$

where $\mathscr{L}_{\ominus,n}^{[21^5],r}$ is of order n. The ϵ^0 term factorises as

$$\mathscr{L}_{\ominus,3}^{[21^5],0} = \mathscr{L}_{a,1}^0 \circ \mathscr{L}_{\ominus}^{[21^5],0}, \tag{133}$$

where $\mathscr{L}_{a,1}^0$ is a first order operator and the second order differential operator $\mathscr{L}_{\ominus}^{[21^5],0}$ matches the mass specialisation of the differential operator derived algorithmically in Section 5.2 of [46] and using Hodge theory in Section 7.3 of [9]. The leading coefficient factorises as

$$\mathscr{L}_{\ominus,4}^{[21^5],\epsilon}\Big|_{(d/dt)^4} = t^3(tk_2^2 - (m_1+m_2)^2)(tk_2^2 - (m_1-m_2)^2)c_1(t)c_2(t)c_3^{[21^5]}(t,\epsilon). \tag{134}$$

Only the positions of the apparent singularities depend on ϵ. They arise from the roots of the polynomial $c_3^{[21^5]}(t,\epsilon)$ of degree 5 in t and 1 in ϵ.

- **Generic masses non-vanishing** $m_1 \neq m_2 \neq \mu_1 \neq \mu_2$ **and generic momenta** $k_1^2 \neq k_2^2 \neq k_3^2 \neq 1$**:** the differential operator is of order 4 and has the ϵ expansion

$$\mathscr{L}_{\ominus}^{[1^7],\epsilon} = \sum_{r=0}^{1} \mathscr{L}_{\ominus,4}^{[1^7],r} \epsilon^r + \sum_{r=0}^{2} \mathscr{L}_{\ominus,2-r}^{[1^7],r} \epsilon^{2+r}. \tag{135}$$

The ϵ^0 term factorises as

$$\mathscr{L}_{\ominus,4}^{[1^7],0} = \mathscr{L}_{a,1}^0 \circ \mathscr{L}_{b,1}^0 \circ \mathscr{L}_{\ominus}^{[1^7],0}, \tag{136}$$

where $\mathscr{L}_{a,1}^0$ and $\mathscr{L}_{b,1}^0$ are first order operators and the second order differential operator $\mathscr{L}_{\ominus}^{[1^7],0}$ matches the mass specialisation of the differential operator derived algorithmically in Section 5.2 of [46] and using Hodge theory in Section 7.3 of [9]. The leading coefficient factorises as

$$\mathscr{L}_{\ominus,4}^{[1^7],\epsilon}\Big|_{(d/dt)^4} = t^4(tk_2^2 - (m_1+m_2)^2)(tk_2^2 - (m_1-m_2)^2)c_1(t)c_2(t)c_3^{[1^7]}(t,\epsilon). \tag{137}$$

The position of the non-apparent singularities is not affected by the ϵ deformation, but the apparent depend on ϵ. They arise from the roots of the polynomial $c_3^{[1^7]}(t,\epsilon)$ of degree 11 in t and 2 in ϵ.

5.4 The three-point non-planar triangle-box graph

For the non-planar triangle-box graph in Fig. 6, setting the internal mass to $m = 1$ and defining the $2k_1 \cdot k_2 = X$ with $k_1^2 = k_2^2 = 0$ and $k_1 + k_2 + k_3 = 0$. We have

$$\mathbf{U}_{\boxtimes} = (x_1 + x_2)(x_3 + x_4 + x_5 + x_6) + (x_3 + x_4)(x_5 + x_6), \tag{138}$$

Fig. 6 The non-planar triangle-box graph. Dashed lines are massless propagators, and solid lines are massive propagators. The external momenta k_i satisfy $k_1 + k_2 + k_3 = 0$ and $k_1^2 = k_2^2 = 0$
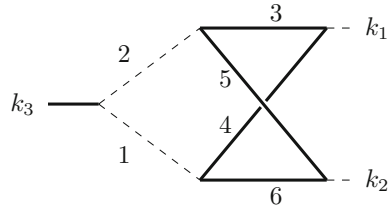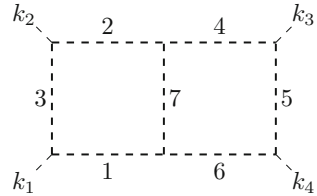
Fig. 7 The planar massless double-box graphs. The massless external momenta k_i satisfy $k_1 + \cdots + k_4 = 0$ and $k_i^2 = 0$. The labels of the graph give the index of the edge variable x_i

$$\mathbf{F}_{\bowtie}(X) = -\left((x_3 + x_4 + x_5 + x_6)x_1 x_2 + x_1 x_3 x_5 + x_2 x_4 x_6\right) X$$
$$+ (x_3 + x_4 + x_5 + x_6)\, \mathbf{U}_{\bowtie}. \quad (139)$$

The integrand of the Feynman integral is the twisted differential form

$$\Omega_{\bowtie} = \frac{\mathbf{U}_{\bowtie}^{6-3\delta}}{(\mathbf{F}_{\bowtie}(X))^{6-2\delta}} \left(\frac{\mathbf{U}_{\bowtie}^3}{\mathbf{F}_{\bowtie}(X)^2}\right)^{\epsilon} \Omega_0^{(6)}. \quad (140)$$

an the differential operator is

$$\mathscr{L}_{\bowtie}^{\epsilon} = (16 + X)X^2 \left(\frac{\mathrm{d}}{\mathrm{d}X}\right)^2 + (2(X+8)\epsilon + 7X + 80)X\frac{\mathrm{d}}{\mathrm{d}X} + 4(X+6)\epsilon + 4(2X+15). \quad (141)$$

5.5 The four-point planar and non-planar double boxes graph

We show how to derive differential equation for the massless box and massless double-box integrals in dimension $D = 4 - 2\epsilon$. Unlike previous cases, these integrals are divergent in four dimensions so the $\epsilon = 0$ integrals are not defined.

5.5.1 The massless planar double-box graph

The graph polynomials associated with the massless double-box graph in Fig. 7 are given by

$$\mathbf{U}_{\square\square} = (x_1 + x_2 + x_3)(x_4 + x_5 + x_6) + (x_1 + \cdots + x_6)x_7, \quad (142)$$

$$\mathbf{F}_{\square\square}(s,t) = t x_3 x_5 x_7 + s\Big((x_1 + x_2 + x_3)x_4 x_6$$

$$+ (x_4 + x_5 + x_6)x_1 x_2 + (x_2 + x_4)(x_1 + x_6)x_7\Big)$$

Fig. 8 The non-planar massless double-box graphs. The massless external momenta k_i satisfy $k_1 + \cdots + k_4 = 0$ and $k_i^2 = 0$. The labels of the graph give the index of the edge variable x_i
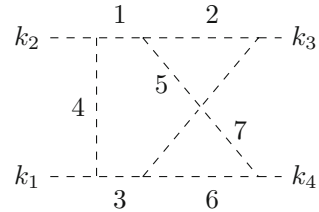
with the twisted differential in $D = 4 - 2\epsilon$ in the projective space $\mathbb{P}^6$

$$\Omega^\epsilon_{\boxempty}(s, t) = \frac{\mathbf{U}_{\boxempty}}{\mathbf{F}_{\boxempty}(s, t)^3} \left(\frac{\mathbf{U}^3_{\boxempty}}{\mathbf{F}^2_{\boxempty}(s, t)} \right)^\epsilon \Omega^{(7)}_0 . \tag{143}$$

We work with the single scale form $\tilde{\Omega}^\epsilon_{\boxempty}(X) = (-s)^{3+2\epsilon} \Omega^\epsilon_{\boxempty}(s, Xs)$ with the result

$$\mathscr{L}^\epsilon_{\boxempty} = (1 + X)X^2 \left(\frac{d}{dX} \right)^2 + (2 + 3X + \epsilon)X \frac{d}{dX} + X - \epsilon - 2\epsilon^2. \tag{144}$$

5.5.2 The massless non-planar double-box graph

The graph polynomials associated with the non-planar massless double-box graph in Fig. 8 are given by

$$\mathbf{U}_{\mathrm{IX}} = (x_1 + x_3 + x_4)(x_2 + x_5 + x_6 + x_7) + (x_2 + x_7)(x_5 + x_6) \tag{145}$$

and

$$\mathbf{F}_{\mathrm{IX}}(s, t) = s \left(x_1 x_3 (x_2 + x_5 + x_6 + x_7) + x_1 x_6 x_7 + x_2 (x_3 x_5 - x_4 x_6) \right)$$
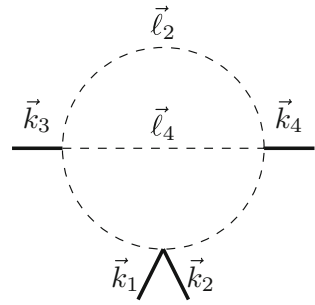$$+ t x_4 (x_5 x_7 - x_2 x_6) \tag{146}$$

we work with the single scale differential form

$$\tilde{\Omega}^\epsilon_{\mathrm{IX}} = (-s)^{3+2\epsilon} \frac{\mathbf{U}_{\mathrm{IX}}}{\mathbf{F}_{\mathrm{IX}}(s, Xs)^3} \left(\frac{\mathbf{U}^3_{\mathrm{IX}}}{\mathbf{F}^2_{\mathrm{IX}}(s, sX)} \right)^\epsilon \Omega^{(7)}_0 . \tag{147}$$

The differential operator we obtain is

$$\mathscr{L}^\epsilon_{\mathrm{IX}} = (1 + X)^2 X^2 \left(\frac{d}{dX} \right)^2 + (1 + X)(1 + 2X)(2 + \epsilon)X \frac{d}{dX} + 2X(X + 1)$$
$$+ (2X(X + 1) - 1)\epsilon - 2\epsilon^2 . \tag{148}$$

5.6 The Witten ice-cream cone diagram

We turn to the Witten ice-cream cone of Fig. 9 in analytic regularisation entering the
two-loop correction to the cosmological correlator between conformally coupled field
analysed in Section 5.3.4 of [27]. The cosmological correlator is the integration over
the energy of the two-loop flat space integral analytically regulated

$$I_\ominus = \int \frac{d^4 L_2 d^4 L_4}{(L_2^2)^{1+\kappa}(L_4^2)^{1+\kappa}((L_4 + L_2 + Q)^2)^{1+\kappa}((L_4 + L_2 + \tilde{Q})^2))^{1+\kappa}} , \quad (149)$$

where $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 = 0$ and $Q = (\omega_3, \vec{k}_3)$ and $\tilde{Q} = (\omega_4, -\vec{k}_4)$. The parametric
representation is given by the integration $I_\ominus = \int_{\Delta_4} \Omega_\ominus^\kappa$ over the domain $\Delta_4 = \{x_i \geq 0, 1 \leq i \leq 4\}$ of the differential form

$$\Omega_\ominus^\kappa = \frac{\pi^4 \Gamma(4\kappa)}{\Gamma(1+\kappa)^4} \frac{1}{\mathbf{U}_\ominus^2} \left(\prod_{i=1}^4 \frac{x_i \mathbf{U}_\ominus}{\mathbf{F}_\ominus} \right)^\kappa \Omega_0^{(4)} , \quad (150)$$

with the graph polynomials

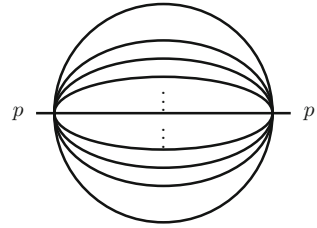$$\mathbf{U}_\ominus = x_1 x_2 + (x_1 + x_2)(x_3 + x_4), \quad (151)$$

and

$$\mathbf{F}_\ominus = x_1 x_2 \left(x_3 Q^2 + x_4 \tilde{Q}^2 \right) + (x_1 + x_2) x_3 x_4 (Q - \tilde{Q})^2. \quad (152)$$

Setting $u = Q^2/(Q - \tilde{Q})^2$ and $v = \tilde{Q}^2/(Q - \tilde{Q})^2$ one finds the following Gröbner
basis of differential operators[3]

$$\mathscr{L}_1 = (1 - u - v) v \left(\frac{\partial}{\partial v} \right)^2 - 2uv \frac{\partial}{\partial u} \frac{\partial}{\partial v} - (3\kappa + 1) u \frac{\partial}{\partial u}$$
$$+ (3\kappa(1 - u - 2v) - v) \frac{\partial}{\partial v} - 8\kappa^2, \quad (153)$$

---

[3] Details are provided in the attached `Mathematica` notebook with this example is accessible at Ice-cream-AdS.nb.

Fig. 10 Multi-loop sunset

$$\mathscr{L}_2 = -u\left(\frac{\partial}{\partial u}\right)^2 + v\left(\frac{\partial}{\partial v}\right)^2 - 3\kappa\frac{\partial}{\partial u} + 3\kappa\frac{\partial}{\partial v}, \tag{154}$$

$$\mathscr{L}_3 = 2u^2\left((u-v)^2 + 1 - 2(u+v)\right)\left(\frac{\partial}{\partial u}\right)^3 \tag{155}$$

$$+ v\left(9\kappa\left((u+v-1)^2 - 2uv\right) - u^2 + 9u^2 - (v-1)^2 - 8u(1+v)\right)\left(\frac{\partial}{\partial v}\right)^2$$

$$+ \left(\kappa^2\left(-29u^2 + u(43v+29) - 18(v-1)^2\right) + \kappa\left(6u(v+3) - 6(v-1)^2\right)\right.$$

$$\left. + u(5u-v+1)\right)\frac{\partial}{\partial u}$$

$$+ \left(\kappa^2\left(27u^2 - 59uv - 54u + 36v^2 - 63v + 27\right)\right.$$

$$- 3\kappa\left(-9u^2 + 6uv + 8u + 3v^2 - 4v + 1\right)$$

$$\left. - v(u+3v-3)\right)\frac{\partial}{\partial v} + 24(\kappa-1)\kappa^2(v-u-1).$$

6 Three- and higher-loop examples

We now turn to the higher-loop cases of Fig. 10. We first discuss the equal-mass case and then a numerical example at three-loop order with all possible mass configurations.

6.1 Minimal differential operator for higher-loop sunset

We now consider the $n-1$-loop sunset integral with $n \geq 4$ in $D = 2 - 2\epsilon$ dimensions. Eq. (9) for this case reads

$$I^\epsilon_{\ominus(n-1)}(\vec{m}, t, \epsilon) = \int_{\Delta_n} \Omega^\epsilon_{\ominus(n-1)}(\vec{m}, t, \epsilon); \quad \Omega^\epsilon_{\ominus(n-1)}(\vec{m}, t, \epsilon) = \frac{\Omega_0^{(n)}}{\mathbf{F}_{\ominus(n-1)}(t)}\left(\frac{\mathbf{U}^n_{\ominus(n-1)}}{\mathbf{F}_{\ominus(n-1)}(t)^{n-1}}\right)^\epsilon \tag{156}$$

with

$$\mathbf{U}_{\ominus(n-1)} = x_1 \cdots x_n \sum_{i=1}^{n} \frac{1}{x_i} \,,$$

$$\mathbf{F}_{\ominus(n-1)}(t) = \mathbf{U}_{n-1} \sum_{i=1}^{n} m_i^2 x_i - t x_1 \cdots x_n \,. \tag{157}$$

Notice that $\mathbf{U}_{\ominus(n-1)}^n / \mathbf{F}_{\ominus(n-1)}(t)^{n-1}$ is a homogeneous rational function of degree 0 in $(x_1, \ldots, x_n)$. As usual the differential form is defined in the complement of the vanishing locus of the denominator in $\mathbb{P}^{n-1} \backslash \{\mathbf{F}_{\ominus(n-1)}(t) = 0\}$. In $D = 2$ dimension ($\epsilon = 0$) we have a rational differential form $\Omega_{\ominus(n-1)}^{\epsilon}(\vec{m}, t, 0)$. The differential operator has been given up to six loops for $\epsilon = 0$, and it is in agreement with the Feynman integral being a (relative) period of a Calabi–Yau manifold of complex dimension $n - 2$ [11, 12, 14, 17, 18, 20, 80, 87].

6.1.1 The equal-mass case

Already in $D = 2$ dimensions, for the sunset integral from three loops on the Griffiths–Dwork algorithm had to be adapted because of the non-isolated singularities of integrand. This was achieved in [46] by using syzygies. The resolution of the linear system in Eq. (29) also takes into account the syzygies when including the ϵ dependent factor.

For the equal-mass case $m_1 = \cdots = m_{l+1} = 1$, we find the sunset Feynman integral satisfies the differential equation

$$\mathscr{L}_{\ominus(l)}^{\epsilon} I_{\ominus}(\{1, \ldots, 1\}, t, \epsilon) = -(l+1)! \frac{\Gamma(1+\epsilon)^l}{\Gamma(1+l\epsilon)} \tag{158}$$

with

$$\mathscr{L}_{\ominus(l)}^{\epsilon} = \sum_{r=0}^{l} \mathscr{L}_{\ominus(l)}^{r} \epsilon^r \,, \tag{159}$$

where the differential operator is $\mathscr{L}_{\ominus(l)}^{r}$ is of order $l - r$. In the all equal mass case, the ϵ dependence takes the particular form $\mathscr{L}_{\ominus(l)}^{\epsilon} = \mathscr{L}_{\ominus(l)}^{0} + O(\epsilon)$ where $\mathscr{L}_{\ominus(l)}^{0}$ is the differential operator of order l for the all-equal-mass sunset integral in $D = 2$ derived in [53] and the ϵ dependent differential operators have an order $l - r$ where r is the power of ϵ. The order ϵ^0 differential operator $\mathscr{L}_{\ominus(l)}^{0}$ reproduces the one derived in [53] up to five-loops using the properties of the Feynman integral in $D = 2$ dimensions (see as well [17, 88–92]). By applying the algorithm presented in Sect. 3.1 we derived the differential equations for the all-equal-mass sunset integral up to 20 loop orders. The explicit results are given on the SageMath worksheet Sunset-1mass-Epsilon.ipynb. We notice that the algorithm presented in this work produces the minimal differential operator and does not need any factorisation of the differential operator, contrary to the procedure presented in [90].

6.1.2 The three-loop generic mass cases

For the three-loop sunset with different masses, we find the following results which are given as well on the `SageMath` worksheet Sunset-Threeloop-Epsilon.ipynb. With the notation of Section 4 of [46]:

- **Case [4]:** The equal-mass case $m_1 = m_2 = m_3 = m_4$ has already been discussed in the previous section. The ϵ^0 operator was derived and analysed in [11, 53, 88]. For $\epsilon \neq 0$, the differential operator reads

$$
\begin{aligned}
\mathcal{L}^{[4],\epsilon}_{\ominus(3)} = & -(t-16)(t-4)t^2 \left(\frac{d}{dt}\right)^3 - 6(t^3 - 15t^2 + 32t) \left(\frac{d}{dt}\right)^2 \\
& - (7t^2 - 68t + 64) \left(\frac{d}{dt}\right) - t + 4 \\
& + \epsilon \left(-6(t-10)t^2 \left(\frac{d}{dt}\right)^2 - 6(3t - 20)t \left(\frac{d}{dt}\right) + 18 - 6t \right) \\
& + \epsilon^2 \left(-(11t^2 - 28t - 64) \left(\frac{d}{dt}\right) - 11t + 14 \right) + \epsilon^3 (-6t - 12) .
\end{aligned}
\tag{160}
$$

- **Case [31]:** For two different masses $m_1 = m_2 = m_3 \neq m_4$ the differential operator is of order 5 and has the following ϵ dependence

$$
\mathcal{L}^{[31],\epsilon}_{\ominus(3)} = \sum_{r=0}^{1} \epsilon^r \mathcal{L}^{[31],r}_5 + \sum_{r=0}^{4} \epsilon^{2+r} \mathcal{L}^{[31],r}_{4-r} ,
\tag{161}
$$

where $\mathcal{L}^{[31],r}_n$ are of order n. The order ϵ^0 operator factorises as

$$
\mathcal{L}^{[31],0}_5 = \mathcal{L}^{[31],0}_{a,1} \circ \mathcal{L}^{[31],\epsilon}_{\ominus(3)} ,
\tag{162}
$$

where $\mathcal{L}^{[31],0}_{a,1}$ is a first order operator and $\mathcal{L}^{[31],0}_{\ominus(3)}$ is the fourth order differential operator of the three-loop sunset integral with mass configuration [31] (see in Section 4.3 of [46]). The coefficient of the highest order term $(d/dt)^5$ is given by

$$
\begin{aligned}
\mathcal{L}^{[31],\epsilon}_{\ominus(3)} \Big|_{(d/dt)^5} = & \, t^3 (t - (m_1 - m_4)^2)(t - (m_1 + m_4)^2)(t - (3m_1 + m_4)^2) \\
& \times (t - (3m_1 - m_4)^2) q^{[31]}(t, \epsilon).
\end{aligned}
\tag{163}
$$

The ϵ dependence appears only in the apparent singularities determined by the polynomial $q^{[31]}(t, \epsilon)$ of degree 3 in t and 1 in ϵ.

- **Case [22]:** For two different masses $m_1 = m_2 \neq m_3 = m_4$ the differential operator has order 6 and the ϵ expansion

$$\mathscr{L}_{\ominus(3)}^{[22],\epsilon} = \sum_{r=0}^{3} \epsilon^r \mathscr{L}_6^{[22],r} + \sum_{r=0}^{5} \epsilon^{4+r} \mathscr{L}_{5-r}^{[22],r} , \tag{164}$$

where $\mathscr{L}_n^{[22],r}$ are operators of order n. The order ϵ^0 operator factorises as

$$\mathscr{L}_6^{[22],0} = \mathscr{L}_{a,1}^{[22],0} \circ \mathscr{L}_{b,1}^{[22],0} \circ \mathscr{L}_{\ominus(3)}^{[22],0} , \tag{165}$$

where $\mathscr{L}_{a,1}^{[22],0}$ and $\mathscr{L}_{b,1}^{[22],0}$ are first order operators. $\mathscr{L}_{\ominus(3)}^{[22],0}$ is the fourth order operator for the three-loop sunset integral with mass configuration [22] given Section 4.3 of [46]. The coefficient of the highest order term $(d/dt)^6$ is given by

$$\mathscr{L}_{\ominus(3)}^{[22],\epsilon}\bigg|_{(d/dt)^6} = t^4(t - (2m_1)^2)(t - (2m_4)^2)(t - (2m_1 + 2m_4)^2)$$
$$\times (t - (2m_1 - 2m_4)^2)\, q^{[22]}(t,\epsilon). \tag{166}$$

The ϵ dependence appears only in the apparent singularities determined by the polynomial $q^{[22]}(t,\epsilon)$ of degree 4 in t and 3 in ϵ.

- **Case [211]:** For three different masses $m_1 = m_2 \neq m_3 \neq m_4$ the differential operator has order 7 and has the ϵ expansion

$$\mathscr{L}_{\ominus(3)}^{[211],\epsilon} = \sum_{r=0}^{8} \epsilon^r \mathscr{L}_7^{[211],r} + \sum_{r=0}^{6} \epsilon^{9+r} \mathscr{L}_{6-r}^{[22],r} , \tag{167}$$

where $\mathscr{L}_n^{[211],r}$ are operators of order n. The order ϵ^0 operator factorises as

$$\mathscr{L}_7^{[211],0} = \mathscr{L}_{a,1}^{[211],0} \circ \mathscr{L}_{b,1}^{[211],0} \circ \mathscr{L}_{c,1}^{[211],0} \circ \mathscr{L}_{\ominus(3)}^{[211],0} , \tag{168}$$

where $\mathscr{L}_{a,1}^{[211],0}$, $\mathscr{L}_{b,1}^{[211],0}$ and $\mathscr{L}_{c,1}^{[211],0}$ are first order operators and $\mathscr{L}_{\ominus(3)}^{[211],0}$ is the fifth order differential operator the three-loop sunset integral with mass configuration [211] given Section 4.3 of [46]. The coefficient of the highest order term $(d/dt)^6$ is given by

$$\mathscr{L}_{\ominus(3)}^{[211],\epsilon}\bigg|_{(d/dt)^6} = t^5 \left(t - (m_1 - m_2)^2\right)\left(t - (m_1 + m_2)^2\right)$$
$$\times \left(t - (m_1 + m_2 - 2m_4)^2\right)\left(t - (m_1 - m_2 + 2m_4)^2\right)\left(t - (-m_1 + m_2 + 2m_4)^2\right)$$
$$\times \left(t - (m_1 + m_2 + 2m_4)^2\right) q^{[211]}(t,\epsilon). \tag{169}$$

The ϵ dependence appears only in the apparent singularities determined by the polynomial $q^{[211]}(t,\epsilon)$ of degree 9 in t and 7 in ϵ.

- **Case [1111]:** For four different masses $m_1 \neq m_2 \neq m_3 \neq m_4$, the differential operator has order 11 and has the ϵ expansion

$$\mathscr{L}_{\ominus(3)}^{[1111],\epsilon} = \sum_{r=0}^{16} \epsilon^r \mathscr{L}_{11}^{[1111],r} + \sum_{r=0}^{11} \epsilon^{16+r} \mathscr{L}_{11-r}^{[1111],\epsilon} . \tag{170}$$

The order ϵ^0 operator factorises as

$$\mathscr{L}_{0,11}^{[1111],0} = \mathscr{L}_{a_1,1}^{[1111],0} \circ \cdots \circ \mathscr{L}_{a_5,1}^{[1111],0} \circ \mathscr{L}_{\ominus(3)}^{[1111],0} , \tag{171}$$

where $\mathscr{L}_{a_1,1}^{[1111],\epsilon}, \ldots, \mathscr{L}_{a_5,1}^{[1111],\epsilon}$ are first order operators and $\mathscr{L}_{\ominus(3)}^{[1111],0}$ is the sixth order differential operator for the three-loop sunset integral with mass configuration [1111] given in [46]. The coefficient of the highest order term $(\mathrm{d}/\mathrm{d}t)^{11}$ is given by

$$\begin{aligned}
\mathscr{L}_{\ominus(3)}^{[1111],\epsilon}\Big|_{(\mathrm{d}/\mathrm{d}t)^{11}} = {}& t^{11} \left(t - (m_1 + m_2 - m_3 - m_4)^2 \right) \\
& \times \left(t - (m_1 - m_2 + m_3 - m_4)^2 \right) \left(t - (m_1 + m_2 + m_3 - m_4)^2 \right) \\
& \times \left(t - (m_1 - m_2 - m_3 + m_4)^2 \right) \left(t - (m_1 + m_2 - m_3 + m_4)^2 \right) \\
& \times \left(t - (m_1 - m_2 + m_3 + m_4)^2 \right) \left(t - (-m_1 + m_2 + m_3 + m_4)^2 \right) \\
& \times \left(t - (m_1 + m_2 + m_3 + m_4)^2 \right) q^{[1111]}(t, \epsilon). \tag{172}
\end{aligned}$$

The ϵ dependence appears only in the apparent singularities determined by the polynomial $q^{[1111]}(t, \epsilon)$ of degree 17 in t and 16 in ϵ.

7 Summary and discussion

In Sect. 3.1, we have presented an algorithm for deriving inhomogeneous differential equations satisfied by Feynman integrals. At each derivative order the procedure consists of solving the linear systems (29) in order to determine the coefficients $c_{a_1,\ldots,a_r}(\underline{z})$ and the inhomogeneous term β_Γ in (10). Our work introduces an explicit dependence on the regulators ϵ or κ, so it is worth discussing their effect on the minimal differential operator and on the singularities of the differential equations.

7.1 Minimal order of the differential operator and number of master integrals

The minimal order differential operator gives direct information about the underlying algebraic-geometry associated with the Feynman integral. The knowledge of this operator is an essential ingredient for identifying if a given Feynman integral is a (relative) period associated with a genus 0, 1 or 2 curve, a Calabi-Yau manifold or other object.

We start by summarising what we have found with the examples studied in this paper:

- For the L-loop sunset integrals, the number of irreducible master integrals is $2^{L+1} - L - 2$ [72, 93, 94].
  - The minimal differential operator for the all equal-mass case has order the number of loops L.
  - For generic mass configuration at one-loop we have an operator of order 1, at two-loop an operator of order 4 and at three-loop an operator has order 11.
  - In integer dimensions $\epsilon = 0$ the order of the minimal order differential operator for generic kinematics can be less than the number of master integrals. One typical example is the one of the Picard–Fuchs operator $\mathscr{L}_{\ominus(l)}$ for the multi-loop sunset in $D = 2$ dimensions which has for minimal order $2^{L+1} - \left(\begin{smallmatrix} L+2 \\ \lfloor \frac{L+2}{2} \rfloor \end{smallmatrix} \right)$ for generic mass configurations [46]. The order of the ϵ-dependent differential operator is the same as the number of irreducible master integrals but its ϵ-independent piece $\mathscr{L}^0_{\ominus(l)}$ is factorisable as $\mathscr{L}^0_{\ominus(l)} = \mathscr{L}_1 \circ \mathscr{L}_{\ominus(l)}$.

- For the two-loop ice-cream cone, the number of irreducible masters is four.
  - In the generic mass and kinematics case, the minimal order of the ϵ-deformed minimal differential operator is four.
  - For special kinematic configurations, the ice-cream cone differential operator is three which is smaller than the number of masters.

We thus see that when $\epsilon \neq 0$ and generic kinematics the order of the differential operator saturates the bound given by the number of irreducible masters. This leads us to formulate the following observation: For general kinematics the minimal (i.e. not factorisable) ϵ-deformed differential operator has the same order as the number of independent master integrals.

7.2 Order reduction

For special kinematics, the minimal order is smaller than the number of master integrals. This reduction of order can be understood by the factorisation of the differential operator

$$\mathscr{L}_{\text{generic}} \overset{\text{restriction}}{\longrightarrow} \hat{\mathscr{L}}_{\text{restricted}} = \mathscr{L}_1 \circ \mathscr{L}_{\text{minimal}} . \tag{173}$$

The reduction of order arises when the integrand has more singularities or more symmetries:

- **More singularities:** In the case of massless internal lines or massless external kinematics, extra singularities arise thus reducing the genus of the singular locus of the integral. This has for consequence a lowering of the order minimal differential operator.
- **More symmetries:** Another situation is when special choices of kinematic configurations the integrand of the Feynman integral produce extra symmetries in the space of projective variables $\underline{x}$. This leads to new relations between independent

(period) integrals, which reduce the number of independent integrals. A typical case is the reduction order of the sunset integral according the mass configurations as given in [11, 17, 18, 46, 90]. These cases are not associated with the appearance of new singularities of the Feynman integral.

In the case of special kinematic configurations that do not lead to new singularities of the integral, the order drop is not detected by either the computation of the Euler characteristic of complement of graph hypersurface in the projective space of the edge variables, nor the computation using the critical points of the Lee–Pomeransky representation [33, 74, 95] nor the Baikov representation [62, 65, 96]. For instance in [97] it was explained how a hidden involution symmetry of the two-loop non-planar double-box allows identifying the hyperelliptic curve of genus 2 when the use of the Baikov representation gave a curve of genus 3. It is shown in [9], using a detailed algebraic-geometrical analysis, that all planar two-loop integrals are either period of rational curves, elliptic curve or genus 2 hyperelliptic curves or minimal order two or four, respectively.

7.3 The regulator dependence

For a differential equation

$$c_N(z)\frac{\mathrm{d}^N f(z)}{\mathrm{d}z^N} + \cdots + c_0(z)f(z) = 0, \tag{174}$$

the roots of $c_N(z)$ are the singularities of the differential equation. A root of $c_N(z)$ where the solution $f(z)$ is regular is called an apparent singularity. A root of $c_N(z)$ where the solution has a singularity is a real singularity (See Section 16.4 of [98] for details). For the case of Feynman integrals, the non-apparent singularities of (174) are the roots of the discriminant of the singular locus of the integrand of Feynman integrals [9].

We have noticed that the dimensional regulator ϵ appears only in the apparent singularities of the differential operator $\mathscr{L}_\Gamma^\epsilon$. This means that the ϵ deformation does not change the position of the real singularities, but it affects the local behaviour (the monodromy) of the solution near the singularity. The physical interpretation of this is that the kinematic singularities (the position of the thresholds and pseudo-thresholds) of a Feynman integral are independent of the space-time dimension. However, the local behaviour of the integral near the thresholds and pseudo-thresholds does change with the space-time dimension. The latter has been used with great success when decomposing amplitudes using the generalised unitarity method [99].

7.4 Outlook

We have presented a generalisation of the Griffiths–Dwork reduction for deriving the differential operator acting on Feynman integrals in dimensional regularisation or analytic regularisation. The algorithm makes a special use of the fact that the twist from the regularisation is the power of a degree zero homogeneous rational function

in the edge variables. The procedure amounts to solving linear systems which is done using `FiniteFlow` routines [51].

We have applied the algorithm to various cases and derived the inhomogeneous partial differential equations satisfied by the integrand of the Feynman integral in parametric representation. In dimensional regularisation we have confirmed that the order the differential operators is smaller or equal to the number of master integrals. The order of the differential operators is lower for the cases of the kinematic invariants where the integrand presents more symmetry (equal-masses, special kinematics,…) or more singularities (massless internal or external states). Something that was already noticed in the case of finite integrals [46] but stays true for the regulated integrals.

One motivation for presenting this algorithm is its application to Feynman integrals in analytical regularisation which arise in the evaluation of the cosmological correlators [25, 27, 100], since the commonly used integration-by-part algorithms need to be adapted to the case of analytic regularisation with several propagators with generalised powers.

We have shown as well how to derive a Gröbner basis of partial differential operators in some multiple scale cases. The differential operators produced by the algorithm of this paper might arise as specialisation of the system of partial differential operators obtained by GKZ approach. The restriction of the GKZ D-module is a difficult open problem, which we leave for further investigations.

Declarations

A The Bessel representation for the sunset graphs

Following the steps in Section 8 of [53] gives the following Bessel integral representation for the multi-loop sunset integrals in $D = 2 - 2\epsilon$ dimensions

$$
I_{\ominus(n-1)}^{\epsilon}(\vec{m}, t, \epsilon) = \frac{2^{(n-1)(1-\epsilon)} t^{\frac{\epsilon}{2}}}{(m_1 \cdots m_{L+1})^{\epsilon} \Gamma(1 + (n-1)\epsilon)}
$$
$$
\times \int_0^{\infty} I_{-\epsilon}(\sqrt{t}x) \prod_{i=1}^{n} K_{-\epsilon}(m_i x) \, x^{1+\epsilon(n-1)} \mathrm{d}x. \quad (175)
$$

This integral representation is valid for $t < (m_1 + \cdots + m_n)^2$. For $x \to 0$ we have

$$\lim_{x \to 0} I_{-\epsilon}(x) \simeq \left(\frac{x}{2}\right)^\epsilon ; \qquad \lim_{x \to 0} K_{-\epsilon}(x) \simeq \left(\frac{x}{2}\right)^\epsilon \qquad (176)$$

and the integral converges as long as $2 + (n-1)\epsilon > 0$ which is the condition $D = 2 - 2\epsilon < 2n/(n-1)$ for the absence of ultraviolet divergences for the $n-1$-loop massive sunset.

Using this representation and applying the creative telescoping algorithm [47–50], we have checked the results obtained the extended Griffiths–Dwork reduction. The creative telescoping algorithm builds an annihilator

$$\mathscr{T}(t, \partial_t; x, \partial_x) = \mathscr{L}(t, \partial_t) + \mathscr{C}(t, \partial_t; x, \partial_x) \qquad (177)$$

of the integrand $f(t, x) := I_{-\epsilon}(\sqrt{t}x) \prod_{i=1}^{L+1} K_{-\epsilon}(m_i x) \, x^{1+\epsilon(n-1)}$ such that

$$\mathscr{T}(t, \partial_t; x, \partial_x) f(t, x) = \mathscr{L}(t, \partial_t) i(t, x) + \mathscr{C}(t, \partial_t; x, \partial_x) f(t, x) = 0 \qquad (178)$$

implying that $\mathscr{L}(t, \partial_t) f(t, x)$ is the operator acting on the integral because the domain of integration is independent of t. On an ordinary laptop, the results for the equal-mass case and up to twenty loops order are obtained in a few second to a few minutes, which is of the same order of time as the generalised Griffiths–Dwork reduction presented in the main text of this work.

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